WHAT IS GOING ON WITH $f(x, y) = \frac{x^3 + 3xy^2 + 4xyz}{x^2 + y^2 + z^2}$

JINCHENG YANG

Abstract. We've seen this type of function a couple of times: a rational function except the origin, and the function is somehow redefined at the origin. In this notes, we introduce how to deal with this type of function systematically.

I believe this has been a headache for many of you.

(1) In quiz 4, question 2,

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is a function that is continuous, with well-defined directional derivatives, differentiable everywhere, but partial derivatives are discontinuous at the origin.

(2) In quiz 5, question 2,

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is a function that is discontinuous at the origin, non-differentiable, but has discontinuous partial derivatives and directional derivatives at the origin.

(3) In midterm 1, question 1-(5),

$$f(x, y) = \begin{cases} \frac{x^3+x^2 y}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous, has partial derivatives, but not differentiable.

(4) In homework 3, question 4,

$$f(x, y, z) = \begin{cases} \frac{xy+5yz^2+6xz^2}{x^2+y^2+z^4} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

is not continuous.

What are the features shared by these functions? Well, away from the origin, they are defined with a possible singularity by dividing something tends to zero ($x^2 + y^2$, $x^4 + y^2$ etc.), and they are claimed to be some number at the origin. Before we start our fun journey of discovering the properties of these functions, we first review some basic concepts of a function.

Date: October 23, 2018.

1. Preliminaries

First, let's recall when a function is called continuous, differentiable, etc.

Definition 1.1. The function $f(x, y)$ is said to be

(1) *continuous* at (0, 0): if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0). \tag{1.1}$$

(2) having *partial derivatives* at (0, 0): if

$$\frac{\partial f}{\partial x}(0,0) := \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} \tag{1.2}$$

exists, and

$$\frac{\partial f}{\partial y}(0,0) := \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} \tag{1.3}$$

exists.

(3) having all *directional derivatives* at (0, 0): if

$$\nabla_{\vec{v}} f(0,0) := \lim_{t\to 0} \frac{f(at,bt)-f(0,0)}{t} \tag{1.4}$$

exists, for any unit vector $\vec{v} = (a, b)$.

(4) *differentiable* at (0, 0): if $|f(x, y) - L(x, y)| = o\left(\sqrt{x^2 + y^2}\right)$ as $(x, y) \to (0, 0)$ for some affine function L (also known as the linear approximation), i.e.

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} = 0. \tag{1.5}$$

(5) *continuously differentiable* (aka. C^1) near $(0, 0)$: if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist and are continuous near $(0, 0)$.

They are not just some isolated properties. Actually they are strongly correlated. Some are stronger in the sense that they may imply other properties.

Proposition 1.2. *The properties of a function f can be compared in the following diagram*[1]*:*

continuously differentiable

⇓

differentiable ⇒ continuous

⇓

having all directional derivatives

⇓

having both partial derivatives

Proof. Try to prove by yourself. Come to my office hour Wed. 12-2 at RLM 11.130 if you are interested in the proof but cannot find it on the textbook or on the internet. □

[1] Each arrow stands for "implies". All the descriptions about f are "local", i.e. at/near $(0, 0)$.

Always bear this proposition in mind. This can save you a lot of time and help you to check your answer in doing multiple choices problems.

Remark 1.3. Each arrow here is not reversible. If a relation cannot be obtained by tracing this diagram, it is not true (e.g. continuity and having partial derivatives cannot imply each other). As a good practice, you can try to come up with counterexamples to show why only the arrows above are true.

Remark 1.4. Sometimes it is useful to use the converse — this proposition immediately implies

NOT continuously differentiable

$\Uparrow$

NON-differentiable $\Leftarrow$ DIScontinuous

$\Uparrow$

NOT having some directional derivatives

$\Uparrow$

NOT having either partial derivatives

by negating each statement and reversing arrows.

Suppose you are given a function. What properties should you check the first? Obviously, we should start from the easiest ones "continuity" and "having partial derivatives". If either one of them does not hold true, you can trace the diagram in Remark 1.4 to see what properties it must also fail. If they are true, you need to go up the tree diagram and check the next property. Alright, now let's look at these functions.

2. Limit & Continuity

2.1. **What is on the bottom?** All these functions contain fractions, and their denominators share some similarities. You see they are always the sum of squares or 4-th powers, while no question gives a function that has x^3 in the bottom. Have you thought about why?

The real reason is: we are lazy. We would like the function to be undefined only at a single point, so it is not too much trouble to redefine it. Imagine if the function has $x^2 + y^3$ at the bottom, then I have to redefine it at every points on the curve $y = x^{2/3}$ where the denominator is zero, to make the question complete, which is apparently way more arduous.

So you will always see even powers in the denominator. You may see $x^{12} + y^{40} + z^{18}$ in the denominator, but you never see $x^3 + y$. In principle, we can come up with denominators like $2x^2 + 4xy + 5y^2$ (which is equal to $2(x + y)^2 + 3y^2$), but we don't want it to be too complicated.

2.2. **Is the function continuous at the origin?** To answer this question, one is supposed to take the limit $(x, y) \to (0, 0)$, which requires us to consider all the possible paths that approaches to the origin. We've seen in class that when dealing with question 2 from quiz 5,

$$f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases},$$

we chose the path $y=kx^2$, and found out the limit doesn't match the definition. How on earth do we think of this particular parabolic path to test? Why can't we just take radial paths, like $y=kx$?

The reason lies in the denominator. In this example, we have x^4+y^2 in the denominator. If we take $y=kx$, then the denominator will become $x^4+k^2x^2$, and you can see when $x\to 0$, k^2x^2 is much larger than x^4, so x^4 is so small comparing to y^2, that we can just drop it, and replace x^4+y^2 by y^2. This strongly reduces the "dimension" of the denominator, and we will lose a lot of information. By the same reason, if we walk along $y=kx^3$, then y^2 will be too small comparing to x^4, and we lose information again. Only when y^2 are comparable to x^4, i.e. $y=kx^2$, both terms in the denominator have a similar size, so they can both contribute to the result, nobody will be left out.

On the one hand, to show the discontinuity, one should try to test paths on which each term in the denominator is not too smaller nor too larger than others. For example in homework 3 question 4, when you have $x^2+y^2+z^4$ in the denominator, a good candidate is the path (at^2,bt^2,ct). On the other hand, to show a function is indeed continuous, we should also use the structure of the denominator. Recall that to show $\frac{xy^2}{x^2+y^2}$ approaches to zero, we simply define $r^2=x^2+y^2$ so that $|x|\le r$ and $|y|\le r$, and the limit follows quickly after squeeze theorem. Thus if we want to show the limit of something divide x^2+y^4 is zero, we should treat y^2 as our new y, say we set $Y=y^2$, then $(x,y)\to(0,0)$ is no different from $(x,Y)=(x,y^2)\to(0,0)$. And now set $r^2=x^2+y^4=x^2+Y^2$, so that $|x|\le r$ and $|y^2|=|Y|\le r$. One can now replacing all the y^2 in the numerator by Y and try squeeze theorem.

Example 2.1.

$$\begin{aligned}\frac{x^4y+x^6+y^2z}{x^4+y^2+z^2} &\xlongequal{\text{Set } X=x^2} \frac{X^2y+X^3+y^2z}{X^2+y^2+z^2}\\ &\xlongequal{\text{Set } r=\sqrt{X^2+y^2+z^2}} \frac{X^2y+X^3+y^2z}{r^2}\\ &=\left(\frac{X}{r}\right)\left(\frac{X}{r}\right)y+\left(\frac{X}{r}\right)\left(\frac{X}{r}\right)X+\left(\frac{y}{r}\right)\left(\frac{y}{r}\right)z\\ &\to 0\end{aligned}$$

as $(x,y,z)\to(0,0,0)$, because $\left|\frac{X}{r}\right|$, $\left|\frac{y}{r}\right|$ and $\left|\frac{z}{r}\right|$ are bounded by 1.

Example 2.2.

$$\begin{aligned}\frac{x^3+xy^2+y^2z^2}{\sqrt{x^4+y^4+z^8}} &\xlongequal{\text{Set } Z=z^2} \frac{x^3+xy^2+y^2Z}{\sqrt{x^4+y^4+Z^4}}\\ &\xlongequal{\text{Set } r=\sqrt[4]{x^4+y^4+Z^4}} \frac{x^3+xy^2+y^2Z}{r^2}\\ &=\left(\frac{x}{r}\right)\left(\frac{x}{r}\right)x+\left(\frac{x}{r}\right)\left(\frac{y}{r}\right)y+\left(\frac{y}{r}\right)\left(\frac{y}{r}\right)Z\\ &\to 0\end{aligned}$$

as $(x,y,z) \to (0,0,0)$, because $\left|\frac{x}{r}\right|$, $\left|\frac{y}{r}\right|$ and $\left|\frac{Z}{r}\right|$ are bounded by 1.

In general, make change of variables so that each coordinate have the same order in the denominator. If the numerator has higher order than the denominator, then the limit must be 0 by squeeze theorem. If the numerator has the same or lower order than the denominator, then probably we will have non-zero limit if we walk along paths on which each coordinate have similar size. You will see that after change of variable, the paths you want to test is just radials in the new variable: setting $X = x^2$ gives $y = kx^2 \to y = kX$.

Remark 2.3. Only use change of variable when finding limits / determining continuity. Do not use change of variable when calculating partial / directional derivatives / verifying (continuously) differentiability. Change of variable will result in things like chain rule which will lose information.

3. Partial & Directional Derivatives

How to find the partial and directional derivatives? This is important so I will use bold font and capital letters and center it and box it.

USE DEFINITION

Why we need to use definition when we have all sorts of derivative tricks? Well, almost all the derivative tricks only work if the function is defined by the same expression in a region. However the function is defined differently at the origin, so we have no choice but to humbly follow the definition of partial and directional derivatives. In particular, you should NOT take the partial derivative of the expression of the function and then take the limit, which only works for continuously differentiable function; you should NOT use gradient to compute the directional derivatives, which only works for differentiable functions.

Well, the good news is, almost always the function will be easier if you use definition to find partial and directional derivatives. Again, take question 2 from quiz 5 for example.

Example 3.1.

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

By definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^2\cdot 0}{x^4+0^2} - 0}{x} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{\frac{0^2\cdot y}{0^4+y^2} - 0}{y} = 0.$$

Directional derivative in unit vector $\vec{v} = (a,b)$ direction is

$$\nabla_{\vec{v}} f(0,0) = \lim_{t\to 0} \frac{f(at,bt)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{(at)^2(bt)}{(at)^4+(bt)^2} - 0}{t} = \begin{cases} \frac{a^2}{b^2} & b \neq 0 \\ 0 & b = 0 \end{cases}.$$

4. Differentiability & Continuous Differentiability

So far we know how to process a function. First check continuity as in Section 2, then check derivatives as in Section 3. If either one fails, then the function is automatically non-differentiable and not continuously differentiable by Remark 1.4. However, if the function is continuous and all the directional derivative exists, it is not enough to conclude differentiability. We still need to use definition

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} = 0. \tag{4.1}$$

Here the linearization is always

$$L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y \tag{4.2}$$

where f_x and f_y are partial derivatives we obtained earlier. So this will again be an old question of finding limit. Using strategy in Section 2 to deal with it.

To see if the function is continuously differentiable, find f_x and f_y away from the origin by simply applying all sorts of derivative tricks you know, because the function is defined by a single expression away from the origin. Now f_x and f_y are again two functions defined by an expression with possible singularity away from the origin, and to be a number you calculated in Section 3 at the origin. So this will again be an old question of determining continuity. Using strategy in Section 2 to deal with it.

5. Summary

To sum up, given a function f that looks like the examples at the beginning,

- Step 1. Check continuity.
  (1) Observe denominator, make change of variables.
  (2) If numerator has higher order than denominator, apply squeeze theorem.
  (3) Otherwise, test radials in the new variable.
- Step 2. Check derivative.
  (1) Partial derivatives, use definition (1.2)-(1.3).
  (2) Directional derivatives, use definition (1.4).
- Step 3. Check differentiability.
  (1) If Step 1 or Step 2 fail, then non-differentiable.
  (2) Otherwise, write linearization by partial derivatives from Step 2(1).
  (3) Use definition (4.1). Refer to Step 1.
- Step 4. Check continuity of partial derivatives.
  (1) If Step 3 fail, then not continuously differentiable.
  (2) Otherwise, find partial derivatives away from origin.
  (3) Check if the limits match partial derivatives from Step 2(1). Refer to Step 1.

Exercise 5.1. What is going on with $f(x,y) = \dfrac{x^3+3xy^2+4xyz}{x^2+y^2+z^2}$?

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712
E-mail address: jcyang@math.utexas.edu
URL: http://ma.utexas.edu/~jcyang