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Partial regularity results for the three-dimensional incompressible Navier–Stokes equation

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Partial regularity results for the three-dimensional incompressible Navier–Stokes equation

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Dedicated to all the feminists for their fight against gender inequality.

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We show a series of works of some regularity results on the incompressible Navier–Stokes equation in dimension three. Using the blow-up method, we estimate the higher regularity in the Lorentz norm for smooth solutions to the Navier–Stokes equation. In particular, we show a second derivative estimate for suitable weak solutions, which improves the currently known regularity. We construct a maximal function associated with geometric objects that we call skewed cylinders, appearing in inviscid flows like the Eulerian cylinders around the Lagrangian trajectories. We also apply the blow-up method to estimate the boundary vorticity, which enables us to achieve an unconditional control of the layer separation of Leray–Hopf solutions from a steady shear flow in a finite periodic channel.

Contents

List of Figures

Chapter 1

Introduction

This thesis is dedicated to the study of the incompressible Navier–Stokes equation in dimension $d = 2, 3$:

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \qquad \operatorname{div} u = 0. \tag{1.1}$$

Systems (1.1) models the motion of an incompressible fluid with viscosity constant $\nu > 0$. In this equation, the unknown quantities u and p represent the velocity field and the pressure field of the fluid. The purpose of this thesis is to present results in [Yan20], [VY21b], and [VY21a], which cover the following three subjects:

(a) Maximal functions associated with skewed cylinders and incompressible flows,

(b) Second derivatives of suitable solutions to the 3D Navier–Stokes equation,

(c) Boundary vorticity of Navier–Stokes and applications to the inviscid limit.

1.1 Navier–Stokes equation

The Navier–Stokes equation (1.1) can be used to model and predict many natural phenomena such as ocean currents, atmospheric flow, hurricanes, or tsunamis.

It also has a wide range of applications in engineering practices, for instance, oil extraction and transportation and aircraft design. This thesis aims to study some regularity results for the weak solutions and their behaviors at the zero-viscosity limit, and partially addresses the following two questions:

1. How regular can our constructed global-in-time weak solutions be?

2. How stable is the system with regards to perturbation in the initial value and the parameter ν as ν degenerates to zero?

To be more precise, let Ω be an open subset of $\mathbb{R}^d$, which for instance may be a bounded domain with smooth boundary, a periodic channel, a half-space, or the entire space $\mathbb{R}^d$. The motion of an incompressible, homogeneous, viscous fluid is prescribed by the following Navier–Stokes equation,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \\ u\big|_{t=0} = u_0 & \text{in } \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \end{cases} . \tag{NSE}$$

In this equation, the unknowns are $u : (0,T) \times \Omega \to \mathbb{R}^3$ and $p : (0,T) \times \Omega \to \mathbb{R}$, which record the velocity and the pressure of a fluid parcel at time $t \in (0,T)$ and at position $x \in \Omega$. $f : (0,T) \times \Omega \to \mathbb{R}^3$ is a given force function. $T > 0$ is the duration of the solution, and $\nu > 0$ is the kinematic viscosity constant. div denotes the divergence operator, and the second equality $\operatorname{div} u = 0$ is known as the incompressibility constraint. $u_0 : \Omega \to \mathbb{R}^3$ is a given initial velocity profile, which satisfies the incompressibility condition (i.e. $\operatorname{div} u_0 = 0$) and has a finite kinetic energy:

$$\int_\Omega |u_0|^2 \, \mathrm{d}x < \infty.$$

With the presence of a nonempty boundary, the no-slip boundary condition $u = 0$ on $\partial\Omega$ is a physical constraint due to viscosity.

In the low viscosity regime, we will compare the solutions to (NSE) to the following Euler equation, which models the motion of an ideal fluid:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = f & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \\ u\big|_{t=0} = u_0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } (0,T) \times \partial\Omega \end{cases} . \tag{EE}$$

Here n is the unit outer normal vector on $\partial\Omega$. Compared with (NSE), the differences are the absence of the dissipation term Δu and the discrepancy in the boundary condition. Due to the vanish of viscosity, the no-slip boundary condition is replaced by $u \cdot n = 0$, called the impermeability (or no-penetration/no-flux) boundary condition.

1.2 Literature Review

The Navier–Stokes equation is named after physicists Claude-Louis Navier and George G. Stokes. During the past two centuries, engineers have been using Navier–Stokes equation to successfully model and predict fluid motion. However, we still have a limited understanding of this equation mathematically. A major open question that has puzzled generations of mathematicians is the following: in dimension three, does the Cauchy problem (NSE) admits a unique smooth solution corresponding to every smooth initial velocity u_0? This is one of the seven Millennium Prize

Problems.

1.2.1 Regularity and Partial Regularity

Although the full regularity of the Navier–Stokes equation is still unknown to us, there have been many partial results in this direction. In this section, we summarize the main developments in the literature on the following three topics: smoothness criteria, partial regularity, and higher regularity.

Smoothness criteria. It is proven that once the solution has enough regularity, it is automatically smooth and unique. For instance, the Ladyženskaya–Prodi–Serrin criteria [KL57, Pro59, Ser62, Ser63, FJR72] states that: if the velocity belongs to any space interpolating $L_t^2 L_x^\infty$ and $L_t^\infty L_x^3$, i.e.

$$u \in L_t^{\frac{2}{\alpha}} L_x^{\frac{3}{1-\alpha}} \text{ for some } 0 < \alpha \leq 1,$$

then it is actually smooth, hence unique. The endpoint case $L_t^\infty L_x^3$ came much later by Escauriaza, Seregin and Šverák [ESŠ03]. When $d = 3$, these spaces require $\frac{1}{6}$ higher spatial integrability (or $\frac{1}{4}$ higher temporal integrability) than the a priori energy bound can provide, which is $L_t^\infty L_x^2 \cap L_t^2 L_x^6$.

Partial regularity. Scheffer began to study the partial regularity for a class of Leray–Hopf solutions, called suitable weak solutions [Sch76, Sch77, Sch78, Sch80]. These solutions exist globally and satisfy the following local energy inequality. Scheffer showed the singular set, at which the solution is unbounded nearby, has a time-space Hausdorff dimension at most $\frac{5}{3}$. This result was later improved by Caffarelli, Kohn, and Nirenberg in [CKN82] (see also [Lin98, Vas07]), where they showed the 1-dimensional Hausdorff measure of the singular set is zero. There is also a series of works on the box-counting (Minkowski) dimension of the singular set [Kuk09, RS09, KP12, KY16, WW17, WY19], as well as Hausdorff and Minkowski dimensions of hyperdissipative and hypodissipative Navier–Stokes equations [KsP02, TY15, Oża20, CLM20, KO22].

Higher regularity. We will investigate the regularity of suitable weak solutions. In the periodic setting, Constantin constructed suitable weak solutions whose second derivatives have space-time integrability $L^{\frac{4}{3}-\varepsilon}$ for any $\varepsilon > 0$, provided the initial vorticities are bounded measures [Con90]. This was improved by Lions to a slightly better space $L^{\frac{4}{3},\infty}$, a Lorentz space which corresponds to weak $L^{\frac{4}{3}}$ space [Lio96]. These estimates are extended to higher derivatives of smooth solutions by one of the authors and Choi using blow-up arguments: $L^{p,\infty}_{\mathrm{loc}}$ space-time boundedness for $(-\Delta)^{\frac{\alpha}{2}}\nabla^n u$, where $p = \frac{4}{n+\alpha+1}$, $n \geq 1$, $0 \leq \alpha < 2$ [Vas10, CV14]. They also constructed suitable weak solutions satisfying these bounds for $n + \alpha < 3$. See also higher regularity of the hypodissipative Navier–Stokes equation [KO22], and spatial-temporal anisotropic regularity results [Sol77, GS91, MS95, FGT⁺07].

1.2.2 Nonuniqueness of Weak Solution.

We now turn to negative results. If an Euler solution is smooth enough, then it should conserve the kinetic energy simply by integration by part since the flow is frictionless. However, for weak solutions, energy does not necessarily conserve, which means the nonexistence of smooth solutions or the nonuniqueness of weak solutions.

Onsager conjecture. Chemist Lars Onsager conjectured that the energy dissipation of an ideal flow depends on the spatial regularity [Ons49]: if a solution to (EE) is $C^0_t C^\alpha_x$ for $\alpha > \frac{1}{3}$ then the solution conserves energy, otherwise there is a possibility for the *anomalous energy dissipation.* What happens in anomalous energy dissipation is that the energy is able to escape to infinite high frequency in the spectrum and disappear in the form of turbulence. The positive part of this conjecture was proven by Constantin, E, and Titi [CET94], and the negative part is solved using convex integration.

Nonuniqueness of the Euler equation. The convex integration is a powerful tool introduced by De Lellis and Szekelyhidi [DLS09] to construct spurious solutions to the Euler equation and model turbulence. Szekelyhidi's construction [Szé11] based on convex integration provides infinitely many solutions to (EE) with the same shear flow as an initial value (see also Bardos, Titi, Wiedemann [BTW12] for a different boundary geometry). This technique was successfully applied by Isett [Ise18] to prove the Onsager theorem (see [BDLSV19] for the construction of admissible solutions, and [LK20] for dissipative solutions). Via a different method, Vishik showed nonuniqueness and instability of the Euler equation [Vis18a, Vis18b] (see also the recent lecture note [ABC+21b]).

Nonuniqueness of the Navier–Stokes equation. The nonuniqueness of mild solutions in $C_t H_x^\varepsilon$ was proven by Buckmaster and Vicol using a convex integration scheme [BV19]. See also the review article [BV21] and the references therein. Very recently, Albritton, Brué, and Colombo [ABC21a] constructed a family of nonunique Leray–Hopf solutions to a forced Navier–Stokes equation, based on the work of Vishik of [Vis18a, Vis18b].

1.2.3 Inviscid limit and Prandtl layer

Denote u^ν to be a solution to (NSE) with viscosity $\nu > 0$ and let $\bar{u}$ be a solution to (EE). The question of inviscid limit asks whether $u^\nu \to \bar{u}$ in appropriate norm if $\nu \to 0^+$ and the initial condition $u^\nu(0) \to \bar{u}(0)$ also in appropriate norm. The difficulty of proving or disproving the inviscid limit stems from the discrepancy in boundary conditions.

Kato's Criteria. In 1984, Kato [Kat84] showed a conditional result ensuring the convergence of $u^\nu \to \bar{u}$ strongly in $L_t^\infty L_x^2$, under the a priori assumption that the energy dissipation rate in a very thin boundary layer Γ_ν of width proportional to ν vanishes:

$$\lim_{\nu \to 0} \int_0^T \int_{\Gamma_\nu} \nu |\nabla u^\nu|^2 \, \mathrm{d}x \, \mathrm{d}t = 0.$$

This condition has been sharpened in a variety of ways (see, for instance [TW97, Wan01, Kel07, Kel08] and Kelliher [Kel17], for a general review), and similar other conditional results have been derived (see for instance [BTW12, CKV15, CEIV17, CV18]). Non-conditional results of strong inviscid limits have been obtained only for real analytic initial data [SC98], vanishing vorticity near the boundary [Mae14, FTZ18], analyticity near the boundary [KVW20], or symmetries [LFMNLT08, MT08].

Prandtl Layer. Inviscid limit postulates that the behavior of a low viscosity fluid should behave similarly to an ideal fluid. However, the d'Alembert paradox claims that the net drag force should be zero for an object moving at constant speed in an ideal fluid, which is counterintuitive. Prandtl [Pra04] claims that in a thin layer near the boundary, the fluid behaves much differently from its behavior in the interior. It is expected that in favorable cases, the Prandtl boundary layer describes the behavior of the solution u^ν up to a distance proportional to $\sqrt{\nu}$. However, even in the simple shear flow case, it is possible to engineer families of initial values $u^\nu(0)$ converging to the shear flow, but associated to Prandtl boundary layers which are either strongly unstable [Gre00], blow up in finite time [E00], or even ill-posed in the Sobolev framework [GVD10, GVN12].

1.3 Main results

Here we present the main results in this thesis. The first is the content of Chapter 4 which states that the second derivative of a suitable solution is locally in the Lorentz space $L^{\frac{4}{3},\frac{4}{3}+\varepsilon}$, which is an improvement from $L^{\frac{4}{3}-\varepsilon}$ of Constantin [Con90, Vas10] and $L^{\frac{4}{3},\infty}$ of Lions [Lio96, CV14].

Theorem 1.1. *Let u be a suitable weak solution to* (NSE) *in* $(0,\infty)\times\mathbb{R}^3$ *with initial data $u_0 \in L^2$. Then for any $q > \frac{4}{3}$, $K \subset\subset (0,\infty)\times\mathbb{R}^3$, there exists a constant $C_{q,K}$ depending on q and K such that the following holds,*

$$\|\nabla^2 u\|_{L^{\frac{4}{3},q}(K)} \leq C_{q,K}\left(\|u_0\|_{L^2}^{\frac{3}{2}} + 1\right).$$

The improvement is achieved based on a blow-up argument, a subquadratic local theorem, and a new maximal function for skewed cylinders.

1. The blow-up argument was an idea used in [Vas10, CV14] to show higher regularity and fractional regularity, which utilizes the scaling of the Navier–Stokes equation. This method first blows up the equation near a spacetime point by the parabolic scaling, and then a local theorem will provide estimates using a local "pivot" quantity, finally the local estimates yield a quantitative global estimate via scaling, usually with the help of a nonlocal operator like the maximal function. The blow-up argument can be adapted to the fluid equation by considering the "skewed cylinder" along with the flow, so the parabolic regularization can apply when the drift is large.

2. The local theorem is an ε-regularity theorem, stating that if a solution is sufficiently small in a certain norm, the solution is smooth in the interior. Partial regularity results were obtained with the quadratic norm of ∇u [CKN82]. We weakened the norm to almost sub-quadratic, using a mixture of subquadratic norm $|\nabla u|^p$ with $p < 2$ and a small percentage of quadratic norm $\delta|\nabla u|^2$.

3. Since the blow-up argument needs to be adapted to the flow, we need to work out a new maximal function with cylinders which mix Euclidean and Lagrangian description of the flow. We show that this new maximal function is of weak-type $(1,1)$ and strong-type (p,p) for $p>1$, same as the classical maximal function. Since $|\nabla u|^2$ is L^1 in space-time and $|\nabla u|^p$ is $L^{\frac{2}{p}}$, we bound $\nabla^2 u$ in a space which interpolates $L^{\frac{4}{3},\infty}$ and $L^{\frac{4}{3}}$, i.e. the Lorentz space $L^{\frac{4}{3},q}$ for $q>\frac{4}{3}$. This part of work is accomplished in Chapter 3.

The second result finds a uniform bound for the layer separation of any weak inviscid limit of Leray–Hopf solution, and it will be presented in detail later in Chapter 5.

Theorem 1.2. *Let $\Omega=\mathbb{T}^{d-1}\times[0,1]$ with $d=2$ or 3. Let u^ν be a family of a Leray–Hopf solutions to* (NSE$_\nu$) *such that $u^\nu(0)$ converges strongly in $L^2(\Omega)$ to Ae_1, and $u^\nu \rightharpoonup u^\infty$ weakly, then*

$$\|u^\infty(T)-Ae_1\|^2_{L^2(\Omega)} \le CA^3T \qquad \text{for} \;\; a.e. \; T>0$$

where C is a universal constant.

This shows a layer separation with an energy proportional to A^3T, while the background flow has a kinetic energy of size A^2. That means for $T \ll 1/A$ the flow is "stable" even though the uniqueness and the inviscid limit are a priori unknown. This is a new notion of pattern predictability and could have potential applications to various other models. It should also be noted that this is an unconditional result that is consistent with the nonunique weak solutions constructed using the convex integration technique. See Proposition 5.1, where we constructed an example of a weak solution with layer separation energy equal to A^3T, based on the example of Szekelyhidi [Szé11].

The proof is based on a new boundary vorticity estimate, which also uses the blow-up argument near the boundary. The reason for choosing boundary vorticity as our breakthrough is two folds: on the one hand, it is known that the problem of inviscid limit is due to the difficulty of generation of vorticity on the boundary (see [MM18]); on the other hand, it is due to a dimensional consideration. We need to find an estimate for the unit viscosity Navier–Stokes equation that can survive under the inviscid scaling. A detailed discussion on the scaling of the equation will be presented in Section 2.3.

Chapter 2

Preliminary

2.1 Notation

Let us begin by introducing notations in this manuscript.

2.1.1 Vectorial notations

For two d dimensional vectors u and v, the tensor product is a d by d matrix $u \otimes v = uv^\top$, with elements

$$(u \otimes v)_{ij} = u_i v_j.$$

For two d by d matrices $A = (a_{ij})$ and $B = (b_{ij})$, we denote the colon product to be

$$A : B = \sum_{ij} a_{ij} b_{ij}.$$

For two vector fields u, v, denote the directional derivative by

$$(u \cdot \nabla)v = (\nabla v)u = \sum_i u_i \partial_i v.$$

For a vector field u, the divergence operator is defined by

$$\operatorname{div} u = \sum_i \partial_i u_i$$

while the curl is denoted by

$$\operatorname{curl} u = \begin{cases} \nabla \times u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^\top & d = 3 \\ \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1 & d = 2 \end{cases}.$$

The Jacobian matrix ∇u has entries

$$(\nabla u)_{ij} = \partial_j u_i$$

For a 2-tensor (i.e. matrix-valued) field A, we denote the divergence to be

$$\operatorname{div} A = \sum_j \partial_j a_j$$

where a_j is the jth column of A. In particular, when u and v have sufficient regularity we have the Leibniz's law

$$\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + v \cdot \nabla u.$$

For a scalar field p, the Hessian matrix is denoted by

$$\nabla^2 p = \nabla(\nabla p),$$

and the Laplacian is denoted by

$$\Delta p = \operatorname{div}(\nabla p).$$

2.1.2 Function spaces

For an open subset $\Omega \subset \mathbb{R}^d$ equipped with the standard Lebesgue measure, for $0 < p \leq \infty$, denote $L^p(\Omega)$ to be the set of measurable functions f for which the following integral is finite:

$$\int_\Omega |f|^p \, \mathrm{d}x < \infty.$$

When $p \geq 1$, $L^p(\Omega)$ is a Banach space equipped with norm

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f|^p \, \mathrm{d}x\right)^{\frac{1}{p}} & p < \infty \\ \operatorname{ess\,sup}_\Omega f & p = \infty \end{cases}$$

For integer $k > 0$, denote $W^{k,p}(\Omega)$ to be the Sobolev space which consist of $L^p(\Omega)$ functions whose distributional partial derivatives up to kth order also belong to $L^p(\Omega)$. When $p \geq 1$, it is a Banach space with norm

$$\|f\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq r} \|\nabla^\alpha f\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} & p < \infty \\ \max_{|\alpha| \leq r} \|\nabla^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases}.$$

We denote $W_0^{k,p}(\Omega)$ to be the closure of compactly supported, infinitely differentiable functions $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ norm. The dual of space of $W_0^{k,p}(\Omega)$ is denoted by $W^{-k,p'}(\Omega)$ where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. We denote $\dot{W}^{k,p}(\Omega)$ to be the homogeneous spaces with seminorm

$$\|f\|_{\dot{W}^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| = r} \|\nabla^\alpha f\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} & p < \infty \\ \max_{|\alpha| = r} \|\nabla^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases}.$$

In particular, if $p = 2$, we denote $H^k(\Omega) = W^{k,2}(\Omega)$, which is a Hilbert space.

$H_0^k(\Omega)$, $H^{-k}(\Omega)$, $\dot{H}^k(\Omega)$ are defined similarly.

2.1.3 Spaces involving time

For vector fields or tensor fields, we employ the same notations to mean the norm is bounded component-wise. For a Banach space X and $T > 0$, $1 \leq p \leq \infty$

$$L^p(0, T; X)$$

contains all strongly measurable functions $f : [0, T] \to X$ whose value has a norm which is L^p integrable in time, i.e.

$$\|f\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|f(t)\|_X^p \, \mathrm{d}t\right)^{\frac{1}{p}} & p < \infty \\ \operatorname{ess\,sup}_{0 \leq t \leq T} \|f(t)\|_X & p = \infty \end{cases}.$$

If $X = L^q(\Omega)$ or $X = W^{k,q}(\Omega)$, we may abbraviate

$$L^p(0, T; L^q(\Omega)) = L_t^p L_x^q((0, T) \times \Omega), \quad L^p(0, T; W^{k,q}(\Omega)) = L_t^p W_x^{k,q}((0, T) \times \Omega).$$

Finally, $C(0, T; X)$ is the set of functions that are continuous from $[0, T]$ to X, and $C_w(0, T; X)$ is the set of functions that are continuous in the weak (star) topology. In other words, for any function $f \in C_w(0, T; X)$ and for any test function $\varphi \in X'$, $\langle f(\cdot), \varphi \rangle$ is a continuous real-valued function defined in $[0, T]$.

2.2 Weak solutions to the Navier–Stokes equation

In this section, we introduce various types of solutions to the Navier–Stokes equation that are weaker than the classical solution.

2.2.1 Distributional solution

We say $u \in L^2\big(0,T;L^2(\Omega)\big)$ is a *distributional solution* if it satisfies (1.1) in distribution, that is, for any smooth incompressible flow $\varphi \in C_c^\infty((0,T)\times\Omega)$ with $\operatorname{div}\varphi = 0$, for any smooth function $\phi \in C_c^\infty((0,T)\times\Omega)$, it holds that

$$\int_0^T \int_\Omega u \cdot (\partial_t \varphi + (u\cdot\nabla)\varphi + \nu\Delta\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0, \qquad \int_0^T \int_\Omega u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.1}$$

Note that for merely distributional solution, the initial value and the boundary conditions are ill-defined, because there is not sufficient continuity to define $u(0)$ or trace $u\big|_{\partial\Omega}$. However, the "normal trace" $u\cdot n$ is still a well-defined quantity in $L^2(0,T;H^{-\frac{1}{2}}(\Omega))$. This quantity will be useful for weak solutions to the Euler equation, in which a non-penetration boundary condition is prescribed instead of non-slip boundary condition.

2.2.2 Leray–Hopf solution

Leray–Hopf weak solution (or Leray–Hopf solution for short), named after mathematicians Jean Leray and Eberhard Hopf, refer to distributional solutions in the energy space

$$u \in C_w(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1(\Omega)) \tag{2.2}$$

satisfying the *energy inequality*: for every $t\in(0,T)$, it holds

$$\frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 + \nu\|\nabla u\|_{L^2((0,t)\times\Omega)}^2 \le \frac{1}{2}\|u(0)\|_{L^2(\Omega)}^2. \tag{EI}$$

It is easy to verify that for any divergence-free test function $\varphi \in C^1([0,T]\times\bar{\Omega})$ vanishing on $\partial\Omega$, it holds that

$$(u(t),\varphi(t)) = (u(0),\varphi(0)) + \int_0^t \int_\Omega (u\otimes u - \nu\nabla u) : (\nabla\varphi) + u\cdot\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t. \tag{2.3}$$

In some literatures, people use the following alternative energy inequality: for every $s,t\in(0,T)$, it holds

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u\|^2_{L^2((s,t)\times\Omega)} \le \frac{1}{2}\|u(s)\|^2_{L^2(\Omega)}. \tag{EI'}$$

In this thesis, (EI') will be used instead of (EI).

2.2.3 Mild solution

Using the notion of the strong solution in the ODE theory, the *mild solution* (Oseen) refers solutions that satisfy (1.1) in the integral sense. To be more precise, given $s>0$ and a divergence-free initial velocity profile $u_0\in H^s(\Omega)$, an H^s mild solution (Fujita-Kato) refers to $u\in C(0,T;H^s(\Omega))\cap L^2\big(0,T;H^{s+1}(\Omega)\big)$ which obeys the integral equation for every $t\in(0,T)$:

$$u(t) = e^{\nu t\Delta}u_0 + \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}_{\mathrm{curl}}(\operatorname{div}(u(s)\otimes u(s)))\,\mathrm{d}s. \tag{2.4}$$

Here $\mathbb{P}_{\mathrm{curl}}$ is the Hodge projection to the divergence-free part, and $e^{t\Delta}$ is the heat kernel, meaning that for any $u_0\in L^2(\Omega)$, $u=e^{t\Delta}u_0$ is the unique solution to the heat equation with Dirichlet boundary condition

$$\begin{cases} \partial_t u = \Delta u & \text{in } (0,T)\times\Omega \\ u\big|_{t=0} = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

We include here two well-known results on mild solutions. See for instance [Ose27, Ler34].

Theorem 2.1 (Local well-posedness)**.** *Navier–Stokes equation is local in time well-posed in the space $C_t^0 H_x^s$ for $s > \frac{d}{2}$. That is, for any initial condition $u_0 \in H^s(\mathbb{R}^d)$, there exists a unique solution up to some terminal time T depending on d, s and u_0.*

Theorem 2.2 (Weak-strong uniqueness)**.** *If u is a strong solution, then u is the unique Leray–Hopf solution.*

2.2.4 Suitable solution

In the work of Caffarelli, Kohn, and Nirenberg [CKN82] on partial regularity, they introduced the notion of *suitable weak solution*, which means a Leray–Hopf weak solution u that satisfies the *local energy inequality* in distribution:

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right) + \nu|\nabla u|^2 \leq \nu\Delta\frac{|u|^2}{2}. \tag{LEI}$$

They showed the $\mathscr{H}^1$ Hausdorff measure of the singular set of a suitable weak solution is zero. Note that (LEI) implies (EI') simply by integration in $(s,t) \times \Omega$. It is also refered as a *dissipation solution* in the work of Duchon and Robert [DR00], and the dissipation term $D(u)$ defined below is a nonnegative distribution encoding the lack of smoothness:

$$D(u) = \nu\Delta\frac{|u|^2}{2} - \partial_t \frac{|u|^2}{2} - \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right) - \nu|\nabla u|^2 \geq 0.$$

By setting $\nu = 0$, this can also be used to define the dissipative solution to the Euler equation.

2.3 Scaling and dimensional analysis

There are two scalings that we bear in mind for the Navier–Stokes equation. One is the parabolic scaling:

$$\mathscr{P}_\varepsilon u(t,x) := \varepsilon u(\varepsilon^2 t, \varepsilon x).$$

If u solves (NSE) with viscosity ν in $(0,T)\times\Omega$, then $\mathscr{P}_\varepsilon u$ solves (NSE) with the same viscosity, in $(0,T/\varepsilon^2)\times\Omega/\varepsilon$. The other is the linear scaling:

$$\mathscr{L}_\epsilon u(t,x) := u(\epsilon t, \epsilon x).$$

If u solves (NSE) with viscosity ν in $(0,T)\times\Omega$, then $\mathscr{L}_\epsilon u$ solves (NSE) with viscosity ν/ϵ, in $(0,T/\epsilon)\times\Omega/\epsilon$. In particular, if u^ν solves (NSE) with viscosity ν, then $\mathscr{L}_\nu u^\nu$ solves (NSE) with unit viscosity 1.

For higher regularity, the correct a priori bound can be anticipated using a dimensional analysis. For instance, the kinetic energy of initial value has the scaling

$$\|\mathscr{P}_\varepsilon \mathscr{L}_\epsilon u\|_{L^2(\Omega)}^2 = \varepsilon^{2-d}\epsilon^{-d}\|u\|_{L^2(\epsilon\varepsilon\Omega)}.$$

The total dissipation of kinetic energy has the scaling

$$(\nu/\epsilon)\|\nabla(\mathscr{P}_\varepsilon \mathscr{L}_\epsilon u)\|_{L^2((0,T)\times\Omega)}^2 = \varepsilon^{2-d}\epsilon^{-d}\nu\|\nabla u\|_{L^2((0,\varepsilon^2\epsilon T)\times\varepsilon\epsilon\Omega)}^2.$$

So they have a matching scaling. We also see this because both $\|u\|_{L_x^2}^2$ and $\nu\|\nabla u\|_{L_{t,x}^2}^2$ have dimension $L^{2+d}T^{-2}$, where L is the length unit and T is the time unit. Similarly, for any n if we want to find m,p,n such that $\nu^m\|\nabla^n u\|_{L_{t,x}^p}^p$ has the same scaling as before, then from

$$\begin{aligned}(\nu/\epsilon)^m\|\nabla^n(\mathscr{P}_\varepsilon\mathscr{L}_\epsilon u)\|_{L^p_{t,x}}^p &= \epsilon^{-m}\varepsilon^p(\varepsilon\epsilon)^{np}(\varepsilon^2\epsilon)^{-1}(\varepsilon\epsilon)^{-d}\nu^m\|\nabla^n u\|_{L^p_{t,x}}^p \\ &= \varepsilon^{(n+1)p-2-d}\epsilon^{np-1-m-d}\nu^m\|\nabla^n u\|_{L^p_{t,x}}^p\end{aligned}$$

we find that m, p need to be

$$\begin{cases} p = \frac{4}{n+1} \\ m = np - 1 = \frac{3n-1}{n+1} = 3 - p. \end{cases}$$

The index $p = \frac{4}{n+1}$ is consistent with the higher regularity results in [CV14]. Therefore, a dimensionally correct nonlinear a priori bound one may expect is

$$\nu^{3-\frac{4}{n+1}}\int_{(0,T)\times\Omega}|\nabla^n u^\nu|^{\frac{4}{n+1}}\,\mathrm{d}x\,\mathrm{d}t \le C\|u_0\|_{L^2(\Omega)}^2.$$

1. The case $n = 1$ is known to be the a priori bound of energy dissipation:

$$\nu\int_{(0,T)\times\Omega}|\nabla u^\nu|^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2.$$

2. The case $n = 0$ estimates the L^4 norm of the velocity by

$$\nu^{-1}\int_{(0,T)\times\Omega}|u^\nu|^4\,\mathrm{d}x\,\mathrm{d}t \le C\|u_0\|_{L^2(\Omega)}^2.$$

   This cannot be derived from the Sobolev embedding or the Ladyzenskaya inequality, even in dimension 2. Note that in dimension 2 we have

$$\int_{\mathbb{R}^2}|u^\nu|^4\,\mathrm{d}x \le C\int_{\mathbb{R}^2}|u^\nu|^2\,\mathrm{d}x\int_{\mathbb{R}^2}|\nabla u^\nu|^2\,\mathrm{d}x.$$

Integration in time yields

$$\int_{(0,T)\times\mathbb{R}^2} |u^\nu|^4 \,\mathrm{d}x \le C \sup_{t\in(0,T)} \left\{ \int_{\mathbb{R}^2} |u^\nu(t)|^2 \,\mathrm{d}x \right\} \int_{(0,T)\times\mathbb{R}^2} |\nabla u^\nu|^2 \,\mathrm{d}x$$
$$\le \nu^{-1} \|u_0\|_{L^2(\Omega)}^4.$$

The a priori bound can hold under a smallness condition $\|u_0\|_{L^2(\Omega)} \le C\nu$.

3. Take $\nu = 1$. One may expect the higher derivate to have an a priori bound

$$\int_{(0,T)\times\Omega} |\nabla^n u|^{\frac{4}{n+1}} \,\mathrm{d}x \,\mathrm{d}t \le C\|u_0\|_{L^2(\Omega)}^2.$$

However, this is unknown at this time. What we will show in Chapter 4 is a weaker version of this for $n \ge 1$. $n < 1$ is, in general, difficult because of the nonlinearity as we subtract the background mollified flow.

4. When $\nu \to 0$, the estimate degenerates as long as $m > 0$, i.e. $n > \frac{1}{3}$. This is also the threshold for energy conservation.

5. If we consider a priori bound of derivatives on the boundary, then

$$\begin{aligned} (\nu/\epsilon)^m \|\nabla^n(\mathscr{P}_\varepsilon \mathscr{L}_\epsilon u)\|_{L^p_{t,x'}}^p &= \epsilon^{-m} \varepsilon^p (\varepsilon\epsilon)^{np} (\varepsilon^2\epsilon)^{-1} (\varepsilon\epsilon)^{-d+1} \nu^m \|\nabla^n u\|_{L^p_{t,x'}}^p \\ &= \varepsilon^{(n+1)p-1-d} \epsilon^{np-m-d} \nu^m \|\nabla^n u\|_{L^p_{t,x'}}^p \end{aligned}$$

we find that m, p need to be

$$\begin{cases} p = \frac{3}{n+1} \\ m = np = \frac{3n}{n+1} = 3 - p. \end{cases}$$

That is, we may expect an a priori nonlinear bound

$$\nu^{3-\frac{3}{n+1}} \int_{(0,T)\times\partial\Omega} |\nabla^n u^\nu|^{\frac{3}{n+1}} \,\mathrm{d}x\,\mathrm{d}t \leq C\|u_0\|^2_{L^2(\Omega)}.$$

In Chapter 5 we show a weaker version of this estimate for $n = 1$. Due to a limitation of bootstraping higher regularity for linear evolutionary Stokes equation on the boundary, results for $n > 1$ are difficult to obtain.

Chapter 3

Maximal Function

3.1 Introduction

This paper is dedicated to the study of the maximal functions adapted to the Lagrangian description of a flow. When studying the motion of a fluid, there are two different but deeply connected descriptions to work with. The Eulerian formulation records physical quantities such as velocity, temperature, and pressure at fixed positions, while the Lagrangian formulation builds the frame of reference following each moving fluid parcel, and describes their motion and trajectories by a flow map. The transport phenomenon is easier to describe in the Lagrangian formulation, while the diffusion usually suits the Eulerian description better. Let us refer to the works of Constantin ([Con01]), Kukavica and Vicol ([CKV16]) for the connection and distinction between these two descriptions in the context of Euler equations.

For both mathematical study and numerical simulation, sometimes it is necessary to switch between two descriptions. For instance, in computational fluid dynamics, *vortex particle method* treats the fluid as a collection of vortex particles, moving along the trajectories generated by the velocity field, which is in turn recovered from vortex particles. It was early developed by Chorin on the study of the two-dimensional Navier–Stokes equations ([Cho73]). The validity and convergence of this vortex method in three and two dimensions are confirmed by Beale and Majda in [BM82a, BM82b]. We refer interested readers to the books of Raviart ([Rav85]), of Cottet and Koumoutsakos ([CK00]) and of Majda and Bertozzi ([MB02]) for detailed bibliographies. Majda and Bertozzi also used the particle-trajectory method to show existence and uniqueness results for Euler equations. Even recently, hybrid numerical schemes are still a very active area ([KSLH13]). To avoid singularities in the computation, a mollification is applied to the velocity field. Therefore, particles are in fact moving along approximated trajectories of this mollified flow defined in Definition 3.1. Mollification is also needed for this Lagrangian formulation when the velocity field does not have enough regularity to define trajectories and flow maps, for instance, weak solutions to Navier–Stokes equations or Euler equations.

Before introducing our new maximal function, let us recall the classical one. For any real-valued or vector-valued function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ with $d \geq 1$, recall the *classical* maximal function $\mathcal{M}f$ is defined as

$$(\mathcal{M}f)(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\,\mathrm{d}y = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(y)|\,\mathrm{d}y. \tag{3.1}$$

Here $B_r(x)$ is a d-dimensional ball with radius r and center x, and $|B_r|$ stands for its d-dimensional Lebesgue measure $\mathcal{L}^d$. Throughout the article, we may use $|\cdot|$ to represent the spatial Lebesgue measure $\mathcal{L}^d$ or the spacetime Lebesgue measure $\mathcal{L}^{d+1}$ depending on the context. The strength of the maximal function is that it captures the nonlocal information of a function, in the meantime keeps the homogeneity: it commutes with rigid motion and scaling, as well as scalar multiplication. $\mathcal{M}$ is a bounded operator on L^p for $1 < p \leq \infty$, and it is also bounded from L^1 to $L^{1,\infty}$, the weak L^1 space. However, if we include a time variable t in an evolutionary problem, for instance, a transport equation, Euclidean balls in the spacetime are no longer the most natural objects to work with. Instead, we may consider using a spacetime cylinder, or "skewed cylinder" transported in the spacetime to be more rigorously defined below. In this paper, we will study such cylinders and construct a maximal function associated with them.

Consider a vector field $u : (S,T) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$u \in L^1_{\mathrm{loc}}(S,T; \dot{W}^{1,p}(\mathbb{R}^d))$$

for some $1 \le p \le \infty$, where $d \ge 1$ and $-\infty \le S < T \le \infty$ are some finite or infinite initial and terminal time fixed through out this article. Fix a spatial function $\varphi \in C_c^\infty(B_1)$ satisfying $\int \varphi \,\mathrm{d}x = 1, \varphi \ge 0$, where $B_1 \subset \mathbb{R}^d$ is a unit ball of dimension d. Define the usual mollifier function $\varphi_\varepsilon := \varepsilon^{-d}\varphi(\cdot/\varepsilon) \in C_c^\infty(B_\varepsilon)$. We denote a universal constant by C if it depends only on φ and d. Its value may change from line to line. We define the *spatially* mollified velocity $u_\varepsilon : (S,T) \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$u_\varepsilon(t,x) := [u(t,\cdot) * \varphi_\varepsilon](x) = \int_{\mathbb{R}^d} u(t, x-y)\varphi_\varepsilon(y)\,\mathrm{d}y.$$

By convolution, $u_\varepsilon \in L^1_{\mathrm{loc}}(S,T; C^1(\mathbb{R}^d))$. Let us now give the definition for the mollified flow and the skewed cylinders.

Definition 3.1 (Mollified Flow, Skewed Cylinders)**.** For some fixed $\varepsilon > 0$ and $(t,x) \in (S,T) \times \mathbb{R}^d$, define the **mollified flow** $X_\varepsilon(t,x;\cdot)$ to be the unique solution to the following initial value problem

$$\begin{cases} \dot{X}_\varepsilon(t,x;s) = u_\varepsilon(s, X_\varepsilon(t,x;s)) \\ X_\varepsilon(t,x;t) = x \end{cases} \qquad s \in (S,T)$$

where the dot means to take derivative in the last argument s. Moreover, if $S + \varepsilon^2 < t < T - \varepsilon^2$, define the **skewed parabolic** [1] **cylinder** with center (t,x) and radius

[1] Parabolic scaling—ε^2 in time versus ε in space—will not be indispensable in this paper. We only employ it because of its applications to the Navier–Stokes equations, but all the results can

ε by

$$Q_\varepsilon(t,x) := \left\{(s,y) : |s-t| < \varepsilon^2, |y - X_\varepsilon(t,x;s)| < \varepsilon\right\}.$$

Heuristically speaking, skewed cylinders defined in Definition 3.1 are objects appearing in the Lagrangian formulation but written in Eulerian coordinates. Indeed, they are following the mollified flow and capturing particles that are close to the center trajectories. Similar to the difficulty of bridging these two formulations, the difficulty of working with these cylinders comes from the lack of control on the distortion. Without a uniform control on the velocity field, these skewed cylinders following different flows may include nonuniform geometric properties. Despite this technical challenge, the maximal function will provide us a tool for overcoming this conceptual difficulty. Instead of taking the average in balls, now we construct a new maximal function that takes the average in the skewed cylinders that are "admissible".

Definition 3.2 (Admissibility, Maximal Function)**.** Given $\varepsilon > 0$, $x \in \mathbb{R}^d$, $t \in (S+\varepsilon^2, T-\varepsilon^2)$, we define a skewed cylinder $Q_\varepsilon(t,x)$ by Definition 3.1. For $\eta > 0$, we say $Q_\varepsilon(t,x)$ is η-**admissible** if

$$\varepsilon^2 \fint_{Q_\varepsilon(t,x)} \mathcal{M}(\nabla u(s))(y)\,\mathrm{d}y\,\mathrm{d}s = \frac{1}{\varepsilon^d |Q_1|} \int_{Q_\varepsilon(t,x)} \mathcal{M}(\nabla u)\,\mathrm{d}y\,\mathrm{d}s < \eta. \tag{3.2}$$

Here $\mathcal{M}$ is the spatial-only maximal function defined in (3.1), and with a slight abuse of notation, we also use $|Q_1|$ to represent the $(d+1)$-dimensional space-time Lebesgue measure $\mathcal{L}^{d+1}$ of a cylinder with radius 1. For any locally integrable function $f \in L^1_{\mathrm{loc}}((S,T)\times\mathbb{R}^d)$, for every $(t,x) \in (S,T)\times\mathbb{R}^d$ we define a new maximal

be generalized to other time-space scaling.

function $\mathcal{M}_{\mathcal{Q}}$ by

$$\mathcal{M}_{\mathcal{Q}}(f)(t,x) := \sup_{\varepsilon>0} \left\{ \fint_{Q_\varepsilon(t,x)} |f(s,y)| \,\mathrm{d}y\,\mathrm{d}s : Q_\varepsilon(t,x) \text{ is } \eta\text{-admissible} \right\}.$$

Note that in the sup we actually need $\varepsilon^2 < \min\{t-S, T-t\}$ to define $Q_\varepsilon(t,x)$, and we will justify in Section 3.3 that admissible choices of ε exist for almost every (t,x), so that $\mathcal{M}_{\mathcal{Q}}$ is well-defined.

The main result of this paper is the following.

Theorem 3.3. *Let $\eta < \eta_0$ for some small universal constant $\eta_0 > 0$. If u is divergence-free, and $\mathcal{M}(\nabla u) \in L^p((S,T)\times\mathbb{R}^d)$ for some $1 \le p \le \infty$ [2], then $\mathcal{M}_{\mathcal{Q}}$ associated with η-admissible cylinders generated by u satisfies the following.*

(1) *$\mathcal{M}_{\mathcal{Q}}$ is of strong type (∞,∞), i.e. for $f \in L^\infty((S,T)\times\mathbb{R}^d)$, it holds that*

$$\|\mathcal{M}_{\mathcal{Q}} f\|_{L^\infty((S,T)\times\mathbb{R}^d)} \le \|f\|_{L^\infty((S,T)\times\mathbb{R}^d)}.$$

(2) *$\mathcal{M}_{\mathcal{Q}}$ is of weak type $(1,1)$, i.e. for $f \in L^1((S,T)\times\mathbb{R}^d)$, $\lambda > 0$, the Lebesgue measure of the superlevel set satisfies*

$$\mathcal{L}^{d+1}\left(\left\{(t,x) \in (S,T)\times\mathbb{R}^d : (\mathcal{M}_{\mathcal{Q}} f)(t,x) > \lambda\right\}\right) \le \frac{C_1}{\lambda}\|f\|_{L^1((S,T)\times\mathbb{R}^d)}.$$

(3) *$\mathcal{M}_{\mathcal{Q}}$ is of strong type (q,q) for any $1 < q < \infty$, i.e. for $f \in L^q((S,T)\times\mathbb{R}^d)$, it holds that*

$$\|\mathcal{M}_{\mathcal{Q}} f\|_{L^q((S,T)\times\mathbb{R}^d)} \le C_q \|f\|_{L^q((S,T)\times\mathbb{R}^d)}.$$

Let us now explain why we are interested in these skewed cylinders and the

[2] In the case $1 < p \le \infty$, since $\mathcal{M}$ is a bounded operator on $L^p(\mathbb{R}^d)$, this condition is equivalent to $\nabla u \in L^p((S,T)\times\mathbb{R}^d)$.

maximal function related to them. In many scaling-invariant partial differential equations, it is a common technique to zoom in near a point and conduct a local analysis in its neighborhood, and use this obtained local information to deduce global results. This form of argument usually consists of two parts: one is a local theorem, which handles the rescaled problem near a point, and the second is a local-to-global step, which contributes to some global information. For instance, the 3D Navier–Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla P = \Delta u, \qquad \operatorname{div} u = 0 \tag{3.3}$$

are scaling invariant. In particular, u_λ and P_λ defined by

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \qquad P_\lambda(t,x) = \lambda^2 P(\lambda^2 t, \lambda x)$$

are also solutions to (3.3). In [CKN82], Caffarelli, Kohn and Nirenberg investigated the partial regularity of suitable weak solutions to the Navier–Stokes equations by zooming into a so-called *parabolic cylinder*, where *parabolic* refers to the fact that the spatial scale is λ while the temporal scale is λ^2. They showed that if a suitable solution u satisfies

$$\limsup_{r\to 0} \frac{1}{r} \int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \int_{B_r(x)} |\nabla u(s,y)|^2 \, \mathrm{d}y \, \mathrm{d}s \le \eta$$

for some fixed small η, then u is regular at (t,x). From this local theorem, they used a covering argument to conclude a global result, that the parabolic measure $\mathscr{P}^1$ of the singular set is zero. This was an improvement from Scheffer's result ([Sch80]) which stated the singular set has at most Hausdorff dimension $\frac{5}{3}$. The reason for this improvement is that $\iint |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t$ has a stronger scaling than other quantities, which is $\iint |\nabla u_\lambda|^2 \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{\lambda} \iint |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t$.

Quantitative global results can also follow from this kind of scaling arguments. Choi and Vasseur ([Vas10, CV14]) estimated higher derivatives, by locally controlling higher derivatives using the De Giorgi technique applied to quantities with the same strong scaling as $\iint |\nabla u|^2$. In particular, one must avoid using $\iint |u|^{\frac{10}{3}}$, which has a weaker scaling. However, without controlling the flux, the parabolic regularization cannot overcome the nonlinearity. A natural idea would be to utilize the Galilean invariance of Navier–Stokes equations and work in a neighborhood following the flow. Instead of working on parabolic cylinders, they worked on skewed parabolic cylinders as we defined above.

The advantage of using such skewed cylinders is that, by taking out the mean velocity, one can use velocity gradient to control the velocity in the local study. The maximal function associated with these skewed cylinders then will help us better bridge the local study to global results.

Let us mention that a similar construction also appears in the recent development of convex integration for Euler equations by Isett ([Ise17, Ise18]) and the subsequent work of Isett and Oh ([IO16]), where the authors call the mollified flow *coarse scale flow* and skewed cylinders *u_ε-adapted Eulerian cylinders*. The difference from the previous definition is that their apertures of mollification, radii of cylinder bases, and lengths of time spans are chosen differently from here. The purpose is however the same, which is to kill the mean velocity, and to obtain estimates that are dimensionally correct.

Note that Theorem 3.3 has already been used in [VY21b, Corollary 1] to show the following result.

Theorem 3.4. *Let u be a suitable weak solution to the 3D Navier–Stokes equations* (3.3) *with initial data $u|_{t=0} = u_0 \in L^2(\mathbb{R}^3)$. For any $q > \frac{4}{3}$, $K \subset\subset (0,T) \times \mathbb{R}^3$, there* *exists a constant $C_{q,K}$ depending on q and K such that the following holds,*

$$\|\nabla^2 u\|_{L^{\frac{4}{3},q}(K)} \leq C_{q,K}\left(\|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + 1\right).$$

This is an improvement of [Con90] where the result was shown for L^q with $q < \frac{4}{3}$, and of [Lio96] where it was shown for $L^{\frac{4}{3},\infty}$.

In this paper, we provide a first application of Theorem 3.3 to give an alternative proof for the results of Choi and Vasseur in [CV14], as an example of using the maximal function to obtain global results from local estimates.

Theorem 3.5. *Let (u, P) be a smooth solution to (3.3) in $(0,T)$ with initial data $u_0 \in L^2$, let $d \geq 1$, $\alpha \in [0,2)$, denote $f = |(-\Delta)^{\frac{\alpha}{2}}\nabla^d u|$, $p = \frac{4}{d+1+\alpha}$. We have*

$$\left\|f\mathbf{1}_{\{f^p > C_{d,\alpha}t^{-2}\}}\right\|^p_{L^{p,\infty}((0,T)\times\mathbb{R}^3)} \leq C\|u_0\|^2_{L^2(\mathbb{R}^3)}.$$

This paper is organized as follows. Bounds on the maximal function rely on a Vitali-type covering lemma, which is introduced in Section 3.2, where we define admissible cylinders and prove the covering lemma for them. We use this covering lemma to show some properties of the maximal function in Section 3.3.

3.2 Covering Lemma

In this section, we derive some basic properties of the mollified flows and admissible cylinders, then use them to prove the covering lemma.

3.2.1 Preliminaries

We first note the following easy pointwise estimate on the mollified velocity gradient.

Lemma 3.6 (Pointwise Estimate on ∇u_ε)**.** *For $(t,x) \in (S,T) \times \mathbb{R}^d$, $y \in \mathbb{R}^d$, and*

$\varepsilon, r > 0$, *we have*

$$|\nabla u_\varepsilon(t,x)| \le C\varepsilon^{-d}\|\nabla u(t)\|_{L^1(B_\varepsilon(x))}, \tag{3.4}$$

$$|\nabla u_\varepsilon(t,x)| \le C\varepsilon^{-d}\left(\frac{|y-x|}{\varepsilon}+2\right)^d \|\mathcal{M}(\nabla u(t))\|_{L^1(B_\varepsilon(y))}, \tag{3.5}$$

$$|\nabla u_\varepsilon(t,x)| \le C\varepsilon^{-d}\left(\frac{|y-x|+r+\varepsilon}{r}\right)^d \|\mathcal{M}(\nabla u(t))\|_{L^1(B_r(y))}. \tag{3.6}$$

Proof. The first estimate follows easily from the scaling that

$$\nabla u_\varepsilon(t,x) = \int_{\mathbb{R}^d} \nabla u(t,x-y)\varphi_\varepsilon(y)\,\mathrm{d}y \le \|\nabla u(t)\|_{L^1(B_\varepsilon(x))}\|\varphi_\varepsilon\|_{L^\infty}.$$

This indicates that by controlling the average of ∇u in a small ball $B_\varepsilon(x)$, we can control the size of mollified gradient at the center x. To control the mollified gradient from elsewhere, we need a maximal function to gather nonlocal information. For any $x' \in B_\varepsilon(x)$, $y' \in B_r(y)$, we have $|y'-x'| \le |y-x|+r+\varepsilon =: Kr$, so $B_\varepsilon(x) \subset B_{Kr}(y')$ and the integral of ∇u can be bounded by

$$\begin{aligned}\int_{B_\varepsilon(x)} |\nabla u(t,z)|\,\mathrm{d}z \le \int_{B_{Kr}(y')} |\nabla u(t,z)|\,\mathrm{d}z &= \left|B_{Kr}(y')\right| \fint_{B_{Kr}(y')} |\nabla u(t,z)|\,\mathrm{d}z \\ &\le K^d|B_r|\mathcal{M}(\nabla u(t))(y').\end{aligned}$$

Since the above holds for any $y' \in B_r(y)$, by taking the average of right-hand side in $B_r(y)$ we have

$$\begin{aligned}\|\nabla u(t)\|_{L^1(B_\varepsilon(x))} &\le K^d|B_r|\fint_{B_r(y)} \mathcal{M}(\nabla u(t))(y')\,\mathrm{d}y' \\ &= K^d\|\mathcal{M}(\nabla u(t))\|_{L^1(B_r(y))}.\end{aligned} \tag{3.7}$$

This bound and estimate (3.4) yield the third estimate, and the second estimate is a special case of the third when $r = \varepsilon$. □

As can be seen here, $\mathcal{M}(\nabla u)$ controls how mollified velocities alter in space. This observation motivates us to introduce the notion of admissibility in Definition 3.2. Let us provide a heuristic explanation for the choice of homogeneity in (3.2). Consider two skewed cylinders, both with radius of order ε, starting at the same time with distance also of order ε. If ∇u is of order $\varepsilon^{-2}\eta$, then their velocities roughly differ by $\varepsilon^{-1}\eta$, so in a time span of length ε^2, they at most diverge $\varepsilon\eta$ further away, so their distance will remain of order ε. This ensures cylinders do not deviate relatively too far away, and will be crucial in the covering lemma.

Remark 3.7. For $1 < p < \infty$, (3.2) is weaker than the L^p analogue

$$\varepsilon^2 \left(\fint_{Q_\varepsilon(t,x)} \mathcal{M}(|\nabla u|^p) \,\mathrm{d}y \,\mathrm{d}s \right)^{\frac{1}{p}} < \eta.$$

This is because Jensen's inequality implies that

$$\left(\fint_{Q_\varepsilon(t,x)} \mathcal{M}(\nabla u) \,\mathrm{d}y \,\mathrm{d}s \right)^p \le \fint_{Q_\varepsilon(t,x)} [\mathcal{M}(\nabla u)]^p \,\mathrm{d}y \,\mathrm{d}s$$

and

$$[\mathcal{M}(\nabla u)]^p(x) = \sup_{r>0} \left(\fint_{B_r(x)} |\nabla u| \,\mathrm{d}y \right)^p \le \sup_{r>0} \fint_{B_r(x)} |\nabla u|^p \,\mathrm{d}y = [\mathcal{M}(|\nabla u|^p)](x).$$

Next, let us discuss the trajectories of the mollified flow that pass through an admissible cylinder.

Lemma 3.8. *There exists a universal constant $\eta_1 > 0$ such that the following is true. Given $\varepsilon > 0$, $t_0 \in (S + \varepsilon^2, T - \varepsilon^2)$ and $x_0 \in \mathbb{R}^d$, suppose $Q_\varepsilon(t_0, x_0)$ is η-admissible as defined in Definition 3.2 with $\eta < \eta_1$. For any $(t_*, x_*) \in Q_\varepsilon(t_0, x_0)$,* *we have*

$$|X_\varepsilon(t_*, x_*; t) - X_\varepsilon(t_0, x_0; t)| \leq 2\varepsilon \tag{3.8}$$

at any given time $t \in (t_0 - \varepsilon^2, t_0 + \varepsilon^2)$.

Proof. To ease the notation, we denote

$$X^*(t) := X_\varepsilon(t_*, x_*; t), \quad X^0(t) := X_\varepsilon(t_0, x_0; t), \quad \Delta X(t) := X^*(t) - X^0(t),$$

thus we need to show $|\Delta X(t)| \leq 2\varepsilon$. We argue by contradiction and suppose $|\Delta X(s_*)| > 2\varepsilon$ at some $s_* \in (S^\alpha, T^\alpha)$. Without loss of generality, suppose $s_* > t_*$. Note that

$$|\Delta X(t_*)| = |X^*(t_*) - X^0(t_*)| = |x_* - X_\varepsilon(t_0, x_0; t_*)| < \varepsilon < 2\varepsilon$$

because $(t_*, x_*) \in Q_\varepsilon(t_0, x_0)$. Since ΔX is absolute continuous, there must exist an $r_* \in (t_*, s_*)$ such that

$$|\Delta X(t)| \leq 2\varepsilon \text{ for any } t \in [t_*, r_*], \qquad |\Delta X(r_*)| = 2\varepsilon. \tag{3.9}$$

For almost every $t \in [t_*, r_*]$, the growth rate of the difference ΔX can be bounded by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta X(t)| \leq \left|\frac{\mathrm{d}}{\mathrm{d}t}\Delta X(s)\right| &= \left|\dot{X}^*(t) - \dot{X}^0(t)\right| \\ &= \left|u_\varepsilon(t, X^*(t)) - u_\varepsilon(t, X^0(t))\right| \\ &\leq |\nabla u_\varepsilon(t, \xi_t)||\Delta X(t)|\end{aligned}$$

for some ξ_t between $X^*(t)$ and $X^0(t)$. We can bound the gradient term by

$$\begin{aligned}
|\nabla u_\varepsilon(t,\xi_t)| &\le C\varepsilon^{-d}\left(\frac{|\xi_t - X^0(t)|}{\varepsilon}+2\right)^d \|\mathcal{M}(\nabla u(t))\|_{L^1(B_\varepsilon(X^0(t)))} \\
&\le C\varepsilon^{-d}\left(\frac{|\Delta X(t)|}{\varepsilon}+2\right)^d \|\mathcal{M}(\nabla u(t))\|_{L^1(B_\varepsilon(X^0(t))}
\end{aligned}$$

using (3.5) for $x = \xi_t$ and $y = X^0(t)$. By (3.9), $|\Delta X(t)| \le 2\varepsilon$ for any $t \in [t_*, r_*]$, so in the above coefficient $C(\frac{|\Delta X(t)|}{\varepsilon}+2)^d \le C(2+2)^d = C$, thus for almost every $t \in [t_*, r_*]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta X(t)| \le \frac{C}{\varepsilon^d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B_\varepsilon(X^0(t)))}|\Delta X(t)|.$$

By Grönwall's inequality, we reach a conclusion that

$$\begin{aligned}
|\Delta X(r_*)| &\le |\Delta X(t_*)| \exp\left(\int_{t_*}^{r_*} \frac{C}{\varepsilon^d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B_\varepsilon(X^0(t)))}\,\mathrm{d}t\right) \\
&\le \varepsilon \exp\left(\int_{t_0-\varepsilon^2}^{t_0+\varepsilon^2} \frac{C}{\varepsilon^d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B_\varepsilon(X^0(t)))}\,\mathrm{d}t\right) \\
&= \varepsilon \exp\left(\frac{C}{\varepsilon^d}\|\mathcal{M}(\nabla u)\|_{L^1(Q_\varepsilon(t_0,x_0))}\right) \\
&\le \varepsilon \exp\left(C\eta\right)
\end{aligned}$$

which contradicts (3.9) when choosing $\eta < \eta_1 = \frac{1}{C}\log 2$. $\square$

To conclude this subsection, we discuss two streamlines with different ε that start from the same location. Before that, we introduce some notations. Let α be an index. Given $\varepsilon_\alpha > 0$, $t^\alpha \in (S+{\varepsilon_\alpha}^2, T-{\varepsilon_\alpha}^2)$, $x^\alpha \in \mathbb{R}^d$, we abbreviate

$$\begin{aligned} X^\alpha(t) &:= X_{\varepsilon_\alpha}(t^\alpha, x^\alpha; t), \qquad & B^\alpha(t) &:= B_{\varepsilon_\alpha}\left(X^\alpha(t)\right) \subset \mathbb{R}^d, \\ S^\alpha &:= t^\alpha - {\varepsilon_\alpha}^2, & T^\alpha &:= t^\alpha + {\varepsilon_\alpha}^2, \\ Q^\alpha &:= Q_{\varepsilon_\alpha}(t^\alpha, x^\alpha) = \left\{(t,x) : S^\alpha < t < T^\alpha, x \in B^\alpha(t)\right\}. \end{aligned} \tag{3.10}$$

For $\lambda > 0$, we denote the spatial dilation of a cylinder Q^α by

$$\lambda Q^\alpha := \left\{(t,x) : S^\alpha < t < T^\alpha, x \in \lambda B^\alpha(t) = B_{\lambda\varepsilon_\alpha}\left(X^\alpha(t)\right)\right\}. \tag{3.11}$$

Notice that different from upright cylinders or cubes, for $\varepsilon_1 < \varepsilon_2$, it is not known that $Q_{\varepsilon_1}(t,x) \subset Q_{\varepsilon_2}(t,x)$, because their center streamlines $X_{\varepsilon_{1,2}}$ solve different equations. As we will see later, this lack of monotonicity only poses a minor technical difficulty. For the same reason, note that $\lambda Q_\varepsilon(t,x) \neq Q_{\lambda\varepsilon}(t,x)$, and neither is necessarily contained in the other.

Lemma 3.9. *Recall that η_1 is a universal constant defined in Lemma 3.8. There exists a universal constant $\eta_0 < \eta_1$ such that the following is true. Given $\varepsilon_\alpha > \frac{1}{2}\varepsilon_\beta > 0$, $t^\alpha \in (S+{\varepsilon_\alpha}^2, T-{\varepsilon_\alpha}^2)$, $t^\beta \in (S+{\varepsilon_\beta}^2, T-{\varepsilon_\beta}^2)$ and $x^\alpha, x^\beta \in \mathbb{R}^d$, suppose $Q^\alpha = Q_{\varepsilon_\alpha}(t^\alpha, x^\alpha)$, $Q^\beta = Q_{\varepsilon_\beta}(t^\beta, x^\beta)$ are η-admissible as defined in Definition 3.2 with $\eta < \eta_0$. For any $(t_*, x_*) \in Q^\alpha \cap Q^\beta$, we have*

$$|X_{\varepsilon_\beta}(t_*, x_*; t) - X_{\varepsilon_\alpha}(t_*, x_*; t)| \le \varepsilon_\alpha \tag{3.12}$$

at any given time $t \in (S^\alpha, T^\alpha) \cap (S^\beta, T^\beta)$.

Proof. Denote

$$X^1(t) = X_{\varepsilon_\alpha}(t_*, x_*; t), \qquad X^2(t) = X_{\varepsilon_\beta}(t_*, x_*; t), \qquad \Delta X(t) = X^1(t) - X^2(t),$$

thus we need to show $|\Delta X(t)| \le \varepsilon_\alpha$. Note that

$$\Delta X(t_*) = X^2(t_*) - X^1(t_*) = X_{\varepsilon_\beta}(t_*, x_*; t_*) - X_{\varepsilon_\alpha}(t_*, x_*; t_*) = x_* - x_* = 0.$$

Similar as in the last lemma, we argue by contradiction and suppose there exists $r_* \in (t_*, \min\{T^\alpha, T^\beta\})$, such that

$$|\Delta X(t)| \le \varepsilon_\alpha \text{ for any } t \in [t_*, r_*], \qquad |\Delta X(r_*)| = \varepsilon_\alpha. \tag{3.13}$$

For almost every $t \in [t_*, r_*]$, the time derivative of ΔX is calculated as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Delta X(t) &= \dot{X}^2(t) - \dot{X}^1(t) \\
&= u_{\varepsilon_\beta}(t, X^2(t)) - u_{\varepsilon_\alpha}(s, X^1(t)) \\
&= u_{\varepsilon_\beta}(t, X^2(t)) - u_{\varepsilon_\alpha}(s, X^2(t)) + u_{\varepsilon_\alpha}(s, X^2(t)) - u_{\varepsilon_\alpha}(s, X^1(t)) \\
&= \int_{\varepsilon_\alpha}^{\varepsilon_\beta} \frac{\partial}{\partial \varepsilon} u_\varepsilon(t, X^2(t))\,\mathrm{d}\varepsilon + u_{\varepsilon_\alpha}(t, X^2(t)) - u_{\varepsilon_\alpha}(t, X^1(t)).
\end{aligned} \tag{3.14}$$

We will use Q^β to control the first integral term and use Q^α to control the rest. Note that

$$\frac{\partial}{\partial \varepsilon} u_\varepsilon(t,x) = \frac{\partial}{\partial \varepsilon}\int_{\mathbb{R}^d} u(t, x - \varepsilon y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \nabla_x u(t, x - \varepsilon y) \cdot -y\varphi(y)\,\mathrm{d}y,$$

thus we can control its absolute value by

$$\begin{aligned}
\left|\frac{\partial}{\partial\varepsilon}u_\varepsilon(t,x)\right| &\le \varepsilon^{-d}\|\nabla u(t)\|_{L^1(B_\varepsilon(x))}\|y\varphi(y)\|_{L^\infty} \\
&= C\varepsilon^{-d}\|\nabla u(t)\|_{L^1(B_\varepsilon(x))} \\
&\le C\varepsilon^{-d}\left(\frac{\left|x-X^\beta(t)\right|+\varepsilon_\beta+\varepsilon}{\varepsilon_\beta}\right)^d\|\mathcal{M}(\nabla u(t))\|_{L^1(B_{\varepsilon_\beta}(X^\beta(t)))} \\
&= C{\varepsilon_\beta}^{-d}\left(\frac{\left|x-X^\beta(t)\right|+\varepsilon_\beta+\varepsilon}{\varepsilon}\right)^d\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))}. \qquad (3.15)
\end{aligned}$$

Here we use (3.7) with $r=\varepsilon_\beta$ and $y=X^\beta(t)$ in the last inequality to control $\|\nabla u(t)\|_{L^1(B_\varepsilon(x))}$. Thanks to Lemma 3.8, $|X^2(t)-X^\beta(t)|\le 2\varepsilon_\beta$. Since ε is between ε_β and $\varepsilon_\alpha>\frac{1}{2}\varepsilon_\beta$, we have

$$\frac{\left|X^2(t)-X^\beta(t)\right|+\varepsilon_\beta+\varepsilon}{\varepsilon}\le\frac{3\varepsilon_\beta+\varepsilon}{\varepsilon}\le 7.$$

Hence if we set $x=X^2(t)$ in (3.15), we would get

$$\left|\frac{\partial}{\partial\varepsilon}u_\varepsilon(t,X^2(t))\right|\le C{\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))},$$

thus we can bounded the ∂_ε term in (3.14) by

$$\begin{aligned}
\left|\int_{\varepsilon_\alpha}^{\varepsilon_\beta}\frac{\partial}{\partial\varepsilon}u_\varepsilon(t,X^2(t))\,\mathrm{d}\varepsilon\right| &\le C|\varepsilon_\beta-\varepsilon_\alpha|{\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))} \\
&\le C\varepsilon_\alpha{\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))}.
\end{aligned}$$

The remaining terms in (3.14) can be bounded similar as in Lemma 3.8 as

$$\begin{aligned}
|u_{\varepsilon_\alpha}(t,X^2(t))-u_{\varepsilon_\alpha}(t,X^1(t))| &\le |\nabla u_{\varepsilon_\alpha}(t,\xi_t)||\Delta X(t)| \\
&\le C{\varepsilon_\alpha}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\alpha(t))}|\Delta X(t)|.
\end{aligned}$$

Combining these two bounds in (3.14), for almost every $t \in [t_*, r_*]$, the growth rate of ΔX is bounded by

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta X(t)| \le C\left({\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))} + {\varepsilon_\alpha}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\alpha(t))}\right) \times (\varepsilon_\alpha + |\Delta X(t)|)\,.$$

By Grönwall's inequality, we would reach

$$\begin{aligned}\varepsilon_\alpha + |\Delta X(r_*)| &\le (\varepsilon_\alpha + |\Delta X(t_*)|)\exp\left(C\int_{t_*}^{r_*} {\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))}\right. \\ &\qquad\qquad \left. + {\varepsilon_\alpha}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\alpha(t))}\,\mathrm{d}t\right) \\ &\le \varepsilon_\alpha \exp(2C\eta) \end{aligned} \tag{3.16}$$

which contradicts (3.13) when choosing $\eta < \eta_0 = \min\{\eta_1, \frac{1}{2C}\log 2\}$. $\square$

3.2.2 Covering Lemma for Admissible Cylinders

The goal of this section is to prove a Vitali-type covering lemma for η-admissible cylinders, provided $\eta < \eta_0$. The key ingredient is Proposition 3.10, which shows that if two cylinders intersect, then during their shared life span, they are uniformly close to each other. Based on this proposition, we conclude in Lemma 3.11 that for an η-admissible cylinder Q^α, the union of all η-admissible cylinders with comparable or less radius that intersect Q^α has a comparable total measure as Q^α. The covering lemma will be a consequence of Lemma 3.11.

Throughout this subsection, we employ the notations introduced in (3.10).

Proposition 3.10. *For any pair of intersecting η-admissible cylinders Q^α, Q^β as in* (3.10) *with $\varepsilon_\beta < 2\varepsilon_\alpha$ and $\eta < \eta_0$ chosen in Lemma 3.9, at any $t \in (S^\alpha, T^\alpha) \cap (S^\beta, T^\beta)$, we have $B^\beta(t) \subset 9B^\alpha(t)$.*

That is, if Q^α intersects Q^β with $\varepsilon_\beta < 2\varepsilon_\alpha$, then $Q^\beta \cap \{S^\alpha < t < T^\alpha\} \subset 9Q^\alpha$.

Recall that λQ^α is the spatial dilation defined in (3.11). The proof is based on Lemma 3.8 and Lemma 3.9 which control the trajectories at the level of Q_ε. See Figure 3.1 for our strategy.

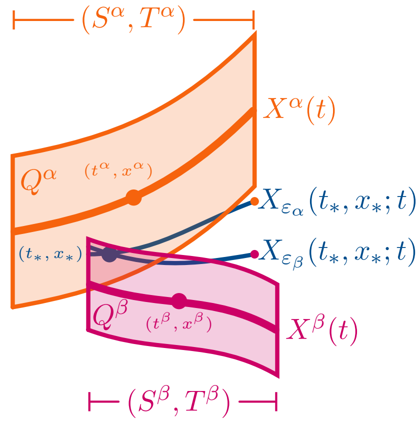

Figure 3.1: Q^α and Q^β intersect

Proof. Let η_0 be chosen as in Lemma 3.9. Fix some $(t_*, x_*) \in Q^\alpha \cap Q^\beta$. For any $(t, x) \in Q^\beta$ with $S^\alpha < t < T^\alpha$, we apply the triangle inequality to estimate

$$\begin{aligned}|x - X^\alpha(t)| \le\ & |x - X^\beta(t)| \\ & + |X^\beta(t) - X_{\varepsilon_\beta}(t_*, x_*; t)| \\ & + |X_{\varepsilon_\beta}(t_*, x_*; t) - X_{\varepsilon_\alpha}(t_*, x_*; t)| \\ & + |X_{\varepsilon_\alpha}(t_*, x_*; t) - X^\alpha(t)| \\ \le\ & \varepsilon_\beta + 2\varepsilon_\beta + \varepsilon_\alpha + 2\varepsilon_\alpha.\end{aligned}$$

Here the first term is because $x \in B^\beta(t)$, the second and the fourth are due to Lemma 3.8, and the third term is controlled by Lemma 3.9. Since $\varepsilon_\beta < 2\varepsilon_\alpha$, we have

$$|x - X^{\alpha}(t)| < 9\varepsilon_{\alpha}.$$

$\square$

We remark here that if we take a sharper estimate in each step of Lemma 3.8 and Lemma 3.9 (and require a smaller η), the factor 9 can be easily improved to $5+\delta$ for any $\delta > 0$. 5 is also the factor that appeared in the original Vitali covering lemma for balls. Recall that in the proof of Vitali lemma, an important reason why we get a comparable volume is because if two balls $B_{r_1}(x_1) \cap B_{r_2}(x_2) \neq \varnothing$ with $r_2 < 2r_1$, then $B_{r_2}(x_2) \subset 5B_{r_1}(x_1)$. Unfortunately, this geometric property cannot be realized in our case, because an admissible cylinder with (3.2) has no control on the past and the future velocities. As a consequence, it is unlikely to cover Q^{β} by a dilation of Q^{α} in space-time. However, this requirement can be relaxed as the following. See Section 1.1 of [Ste93] for a more general setting.

Lemma 3.11. *Given a fixed Q^{α} and a family of $\{Q^{\beta}\}_{\beta \subset \Lambda}$ as in* (3.10) *such that for each Q^{β}, $Q^{\alpha} \cap Q^{\beta} \neq \varnothing$, $\varepsilon_{\beta} < 2\varepsilon_{\alpha}$, and they are η-admissible for $\eta < \eta_0$. Let $Q^{\alpha}_{*} = \bigcup_{\beta \in \Lambda} Q^{\beta}$ denote the union of this family. Then there exists a universal constant C such that*

$$|Q^{\alpha}_{*}| \leq C|Q^{\alpha}|.$$

Proof. Without loss of generality, we may assume that $\{Q^{\beta}\}_{\beta \subset \Lambda}$ is a finite collection. The general case can be proven using the finite case. Note that each Q^{β} is an open set. For any compact subset $K \subset\subset Q^{\alpha}_{*}$, K admits a finite open cover, thus $|K| \leq C|Q^{\alpha}|$ using the finite case. Since the inequality holds for any compact subset K, it must also be true for Q^{α}_{*}.

For each Q^β, we can break it into $Q^\beta = Q^\beta_+ \cup Q^\beta_- \cup Q^\beta_\circ$, where

$$\begin{aligned} Q^\beta_+ &= Q^\beta \cap \{t \geq T^\alpha\}, \\ Q^\beta_- &= Q^\beta \cap \{t \leq S^\alpha\}, \\ Q^\beta_\circ &= Q^\beta \cap \{S^\alpha < t < T^\alpha\}. \end{aligned}$$

From Proposition 3.10, we can conclude that

$$\bigcup_{\beta\in\Lambda} Q^\beta_\circ \subset 9Q^\alpha \Rightarrow \left|\bigcup_{\beta\in\Lambda} Q^\beta_\circ\right| \leq 9^d|Q^\alpha|. \tag{3.17}$$

As mentioned in the remark, we cannot bound the size of $\bigcup_{\beta\in\Lambda} Q^\beta_+$ or $\bigcup_{\beta\in\Lambda} Q^\beta_-$ directly by Q^α, as their center streamlines can diverge away from X^α after T^α. Fortunately, we do not need them to be close to X^α, as long as we can show they remain a small distance to each other.

Let us measure $\bigcup_{\beta\in\Lambda} Q^\beta_+$. First, we group the cylinders by their radii. Denote

$$\Lambda_i = \{\beta \in \Lambda : 2^{-i}\varepsilon_\alpha \leq \varepsilon_\beta < 2^{-i+1}\varepsilon_\alpha\}. \tag{3.18}$$

Because each $\varepsilon_\beta < 2\varepsilon_\alpha$, we have $\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i$, hence we can write the union as

$$\bigcup_{\beta\in\Lambda} Q^\beta_+ = \bigcup_{i\in\mathbb{N}} \bigcup_{\beta\in\Lambda_i} Q^\beta_+. \tag{3.19}$$

Now we fix i and estimate the size of $\bigcup_{\beta\in\Lambda_i} Q^\beta_+$. Clearly we can disregard the empty ones, and assume $T^\alpha < T^\beta$ for each $\beta \in \Lambda_i$. To begin with, set $\mathcal{Q}_i^{(0)} = \{Q^\beta_+\}_{\beta\in\Lambda_i}$. Then we repeat the following two steps: at the j-th iteration ($j \geq 1$),

`Step 1.` Select some β_j such that $T^{\beta_j} = \max\left\{T^\beta : Q^\beta_+ \in \mathcal{Q}_i^{(j-1)}\right\}$.

`Step 2.` From $\mathcal{Q}_i^{(j-1)}$ we remove any Q^β_+ such that $B^\beta(T^\alpha) \cap B^{\beta_j}(T^\alpha) \neq \varnothing$, and denote the rest by $\mathcal{Q}_i^{(j)}$.

After finitely many iterations, $\mathcal{Q}_i^{(n+1)}$ will be empty, and we have a list of $Q_+^{\beta_1}$, ..., $Q_+^{\beta_n}$. We claim that

$$\bigcup_{\beta\in\Lambda_i} Q_+^\beta \subset \bigcup_{j=1}^n 9Q_+^{\beta_j}. \tag{3.20}$$

To see why this is true, take any $Q_+^\beta \in \mathcal{Q}_i^{(0)}$. It must have been removed from $\mathcal{Q}_i^{(j-1)}$ at some step j in the above process. This implies $B^\beta(T^\alpha) \cap B^{\beta_j}(T^\alpha) \neq \varnothing$, and $T^\beta \leq T^{\beta_j}$. Also, we have $\varepsilon_\beta < 2\varepsilon_{\beta_j}$, which is actually true for any pair of cylinders by our selection of Λ_i according to (3.18). Therefore, by Proposition 3.10 we have $B^\beta(t) \subset 9B^{\beta_j}(t)$ at any $t \in (S^\beta, T^\beta) \cap (S^{\beta_j}, T^{\beta_j})$. Because $S^\beta, S^{\beta_j} \leq T^\alpha \leq T^\beta \leq T^{\beta_j}$, we have $Q_+^\beta \subset 9Q_+^{\beta_j}$ and this proves the claim (3.20).

Note that by our construction, $\{B^{\beta_j}(T^\alpha)\}_{j=1}^n$ are pairwise disjoint, and they are all inside $9B^\alpha(T^\alpha)$ by the Proposition 3.10. Therefore their total measure is

$$\begin{aligned}
\sum_{j=1}^n |Q_+^{\beta_j}| &\leq \sum_{j=1}^n |B^{\beta_j}(T^\alpha)| \cdot 2(\varepsilon_{\beta_j})^2 \\
&\leq \sum_{j=1}^n 2 \cdot |B^{\beta_j}(T^\alpha)| \cdot (2^{-i+1}\varepsilon_\alpha)^2 \\
&= 2 \cdot 4^{-i+1}{\varepsilon_\alpha}^2 \left| \bigcup_{j=1}^n B^{\beta_j}(T^\alpha) \right| \\
&\leq 2 \cdot 4^{-i+1}{\varepsilon_\alpha}^2 |9B^\alpha(T^\alpha)| = 4^{-i+1} \cdot 9^d |Q^\alpha|.
\end{aligned}$$

Combining with the claim (3.20), we have

$$\left| \bigcup_{\beta\in\Lambda_i} Q_+^\beta \right| \leq \left| \bigcup_{j=1}^n 9Q_+^{\beta_j} \right| \leq 9^d \sum_{j=1}^n |Q_+^{\beta_j}| \leq 4^{-i+1} \cdot 9^{2d} |Q^\alpha|.$$

Finally, take the summation over i, and (3.19) yields

$$\left|\bigcup_{\beta\in\Lambda} Q_+^\beta\right| \le \sum_{i=0}^{\infty} \left|\bigcup_{\beta\in\Lambda_i} Q_+^\beta\right| \le \sum_{i=0}^{\infty} 4^{-i+1}\cdot 9^{2d}|Q^\alpha| = \frac{16}{3}\cdot 9^{2d}|Q^\alpha|.$$

The same proof also applies to $\bigcup_{\beta\in\Lambda} Q_-^\beta$. Therefore, together with estimate (3.17), we have proven that

$$\begin{aligned}|Q_*^\alpha| = \left|\bigcup_{\beta\in\Lambda} Q^\beta\right| &\le \left|\bigcup_{\beta\in\Lambda} Q_+^\beta\right| + \left|\bigcup_{\beta\in\Lambda} Q_-^\beta\right| + \left|\bigcup_{\beta\in\Lambda} Q_\circ^\beta\right| \\ &\le \frac{16}{3}\cdot 9^{2d}|Q^\alpha| + \frac{16}{3}\cdot 9^{2d}|Q^\alpha| + 9^d|Q^\alpha| = C|Q^\alpha|.\end{aligned}$$

□

We are finally ready to show the Vitali-type covering lemma.

Proposition 3.12 (Covering Lemma)**.** *Let $\mathcal{A}$ be an index set and let*

$$\mathcal{Q} = \{Q^\alpha = Q_{\varepsilon_\alpha}(t^\alpha, x^\alpha) : \alpha\in\mathcal{A}\}$$

be a collection of η-admissible cylinders, where $\eta < \eta_0$ defined in Lemma 3.9 and ε_α are uniformly bounded. Then there is a pairwise disjoint sub-collection $\mathcal{P} = \{Q^{\alpha_1}, Q^{\alpha_2}, \dots, Q^{\alpha_n}, \dots\}$ (finite or infinite) such that

$$\sum_j |Q^{\alpha_j}| \ge \frac{1}{C}\left|\bigcup_{\alpha\in\mathcal{A}} Q^\alpha\right|,$$

where C is a universal constant.

Proof. With the help of the previous lemma, the proof of the covering lemma is the same as the classical one in [Ste70]. We select the sub-collection $\mathcal{P}$ by the following procedure. To begin with, set $\mathcal{Q}^{(0)} = \mathcal{Q}$. Then repeat the following two steps: at the j-th iteration ($j \ge 1$), `Step 1.` Select some α_j such that $\varepsilon_{\alpha_j} > \frac{1}{2} \sup_{Q^\alpha \in \mathcal{Q}^{(j-1)}} \{\varepsilon_\alpha\}$.

`Step 2.` From $\mathcal{Q}^{(j-1)}$ we remove any Q^α that intersects with Q^{α_j}, and denote the rest by $\mathcal{Q}^{(j)}$.

This procedure may stop after a certain step if $\mathcal{Q}^{(n+1)} = \varnothing$, or it can continue indefinitely. We denote the chosen ones by $\mathcal{P} := \{Q^{\alpha_1}, \dots, Q^{\alpha_n}, \dots\}$ (finite or infinite). They are pairwise disjoint due to our strategy.

Suppose that $\sum_j |Q^{\alpha_j}| < \infty$, otherwise the conclusion is automatically true. Thus either $\mathcal{P}$ is a finite collection, or $\mathcal{P}$ is infinite and $\varepsilon_{\alpha_j} \to 0$ as $j \to \infty$. In either case, each Q^α must be removed from $\mathcal{Q}^{(j)}$ at some iteration. Otherwise, we would have $Q^\alpha \in \mathcal{Q}^{(j-1)}$ for all j, then `Step 1` would imply that $\varepsilon_{\alpha_j} > \frac{1}{2}\varepsilon_\alpha$ for all j, thus the sequence ε_{α_j} cannot converges to zero. Now suppose $Q^\alpha \in \mathcal{Q}^{(j-1)} \setminus \mathcal{Q}^{(j)}$, then we have $Q^\alpha \cap Q^{\alpha_j} \neq \varnothing$, and $\varepsilon_\alpha < 2\varepsilon_{\alpha_j}$. This implies

$$Q^\alpha \subset Q_*^{\alpha_j} := \bigcup_{\alpha \in \mathcal{A}} \left\{ Q^\alpha \in \mathcal{Q} : \varepsilon_\alpha < 2\varepsilon_{\alpha_j}, Q^\alpha \cap Q^{\alpha_j} \neq \varnothing \right\}.$$

Thus $\bigcup_{\alpha \in \mathcal{A}} Q^\alpha \subset \bigcup_{j=1}^n Q_*^{\alpha_j}$, and finally we control the measure of the union by

$$\left| \bigcup_{\alpha \in \mathcal{A}} Q^\alpha \right| \leq \left| \bigcup_{j=1}^n Q_*^{\alpha_j} \right| \leq \sum_{j=1}^n |Q_*^{\alpha_j}| \leq C \sum_{j=1}^n |Q^{\alpha_j}|$$

thanks to Lemma 3.11. □

3.3 Construction of the Maximal Function

In this section, we use the covering lemma to generalize some results from the classical harmonic analysis to our situation. First, we confirm the existence of η-admissible cylinders centering almost everywhere under some assumptions on u. Then we prove the main theorem for the maximal function on these skewed cylinders and show related results similar to the classical case.

3.3.1 Existence of Admissible Cylinders

To begin with, we need some assumptions to guarantee the existence of η-admissible cylinders centering almost everywhere, which are the following. For the entire Section 3.3, we assume

Assumption 3.13. *For some* $1 \le p \le \infty$*,*

(1) $\mathcal{M}(\nabla u) \in L^p((S,T) \times \mathbb{R}^d)$.

(2) $\operatorname{div} u = 0$.

Proposition 3.14. *Let* $\eta > 0$*. For almost every* $(t,x) \in (S,T) \times \mathbb{R}^d$*,* $Q_\varepsilon(t,x)$ *is* η*-admissible for sufficiently small* ε *(depending on* (t,x)*). Moreover, we have*

$$\lim_{\varepsilon \to 0} \operatorname{diam}(Q_\varepsilon(t,x)) = 0,$$

where diam *refers to the* $(d+1)$*-dimensional diameter.*

Before showing the proof of Proposition 3.14, we first give a general lemma on the L^1 boundedness of the map $f \mapsto f_\varepsilon$ defined below. Given $f \in L^1_{\mathrm{loc}}((S,T) \times \mathbb{R}^d)$, for $x \in \mathbb{R}^d$, $t \in (S,T)$, $\varepsilon > 0$, we define

$$f_\varepsilon(t,x) = \begin{cases} \fint_{Q_\varepsilon(t,x)} f(s,y) \,\mathrm{d}y \,\mathrm{d}s & t \in (S+\varepsilon^2, T-\varepsilon^2) \\ 0 & t \in (S, S+\varepsilon^2] \cup [T-\varepsilon^2, T) \end{cases}. \tag{3.21}$$

Then we have the following bound on f_ε.

Lemma 3.15 (L^1 Boundedness)**.** *Given* $f \in L^1((S,T) \times \mathbb{R}^d)$*, we have*

$$\|f_\varepsilon\|_{L^1((S,T)\times\mathbb{R}^d)} \le \|f\|_{L^1((S,T)\times\mathbb{R}^d)}.$$

Proof. A direct computation gives

$$\begin{aligned}
&\int_{S+\varepsilon^2}^{T-\varepsilon^2}\int_{\mathbb{R}^d}|f_\varepsilon(t,x)|\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{S+\varepsilon^2}^{T-\varepsilon^2}\int_{\mathbb{R}^d}\frac{1}{|Q_\varepsilon|}\left|\int_{Q_\varepsilon(t,x)}f(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|\mathrm{d}x\,\mathrm{d}t\\
&\le\frac{1}{|Q_\varepsilon|}\int_S^T\int_{\mathbb{R}^d}\int_{S+\varepsilon^2}^{T-\varepsilon^2}\int_{\mathbb{R}^d}|f(s,y)|\mathbf{1}_{\{(s,y)\in Q_\varepsilon(t,x)\}}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&=\frac{1}{|Q_\varepsilon|}\int_S^T\int_{\mathbb{R}^d}|f(s,y)|\mathcal{L}^{d+1}(\tilde{Q}_\varepsilon(s,y))\,\mathrm{d}y\,\mathrm{d}s \qquad (3.22)
\end{aligned}$$

where we define for any fixed $(s,y)\in(S,T)\times\mathbb{R}^d$ the **dual Lagrangian cylinder** by (see [IO16] for a detailed discussion of these cylinders)

$$\tilde{Q}_\varepsilon(s,y):=\Big\{(t,x)\in(S+\varepsilon^2,T-\varepsilon^2)\times\mathbb{R}^d:(s,y)\in Q_\varepsilon(t,x)\Big\}.$$

Then from the definition of $Q_\varepsilon(t,x)$, we can see that

$$\begin{aligned}
\tilde{Q}_\varepsilon(s,y)&\subset\big\{(t,x):|t-s|<\varepsilon^2,|X_\varepsilon(t,x;s)-y|<\varepsilon\big\}\\
&=\big\{(t,x):|t-s|<\varepsilon^2,x':=X_\varepsilon(t,x;s)\in B_\varepsilon(y)\big\}\\
&=\big\{(t,x):|t-s|<\varepsilon^2,x'\in B_\varepsilon(y),x=X_\varepsilon(s,x';t)\big\}.
\end{aligned}$$

Because u_ε is also divergence free, measure of a set is invariant under the flow, so we have

$$\mathcal{L}^d\left(\{X_\varepsilon(s,x';t):x'\in B_\varepsilon(y)\}\right)=\mathcal{L}^d(B_\varepsilon(y)).$$

Thus the measure of the dual cylinder is

$$\begin{aligned}\mathcal{L}^{d+1}(\tilde{Q}_\varepsilon(s,y)) &\le \mathcal{L}^{d+1}\left(\left\{(t,x) : |t-s| < \varepsilon^2, x' \in B_\varepsilon(y), x = X_\varepsilon(s,x';t)\right\}\right)\\ &= \int_{\max(S,s-\varepsilon^2)}^{\min(T,s+\varepsilon^2)} \mathcal{L}^d\left(\left\{X_\varepsilon(s,x';t) : x' \in B_\varepsilon(y)\right\}\right)\mathrm{d}t\\ &\le 2\varepsilon^2|B_\varepsilon| = |Q_\varepsilon|.\end{aligned}$$

Plugging into (3.22), we conclude that

$$\begin{aligned}\int_{S+\varepsilon^2}^{T-\varepsilon^2}\int_{\mathbb{R}^d}|f_\varepsilon(t,x)|\,\mathrm{d}x\,\mathrm{d}t &\le \frac{1}{|Q_\varepsilon|}\int_S^T\int_{\mathbb{R}^d}|f(s,y)|\mathcal{L}^{d+1}(\tilde{Q}_\varepsilon(s,y))\,\mathrm{d}y\,\mathrm{d}s\\ &\le \int_S^T\int_{\mathbb{R}^d}|f(t,x)|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

□

Proof of Proposition 3.14. If $p=\infty$ in the Assumption 3.13, the conclusions follow naturally from the Definition 3.1 and 3.2, as now both the velocity field and its gradient are locally bounded. We shall only focus on the case $p<\infty$ from now.

Without loss of generality, assume $\eta \le \eta_0$. For $S<t<T$, $x\in\mathbb{R}^d$, define

$$F(t,x) := [\mathcal{M}(\nabla u(t))(x)]^p \in L^1((S,T)\times\mathbb{R}^d).$$

$F_\varepsilon(t,x)$ is defined same as in (3.21). Lemma 3.15 shows that $\|F_\varepsilon\|_{L^1} \le \|F\|_{L^1}$. We want to show that for sufficiently small ε,

$$F_\varepsilon(t,x) \le \eta^p\varepsilon^{-2p}.$$

By Remark 3.7, this implies that $Q_\varepsilon(t,x)$ is η-admissible. Define the set of non-admissible points by

$$\Omega_\varepsilon = \left\{(t,x) \in (S+\varepsilon^2, T-\varepsilon^2) \times \mathbb{R}^d : F_\varepsilon(t,x) > \eta^p \varepsilon^{-2p}\right\}.$$

By Chebyshev's inequality, its measure is bounded by

$$|\Omega_\varepsilon| \le |\{F_\varepsilon > \eta^p \varepsilon^{-2p}\}| \le \frac{\|F_\varepsilon\|_{L^1}}{\eta^p \varepsilon^{-2p}} \le \frac{\|F\|_{L^1}}{\eta^p} \varepsilon^{2p} \to 0$$

as $\varepsilon \to 0$. Therefore, $| \cap_{\varepsilon > 0} \Omega_\varepsilon| = 0$, that is, the set of points at which no η-admissible cylinder centers has measure zero. In other words, for almost every point (t,x), there exists $\varepsilon > 0$ such that $Q_\varepsilon(t,x)$ is η-admissible.

This is not enough to show the conclusion, because Ω_ε may not be monotone in ε. To see that $Q_\varepsilon(t,x)$ is η-admissible for all sufficiently small ε, let us define

$$\Omega'_\varepsilon = \left\{(t,x) \in (S+\varepsilon^2, T-\varepsilon^2) \times \mathbb{R}^d : F_\varepsilon(t,x) > \eta^p (2^{d+1}\varepsilon)^{-2p}\right\}.$$

Similar as before, Chebyshev's inequality implies

$$|\Omega'_\varepsilon| \le \frac{\|F\|_{L^1}}{\eta^p} (2^{d+1}\varepsilon)^{2p}.$$

In particular, for each $i \ge 1$, we have a geometric decaying upper bound as

$$|\Omega'_{2^{-i}}| \le \frac{\|F\|_{L^1}}{\eta^p} (2^{d+1} 2^{-i})^{2p}.$$

It is a summable geometric series in i, thus by Borel-Cantelli lemma, we have

$$\left| \limsup_{i \to \infty} \Omega'_{2^{-i}} \right| = \left| \bigcap_{I>0} \bigcup_{i>I} \Omega'_{2^{-i}} \right| = 0.$$

That is, for almost every $(t,x) \in (S,T) \times \mathbb{R}^d$, there exists $I > 0$ such that for all

$i > I$, $(t,x) \notin \Omega'_{2^{-i}}$, i.e., for $\varepsilon_i = 2^{-i}$, we have

$$F_{\varepsilon_i}(t,x) = \fint_{Q_{\varepsilon_i}(t,x)} F(s,y)\,\mathrm{d}y\,\mathrm{d}s \le \eta^p (2^{d+1}\varepsilon_i)^{-2p}.$$

By Remark 3.7, Jensen's inequality implies

$$\varepsilon_i^2 \fint_{Q_{\varepsilon_i}(t,x)} \mathcal{M}(\nabla u)\,\mathrm{d}y\,\mathrm{d}s \le \varepsilon_i^2 \left(\fint_{Q_{\varepsilon_i}(t,x)} [\mathcal{M}(\nabla u)]^p\,\mathrm{d}y\,\mathrm{d}s \right)^{\frac{1}{p}} \le \frac{\eta}{4^{d+1}}.$$

That is, $Q_{\varepsilon_i}(t,x)$ is $(4^{-d-1}\eta)$-admissible.

We claim that if $Q_{\varepsilon_\alpha}(t_0,x_0)$ is $(4^{-d-1}\eta)$-admissible, then for every ε_β within $\frac{\varepsilon_\alpha}{4} \le \varepsilon_\beta \le \frac{\varepsilon_\alpha}{2}$, $Q_{\varepsilon_\beta}(t_0,x_0) \subset \frac{3}{4} Q_{\varepsilon_\alpha}(t_0,x_0)$. This can be proven by the claim

$$|X_{\varepsilon_\beta}(t_0,x_0;t) - X_{\varepsilon_\alpha}(t_0,x_0;t)| \le \frac{\varepsilon_\alpha}{4}, \qquad \text{for all } t \in (t_0 - {\varepsilon_\beta}^2, t_0 + {\varepsilon_\beta}^2) \tag{3.23}$$

whose proof is a slight modification of Lemma 3.9. Define $Q^\alpha = Q_{\varepsilon_\alpha}(t_0,x_0)$ and $Q^\beta = Q_{\varepsilon_\beta}(t_0,x_0)$. If we proceed the proof of Lemma 3.9, without knowing Q^β is η-admissible, the only difficulty will arise at the last step (3.16), when we want to bound the integral of ${\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))}$ in the Grönwall's inequality. However, as long as (3.23) holds at time t, $B^\beta(t)$ is contained in $B^\alpha(t)$, thus

$${\varepsilon_\beta}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\beta(t))} \le 4^d {\varepsilon_\alpha}^{-d}\|\mathcal{M}(\nabla u(t))\|_{L^1(B^\alpha(t))}$$

while the integral of the latter is bounded by $4^d\eta$. Following the same continuity argument we conclude (3.23) in the end.

By this claim, for every ε between $\frac{\varepsilon_i}{4}$ and $\frac{\varepsilon_i}{2}$, we have

$$Q_\varepsilon(t,x) \subset \frac{3}{4} Q_{\varepsilon_i}(t,x) \subset \left(\frac{3}{4}\right)^2 Q_{\varepsilon_{i-1}}(t,x) \subset \cdots$$

which implies $\operatorname{diam}(Q_\varepsilon(t,x)) \to 0$ as $\varepsilon \to 0$. Although we do not have monotonicity for $Q_\varepsilon(t,x)$ in ε, we have this "monotonicity with gaps". Moreover, since $Q_\varepsilon(t,x) \subset Q_{\varepsilon_i}(t,x)$, $\varepsilon > \frac{\varepsilon_i}{4}$, we can bound F_ε by

$$F_\varepsilon(t,x) = \frac{1}{|Q_\varepsilon|}\int_{Q_\varepsilon(t,x)} F \,\mathrm{d}y\,\mathrm{d}s \le \frac{|Q_{\varepsilon_i}|}{|Q_\varepsilon|} \fint_{Q_{\varepsilon_i}(t,x)} F \,\mathrm{d}y\,\mathrm{d}s \le 4^{d+1}\eta^p(2^{d+1}\varepsilon_i)^{-2p}$$
$$\le \eta^p \varepsilon_i^{-2p} \le \eta^p \varepsilon^{-2p}.$$

Thus $(t,x) \notin \Omega_\varepsilon$ for every $\varepsilon \in \left[\frac{\varepsilon_i}{4}, \frac{\varepsilon_i}{2}\right]$ and for every $i > I$, that is, for every $\varepsilon \le 2^{-I-1}$. This means $Q_\varepsilon(t,x)$ is admissible for all ε sufficiently small. □

Following this existence proposition, we furthermore have the following corollary on the L^1 convergence.

Corollary 3.16 (L^1 Convergence)**.** *Let $f \in L^1((S,T)\times\mathbb{R}^d)$, and define f_ε by* (3.21)*, then*

$$f_\varepsilon \to f \text{ in } L^1((S,T)\times\mathbb{R}^d) \text{ as } \varepsilon \to 0.$$

Proof. For any $\delta > 0$, we can find $g \in C_c^\infty((S,T)\times\mathbb{R}^d)$ such that $\|f-g\|_{L^1} < \frac{\delta}{3}$. Denote $h = f - g$, then $\|h\|_{L^1} < \frac{\delta}{3}$, and by Lemma 3.15, also $\|h_\varepsilon\|_{L^1} < \frac{\delta}{3}$ (we define h_ε in the same way as (3.21)). Since g is uniformly continuous, it is clear that as $\mathrm{diam}(Q_\varepsilon(t,x)) \to 0$,

$$\|g - g_\varepsilon\|_{L^1} \le \int_{(S+\varepsilon^2, T-\varepsilon^2)\times\mathbb{R}^d} \fint_{Q_\varepsilon(t,x)} |g(t,x) - g(s,y)| \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t < \frac{\delta}{3}$$

for sufficiently small ε such that $g(t,\cdot) = 0$ in $(S, S+\varepsilon^2) \cup (T-\varepsilon^2, T)$. Thus

$$\|f - f_\varepsilon\|_{L^1} = \|g + h - g_\varepsilon - h_\varepsilon\|_{L^1} \le \|g - g_\varepsilon\|_{L^1} + \|h\|_{L^1} + \|h_\varepsilon\|_{L^1} < \delta$$

provided ε is small enough. □

3.3.2 Maximal Function

The Existence Proposition 3.14 ensures the maximal function is well-defined almost everywhere. With the help of covering lemma, we can prove the bounds for the maximal function. A lot of ideas are borrowed from [Ste70]. We do not claim any originality for results in this section, but only put them here for the sake of completeness.

Proof of Theorem 3.3. By the Existence Proposition 3.14, for almost every $(t,x) \in (S,T) \times \mathbb{R}^d$, the set $\{\varepsilon > 0 : Q_\varepsilon(t,x) \text{ is } \eta \text{ admissible}\}$ is nonempty, so the maximal function $\mathcal{M}_{\mathcal{Q}}(f)$ is well-defined almost everywhere.

(1) This is evident from the definition, since for any (t,x) it holds that

$$\fint_{Q_\varepsilon(t,x)} |f(s,y)| \,\mathrm{d}y \,\mathrm{d}s \leq \|f\|_{L^\infty}.$$

(2) For any $\lambda > 0$, let $E_\lambda = \{(t,x) : (\mathcal{M}_{\mathcal{Q}} f)(t,x) > \lambda\}$ be the superlevel set. Then by definition, there is an η-admissible Q_ε centered at each point $(t,x) \in E_\lambda$, such that

$$|Q_\varepsilon| < \frac{1}{\lambda} \int_{Q_\varepsilon(t,x)} |f(s,y)| \,\mathrm{d}y \,\mathrm{d}s.$$

Their radii are thus uniformly bounded. Thanks to the Covering Lemma Proposition 3.12, we can choose a pairwise disjoint subcollection $\{Q_{\varepsilon_j}(t^j, x^j)\}$, such that

$$\sum_j |Q_{\varepsilon_j}| \geq \frac{1}{C} \left| \bigcup\nolimits_{(t,x)\in E_\lambda} Q_\varepsilon(t,x) \right|.$$

Therefore the measure of the superlever set can be bounded by

$$|E_\lambda| \leq C \sum_j |Q_{\varepsilon_j}| \leq \frac{C}{\lambda} \sum_j \int_{Q_{\varepsilon_j}(t^j, x^j)} |f| \, \mathrm{d}x \, \mathrm{d}t \leq \frac{C}{\lambda} \int_{(S,T)\times\mathbb{R}^d} |f| \, \mathrm{d}x \, \mathrm{d}t.$$

(3) For the type (q, q) part, we use Marcinkiewicz interpolation. Note that $\mathcal{M}_{\mathcal{Q}}$ is subadditive: $\mathcal{M}_{\mathcal{Q}}(f+g) \leq \mathcal{M}_{\mathcal{Q}}(f) + \mathcal{M}_{\mathcal{Q}}(g)$. We can split $f = f_1 + f_2$ where $f_1 = f\chi_{|f| \leq \frac{\lambda}{2}}$ and $f_2 = f\chi_{|f| > \frac{\lambda}{2}}$. First, the strong type (∞, ∞) estimate applied to f_1 yields

$$\|\mathcal{M}_{\mathcal{Q}}(f_1)\|_{L^\infty} \leq \frac{\lambda}{2}.$$

Thus we have

$$\mathcal{M}_{\mathcal{Q}}(f) \leq \mathcal{M}_{\mathcal{Q}}(f_1) + \mathcal{M}_{\mathcal{Q}}(f_2) \leq \frac{\lambda}{2} + \mathcal{M}_{\mathcal{Q}}(f_2).$$

So $\mathcal{M}_{\mathcal{Q}}(f) > \lambda$ implies $\mathcal{M}_{\mathcal{Q}}(f_2) > \frac{\lambda}{2}$. Next, the weak type $(1, 1)$ estimate applied to f_2 yields

$$\mu(E_\lambda) \leq \mu\left(\left\{\mathcal{M}_{\mathcal{Q}}(f_2) > \frac{\lambda}{2}\right\}\right) \leq \frac{2C}{\lambda}\|f_2\|_{L^1}.$$

By the layer cake representation, we have that

$$\begin{aligned}
\int_{(S,T)\times\mathbb{R}^d} [\mathcal{M}_{\mathcal{Q}}(f)]^q \, \mathrm{d}t \, \mathrm{d}x &= q \int_0^\infty \mu(E_\lambda)\lambda^{q-1} \, \mathrm{d}\lambda \\
&\leq 2Cq \int_0^\infty \frac{1}{\lambda} \int_{(S,T)\times\mathbb{R}^d} |f|\chi_{|f| > \frac{\lambda}{2}} \lambda^{q-1} \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}\lambda \\
&= 2Cq \int_{(S,T)\times\mathbb{R}^d} |f| \int_0^{2|f|} \lambda^{q-2} \, \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{2Cq \cdot 2^{q-1}}{q-1} \int_{(S,T)\times\mathbb{R}^d} |f|^q \, \mathrm{d}x \, \mathrm{d}t = C_q \|f\|_{L^q}^q.
\end{aligned}$$

This finishes the proof of the theorem.

□

This theorem, together with the L^1 convergence will imply the almost everywhere convergence of f_ε.

Corollary 3.17 (a.e. Convergence)**.** *Given $f \in L^1_{\mathrm{loc}}((S,T)\times\mathbb{R}^d)$, for almost every $(t,x) \in (S,T)\times\mathbb{R}^d$, we have $f_\varepsilon(t,x) \to f(t,x)$ as $\varepsilon \to 0$, where f_ε is defined in* (3.21).

Proof. According to the Proposition 3.14, $\operatorname{diam}(Q_\varepsilon(t,x)) \to 0$ for almost every (t,x), so we can assume f is compactly supported and thus integrable without loss of generality. By Corollary 3.16 L^1 convergence, we can find a subsequence which converges to f almost everywhere, hence it suffices to show the following oscillation function is zero almost everywhere: for $f \in L^1_{\mathrm{loc}}((S,T)\times\mathbb{R}^d)$, define the oscillation function by

$$\Omega f(t,x) = \limsup_{\varepsilon\to 0} f_\varepsilon(t,x) - \liminf_{\varepsilon\to 0} f_\varepsilon(t,x).$$

For a uniformly continuous function g, we have $\Omega g \equiv 0$ almost everywhere, again using the fact that $\operatorname{diam}(Q_\varepsilon(t,x)) \to 0$ by Proposition 3.14. Moreover, notice that as $\varepsilon \to 0$, $Q_\varepsilon(t,x)$ is η-admissible, so we have

$$\limsup_{\varepsilon\to 0} f_\varepsilon(t,x) \le \limsup_{\varepsilon\to 0} |f_\varepsilon(t,x)| \le \mathcal{M}_{\mathcal{Q}}(f)(t,x),$$
$$-\liminf_{\varepsilon\to 0} f_\varepsilon(t,x) \le \limsup_{\varepsilon\to 0} |f_\varepsilon(t,x)| \le \mathcal{M}_{\mathcal{Q}}(f)(t,x),$$

so $\Omega f \le 2\mathcal{M}_{\mathcal{Q}}(f)$ almost everywhere. Now we fix $\lambda > 0$. For any given $\delta > 0$, we split $f = g + h$ with $g \in C_c^\infty((S,T)\times\mathbb{R}^d)$ and $\|h\|_{L^1} < \delta$, we have

$$\Omega f \le \Omega h + \Omega g = \Omega h \le 2\mathcal{M}_{\mathcal{Q}}(h).$$

By Theorem 3.3, weak type $(1,1)$ estimate gives

$$\mu(\{\Omega f > \lambda\}) \leq \mu\left(\left\{\mathcal{M}_{\mathcal{Q}}(h) > \frac{\lambda}{2}\right\}\right) \leq \frac{2C}{\lambda}\|h\|_{L^1} = \frac{2C}{\lambda}\delta.$$

Set $\delta \to 0$ we obtain

$$\mu(\{\Omega f > \lambda\}) = 0.$$

This is true for any $\lambda > 0$, therefore we actually have

$$\mu(\{\Omega f > 0\}) = 0.$$

This means, for almost every $(t,x) \in (S,T) \times \mathbb{R}^d$, the oscillation is zero and

$$\limsup_{\varepsilon \to 0} f_\varepsilon(t,x) = \liminf_{\varepsilon \to 0} f_\varepsilon(t,x) = \lim_{\varepsilon \to 0} f_\varepsilon(t,x) = f(t,x).$$

□

Using the definition of $\mathcal{M}_{\mathcal{Q}}$, it is easy to deduce the following.

Corollary 3.18. *For $f \in L^1_{\text{loc}}((S,T) \times \mathbb{R}^d)$, $f \leq \mathcal{M}_{\mathcal{Q}}(f)$ almost everywhere.*

To conclude this section, we present a slightly stronger result than the almost everywhere convergence.

Theorem 3.19 ($\mathcal{Q}$-Lebesgue Differentiation Theorem)**.** *Under the same assumption of Corollary 3.17, for almost every $(t,x) \in (S,T) \times \mathbb{R}^d$, we have*

$$\lim_{\varepsilon \to 0} \fint_{Q_\varepsilon(t,x)} |f(s,y) - f(t,x)| \, \mathrm{d}y \, \mathrm{d}s = 0. \tag{3.24}$$

If (3.24) is true for (t,x), we call it a **$\mathcal{Q}$-Lebesgue point of** f, and define **$\mathcal{Q}$-Lebesgue set of** f to be the set of all $\mathcal{Q}$-Lebesgue points of f.

Proof. Consider any rational number $q \in \mathbb{Q}$. Then $f - q \in L^1_{\mathrm{loc}}$, thus by Corollary 3.17, we have

$$|f-q|_\varepsilon(t,x) = \fint_{Q_\varepsilon(t,x)} |f-q|(s,y)\,\mathrm{d}y\,\mathrm{d}s \to |f-q|(t,x), \qquad \text{a.e. as } \varepsilon \to 0.$$

By taking a countable intersection over $q \in \mathbb{Q}$ of all the sets where the convergence $|f-q|_\varepsilon \to |f-q|$ happens, we have

$$|f-q|_\varepsilon(t,x) \to |f-q|(t,x), \qquad \text{a.e. as } \varepsilon \to 0 \text{ for all } q \in \mathbb{Q}.$$

By the density of rational numbers, it holds that

$$|f-r|_\varepsilon(t,x) \to |f-r|(t,x), \qquad \text{a.e. as } \varepsilon \to 0 \text{ for all } r \in \mathbb{R}.$$

In particular, letting $r = f(t,x)$ gives

$$|f-f(t,x)|_\varepsilon(t,x) \to |f(t,x)-f(t,x)| = 0, \qquad \text{a.e. as } \varepsilon \to 0.$$

This is equivalent to (3.24). □

Chapter 4

Higher Regularity

4.1 Introduction

We study the three dimensional incompressible Navier–Stokes equations,

$$\partial_t u + u \cdot \nabla u + \nabla P = \Delta u, \qquad \operatorname{div} u = 0. \tag{4.1}$$

Here $u : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ and $P : (0,T) \times \mathbb{R}^3 \to \mathbb{R}$ represent the velocity field and the pressure field of a fluid in $\mathbb{R}^3$, within a finite or infinite timespan of length T. Initial condition

$$u(0,\cdot) = u_0 \in L^2(\mathbb{R}^3)$$

is given by a divergence-free velocity profile u_0 of finite energy.

Leray [Ler34] and Hopf [Hop51] proved the existence of weak solutions for all time. They constructed solutions $u \in C_w(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$ corresponding to each aforementioned initial value, and satisfying (4.1) in the sense of distribution. A weak solution is called a Leray–Hopf solution if it satisfies energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u\|_{L^2((0,t)\times\mathbb{R}^3)}^2 \le \frac{1}{2}\|u_0\|_{L^2(\mathbb{R}^3)}^2$$

for every $t > 0$. Since then, much work has been developed in regard to the uniqueness and regularity of weak solutions. Nonuniqueness of weak solutions was proven very recently by Buckmaster and Vicol using convex integration scheme [BV19]. However, the question of the uniqueness of Leray–Hopf solutions still remains open. The uniqueness is related with the regularity of solutions by the Ladyženskaya–Prodi–Serrin criteria [KL57, Pro59, Ser62, Ser63, FJR72]: if the velocity belongs to any space interpolating $L_t^2 L_x^\infty$ and $L_t^\infty L_x^3$ then it is actually smooth, hence unique. The endpoint case $L_t^\infty L_x^3$ came much later by Eskauriaza, Serëgin and Shverak [ESŠ03]. These spaces require $\frac{1}{6}$ higher spatial integrability than the energy space provides, which is $\mathcal{E} = L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$.

At the level of energy space, Scheffer began to study the partial regularity for a class of Leray–Hopf solutions, called suitable weak solutions [Sch76, Sch77, Sch78, Sch80]. These solutions exist globally and satisfy the following local energy inequality,

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + P\right)\right) + |\nabla u|^2 \le \Delta \frac{|u|^2}{2}.$$

Scheffer showed the singular set, at which the solution is unbounded nearby, has time-space Hausdorff dimension at most $\frac{5}{3}$. This result was later improved by Caffarelli, Kohn and Nirenberg in [CKN82] (see also [Lin98, Vas07]), where they showed the 1-dimensional Hausdorff measure of the singular set is zero. We will investigate the regularity of suitable weak solutions. In the periodic setting, Constantin constructed suitable weak solutions whose second derivatives have space-time integrability $L^{\frac{4}{3}-\varepsilon}$ for any $\varepsilon > 0$, provided the initial vorticities are bounded measures

[Con90]. This was improved by Lions to a slightly better space $L^{\frac{4}{3},\infty}$, a Lorentz space which corresponds to weak $L^{\frac{4}{3}}$ space [Lio96]. These estimates are extended to higher derivatives of smooth solutions by one of the authors and Choi using blow-up arguments: $L^{p,\infty}_{\text{loc}}$ space-time boundedness for $(-\Delta)^{\frac{\alpha}{2}}\nabla^n u$, where $p = \frac{4}{n+\alpha+1}$, $n \geq 1$, $0 \leq \alpha < 2$ [Vas10, CV14]. They also constructed suitable weak solutions satisfying these bounds for $n + \alpha < 3$.

The aim of this paper is to improve these regularity results in Lorentz space. The main result is the following. Note that the estimate does not rely on the size of the pressure.

Theorem 4.1. *Suppose we have a smooth solution u to the Navier–Stokes equations in $(0,T)\times\mathbb{R}^3$ for some $0 < T \leq \infty$ with smooth divergence free initial data $u_0 \in L^2$. Then for any integer $n \geq 0$, for any real number $q > 1$, the vorticity $\omega = \operatorname{curl} u$ satisfies*

$$\left\| |\nabla^n \omega|^{\frac{4}{n+2}} \mathbf{1}_{\{|\nabla^n\omega|^{\frac{4}{n+2}} > C_n t^{-2}\}} \right\|_{L^{1,q}((0,T)\times\mathbb{R}^3)} \leq C_{q,n}\|u_0\|^2_{L^2} \tag{4.2}$$

for some constant C_n depending on n and $C_{q,n}$ depending only on q and n, uniform in T. The above estimate (4.2) also holds for suitable weak solutions with only L^2 divergence free initial data in the case $n = 1$.

This theorem gives the following improvement on the second derivatives.

Corollary 4.2. *Let u be a suitable weak solution in $(0,\infty)\times\mathbb{R}^3$ with initial data $u_0 \in L^2$. Then for any $q > \frac{4}{3}$, $K \subset\subset (0,\infty)\times\mathbb{R}^3$, there exists a constant $C_{q,K}$ depending on q and K such that the following holds,*

$$\|\nabla^2 u\|_{L^{\frac{4}{3},q}(K)} \leq C_{q,K}\left(\|u_0\|^{\frac{3}{2}}_{L^2} + 1\right).$$

Let us explain the main ideas of the proof. Similar as previous work on higher derivatives, the proof is also based on blow-up techniques. In particular, we blow up the equation along a trajectory, using the scaling symmetry and the Galilean invariance of the Navier–Stokes equations. That is, if we fix an initial time t_0 and move the frame of reference along some $X(t)$, and zoom in into ε scale, then it is easy to verify that $\tilde{u}(s, y)$, $\tilde{P}(s, y)$ defined by

$$\frac{1}{\varepsilon}\tilde{u}\left(\frac{t-t_0}{\varepsilon^2}, \frac{x-X(t)}{\varepsilon}\right) := u(t,x) - \dot{X}(t) \tag{4.3}$$

$$\frac{1}{\varepsilon^2}\tilde{P}\left(\frac{t-t_0}{\varepsilon^2}, \frac{x-X(t)}{\varepsilon}\right) := P(t,x) + x \cdot \ddot{X}(t)$$

also satisfy the Navier–Stokes equation

$$\partial_s \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{P} = \Delta \tilde{u}, \qquad \operatorname{div} \tilde{u} = 0.$$

We develop the following local theorem for $\tilde{u}$ and $\tilde{P}$. Note that it needs nothing from the pressure. Denote $B_r \subset \mathbb{R}^3$ to be a ball centered at the origin with radius r, and $Q_r = (-r^2, 0) \times B_r \subset \mathbb{R}^4$ to be a space-time cylinder.

Theorem 4.3 (Local Theorem)**.** *There exists a universal constant $\eta_1 > 0$, such that for any suitable weak solution u to the Navier–Stokes equations in $(-4, 0) \times \mathbb{R}^3$ satisfying*

$$\int_{B_1} u(t,x)\phi(x)\,\mathrm{d}x = 0 \qquad \text{a.e. } t \in (-4, 0), \tag{4.4}$$

$$\|\nabla u\|_{L_t^{p_1} L_x^{q_1}(Q_2)} + \|\omega\|_{L_t^{p_2} L_x^{q_2}(Q_2)} \le \eta_1, \tag{4.5}$$

where $\phi \in C_c^\infty(B_1)$ is a non-negative function with $\int \phi = 1$, $\omega = \operatorname{curl} u$ is the vorticity, $\frac{4}{3} \le p_1 \le \infty$, $1 \le p_2 \le \infty$, $1 \le q_1, q_2 < 3$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} < 1, \qquad \frac{1}{q_1} + \frac{1}{q_2} \le \frac{7}{6},$$

then for any integer $n \geq 0$, we have

$$\|\nabla^n \omega\|_{L^\infty(Q_{8^{-n-2}})} \leq C_n$$

for some constant C_n depending only on n.

Let us illustrate the ideas of how to go from this local theorem towards the main result. We want to choose a "pivot quantity", blow up near a point, and use this quantity to control $\nabla^n \omega$. When we patch the local results together, we will obtain a nonlinear bound with the same scaling as the pivot quantity, so we want the pivot quantity to have the best possible scaling. The ideal pivot quantities would be $\int |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t$ and $\int |\nabla^2 P| \, \mathrm{d}x \, \mathrm{d}t$. $\int |u|^{\frac{10}{3}} \, \mathrm{d}x \, \mathrm{d}t$ has a worse scaling and should not be used. However, we still need to control the flux in the local theorem, so we want to take out the mean velocity and control u by ∇u using Poincaré's inequality.

In order to take out the mean velocity, we choose $X(t)$ to be the trajectory of the mollified flow so that (4.4) can be realized. Notice that a cylinder Q_r in the local (s, y) coordinate will be transformed into a "skewed cylinder" growing along $X(t)$ in the global (t, x) coordinate. One of the authors recently constructed a maximal function $\mathcal{M}_{\mathcal{Q}}$ associated with these cylinders [Yan20], which serves as a bridge between the local theorem and the global result, and is one of the main reasons for the improvement in this paper. The idea is, if locally the vorticity gradient can be controlled in L^∞ by the integral of something in the skewed cylinder, and the integral in a skewed cylinder can be controlled by the maximal function $\mathcal{M}_{\mathcal{Q}}$, then vorticity gradient is pointwise bounded by the maximal function.

If one uses $\int |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t$ and $\int |\nabla^2 P| \, \mathrm{d}x \, \mathrm{d}t$ as the pivot quantity, then unfortunately the best possible outcome would just be an $L^{1,\infty}$ bound, as obtained in [Yan20]. The reason is, the maximal function is bounded on L^p for $p > 1$, but for $p = 1$ it is only bounded from L^1 to $L^{1,\infty}$. Unfortunately $|\nabla u|^2$ and $|\nabla^2 P|$ are both L^1 quantities, so $\mathcal{M}_{\mathcal{Q}} \left(|\nabla u|^2 + |\nabla^2 P| \right)$ is only $L^{1,\infty}$. We need two things to improve from $L^{1,\infty}$: replace $\int |\nabla u|^2$ by $\int |\nabla u|^p$, and drop the pressure $\nabla^2 P$.

Suppose we could use $(\int |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t)^{\frac{2}{p}}$ as the pivot quantity for some $p < 2$, then we can majorize it by $\mathcal{M}_{\mathcal{Q}}\,(|\nabla u|^p)^{\frac{2}{p}} \in L^1$, since $\frac{2}{p} > 1$ and $\mathcal{M}_{\mathcal{Q}}$ is bounded in $L^{\frac{2}{p}}$. However, this poses significant difficulties in the local theorem. The nonlinear term $u \cdot \nabla u$ is quadratic, and if we only have a subquadratic integrability to begin with, we cannot treat this quadratic transport term as a source term because it is not integrable. Observe that what we lack is the temporal integrability rather than the spatial one: if p is slightly smaller than two, then $u \cdot \nabla u$ is still $L^{\frac{3}{2}-}$ in space, but L^{1-} in time. To overcome this difficulty, we write $u \cdot \nabla u$ as $\omega \times u$ up to a gradient term, and put $L^{2-}_t L^{6-}_x$ on u and $L^{2+}_t L^{2-}_x$ on ω. We compensate the lower integrability term by pairing with a higher integrability term to make $\omega \times u$ integrable. $L^{2+}_t L^{2-}_x$ of ω can be interpolated between $L^{2-}_t L^{2-}_x$ and $L^\infty_t L^1_x$, while the latter is controlled by $L^2_{t,x}$ of ∇u. Since $L^{2+}_t L^{2-}_x$ is closer to $L^{2-}_t L^{2-}_x$ than to $L^\infty_t L^1_x$, the pivot quantity that we use is actually $\delta^{-\nu} \|\nabla u\|^2_{L^p} + \delta \|\nabla u\|^2_{L^2}$ for ν close to 0. By using more subquadratic integrability and a tiny bit of the quadratic one, we can complete the task by interpolation. That is why we obtain $L^{1,q}$ in the end: it interpolates L^1 bound from $\|\nabla u\|_{L^p}$ and $L^{1,\infty}$ bound from $\|\nabla u\|_{L^2}$. Unfortunately we still miss the endpoint L^1.

The second task is more subtle and technical. Without any information on the pressure, we don't have any control on the nonlocal effect. However, the role of the pressure is not important at the vorticity level: if we take the curl of the Navier–Stokes equation, the pressure will disappear and we are left with the vorticity equation involving only local quantities:

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega. \tag{4.6}$$

Inspired by Chamorro, Lemarié-Rieusset and Mayoufi [CLRM18], we introduce a new velocity variable $v = -\operatorname{curl} \varphi^\sharp \Delta^{-1} \varphi \omega$ using only local information of vorticity

(φ and $\varphi^\sharp$ are spatial cut-off functions), and this helps us to prove the local theorem. This is another main reason for the improvement in this paper. Consequently, the bounds we obtain in the end are on the vorticity ω rather than on the velocity u.

This paper is organized as follows. In the preliminary Section 2 we introduce the analysis tools to the reader. We show how to rigorously derive the main results from the local theorem in Section 3, and then deal with technicalities of the local theorem in the later sections. The proof of the local theorem consists of three parts. Section 4 introduces the new variables v, and shows the smallness of v in the energy space. Then we use De Giorgi iteration argument in Section 5 to prove boundedness of v. Finally, we inductively bound ω and all its higher derivatives in Section 6.

4.2 Preliminary

In this section, we introduce a few tools that we are going to use in the paper, including the maximal function, Lorentz space, and Helmholtz decomposition.

4.2.1 Maximal Function associated with Skewed Cylinders

This is recently developed for incompressible flows in [Yan20]. We quote useful results here without proof.

Suppose $u \in L^p(0,T;\dot{W}^{1,p}(\mathbb{R}^3;\mathbb{R}^3))$ is a vector field in $\mathbb{R}^3$. Fix $\phi \in C_c^\infty(B_1)$ to be a nonnegative function with $\int \phi = 1$ through out the paper. For $\varepsilon > 0$ define $\phi_\varepsilon(x) = \varepsilon^{-3}\phi(-x/\varepsilon)$, and let $u_\varepsilon(t,\cdot) = u(t,\cdot) * \phi_\varepsilon$ be the mollified velocity. For a fixed (t,x) we let $X(s)$ solve the following initial value problem,

$$\begin{cases} \dot{X}(s) = u_\varepsilon(s, X(s)), \\ X(t) = x. \end{cases}$$

The skewed parabolic cylinder $Q_\varepsilon(t,x)$ is then defined to be

$$Q_\varepsilon(t,x) := \left\{(t+\varepsilon^2 s, X(t)+\varepsilon y) : -9 \le s \le 0, y \in B_3\right\}. \tag{4.7}$$

We use $\mathcal{M}$ to denote the spatial Hardy-Littlewood maximal function, which is defined by

$$\mathcal{M}(f)(t,x) = \sup_{r>0} ⨍_{B_r(x)} |f(t,y)| \,\mathrm{d}y.$$

Then we construct the space-time maximal function adapted to the flow.

Theorem 4.4 ($\mathcal{Q}$-Maximal Function)**.** *There exists a universal constant η_0 such that the following is true. We say $Q_\varepsilon(t,x)$ is admissible if $Q_\varepsilon(t,x) \subset (0,T)\times\mathbb{R}^3$ and*

$$\varepsilon^2 ⨍_{Q_\varepsilon(t,x)} \mathcal{M}(|\nabla u|) \,\mathrm{d}x\,\mathrm{d}t \le \eta_0. \tag{4.8}$$

Define the maximal function

$$\mathcal{M}_{\mathcal{Q}}(f)(t,x) := \sup_{\varepsilon>0}\left\{ ⨍_{Q_\varepsilon(t,x)} |f(s,y)| \,\mathrm{d}s\,\mathrm{d}y : Q_\varepsilon(t,x) \text{ is admissible}\right\}.$$

If u is divergence free and $\mathcal{M}(|\nabla u|) \in L^q$ for some $1 \le q \le \infty$, then $\mathcal{M}_{\mathcal{Q}}$ is bounded from $L^1((0,T)\times\mathbb{R}^3)$ to $L^{1,\infty}((0,T)\times\mathbb{R}^3)$ and from $L^p((0,T)\times\mathbb{R}^3)$ to itself for any $p>1$ with norm depending on p.

An important consequence of the weak type $(1,1)$ bound of the Hardy-Littlewood maximal function is the Lebesgue differentiation theorem in $\mathbb{R}^n$. Similarly, we can use the $\mathcal{Q}$-maximal function to prove the $\mathcal{Q}$-Lebesgue differentiation theorem.

Theorem 4.5 ($\mathcal{Q}$-Lebesgue Differentiation Theorem)**.** *Let $f \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$.*

Then for almost every $(t,x) \in (0,T) \times \mathbb{R}^3$,

$$\lim_{\varepsilon \to 0} \fint_{Q_\varepsilon(t,x)} |f(s,y) - f(t,x)| \, \mathrm{d}s \, \mathrm{d}y = 0.$$

In this case we say (t,x) is a $\mathcal{Q}$-Lebesgue point of f.

4.2.2 Lorentz Space

Let (X,μ) be a measure space. Recall that for a measurable function f, its decreasing rearrangement is defined as

$$f^*(\lambda) := \inf\left\{\alpha > 0 : \mu(\{|f| > \alpha\}) < \lambda\right\}, \qquad \lambda \geq 0.$$

For $0 < p < \infty$, $0 < q \leq \infty$, Lorentz space $L^{p,q}(X)$ is defined as the set of functions f for which

$$\|f\|_{L^{p,q}(X)} := \|\lambda^{\frac{1}{p}} f^*\|_{L^q(\frac{\mathrm{d}\lambda}{\lambda})} = \|\lambda^{\frac{1}{p}-\frac{1}{q}} f^*(\lambda)\|_{L^q} < \infty.$$

Now we introduce the interpolation lemma for Lorentz spaces.

Lemma 4.6 (Interpolation of Lorentz Spaces)**.** *Let $\nu > 0$ be a fixed positive number. Assume $f_0 \in L^{p_0,q_0}$, $f_1 \in L^{p_1,q_1}$, where $0 < p_0, p_1 < \infty, 0 < q_0, q_1 \leq \infty$. If f is a measurable function satisfying*

$$2|f| \leq \delta f_0 + \delta^{-\nu} f_1 \qquad \textit{for all } \delta > 0, \tag{4.9}$$

then $f \in L^{p,q}$, where

$$\frac{1}{p} = \frac{\nu}{1+\nu}\frac{1}{p_0} + \frac{1}{1+\nu}\frac{1}{p_1}, \qquad \frac{1}{q} = \frac{\nu}{1+\nu}\frac{1}{q_0} + \frac{1}{1+\nu}\frac{1}{q_1}.$$

Proof. It is easy to check from the definition of decreasing rearrangement that if

$h \le f + g$, then $h^*(2\lambda) \le (f+g)^*(2\lambda) \le f^*(\lambda) + g^*(\lambda)$. Thus (4.9) implies

$$2|f^*(2\lambda)| \le \delta f_0^*(\lambda) + \delta^{-\nu} f_1^*(\lambda), \qquad \text{for all } \lambda \ge 0, \delta > 0.$$

Set $\theta = \frac{1}{1+\nu}$, $\delta = f_0^*(\lambda)^{-\theta} f_1^*(\lambda)^{\theta}$, then

$$\begin{aligned} 2|f^*(2\lambda)| &\le f_0^*(\lambda)^{-\theta} f_1^*(\lambda)^{\theta} f_0^*(\lambda) + f_0^*(\lambda)^{\nu\theta} f_1^*(\lambda)^{-\nu\theta} f_1^*(\lambda) \\ &= 2 f_0^*(\lambda)^{1-\theta} f_1^*(\lambda)^{\theta}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_{L^{p,q}} &= \|\lambda^{\frac{1}{p}-\frac{1}{q}} f^*(\lambda)\|_{L^q} = C\|\lambda^{\frac{1}{p}-\frac{1}{q}} f^*(2\lambda)\|_{L^q} \\ &\le C\|\lambda^{\frac{1-\theta}{p_0}-\frac{1-\theta}{q_0}} f_0^*(\lambda)^{1-\theta} \cdot \lambda^{\frac{\theta}{p_1}-\frac{\theta}{q_1}} f_1^*(\lambda)^{\theta}\|_{L^q} \\ &\le C\|\lambda^{\frac{1-\theta}{p_0}-\frac{1-\theta}{q_0}} f_0^*(\lambda)^{1-\theta}\|_{L^{\frac{q_0}{1-\theta}}} \|\lambda^{\frac{\theta}{p_1}-\frac{\theta}{q_1}} f_1^*(\lambda)^{\theta}\|_{L^{\frac{q_1}{\theta}}} \\ &= C\|\lambda^{\frac{1}{p_0}-\frac{1}{q_0}} f_0^*(\lambda)\|_{L^{q_0}}^{1-\theta} \|\lambda^{\frac{1}{p_1}-\frac{1}{q_1}} f_1^*(\lambda)\|_{L^{q_1}}^{\theta} \\ &= C\|f_0\|_{L^{p_0,q_0}}^{1-\theta} \|f_1\|_{L^{p_1,q_1}}^{\theta}, \end{aligned}$$

where $C = 2^{\frac{1}{p}}$. □

We would also like to mention that Riesz transform is bounded on Lorentz space. The proof can be found in [CF07]. See [Saw90] for general Lorentz spaces.

Lemma 4.7. *For $1 < p < \infty$, $1 \le q \le \infty$, $R_{ij} = \partial_i \partial_j \Delta^{-1}$ is a bounded linear operator from $L^{p,q}(\mathbb{R}^n)$ to itself. As a spatial operator, it is also bounded in time-space from $L^{p,q}((0,T) \times \mathbb{R}^n)$ to itself.*

4.2.3 Helmholtz decomposition

First recall two vector calculus identities:

$$\nabla(u \cdot v) = (u \cdot \nabla)v + (v \cdot \nabla)u + u \times \operatorname{curl} v + v \times \operatorname{curl} u, \tag{4.10}$$

$$\operatorname{curl}(u \times v) = u \operatorname{div} v - v \operatorname{div} u + (v \cdot \nabla)u - (u \cdot \nabla)v. \tag{4.11}$$

For operators A and B, denote $[A, B] = AB - BA$ to be their commutator. Define $\mathbb{P}_{\text{curl}} = -\operatorname{curl}\operatorname{curl}\Delta^{-1}$ and $\mathbb{P}_{\nabla} = \nabla\Delta^{-1}\operatorname{div} = \mathrm{Id} - \mathbb{P}_{\text{curl}}$ to be the Helmholtz decomposition. Then we compute the following commutators.

$$[\varphi, \operatorname{curl}]u = -\nabla\varphi \times u, \tag{4.12}$$

$$[\varphi, \Delta]u = -2\nabla\varphi \cdot \nabla u - (\Delta\varphi)u = -2\operatorname{div}(\nabla\varphi \otimes u) + (\Delta\varphi)u, \tag{4.13}$$

$$[\varphi, \Delta^{-1}]u = \Delta^{-1}\left\{2\nabla\varphi \cdot \nabla\Delta^{-1}u + (\Delta\varphi)\Delta^{-1}u\right\}, \tag{4.14}$$

$$\begin{aligned}[\varphi, \mathbb{P}_{\text{curl}}]u = {} & \nabla\varphi \times \operatorname{curl}\Delta^{-1}u + \nabla\varphi \operatorname{div}\Delta^{-1}u - \Delta^{-1}u\Delta\varphi \\ & + (\Delta^{-1}u \cdot \nabla)\nabla\varphi - (\nabla\varphi \cdot \nabla)\Delta^{-1}u \\ & + \mathbb{P}_{\text{curl}}\left\{2\nabla\varphi \cdot \nabla\Delta^{-1}u + (\Delta\varphi)\Delta^{-1}u\right\}.\end{aligned} \tag{4.15}$$

The first two are straightforward. The third uses

$$[\varphi, \Delta^{-1}] = -\Delta^{-1}[\varphi, \Delta]\Delta^{-1},$$

and the last one is because

$$\begin{aligned}[\varphi, \mathbb{P}_{\text{curl}}] &= [\varphi, -\operatorname{curl}\operatorname{curl}\Delta^{-1}] \\ &= -[\varphi, \operatorname{curl}]\operatorname{curl}\Delta^{-1} - \operatorname{curl}[\varphi, \operatorname{curl}]\Delta^{-1} - \operatorname{curl}\operatorname{curl}[\varphi, \Delta^{-1}], \\ [\varphi, \mathbb{P}_{\text{curl}}]u &= \nabla\varphi \times \operatorname{curl}\Delta^{-1}u + \operatorname{curl}(\nabla\varphi \times \Delta^{-1}u) \\ &\quad - \operatorname{curl}\operatorname{curl}\Delta^{-1}\left\{2\nabla\varphi \cdot \nabla\Delta^{-1}u + (\Delta\varphi)\Delta^{-1}u\right\},\end{aligned}$$

and we can expand $\operatorname{curl}(\nabla\varphi \times \Delta^{-1}u)$ by (4.11).

Lemma 4.8. *$\partial_i[\varphi, \mathbb{P}_{\text{curl}}]$ and $[\varphi, \mathbb{P}_{\text{curl}}]\partial_i$ are both bounded linear operator from L^p to L^p for any $1 < p < \infty$, i.e.*

$$\|\partial_i[\varphi, \mathbb{P}_{\text{curl}}]u\|_{L^p} + \|[\varphi, \mathbb{P}_{\text{curl}}]\partial_i u\|_{L^p} \leq C_{p,\varphi}\|u\|_{L^p}.$$

Proof. First, we observe that by Jacobi identity $[\varphi, \mathbb{P}_{\text{curl}}]\partial_i$ and $\partial_i[\varphi, \mathbb{P}_{\text{curl}}]$ differ by

$$[[\varphi, \mathbb{P}_{\text{curl}}], \partial_i] = [\varphi, [\mathbb{P}_{\text{curl}}, \partial_i]] - [\mathbb{P}_{\text{curl}}, [\varphi, \partial_i]] = 0 - [\mathbb{P}_{\text{curl}}, \partial_i\varphi]$$

which is bounded from L^p to L^p for any p, because both $\mathbb{P}_{\text{curl}}$ and multiplication by $\partial_i\varphi$ are bounded from L^p to L^p, so we can complete the proof by duality. For $1 < p < 3$, set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$, from (4.15) we can see

$$\begin{aligned}
\|[\varphi, \mathbb{P}_{\text{curl}}]\partial_i u\|_{L^p} &\lesssim \|\nabla\Delta^{-1}\partial_i u\|_{L^p(\mathbb{R}^3)} + C_{p,\varphi}\|\Delta^{-1}\partial_i u\|_{L^p(\operatorname{supp}\varphi)} \\
&\lesssim \|u\|_{L^p(\mathbb{R}^3)} + C_{p,\varphi}\|\partial_i\Delta^{-1}u\|_{L^{p^*}(\operatorname{supp}\varphi)} \\
&\leq C\|u\|_{L^p(\mathbb{R}^3)}.
\end{aligned}$$

For $\frac{3}{2} < p < \infty$, set $1 - \frac{1}{p} = \frac{1}{q} = \frac{1}{q^*} + \frac{1}{3}$, then $1 < p, q, q^* < \infty$. Take any $u \in L^p(\mathbb{R}^3)$ and any vector field $v \in L^q(\mathbb{R}^3)$,

$$\begin{aligned}
\int \partial_i[\varphi, \mathbb{P}_{\text{curl}}]u \cdot v \,\mathrm{d}x &= -\int [\varphi, \mathbb{P}_{\text{curl}}]u \cdot \partial_i v \,\mathrm{d}x \\
&= \int u \cdot [\varphi, \mathbb{P}_{\text{curl}}]\partial_i v \,\mathrm{d}x \\
&\leq \|u\|_{L^p(\mathbb{R}^3)}(\|v\|_{L^q(\mathbb{R}^3)} + \|\partial_i\Delta^{-1}v\|_{L^q(\operatorname{supp}\varphi)}) \\
&\leq \|u\|_{L^p(\mathbb{R}^3)}(\|v\|_{L^q(\mathbb{R}^3)} + C_{q,\varphi}\|\partial_i\Delta^{-1}v\|_{L^{q^*}(\operatorname{supp}\varphi)}) \\
&\leq C\|u\|_{L^p(\mathbb{R}^3)}\|v\|_{L^q(\mathbb{R}^3)}.
\end{aligned}$$

$\square$

Corollary 4.9. *$\partial_i[\varphi, \mathbb{P}_\nabla]$ and $[\varphi, \mathbb{P}_\nabla]\partial_i$ are both bounded linear operator from L^p to L^p for any $1 < p < \infty$.*

Proof. $\mathrm{Id} = \mathbb{P}_\nabla + \mathbb{P}_{\mathrm{curl}}$ commutes with φ, so $[\varphi, \mathbb{P}_\nabla] = -[\varphi, \mathbb{P}_{\mathrm{curl}}]$. $\square$

Because of the smoothing effect of the Laplace potential, we have the following.

Lemma 4.10. *Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be supported away from some openset $\Omega \subset \mathbb{R}^3$, that is, $\mathrm{dist}(\operatorname{supp}\varphi, \Omega) = d > 0$. Then for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $k > 0$,*

$$\|\Delta^{-1}(\varphi f)\|_{C^k(\Omega)} \lesssim_{k,d} \|f\|_{L^1(\operatorname{supp}\varphi)}.$$

We also have

$$\|\mathbb{P}_\nabla(\varphi f)\|_{C^k(\Omega)}, \|\mathbb{P}_{\mathrm{curl}}(\varphi f)\|_{C^k(\Omega)} \lesssim_{k,d} \|f\|_{L^1(\operatorname{supp}\varphi)}.$$

4.3 Proof of the Main Results

In this section, we show that the Local Theorem 4.3 leads to the main results. First, we show the pivot quantity is indeed enough to bound $\nabla^n\omega$.

Lemma 4.11. *There exists $\eta_2 > 0$ such that the following holds. Let $\frac{11}{6} < p < 2$, $\frac{2-p}{p-1} < \nu \le \frac{7p-12}{6-p}$. If u is a suitable solution to the Navier–Stokes equations in*

$(-9,0)\times\mathbb{R}^3$ *satisfying the following conditions,*

$$\int_{B_1} u(t,x)\phi(x)\,\mathrm{d}x = 0, \qquad \text{a.e. } t\in(-9,0), \tag{4.16}$$

$$\delta^{-\nu}\left(\int_{Q_3} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \le \eta_2, \tag{4.17}$$

$$\delta\int_{Q_3} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \eta_2, \tag{4.18}$$

for some $\delta\le\eta_2$, then we have for any $n\ge 0$,

$$\|\nabla^n\omega\|_{L^\infty_{t,x}(Q_{8^{-n-2}})} \le C_n.$$

Here C_n is the same constant in Theorem 4.3.

Proof. First, we claim that

$$\delta\|\omega\|_{L^\infty(-4,0;L^1(B_2))} \le C\eta_2. \tag{4.19}$$

Formally, we can take the dot product of both sides of the vorticity equation (4.1) with $\omega^0 := \frac{\omega}{|\omega|}$, and recalling the convexity inequality $\omega^0\cdot\Delta\omega \le \Delta|\omega|$, we have

$$(\partial_t + u\cdot\nabla - \Delta)|\omega| - \omega\cdot\nabla u\cdot\omega^0 \le 0. \tag{4.20}$$

Let $\psi\in C_c^\infty((-9,0]\times\mathbb{R}^3)$ be a cut-off function such that $\mathbf{1}_{Q_2}\le\psi\le\mathbf{1}_{Q_3}$. Multiply (4.20) by ψ and then integrate in space,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \psi|\omega|\,\mathrm{d}x &\le \int \left[(\partial_t + u\cdot\nabla + \Delta)\psi\right]|\omega|\,\mathrm{d}x + \int \psi\omega\cdot\nabla u\cdot\omega^0\,\mathrm{d}x \\ &\le C\int_{B_3} 1 + |u|^2 + |\nabla u|^2\,\mathrm{d}x \le C\left(1 + \int_{B_3}|\nabla u|^2\,\mathrm{d}x\right).\end{aligned}$$

for some large universal constant $C>1$. The last step uses Poincaré's inequality and (4.16). Integrate in time we obtain

$$\|\omega\|_{L^\infty(-4,0;L^1(B_2))} \leq C\left(1+\frac{\eta_2}{\delta}\right) \leq 2C\frac{\eta_2}{\delta}.$$

This proves the claim. A more rigorous proof can be obtained by difference quotient same as in Constantin [Con90] or Lions [Lio96] Theorem 3.6, so we omit the details.

Now we interpolate between (4.17) and (4.19). Let $\theta = \frac{1}{1+\nu}$,

$$\|\omega\|_{L_t^{p_2}L_x^{q_2}(Q_2)} \leq \|\omega\|_{L^p(Q_2)}^{\theta}\|\omega\|_{L_t^\infty L_x^1(Q_2)}^{1-\theta} \leq (2C)^{1-\theta}\eta_2\delta^{\theta\nu+\theta-1} \leq \frac{1}{2}\eta_1,$$

where we choose $\eta_2 = \frac{\eta_1}{4C+1} \leq \frac{1}{2}\eta_1$ from Theorem 4.3, and p_2, q_2 are determined by

$$\frac{1}{p_2} = \frac{\theta}{p}, \qquad \frac{1}{q_2} = \frac{\theta}{p} + 1 - \theta.$$

Combine the above with (4.17) we have

$$\|\nabla u\|_{L_t^pL_x^p(Q_2)} + \|\omega\|_{L_t^{p_2}L_x^{q_2}(Q_2)} \leq \frac{1}{2}\eta_1 + \frac{1}{2}\eta_1 = \eta_1. \tag{4.21}$$

By the choice of θ and the range of ν,

$$\begin{aligned}\frac{1}{p} + \frac{1}{p_2} &= \frac{1}{p} + \frac{1}{p(1+\nu)} = \frac{2+\nu}{p(1+\nu)} < 1,\\ \frac{1}{p} + \frac{1}{q_2} &= \frac{1}{p} + \frac{1+\nu p}{p(1+\nu)} = \frac{2+\nu+\nu p}{p(1+\nu)} \leq \frac{7}{6}.\end{aligned}$$

One can also easily check that $p < 2$ implies $q_2 < 2$, and thus by (4.16) and (4.21) the requirements of the Local Theorem 4.3 are satisfied with $p_1 = q_1 = p$, and it completes the proof of the lemma. □

Now we transform this lemma into the global coordinate. Recall that $Q_\varepsilon(t, x)$ is defined by (4.7).

Corollary 4.12. *There exists* $\eta_3 > 0$ *such that the following holds. If for some* $\delta \leq \eta_2$*,*

$$\delta^{-2\nu} \left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{2}{p}} + \delta \fint_{Q_\varepsilon(t,x)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \eta_3 \varepsilon^{-4}, \tag{4.22}$$

then

$$|\nabla^n \omega(t,x)| \leq C_n \varepsilon^{-n-2}.$$

Proof. Define $\tilde{u}$ by (4.3). Then (4.22) implies

$$\begin{aligned} &\delta^{-2\nu} \left(\fint_{Q_3} |\nabla \tilde{u}|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{2}{p}} \leq \eta_3, \qquad \delta \fint_{Q_3} |\nabla \tilde{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \eta_3 \\ \Rightarrow\ &\delta^{-\nu} \left(\int_{Q_3} |\nabla \tilde{u}|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{p}} \leq \eta_3^{\frac{1}{2}} |Q_3|^{\frac{1}{p}}, \qquad \delta \int_{Q_3} |\nabla \tilde{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \eta_3 |Q_3|. \end{aligned}$$

Moreover, (4.16) is satisfied by $\tilde{u}$. Therefore, if we choose η_3 such that

$$\max \left\{ \eta_3^{\frac{1}{2}} |Q_3|^{\frac{1}{p}}, \eta_3 |Q_3| \right\} = \eta_2,$$

then by Lemma 4.11, $\tilde{\omega} := \operatorname{curl} \tilde{u}$ has bounded derivatives at $(0,0)$, and thus finish the proof of the corollary by scaling. □

Then we use the maximal function to go from the local bound to a global bound.

Proof of Theorem 4.1. First, we fix $\frac{11}{6} < p < 2$, $\frac{2-p}{p-1} < \nu \leq \frac{7p-12}{6-p}$. Let $\eta \ll 1$ be a small constant to be specified later. Finally we fix a $0 < \delta < \infty$. For $(t,x) \in$

$(0,T)\times\mathbb{R}^3$, define

$$I(\varepsilon)=\varepsilon^4\left[\delta^{-2\nu}\left(\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^p\right)^{\frac{2}{p}}+\delta\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^2\right].$$

If (t,x) is both a $\mathcal{Q}$-Lebesgue point of $|\mathcal{M}(\nabla u)|^p$ and of $|\mathcal{M}(\nabla u)|^2$, then we claim that there exists a positive $\varepsilon=\varepsilon_{(t,x)}$ such that one of the two cases is true:

`Case 1.` $3\varepsilon_{(t,x)}<t^{\frac{1}{2}}$, and $I(\varepsilon_{(t,x)})=\eta$.

`Case 2.` $3\varepsilon_{(t,x)}=t^{\frac{1}{2}}$, and $I(\varepsilon_{(t,x)})\le\eta$.

This is because by Theorem 4.5

$$\lim_{\varepsilon\to0}I(\varepsilon)=0^4\left[\delta^{-2\nu}(|\mathcal{M}(\nabla u)(t,x)|^p)^{\frac{2}{p}}+\delta|\mathcal{M}(\nabla u)(t,x)|^2\right]=0,$$

and $I(\varepsilon)$ is a continuous function of ε.

On the one hand, in both cases we have $I(\varepsilon)\le\eta$, which implies that

$$\delta^{-\nu}\varepsilon^2\left(\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^p\right)^{\frac{1}{p}}\le\sqrt{\eta},\qquad \delta^{\frac{1}{2}}\varepsilon^2\left(\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^2\right)^{\frac{1}{2}}\le\sqrt{\eta}.$$

If we set $\eta<\eta_0^2$, then depending on $\delta\ge1$ or $\delta\le1$, one of the two would imply admissibility condition (4.8) by Jensen's inequality. Therefore $Q_\varepsilon(t,x)$ is admissible and

$$I(\varepsilon)\le\varepsilon^4\left[\delta^{-2\nu}\mathcal{M}_{\mathcal{Q}}(\mathcal{M}(\nabla u)^p)^{\frac{2}{p}}+\delta\mathcal{M}_{\mathcal{Q}}(\mathcal{M}(\nabla u)^2)\right],$$

so we can combine two canses and conclude

$$\varepsilon_{(t,x)}^{-4}\le\max\left\{\frac{1}{\eta}\left[\delta^{-2\nu}\mathcal{M}_{\mathcal{Q}}(\mathcal{M}(\nabla u)^p)^{\frac{2}{p}}+\delta\mathcal{M}_{\mathcal{Q}}(\mathcal{M}(\nabla u)^2)\right],81t^{-2}\right\}.\qquad(4.23)$$

On the other hand, if we set $\eta < \eta_3$, then in both cases $I(\varepsilon) \le \eta_3$. If $\delta \le \eta_2$ one would have

$$|\nabla^n \omega(t,x)| \le C_n \varepsilon^{-n-2} \tag{4.24}$$

by Corollary 4.12. If $\delta > \eta_2$, notice that by Jensen's inequality,

$$\left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} \le \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2,$$

so

$$\begin{aligned}
I(\varepsilon) &\ge \varepsilon^4 \left[(\delta^{-2\nu} + \delta - \eta_2) \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right] \\
&\ge \varepsilon^4 \left[(1 - \eta_2) \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right] \\
&\ge (1-\eta_2)\eta_2^{2\nu} \varepsilon^4 \left[\eta_2^{-2\nu} \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right].
\end{aligned}$$

If we require $\eta < (1-\eta_2)\eta_2^{2\nu}\eta_3$, then

$$\varepsilon^4 \left[\eta_2^{-2\nu} \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right] \le \eta_3.$$

again by Corollary 4.12 we would still have (4.24). In conclusion, we choose

$$\eta = \min \left\{ \eta_0^2, (1-\eta_2)\eta_2^{2\nu}\eta_3 \right\},$$

then for any $0 < \delta < \infty$ one would have

$$|\nabla^n \omega(t,x)|^{\frac{4}{n+2}} \le C_n^{\frac{4}{n+2}} \max\left\{ \frac{1}{\eta}\left[\delta^{-2\nu}\mathcal{M}_{\mathcal{Q}}(\mathcal{M}(\nabla u)^p)^{\frac{2}{p}} + \delta \mathcal{M}_{\mathcal{Q}}(\mathcal{M}(\nabla u)^2)\right], 81t^{-2}\right\}$$

by putting (4.24) and (4.23) together. Denote $f = |\nabla^n \omega|^{\frac{4}{n+2}}$, and we denote $f_1 = \mathcal{M}_{\mathcal{Q}}(\mathcal{M}(|\nabla u|)^p)^{\frac{2}{p}}$, $f_2 = \mathcal{M}_{\mathcal{Q}}(\mathcal{M}(|\nabla u|)^2)$. Then we have almost everywhere

$$f \mathbf{1}_{\{f > C_n t^{-2}\}} \lesssim_n \delta^{-2\nu} f_1 + \delta f_2.$$

By Theorem 4.4,

$$\|f_1\|_{L^1} \le C_p \|\mathcal{M}(\nabla u)^p\|_{L^{\frac{2}{p}}}^{\frac{2}{p}} \lesssim C_p \|\mathcal{M}(\nabla u)^2\|_{L^1} \le C_p \|\nabla u\|_{L^2}^2,$$
$$\|f_2\|_{L^{1,\infty}} \le C_1 \|\mathcal{M}(\nabla u)^2\|_{L^1} \le C_1 \|\nabla u\|_{L^2}^2.$$

Finally, by the interpolation between Lorentz spaces Lemma 4.6,

$$\|f \mathbf{1}_{\{f > C_n t^{-2}\}}\|_{L^{1,1+2\nu}} \lesssim_{p,n} \|\nabla u\|_{L^2((0,T)\times\mathbb{R}^3)}^2 \le \|u_0\|_{L^2(\mathbb{R}^3)}^2.$$

This proves the theorem for $q \ge 1 + 2\nu$. Recall that p can be arbitrarily chosen between $\frac{11}{6}$ and 2, and ν can be chosen between $\frac{2-p}{p-1}$ and $\frac{7p-12}{6-p}$, so ν can be arbitrarily small, therefore we prove the theorem for any $q > 1$. □

Estimates on $\nabla^2 u$ can be obtained by a Riesz transform of $\Delta u = -\operatorname{curl}\omega$.

Proof of Corollary 4.2. We can put $K \subset (t_0, T) \times B_R$ for some $t_0, T, R > 0$. Denote $Q = (t_0, T) \times B_{2R}$. Let $\rho \in C_c^\infty(\mathbb{R}^3)$ be a smooth spatial cut-off function between $\mathbf{1}_{B_R} \le \rho \le \mathbf{1}_{B_{2R}}$. Then

$$\|\Delta(\rho u)\|_{L^{\frac{4}{3},q}((t_0,T)\times\mathbb{R}^3)} \lesssim_\rho \|\Delta u\|_{L^{\frac{4}{3},q}(Q)} + \|\nabla u\|_{L^{\frac{4}{3},q}(Q)} + \|u\|_{L^{\frac{4}{3},q}(Q)}.$$

Since $\Delta u = -\operatorname{curl}\omega$, the case $n = 1$ of Theorem 4.1 gives

$$\|\Delta u \mathbf{1}_{\{|\Delta u| > C_1 t^{-\frac{3}{2}}\}}\|_{L^{\frac{4}{3},q}((0,T)\times\mathbb{R}^3)} \le C_q \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}},$$

so

$$\|\Delta u\|_{L^{\frac{4}{3},q}(Q)} \le C_q \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} + C_1 \|t^{-\frac{3}{2}}\|_{L^{\frac{4}{3}}(Q)} \lesssim C_q \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} + C_1 \left(\frac{R^3}{t_0}\right)^{\frac{3}{4}}.$$

As for lower order terms,

$$\|\nabla u\|_{L^{\frac{4}{3}}(Q)} \lesssim \|\nabla u\|_{L^2(Q)},$$
$$\|u\|_{L^{\frac{4}{3}}(Q)} \le \|u\|_{L_t^\infty L_x^2(Q)}.$$

For Leray–Hopf solution, $\|\nabla u\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1((0,T)\times\mathbb{R}^3)} \le \|u_0\|_{L^2}$, so

$$\|\Delta(\rho u)\|_{L^{\frac{4}{3},q}((t_0,T)\times\mathbb{R}^3)} \lesssim_{q,K} \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} + 1 + \|u_0\|_{L^2(\mathbb{R}^3)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} + 1.$$

Because Riesz transform is bounded from $L^{\frac{4}{3},q}((t_0,T)\times\mathbb{R}^3)$ to itself by Lemma 4.7,

$$\|\nabla^2 u\|_{L^{\frac{4}{3},q}(K)} \le \|\nabla^2(\rho u)\|_{L^{\frac{4}{3},q}(Q)} \lesssim_{q,K} \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} + 1.$$

□

Remark 4.13. For smooth solutions to the Navier–Stokes equation, we have $L^{1,q}$ estimate for the third derivatives for any $q > 1$,

$$\left\|\nabla^2 \omega \mathbf{1}_{\{|\nabla^2\omega| > Ct^{-2}\}}\right\|_{L^{1,q}((0,T)\times\mathbb{R}^3)} \le C_q \|u_0\|_{L^2}^2.$$

4.4 Local Study: Part One, Initial Energy

The following three sections are dedicated to the proof of the Local Theorem 4.3. In [Vas10], the proof of the local theorem consists of the following three parts:

`Step 1.` Show the velocity u is locally small in the energy space $\mathcal{E} = L_t^\infty L_x^2 \cap L_t^2 H_x^1$.

`Step 2.` Use De Giorgi iteration and the truncation method developed in [Vas07] to show u is locally bounded in L^∞.

`Step 3.` Bootstrap to higher regularity by differentiating the original equation.

In our case, directly working with u is difficult due to the lack of control on the pressure, which is nonlocal. Therefore, we would like to work on vorticity, whose evolution is governed by (4.6) and only involves local quantities. Since ω is one derivative of u, we have less integrability to do any parabolic regularization, and we don't have the local energy inequality to perform De Giorgi iteration. This motivates us to work on minus one derivative of ω, but instead of ω we use a localization of ω. Similar as [CLRM18], we introduce a new local quantity

$$\boxed{v := -\operatorname{curl} \varphi^\sharp \Delta^{-1} \varphi \operatorname{curl} u = -\operatorname{curl} \varphi^\sharp \Delta^{-1} \varphi \omega.}$$

where φ and $\varphi^\sharp$ are a pair of fixed smooth spatial cut-off functions, which are defined between $\mathbf{1}_{B_{\frac{6}{5}}} \le \varphi \le \mathbf{1}_{B_{\frac{5}{4}}}$, $\mathbf{1}_{B_{\frac{4}{3}}} \le \varphi^\sharp \le \mathbf{1}_{B_{\frac{3}{2}}}$. This v is divergence free and compactly supported. It will help us get rid of the pressure P, while staying in the same space as u: it scales the same as u, has the same regularity, inherit a local energy inequality from u, and its evolution only depends on local information. We will follow the same three steps above, but we will work on v instead of u.

For convenience, from now on we will use η to denote a small universal constant depending only on the smallness of η_1, such that $\lim_{\eta_1 \to 0} \eta = 0$. Similar as the constant C, the value of η may change from line to line. The purpose of this section is to obtain the smallness of v in the energy space $\mathcal{E}$, which is the following proposition.

Proposition 4.14. *Under the same assumptions of the Local Theorem 4.3, we have*

$$\|v\|^2_{\mathcal{E}(Q_1)} = \sup_{t\in(-1,0)} \int_{B_1} |v(t)|^2 \,\mathrm{d}x + \int_{Q_1} |\nabla v|^2 \,\mathrm{d}x \le \eta. \tag{4.25}$$

For convenience, define q_3, q_4, q_5 by

$$\frac{1}{q_3} = \frac{1}{q_1} - \frac{1}{3}, \qquad \frac{1}{q_4} = \frac{1}{q_2} - \frac{1}{3}, \qquad \frac{1}{q_5} = \left(\frac{1}{q_3} - \frac{1}{3}\right)_+.$$

4.4.1 Equations of v

We use (4.10) in (4.1) to rewrite the equation of u, then take the curl to rewrite the equation of ω, finally apply $-\operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi$ on the vorticity equation to obtain the equation of v.

$$\begin{aligned} \partial_t u + \mathbb{P}_{\mathrm{curl}}(\omega\times u) &= \Delta u, \\ \partial_t \omega + \operatorname{curl}(\omega\times u) &= \Delta\omega, \\ \partial_t v - \operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi\operatorname{curl}(\omega\times u) &= -\operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi\Delta\omega. \end{aligned} \tag{4.26}$$

The second term of (4.26) is

$$\operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi\operatorname{curl}(\omega\times u) = \mathbf{B} - \mathbb{P}_{\mathrm{curl}}(\varphi\omega\times u)$$

where $\mathbf{B}$ denotes the quadratic commutator

$$\begin{aligned} \mathbf{B} &:= -\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\operatorname{curl}(\omega\times u) + \operatorname{curl}\Delta^{-1}[\varphi,\operatorname{curl}](\omega\times u) \\ &= -\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\operatorname{curl}(\omega\times u) + \operatorname{curl}\Delta^{-1}(-\nabla\varphi\times(\omega\times u)) \end{aligned}$$

Here we used (4.12). The right hand side of (4.26) is

$$-\operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi\Delta\omega = \Delta v + \mathbf{L}$$

where $\mathbf{L}$ denotes the linear commutator

$$\begin{aligned}\mathbf{L} &:= [-\operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi, \Delta]\omega \\ &= -\operatorname{curl}[\varphi^\sharp\Delta^{-1}\varphi, \Delta]\omega \\ &= -\operatorname{curl}[\varphi^\sharp, \Delta]\Delta^{-1}\varphi\omega - \operatorname{curl}\varphi^\sharp\Delta^{-1}[\varphi, \Delta]\omega \\ &= -\operatorname{curl}[\varphi^\sharp, \Delta]\Delta^{-1}\varphi\omega + \operatorname{curl}\varphi^\sharp\Delta^{-1}\left(2\operatorname{div}(\nabla\varphi\otimes\omega) - (\Delta\varphi)\omega\right).\end{aligned}$$

Here we used (4.13). Therefore we have the equation for v as the following,

$$\partial_t v + \mathbb{P}_{\mathrm{curl}}(\varphi\omega\times u) = \mathbf{B} + \mathbf{L} + \Delta v. \tag{4.27}$$

We observe the following localization decomposition.

Lemma 4.15. *We can decompose*

$$\varphi u = v + w, \qquad \varphi\omega = \operatorname{curl} v + \varpi,$$

where w and ϖ are harmonic inside B_1.

Proof. We can compute v by

$$\begin{aligned}
v &= -\operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi\operatorname{curl}u\\
&= \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\operatorname{curl}u - \operatorname{curl}\Delta^{-1}\varphi\operatorname{curl}u\\
&= \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega - \operatorname{curl}\Delta^{-1}[\varphi,\operatorname{curl}]u + \mathbb{P}_{\operatorname{curl}}(\varphi u)\\
&= \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega + \operatorname{curl}\Delta^{-1}(\nabla\varphi\times u) - \mathbb{P}_\nabla(\varphi u) + \varphi u\\
&= \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega + \operatorname{curl}\Delta^{-1}(\nabla\varphi\times u) - \nabla\Delta^{-1}(\nabla\varphi\cdot u) + \varphi u
\end{aligned}$$

using $\operatorname{div}u = 0$. We denote

$$w := -\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega - \operatorname{curl}\Delta^{-1}(\nabla\varphi\times u) + \nabla\Delta^{-1}(\nabla\varphi\cdot u),$$

which implies the first decomposition $\varphi u = v + w$. By taking the curl,

$$\begin{aligned}
\operatorname{curl}(\varphi u) &= \operatorname{curl}v + \operatorname{curl}w,\\
\nabla\varphi\times u + \varphi\omega &= \operatorname{curl}v - \operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega - \operatorname{curl}\operatorname{curl}\Delta^{-1}(\nabla\varphi\times u)\\
&= \operatorname{curl}v - \operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega + \mathbb{P}_{\operatorname{curl}}(\nabla\varphi\times u).
\end{aligned}$$

We denote

$$\begin{aligned}
\varpi &:= -\operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega - \mathbb{P}_\nabla(\nabla\varphi\times u)\\
&= -\operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega - \nabla\Delta^{-1}\operatorname{div}(\nabla\varphi\times u)\\
&= -\operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega + \nabla\Delta^{-1}(\nabla\varphi\cdot\omega)
\end{aligned}$$

which implies the second decomposition $\varphi\omega = \operatorname{curl}v + \varpi$. We can easily see that Δw and $\Delta\varpi$ are both the sum of a smooth function supported outside $B_{\frac{3}{2}}$ and the Newtonian potential of something supported inside $\operatorname{supp}(\nabla\varphi)\subset B_{\frac{5}{4}}\setminus B_{\frac{6}{5}}$, so they are harmonic inside B_1. $\square$

Using this decomposition, we can continue to expand

$$\begin{aligned}
\mathbb{P}_{\mathrm{curl}}(\varphi\omega \times u) &= \varphi\omega \times u - \mathbb{P}_{\nabla}(\varphi\omega \times u) \\
&= \omega \times v + \omega \times w - \frac{1}{2}\mathbb{P}_{\nabla}\left((\operatorname{curl} v + \varpi) \times u + \omega \times (v + w)\right) \\
&= \omega \times v - \frac{1}{2}\mathbb{P}_{\nabla}\left(\operatorname{curl} v \times u + \omega \times v\right) - \mathbf{W},
\end{aligned}$$

where $\mathbf{W}$ denotes the remainders involving w and ϖ,

$$\mathbf{W} := -\omega \times w + \frac{1}{2}\mathbb{P}_{\nabla}\left(\varpi \times u + \omega \times w\right).$$

By subtracting (4.11) from (4.10), for divergence free u, v we have

$$\operatorname{curl} v \times u + \operatorname{curl} u \times v = -\nabla(u \cdot v) + 2u \cdot \nabla v + \operatorname{curl}(u \times v),$$

so

$$\begin{aligned}
\mathbb{P}_{\mathrm{curl}}(\varphi\omega \times u) &= \omega \times v + \frac{1}{2}\nabla(u \cdot v) - \mathbb{P}_{\nabla} \operatorname{div}(u \otimes v) - \mathbf{W} \\
&= \omega \times v + \nabla\left(\frac{1}{2}u \cdot v - \Delta^{-1} \operatorname{div}\operatorname{div}(u \otimes v)\right) - \mathbf{W}.
\end{aligned}$$

For convenience, denote the Riesz operator

$$\mathbf{R} = \frac{1}{2}\operatorname{tr} - \Delta^{-1}\operatorname{div}\operatorname{div}$$

Finally, we have the equation of v as

$$\partial_t v + \omega \times v + \nabla \mathbf{R}(u \otimes v) = \mathbf{B} + \mathbf{L} + \mathbf{W} + \Delta v, \qquad \operatorname{div} v = 0. \tag{4.28}$$

We now check the spatial integrability of these new terms.

Lemma 4.16. *For any* $1 < p < \infty$,

$$\|v\|_{L^p}, \|\nabla w\|_{L^p}, \|\varpi\|_{L^p} \lesssim \|\omega\|_{L^1(B_2)} + \|u\|_{L^p(B_2)},$$
$$\|\nabla v\|_{L^p}, \|\nabla \varpi\|_{L^p} \lesssim \|\omega\|_{L^p(B_2)},$$
$$\|\nabla^2 w\|_{L^p} \lesssim \|u\|_{W^{1,p}}(B_2).$$

If we denote $q = (\frac{1}{p} - \frac{1}{3})_+^{-1}$, *then*

$$\|\mathbf{B}\|_{L^q(B_2)} \lesssim \|\omega \times u\|_{L^p(B_2)},$$
$$\|\mathbf{L}\|_{L^p(B_2)} \lesssim \|\omega\|_{L^p(B_2)},$$
$$\|\mathbf{W}\|_{L^p(B_2)} \lesssim \|\omega \times w\|_{L^p(B_2)} + \|\varpi \times u\|_{L^p(B_2)}.$$

Proof. v, w, ϖ are all supported inside B_2, so

$$\begin{aligned}
\|v\|_{L^p} &\leq \|\varphi u\|_{L^p} + \|w\|_{L^p} \lesssim \|u\|_{L^p(B_2)} + \|\nabla w\|_{L^p},\\
\|\nabla w\|_{L^p} &\leq \|\nabla \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla \operatorname{curl}\Delta^{-1}(\nabla\varphi \times u)\|_{L^p}\\
&\quad + \|\nabla^2\Delta^{-1}(\nabla\varphi \cdot u)\|_{L^p}\\
&\lesssim \|(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{C^2} + \|\nabla\varphi \times u\|_{L^p} + \|\nabla\varphi \cdot u\|_{L^p}\\
&\lesssim \|\omega\|_{L^1(B_2)} + \|u\|_{L^p(B_2)},\\
\|\varpi\|_{L^p} &\leq \|\operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla\Delta^{-1}(\nabla\varphi \cdot \omega)\|_{L^p}\\
&\leq \|(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{C^2} + \|\nabla\Delta^{-1}\operatorname{div}(\nabla\varphi \times u)\|_{L^p}\\
&\leq \|\omega\|_{L^1(B_2)} + \|\nabla\varphi \times u\|_{L^p}\\
&\leq \|\omega\|_{L^1(B_2)} + \|u\|_{L^p(B_2)}.
\end{aligned}$$

Here we used Lemma 4.10 since φ and $1-\varphi^\sharp$ are supported away from each other, and we also used the boundedness of Riesz transform by Lemma 4.7. Their derivatives are bounded by

$$
\begin{aligned}
\|\nabla v\|_{L^p} &= \|\nabla \operatorname{curl} \varphi^\sharp \Delta^{-1} \varphi\omega\|_{L^p} \\
&\le \|\nabla \operatorname{curl} \Delta^{-1}\varphi\omega\|_{L^p} + \|\nabla \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} \\
&\lesssim \|\omega\|_{L^p(B_2)} + \|\omega\|_{L^1(B_2)} \lesssim \|\omega\|_{L^p(B_2)}, \\
\|\nabla^2 w\|_{L^p} &\le \|\nabla^2 \operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla^2 \operatorname{curl} \Delta^{-1}(\nabla\varphi \times u)\|_{L^p} \\
&\quad + \|\nabla^3 \Delta^{-1}(\nabla\varphi \cdot u)\|_{L^p} \\
&\lesssim \|\omega\|_{L^1(B_2)} + \|u\|_{W^{1,p}(B_2)} \lesssim \|u\|_{W^{1,p}(B_2)}, \\
\|\nabla \varpi\|_{L^p} &\le \|\nabla \operatorname{curl}\operatorname{curl}(1-\varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla^2\Delta^{-1}(\nabla\varphi\cdot\omega)\|_{L^p} \\
&\lesssim \|\omega\|_{L^1(B_2)} + \|\omega\|_{L^p(B_2)} \lesssim \|\omega\|_{L^p(B_2)}.
\end{aligned}
$$

The proof for $\mathbf{B}, \mathbf{L}, \mathbf{W}$ are similar so we omit here. □

Since $u \in \mathcal{E}$ and $\omega \in L_t^\infty L_x^1$, it can be seen from the above lemma that $v, \nabla w, \varpi \in \mathcal{E}$, thus

$$
\begin{aligned}
\|\mathbf{B}\|_{L^3(B_2)} &\lesssim \|\omega \times u\|_{L^{\frac{3}{2}}(B_2)} \in L_t^1, \\
\|\mathbf{L}\|_{L^2(B_2)} &\lesssim \|\omega\|_{L^2(B_2)} \in L_t^2, \\
\|\mathbf{W}\|_{L^{\frac{3}{2}}(B_2)} &\lesssim \|\omega \times w\|_{L^{\frac{3}{2}}(B_2)} + \|\varpi \times u\|_{L^{\frac{3}{2}}(B_2)} \in L_t^2,
\end{aligned}
$$

therefore $\mathbf{B}, \mathbf{L}, \mathbf{W} \in L_t^1 L^3_{\mathrm{loc},x} + L_x^2 L^{\frac{3}{2}}_{\mathrm{loc},x}$. In the appendix we prove the suitability for v: it satisfies the following local energy inequality,

$$
\partial_t \frac{|v|^2}{2} + |\nabla v|^2 + \operatorname{div}\left[v\mathbf{R}(u \otimes v)\right] \le \Delta \frac{|v|^2}{2} + v \cdot (\mathbf{B} + \mathbf{L} + \mathbf{W}). \tag{4.29}
$$

4.4.2 Energy Estimate

Multiply (4.29) by φ^4 then integrate over $\mathbb{R}^3$ yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi^4\frac{|v|^2}{2}\,\mathrm{d}x + \int \varphi^4|\nabla v|^2\,\mathrm{d}x\\ &\le \int \frac{|v|^2}{2}\Delta\varphi^4\,\mathrm{d}x + \int (v\cdot\nabla\varphi^4)\mathbf{R}(u\otimes v)\,\mathrm{d}x\\ &\qquad + \int \varphi^4 v\cdot\mathbf{B}\,\mathrm{d}x + \int \varphi^4 v\cdot\mathbf{L}\,\mathrm{d}x + \int \varphi^4 v\cdot\mathbf{W}\,\mathrm{d}x.\end{aligned}$$

Let us discuss these terms. For the first four terms on the right hand side,

$$I_\Delta := \int \frac{|v|^2}{2}\Delta\varphi^4\,\mathrm{d}x \le C\|\varphi^2 v\|_{L^2}\|v\|_{L^2}, \tag{4.30}$$

$$\begin{aligned}I_\mathbf{R} &:= \int (v\cdot\nabla\varphi^4)\mathbf{R}(u\otimes v)\,\mathrm{d}x \le C\|\varphi^2 v\|_{L^2}\|\mathbf{R}(u\otimes v)\|_{L^2} \\ &\le C\|\varphi^2 v\|_{L^2}\|u\otimes v\|_{L^2},\end{aligned} \tag{4.31}$$

$$\begin{aligned}I_\mathbf{B} &:= \int \varphi^4 v\cdot\mathbf{B}\,\mathrm{d}x \le \|\varphi^2 v\|_{L^2}\|\varphi^2\mathbf{B}\|_{L^2} \\ &\le C\|\varphi^2 v\|_{L^2}\|\omega\times u\|_{L^{\frac{6}{5}}(B_2)},\end{aligned} \tag{4.32}$$

$$\begin{aligned}I_\mathbf{L} &:= \int \varphi^4 v\cdot\mathbf{L}\,\mathrm{d}x \le \|\varphi^{\frac{2}{3}}|v|^{\frac{1}{3}}\|_{L^6}\||v|^{\frac{2}{3}}\|_{L^{q_3}}\|\varphi^2\mathbf{L}\|_{L^{q_2}} \\ &\le \|\varphi^2 v\|_{L^2}^{\frac{1}{3}}\|v\|_{L^{q_3}}^{\frac{2}{3}}\|\omega\|_{L^{q_2}(B_2)}.\end{aligned} \tag{4.33}$$

Here we use Hölder's inequality, φ is compactly supported in B_2 and $\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{6}\le 1$. For the $\mathbf{W}$ term,

$$\begin{aligned}I_\mathbf{W} &:= \int \varphi^4 v\cdot\mathbf{W}\,\mathrm{d}x\\ &= -\int \varphi^4 v\cdot\omega\times w\,\mathrm{d}x + \frac{1}{2}\int \varphi^4 v\cdot\mathbb{P}_\nabla\left(\varpi\times u + \omega\times w\right)\mathrm{d}x\\ &= -I_{\mathbf{W}1} + \frac{1}{2}I_{\mathbf{W}2}.\end{aligned}$$

For the first one, we break it as

$$I_{\mathbf{W}1} = \int \varphi^4 v \cdot \omega \times w \,\mathrm{d}x = \int \varphi^3 v \times \operatorname{curl} v \cdot w \,\mathrm{d}x + \int \varphi^3 v \cdot \varpi \times w \,\mathrm{d}x.$$

Using (4.10),

$$v \times \operatorname{curl} v = \frac{1}{2}\nabla |v|^2 - (v \cdot \nabla) v,$$

we have

$$\begin{aligned}\int \varphi^3 v \times \operatorname{curl} v \cdot w \,\mathrm{d}x &= -\frac{1}{2}\int |v|^2 \operatorname{div}(\varphi^3 w) \,\mathrm{d}x + \int v \cdot \nabla(\varphi^3 w) \cdot v \,\mathrm{d}x \\ &\leq C \|\varphi^2 v\|_{L^2} \left(\|\nabla w \otimes v\|_{L^2} + \|w \otimes v\|_{L^2} \right).\end{aligned}$$

The remaining is of lower order,

$$\int \varphi^3 v \cdot \varpi \times w \,\mathrm{d}x \leq C \|\varphi^2 v\|_{L^2} \|\varpi \times w\|_{L^2}$$

For the second one,

$$\begin{aligned}I_{\mathbf{W}2} &= \int \mathbb{P}_{\nabla}(\varphi^4 v) \cdot (\varpi \times u + \omega \times w) \,\mathrm{d}x \\ &\leq \|\mathbb{P}_{\nabla}(\varphi^4 v)\|_{L^6} \|\varpi \times u + \omega \times w\|_{L^{\frac{6}{5}}}\end{aligned}$$

where

$$\|\mathbb{P}_{\nabla}(\varphi^4 v)\|_{L^6} = \|\nabla \Delta^{-1} \operatorname{div}(\varphi^4 v)\|_{L^6} = \|\nabla \Delta^{-1}(v \cdot \nabla \varphi^4)\|_{L^6} \leq C \|\varphi^2 v\|_{L^2}.$$

So $I_{\mathbf{W}}$ can be bounded by

$$I_{\mathbf{W}} \leq C \|\varphi^2 v\|_{L^2} \left(\|\nabla w \otimes v\|_{L^2} + \|\varpi \times w\|_{L^2} + \|\varpi \times u + \omega \times w\|_{L^{\frac{6}{5}}} \right). \tag{4.34}$$

In summary, we conclude that for $-4 \le t \le 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x + \int \varphi^4 |\nabla v|^2\,\mathrm{d}x \le I_{\boldsymbol{\Delta}} + I_{\mathbf{R}} + I_{\mathbf{B}} + I_{\mathbf{L}} + I_{\mathbf{W}} \tag{4.35}$$

with good estimates on each of the term on the right.

4.4.3 Proof of Proposition 4.14

First we check the integrability of each terms.

Lemma 4.17 (Integrability)**.** *Given conditions* (4.4) *and* (4.5)*, we have*

$$\begin{aligned}
\|u\|_{L_t^{p_1}L_x^{q_3}(Q_2)} &\le \eta, \\
\|\varphi\omega\|_{L_t^{p_1}L_x^{q_1}((-4,0)\times\mathbb{R}^3)} &\le \eta, & \|\varphi\omega\|_{L_t^{p_2}L_x^{q_2}((-4,0)\times\mathbb{R}^3)} &\le \eta, \\
\|\nabla v\|_{L_t^{p_1}L_x^{q_1}((-4,0)\times\mathbb{R}^3)} &\le \eta, & \|\nabla v\|_{L_t^{p_2}L_x^{q_2}((-4,0)\times\mathbb{R}^3)} &\le \eta, \\
\|v\|_{L_t^{p_1}L_x^{q_3}((-4,0)\times\mathbb{R}^3)} &\le \eta, & \|v\|_{L_t^{p_2}L_x^{q_4}((-4,0)\times\mathbb{R}^3)} &\le \eta, \\
\|\nabla w\|_{L_t^{p_1}L_x^{q_3}((-4,0)\times\mathbb{R}^3)} &\le \eta, \\
\|w\|_{L_t^{p_1}L_x^{q_5}((-4,0)\times\mathbb{R}^3)} &\le \eta, & & & (4.36) \\
\|\varpi\|_{L_t^{p_1}L_x^{q_3}((-4,0)\times\mathbb{R}^3)} &\le \eta, & \|\varpi\|_{L_t^{p_2}L_x^{q_4}((-4,0)\times\mathbb{R}^3)} &\le \eta. & & (4.37)
\end{aligned}$$

Proof. Integrability of u is obtained by Sobolev embedding and that φu has average 0. Integrability of $\varphi\omega$ is given. The remaining are consequences of Lemma 4.16 and Sobolev embedding. □

Proof of Proposition 4.14. We prove Proposition 4.14 using a Grönwall argument. Multiply (4.35) by an increasing smooth function $\psi_1(t)$ with $\psi_1(t) = 0$ for $t \le -2$,

$\psi_1(t) = 1$ for $t \geq -1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_1(t)\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x\right) + \psi_1(t)\int \varphi^4 |\nabla v|^2\,\mathrm{d}x$$
$$= \psi_1'(t)\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x + \psi_1(t)\left(I_{\Delta} + I_{\mathbf{R}} + I_{\mathbf{B}} + I_{\mathbf{L}} + I_{\mathbf{W}}\right).$$

Formally, we can integrate from -4 to $t < 0$ and we have

$$\psi_1(t)\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x + \int_{-2}^{t} \psi_1(s)\int \varphi^4 |\nabla v|^2\,\mathrm{d}x$$
$$= \int_{-2}^{t} \psi_1'(s)\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x\,\mathrm{d}t + \int_{-2}^{t} \psi_1(s)\left(I_{\Delta,\mathbf{R},\mathbf{B},\mathbf{L},\mathbf{W}}\right)\mathrm{d}t.$$

This integration is justified since v satisfies the local energy inequality (4.29) in distribution, and $\psi_1(t)\varphi^4(x) \in C_c^{\infty}((-4,0] \times B_2)$. Because of (4.30), (4.31), (4.32), (4.33), (4.34), and

$$\|\varphi^2 v\|_{L^2(B_2)}, \|\varphi^2 v\|_{L^2(B_2)}^{\frac{1}{3}} \leq C\left(1 + \int \varphi^4 |v|^2\,\mathrm{d}x\right),$$

we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_1(t)\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x\right) + \psi_1(t)\int \varphi^4 |\nabla v|^2\,\mathrm{d}x$$
$$\leq C\Phi(t)\left(1 + \psi_1(t)\int \varphi^4 \frac{|v|^2}{2}\,\mathrm{d}x\right),$$

where

$$
\begin{aligned}
\Phi(t) = {} & \psi_1'(t) \int \varphi^4 \frac{|v|^2}{2} \,\mathrm{d}x \\
& + \|v\|_{L^2} + \|u \otimes v\|_{L^2} + \|\omega \times u\|_{L^{\frac{6}{5}}(B_2)} \\
& + \|v\|_{L_x^{q_3}}^{\frac{2}{3}} \|\omega\|_{L_x^{q_2}(B_2)} + \|\nabla w \otimes v\|_{L^2} \\
& + \|\varpi \times w\|_{L^2} + \|\varpi \times u + \omega \times w\|_{L^{\frac{6}{5}}} \\
\leq {} & \|v\|_{L^2}^2 + \|v\|_{L^2} + \|v\|_{L_x^{q_4}} \|u\|_{L_x^{q_3}(B_2)} + \|\omega\|_{L_x^{q_2}(B_2)} \|u\|_{L_x^{q_3}(B_2)} \\
& + \|v\|_{L_x^{q_3}}^{\frac{2}{3}} \|\omega\|_{L_x^{q_2}(B_2)} + \|\nabla w\|_{L_x^{q_3}} \|v\|_{L_x^{q_4}} \\
& + \|\varpi\|_{L_x^{q_4}} \|w\|_{L_x^{q_3}} \\
& + \|\varpi\|_{L_x^{q_4}} \|u\|_{L_x^{q_3}} + \|\omega\|_{L_x^{q_2}} \|w\|_{L_x^{q_3}} \\
\leq {} & \left(\|v\|_{L_x^{q_3}} + \|v\|_{L_x^{q_3}}^{\frac{1}{2}} + \|u\|_{L_x^{q_3}(B_2)} + \|\nabla w\|_{L_x^{q_3}} + \|w\|_{L_x^{q_3}} \right) \\
& \times \left(\|v\|_{L_x^{q_4}} + \|v\|_{L_x^{q_4}}^{\frac{1}{2}} + \|\omega\|_{L_x^{q_2}} + \|\varpi\|_{L_x^{q_4}} \right)
\end{aligned}
$$

Here we used interpolation for $\|v\|_{L^2}^2 \leq \|v\|_{L_x^{q_3}} \|v\|_{L_x^{q_4}}$. Therefore

$$
\begin{aligned}
\|\Phi\|_{L_t^1} \lesssim {} & \left\| \left(\|v\|_{L_x^{q_3}} + \|v\|_{L_x^{q_3}}^{\frac{1}{2}} + \|u\|_{L_x^{q_3}(B_2)} + \|\nabla w\|_{L_x^{q_3}} + \|w\|_{L_x^{q_3}} \right) \right\|_{L_t^{p_1}} \\
& \times \left\| \left(\|v\|_{L_x^{q_4}} + \|v\|_{L_x^{q_4}}^{\frac{1}{2}} + \|\omega\|_{L_x^{q_2}} + \|\varpi\|_{L_x^{q_4}} \right) \right\|_{L_t^{p_2}} \leq \eta.
\end{aligned}
$$

By a Grönwall's lemma, we conclude that for every $-4 \leq t \leq 0$,

$$
1 + \psi_1(t) \int \varphi^4 \frac{|v|^2}{2} \,\mathrm{d}x + \int_{-4}^{t} \psi_1(t) \int \varphi^4 |\nabla v|^2 \,\mathrm{d}x \leq e^{\int_{-4}^{t} C\Phi(s)\,\mathrm{d}s} \leq e^{C\eta}.
$$

Therefore by taking the sup over $-1 \leq t \leq 0$ and $t = 0$ respectively, we conclude

$$
\sup_{-1 \leq t \leq 0} \int |v(t)|^2 \,\mathrm{d}x \leq \eta, \qquad \int_{Q_1} |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \eta.
$$

$\square$

4.5 Local Study: Part Two, De Giorgi Iteration

In this section, we derive the boundedness of v in $Q_{\frac{1}{2}}$ which is the following.

Proposition 4.18. *Let v solves* (4.28)*. If* (4.25) *holds for sufficiently small η, and we have integrability bounds in Lemma 4.17, then we have*

$$\|v\|_{L^\infty(Q_{\frac{1}{2}})} = \sup_{t\in(-1,0)} \|v(t)\|_{L^\infty(B_{\frac{1}{2}})} \leq 1.$$

The proof uses De Giorgi technique and the truncation method. First, we set dyadically shrinking radius,

$$r_k^\flat = \frac{1}{2}(1 + 8^{-k}), \qquad r_k^\natural = \frac{1}{2}(1 + 2 \times 8^{-k}), \qquad r_k^\sharp = \frac{1}{2}(1 + 4 \times 8^{-k}).$$

Then we define dyadically shrinking cylinder Q_k's,

$$\begin{aligned} T_k^\flat &= {r_k^\flat}^2, & B_k^\flat &= B_{r_k^\flat}(0), & Q_k^\flat &= (-T_k^\flat, 0) \times B_k^\flat, \\ T_k^\natural &= {r_k^\natural}^2, & B_k^\natural &= B_{r_k^\natural}(0), & Q_k^\natural &= (-T_k^\natural, 0) \times B_k^\natural, \\ T_k^\sharp &= {r_k^\sharp}^2, & B_k^\sharp &= B_{r_k^\sharp}(0), & Q_k^\sharp &= (-T_k^\sharp, 0) \times B_k^\sharp. \end{aligned}$$

We also introduce positive smooth space-time cut-off functions ρ_k and $\rho_k^\sharp$ with

$$\mathbf{1}_{Q_k^\flat} \leq \rho_k \leq \mathbf{1}_{Q_k^\natural}, \qquad \mathbf{1}_{Q_k^\sharp} \leq \rho_k^\sharp \leq \mathbf{1}_{Q_{k-1}^\flat}.$$

Then, let c_k denote a sequence of rising energy level,

$$c_k = 1 - 2^{-k}, \qquad v_k = (|v| - c_k)_+, \qquad \beta_k = \frac{v_k}{|v|},$$
$$\Omega_k = \{v_k > 0\}, \qquad \mathbf{1}_k = \mathbf{1}_{\Omega_k}, \qquad \alpha_k = 1 - \beta_k.$$

We define analogous of vector derivative d_k and energy quantity U_k:

$$d_k^2 = \mathbf{1}_k \left(\alpha_k |\nabla |v||^2 + \beta_k |\nabla v|^2 \right),$$
$$U_k = \|v_k\|^2_{L^\infty(-T_k^\flat, 0; L^2(B_k^\flat))} + \|d_k\|^2_{L^2(Q_k^\flat)}.$$

We have the following truncation estimates.

Lemma 4.19.

$$\alpha_k v \leq c_k \leq 1,$$
$$\|\beta_k v\|^2_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{k-1}^\flat)} \leq 9 U_{k-1},$$
$$\|\mathbf{1}_k\|^2_{L_t^\infty L_x^2 \cap L_t^2 L_x^6(Q_{k-1}^\flat)} \leq C^k U_{k-1}.$$

Proof. The first estimate follows from the definition. By Lemma 4 in [Vas07], we have $|\nabla v_k| \leq d_k$ and $|\nabla(\beta_k v)| \leq 3 d_k$. Moreover, since $|\nabla |v|| \leq |\nabla v|^2$, we see $d_k \leq d_{k-1}$, as v_k and β_k are monotonously decreasing. So

$$\|\nabla(\beta_k v)\|_{L^2(Q_{k-1}^\flat)} \leq 3 \|d_k\|_{L^2(Q_{k-1}^\flat)} \leq 3 \|d_{k-1}\|_{L^2(Q_{k-1}^\flat)}.$$

Moreover, the truncation gives $|\beta_k v| + 2^{-k}\mathbf{1}_k = v_k + 2^{-k}\mathbf{1}_k = \mathbf{1}_k v_{k-1}$, so

$$\begin{aligned}
\|\beta_k v\|_{L_t^\infty L_x^2(Q_{k-1}^\flat)} &\leq \|v_{k-1}\|_{L_t^\infty L_x^2(Q_{k-1}^\flat)}, \\
2^{-k}\|\mathbf{1}_k\|_{L_t^\infty L_x^2(Q_{k-1}^\flat)} &\leq \|v_{k-1}\|_{L_t^\infty L_x^2(Q_{k-1}^\flat)}, \\
2^{-k}\|\mathbf{1}_k\|_{L_t^2 L_x^6(Q_{k-1}^\flat)} &\leq \|v_{k-1}\|_{L_t^2 L_x^6(Q_{k-1}^\flat)} \\
&\leq \|v_{k-1}\|_{L_t^\infty L_x^2(Q_{k-1}^\flat)} + \|\nabla v_{k-1}\|_{L^2(Q_{k-1}^\flat)} \\
&\leq \|v_{k-1}\|_{L_t^\infty L_x^2(Q_{k-1}^\flat)} + \|d_{k-1}\|_{L^2(Q_{k-1}^\flat)}.
\end{aligned}$$

□

Corollary 4.20 (Nonlinearization)**.** *If* $f \in L_t^p L_x^q(Q_{k-1})$, *with*

$$\frac{1}{p} + \gamma\left(\frac{\theta}{2} + \frac{1-\theta}{\infty}\right) = 1, \qquad \frac{1}{q} + \gamma\left(\frac{\theta}{6} + \frac{1-\theta}{2}\right) = 1,$$

for some $0 \leq \theta \leq 1$, $0 < \sigma \leq \gamma$, *then uniformly in* σ,

$$\int_{Q_{k-1}^\flat} |\beta_k v|^\sigma |f| \, \mathrm{d}x \, \mathrm{d}t \leq C^k \|f\|_{L_t^p L_x^q(Q_{k-1})} U_{k-1}^{\frac{\gamma}{2}}.$$

Proof. By interpolation,

$$\|\beta_k v\|, \|\mathbf{1}_k\|_{L_t^{p_\theta} L_x^{q_\theta}(Q_{k-1})} \leq U_{k-1}^{\frac{1}{2}},$$

where

$$\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{\infty}, \qquad \frac{1}{q_\theta} = \frac{\theta}{6} + \frac{1-\theta}{2}.$$

Therefore, using Hölder's inequality,

$$\int_{Q_{k-1}} |\beta_k v|^\sigma |f| \,\mathrm{d}x\,\mathrm{d}t \le \|f\|_{L_t^p L_x^q} \|\beta_k v\|_{L_t^{p\theta} L_x^{q\theta}}^\sigma \|\mathbf{1}_k\|_{L_t^{p\theta} L_x^{q\theta}}^{\gamma-\sigma} \le \|f\|_{L_t^p L_x^q} U_{k-1}^{\frac{\gamma}{2}}.$$

□

First, we recall the following identities from [Vas07].

$$\alpha_k v \cdot \partial_\bullet v = \partial_\bullet \left(\frac{|v|^2 - v_k^2}{2} \right), \tag{4.38}$$

$$\alpha_k v \cdot \Delta v = \Delta \left(\frac{|v|^2 - v_k^2}{2} \right) + d_k^2 - |\nabla v|^2. \tag{4.39}$$

Since $\alpha_k v$ is bounded, we can multiply equation (4.28) by $\alpha_k v$ and obtain

$$\begin{aligned} \partial_t \left(\frac{|v|^2 - v_k^2}{2} \right) &+ \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \\ &= \Delta \left(\frac{|v|^2 - v_k^2}{2} \right) + d_k^2 - |\nabla v|^2 + \alpha_k v \cdot (\mathbf{B} + \mathbf{L} + \mathbf{W}). \end{aligned} \tag{4.40}$$

using (4.38) and (4.39). Denote $\mathbf{C}_v = \mathbf{B} + \mathbf{L} + \mathbf{W}$. Subtracting (4.40) from (4.29), we have

$$\partial_t \frac{v_k^2}{2} + d_k^2 + \operatorname{div}(v\mathbf{R}(u \otimes v)) - \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \le \Delta \frac{v_k^2}{2} + \beta_k v \cdot \mathbf{C}_v.$$

Multiply by ρ_k, then integrate in space and from σ to τ in time,

$$\begin{aligned} \left[\int \rho_k \frac{v_k^2}{2} \,\mathrm{d}x \right]_\sigma^\tau &+ \int_\sigma^\tau \int \rho_k d_k^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_\sigma^\tau \int (\partial_t \rho_k + \Delta \rho_k) \frac{v_k^2}{2} \,\mathrm{d}x\,\mathrm{d}t - \int_\sigma^\tau \int \rho_k \operatorname{div}(v\mathbf{R}(u \otimes v)) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\sigma^\tau \int \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \,\mathrm{d}x\,\mathrm{d}t + \int_\sigma^\tau \int \rho_k \beta_k v \cdot \mathbf{C}_v \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Take the sup over $\tau > -T_k^\flat$, and set $\sigma < -T_{k-1}^\flat$, we obtain

$$\begin{aligned} U_k \le & \sup_{\tau\in(-T_k^\flat,0)} \int \rho_k \frac{v_k^2}{2}\,\mathrm{d}x + \int_{-T_{k-1}^\flat}^0 \int \rho_k d_k^2 \,\mathrm{d}x\,\mathrm{d}t \\ \le & C^k \int_{Q_k^\natural} v_k^2 \,\mathrm{d}x\,\mathrm{d}t + \sup_{\tau\in(-T_k^\flat,0)} \Big\{ \int_{-T_k^\natural}^\tau \int \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u\otimes v)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_{-T_k^\natural}^\tau \int \rho_k \operatorname{div}(v\mathbf{R}(u\otimes v))\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{-T_k^\natural}^\tau \int \rho_k \beta_k v \cdot \mathbf{C}_v \,\mathrm{d}x\,\mathrm{d}t \Big\}. \end{aligned} \tag{4.41}$$

Using Corollary 4.20, the first one is bounded by

$$\int_{Q_k^\natural} v_k^2 \,\mathrm{d}x\,\mathrm{d}s \le \int_{Q_{k-1}^\flat} |\beta_k v|^2 \,\mathrm{d}x\,\mathrm{d}s \le U_{k-1}^{\frac{5}{3}}. \tag{4.42}$$

Now let's deal with the last few terms. For simplicity, we use $\iint \mathrm{d}x\,\mathrm{d}t$ to denote $\int_{-T_k^\natural}^\tau \int_{\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}t$ in the rest of this section.

4.5.1 Highest Order Nonlinear Term

Define three trilinear forms,

$$\begin{aligned} \mathbf{T}_\circ[v_1,v_2,v_3] &= \iint \rho_k \operatorname{div}(v_1 \mathbf{R}(v_2\otimes v_3))\,\mathrm{d}x\,\mathrm{d}t, \\ \mathbf{T}_\nabla[v_1,v_2,v_3] &= \iint \rho_k v_1 \cdot \nabla \mathbf{R}(v_2\otimes v_3)\,\mathrm{d}x\,\mathrm{d}t, \\ \mathbf{T}_{\mathrm{div}}[v_1,v_2,v_3] &= \iint \rho_k \operatorname{div} v_1 \mathbf{R}(v_2\otimes v_3)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

They are symmetric on v_2, v_3 positions. When we have enough integrability, that is, when

$$|\nabla v_1||v_2||v_3|, |v_1||\nabla v_2||v_3|, |v_1||v_2||\nabla v_3| \in L^1_{t,x},$$

we have Leibniz rule

$$\mathbf{T}_\circ = \mathbf{T}_\nabla + \mathbf{T}_{\mathrm{div}}.$$

The goal is to estimate the first two double integrals in (4.41),

$$\begin{aligned}\iint \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \, \mathrm{d}x \, \mathrm{d}t - \iint \rho_k \operatorname{div}(v \mathbf{R}(u \otimes v)) \, \mathrm{d}x \, \mathrm{d}t \\ = \mathbf{T}_\nabla[\alpha_k v, u, v] - \mathbf{T}_\circ[v, u, v].\end{aligned}$$

We first separate $w \otimes v$ from $u \otimes v$, and we will have

$$\begin{aligned}\mathbf{T}_\nabla[\alpha_k v, w, v] - \mathbf{T}_\circ[v, w, v] &= \mathbf{T}_\nabla[\alpha_k v, w, v] - \mathbf{T}_\nabla[v, w, v] - \mathbf{T}_{\mathrm{div}}[v, w, v] \\ &= -\mathbf{T}_\nabla[\beta_k v, w, v] \\ &= -\iint \rho_k \beta_k v \cdot \nabla \mathbf{R}(w \otimes v) \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

Denote $-\nabla \mathbf{R}(w \otimes v) =: \mathbf{W}_2$ and we will deal with it later. The remaining $(u-w) \otimes v$ can be separated into interior part and exterior part,

$$(u - w) \otimes v = \rho_k^\sharp v \otimes v + (1 - \rho_k^\sharp)(u - w) \otimes v.$$

The exterior part is bounded and smooth in space over the support of ρ_k.

$$\begin{aligned}\|\rho_k \mathbf{R}((1 - \rho_k^\sharp)(u - w) \otimes v)\|_{L_t^{p_3} C_x^\infty} &\le C \|(u - w) \otimes v\|_{L_t^{p_3} L_x^2} \\ &\le C \|u - w\|_{L_t^{p_1} L_x^{q_3}(Q_2)} \|v\|_{L_t^{p_2} L_x^{q_4}} \le \eta.\end{aligned}$$

Here, we denote

$$\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} < 1.$$

Therefore we can use Leibniz rule similar as w and

$$\begin{aligned}
&\mathbf{T}_{\nabla}[\alpha_k v, (1-\rho_k^{\sharp})(u-w), v] - \mathbf{T}_{\circ}[v, (1-\rho_k^{\sharp})(u-w), v] \\
&= \mathbf{T}_{\nabla}[\alpha_k v, (1-\rho_k^{\sharp})(u-w), v] - \mathbf{T}_{\nabla}[v, (1-\rho_k^{\sharp})(u-w), v] \\
&= -\mathbf{T}_{\nabla}[\beta_k v, (1-\rho_k^{\sharp})(u-w), v] \\
&= -\iint \rho_k \beta_k v \cdot \nabla \mathbf{R}(v \otimes (1-\rho_k^{\sharp})(u-w)) \, \mathrm{d}x \, \mathrm{d}t \\
&\leq C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_3}}
\end{aligned}$$

by nonlinearization Corollary 4.20. The interior part is

$$\begin{aligned}
&\mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} v, v] - \mathbf{T}_{\circ}[v, \rho_k^{\sharp} v, v] \\
&= \mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} \beta_k v, \beta_k v] + 2\mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \beta_k v] \\
&\qquad + \mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \alpha_k v] - \mathbf{T}_{\circ}[v, \rho_k^{\sharp} v, v] \\
&= \mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} \beta_k v, \beta_k v] \\
&\qquad + 2\mathbf{T}_{\circ}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \beta_k v] - 2\mathbf{T}_{\mathrm{div}}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \beta_k v] \\
&\qquad + \mathbf{T}_{\circ}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \alpha_k v] - \mathbf{T}_{\mathrm{div}}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \alpha_k v] \\
&\qquad - \mathbf{T}_{\circ}[v, \rho_k^{\sharp} v, v] \\
&= \mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} \beta_k v, \beta_k v] \\
&\qquad + 2\mathbf{T}_{\mathrm{div}}[\beta_k v, \rho_k^{\sharp} \alpha_k v, \beta_k v] + \mathbf{T}_{\mathrm{div}}[\beta_k v, \rho_k^{\sharp} \alpha_k v, \alpha_k v] \\
&\qquad + 2\mathbf{T}_{\circ}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \beta_k v] + \mathbf{T}_{\circ}[\alpha_k v, \rho_k^{\sharp} \alpha_k v, \alpha_k v] \\
&\qquad - \mathbf{T}_{\circ}[v, \rho_k^{\sharp} v, v] \\
&= \mathbf{T}_{\nabla}[\alpha_k v, \rho_k^{\sharp} \beta_k v, \beta_k v] + \mathbf{T}_{\mathrm{div}}[\beta_k v, \rho_k^{\sharp} \alpha_k v, (\beta_k + 1) v] \\
&\qquad - \mathbf{T}_{\circ}[\alpha_k v, \rho_k^{\sharp} \beta_k v, \beta_k v] - \mathbf{T}_{\circ}[\beta_k v, \rho_k^{\sharp} v, v].
\end{aligned}$$

Notice that the boundedness of $\alpha_k v$ guarantees enough integrability to switch between trilinear forms. Then

$$
\begin{aligned}
&|\mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v]|, |\mathbf{T}_{\mathrm{div}}[\beta_k v, \rho_k^\sharp \alpha_k v, (\beta_k+1)v]| \\
&\qquad \lesssim \|\nabla(\beta_k v)\|_{L^2(Q_{k-1})} U_{k-1}^{\frac{5}{6}} \le U_{k-1}^{\frac{4}{3}}, \\
&|\mathbf{T}_\circ[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v]|, |\mathbf{T}_\circ[\beta_k v, \rho_k^\sharp v, v]| \lesssim U_{k-1}^{\frac{5}{3}}.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
\bigg| \iint \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \, \mathrm{d}x \, \mathrm{d}t - \iint \rho_k \operatorname{div}(v \mathbf{R}(u \otimes v)) \, \mathrm{d}x \, \mathrm{d}t \\
- \iint \rho_k \beta_k v \cdot \mathbf{W}_2 \, \mathrm{d}x \, \mathrm{d}t \bigg| \lesssim C^k U_{k-1}^{\min\{\frac{4}{3}, \frac{5}{3} - \frac{2}{3}p_3\}}.
\end{aligned} \tag{4.43}
$$

4.5.2 Lower Order Terms

For the bilinear and linear term, recall that inside B_1,

$$
\begin{aligned}
\mathbf{B} &= -\operatorname{curl} \Delta^{-1}(\nabla\varphi \times (\omega \times u)), \\
\mathbf{L} &= \operatorname{curl} \Delta^{-1} \left(2 \operatorname{div}(\nabla \varphi \otimes \omega) - (\Delta\varphi)\omega \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\rho_k \mathbf{B}\|_{L_t^{p_3} L_x^\infty} \le \|\omega \times u\|_{L_t^{p_3} L_x^{\frac{6}{5}}(Q_2)} \le \|u\|_{L_t^{p_1} L_x^{q_3}} \|\omega\|_{L_t^{p_2} L_x^{q_2}} \le \eta, \\
&\|\rho_k \mathbf{L}\|_{L_t^{p_2} L_x^\infty} \le \|\omega\|_{L_t^{p_2} L_x^{q_2}(Q_2)} \le \eta,
\end{aligned}
$$

Thus

$$
\iint \mathbf{B} \cdot \rho_k \beta_k v \, \mathrm{d}x \, \mathrm{d}t \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_3}}, \tag{4.44}
$$

$$
\iint \mathbf{L} \cdot \rho_k \beta_k v \, \mathrm{d}x \, \mathrm{d}t \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_2}}. \tag{4.45}
$$

4.5.3 W Terms

Finally, let us deal with

$$\mathbf{W} + \mathbf{W}_2 = -\omega \times w + \frac{1}{2}\mathbb{P}_\nabla\left(\varpi \times u + \omega \times w\right) - \nabla\mathbf{R}(w \otimes v).$$

Here $\nabla\mathbf{R} = \frac{1}{2}\nabla\operatorname{tr} - \mathbb{P}_\nabla \operatorname{div}$, so

$$\begin{aligned}
\nabla\mathbf{R}(w \otimes v) &= \frac{1}{2}\nabla(w \cdot v) - \mathbb{P}_\nabla \operatorname{div}(v \otimes w) \\
&= \frac{1}{2}\left(w \cdot \nabla v + v \cdot \nabla w + w \times \operatorname{curl} v + v \times \operatorname{curl} w\right) - \mathbb{P}_\nabla(v \cdot \nabla w) \\
&= \frac{1}{2}\left(w \cdot \nabla v - v \cdot \nabla w\right) + \mathbb{P}_{\text{curl}}(v \cdot \nabla w) \\
&\quad + \frac{1}{2}\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right), \\
\nabla\mathbf{R}(w \otimes v) &= \mathbb{P}_\nabla(\nabla\mathbf{R}(w \otimes v)) \\
&= \frac{1}{2}\mathbb{P}_\nabla\left(w \cdot \nabla v - v \cdot \nabla w\right) + \frac{1}{2}\mathbb{P}_\nabla\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right) \\
&= \frac{1}{2}\mathbb{P}_\nabla\left(\operatorname{curl}(v \times w) - v \operatorname{div} w + w \operatorname{div} v\right) \\
&\quad + \frac{1}{2}\mathbb{P}_\nabla\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right) \\
&= -\frac{1}{2}\mathbb{P}_\nabla\left(v(u \cdot \nabla\varphi)\right) + \frac{1}{2}\mathbb{P}_\nabla\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{W} + \mathbf{W}_2 = -\omega \times w &+ \frac{1}{2}\mathbb{P}_\nabla\left(v(u \cdot \nabla\varphi)\right) \\
&+ \frac{1}{2}\mathbb{P}_\nabla\left(\varpi \times u + \omega \times w + \operatorname{curl} v \times w + \operatorname{curl} w \times v\right).
\end{aligned}$$

Again, we separate $\mathbf{W} + \mathbf{W}_2$ into exterior and interior part, with

$$\mathbf{W} + \mathbf{W}_2 = \mathbf{W}_{\text{ext}} + \mathbf{W}_{\text{int}}$$

where

$$
\begin{aligned}
\mathbf{W}_{\text{ext}} &= -(1-\rho_k^\sharp)\omega \times w + \frac{1}{2}\mathbb{P}_\nabla\left(v(u\cdot\nabla\varphi)\right) \\
&\quad + \frac{1}{2}\mathbb{P}_\nabla\left((1-\rho_k^\sharp)\left(\varpi \times u + \omega \times w + \operatorname{curl} v \times w + \operatorname{curl} w \times v\right)\right), \\
\mathbf{W}_{\text{int}} &= -\rho_k^\sharp \omega \times w \\
&\quad + \frac{1}{2}\mathbb{P}_\nabla\left(\rho_k^\sharp\left(\varpi \times u + \omega \times w + \operatorname{curl} v \times w + \operatorname{curl} w \times v\right)\right) \\
&= -\rho_k^\sharp \operatorname{curl} v \times w - \rho_k^\sharp \varpi \times w \\
&\quad + \frac{1}{2}\mathbb{P}_\nabla\left(\rho_k^\sharp\left(\varpi \times u + \operatorname{curl} w \times v + \varpi \times w\right)\right) \\
&\quad + \frac{1}{2}\mathbb{P}_\nabla\left(\rho_k^\sharp\left(\omega \times w + \operatorname{curl} v \times w - \varpi \times w\right)\right) \\
&= -\rho_k^\sharp \operatorname{curl} v \times w - \rho_k^\sharp \varpi \times w \\
&\quad + \mathbb{P}_\nabla\left(\rho_k^\sharp \varpi \times u\right) + \mathbb{P}_\nabla\left(\rho_k^\sharp \operatorname{curl} v \times w\right) \\
&= -\mathbb{P}_{\text{curl}}(\rho_k^\sharp \operatorname{curl} v \times w) - \mathbb{P}_{\text{curl}}(\rho_k^\sharp \varpi \times w) + \mathbb{P}_\nabla\left(\rho_k^\sharp \varpi \times v\right).
\end{aligned}
$$

Similar as bilinear terms, $\rho_k \mathbf{W}_{\text{ext}}$ is small in $L_t^{p_3} L_x^\infty$. Among the three terms in $\mathbf{W}_{\text{int}}$, $\rho_k^\sharp \varpi \times w$ is bounded in $L_t^{p_3} L_x^\infty$, and $\rho_k^\sharp \varpi$ is in $L_t^{p_2} L_x^\infty$. Finally, for the first term,

$$
\begin{aligned}
\mathbb{P}_{\text{curl}}(\operatorname{curl} v \times \rho_k^\sharp w) &= -\mathbb{P}_{\text{curl}}(\operatorname{curl} \rho_k^\sharp w \times v) + \mathbb{P}_{\text{curl}}(v \cdot \nabla \rho_k^\sharp w + \rho_k^\sharp w \cdot \nabla v), \\
\mathbb{P}_{\text{curl}}(\rho_k^\sharp w \cdot \nabla v) &= \mathbb{P}_{\text{curl}}(\operatorname{curl}(v \times \rho_k^\sharp w) + v \cdot \nabla \rho_k^\sharp w - v \operatorname{div} \rho_k^\sharp w) \\
&= \operatorname{curl}(v \times \rho_k^\sharp w) + \mathbb{P}_{\text{curl}}(v \cdot \nabla \rho_k^\sharp w - v \operatorname{div} \rho_k^\sharp w), \\
\operatorname{curl}(v \times \rho_k^\sharp w) &= v \operatorname{div} \rho_k^\sharp w + \rho_k^\sharp w \cdot \nabla v - v \cdot \nabla \rho_k^\sharp w.
\end{aligned}
$$

Every term is a product of v and $\nabla\rho_k^\sharp w$ (possibly with a Riesz transform) except $\rho_k^\sharp w\cdot\nabla v$. Because in Ω_k, $\nabla|v| = \nabla v_k$ are the same, we have

$$\begin{aligned}\int \rho_k\beta_k v\cdot(\rho_k^\sharp w\cdot\nabla)v\,\mathrm{d}x &= \int \rho_k\beta_k(w\cdot\nabla)\frac{|v|^2}{2}\,\mathrm{d}x\\ &= \int \rho_k\beta_k|v|(w\cdot\nabla)|v|\,\mathrm{d}x\\ &= \int \rho_k v_k(w\cdot\nabla)v_k\,\mathrm{d}x\\ &= \int \rho_k(w\cdot\nabla)\frac{v_k^2}{2}\,\mathrm{d}x\\ &= -\int \frac{v_k^2}{2}\operatorname{div}(\rho_k w)\,\mathrm{d}x.\end{aligned}$$

Therefore, every term of $\mathbb{P}_{\mathrm{curl}}(\operatorname{curl} v\times\rho_k^\sharp w)$ is a product of v and $\nabla\rho_k w$ or $\nabla\rho_k^\sharp w$. Inside B_1, $w\in L_t^{p_1}C_x^\infty$. In conclusion,

$$\begin{aligned}\iint \rho_k\beta_k v\cdot\mathbf{W}_{\mathrm{ext}}\,\mathrm{d}x\,\mathrm{d}t &\le C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_3}},\\ \iint \rho_k\beta_k v\cdot\mathbb{P}_{\mathrm{curl}}(\rho_k^\sharp\operatorname{curl} v\times w)\,\mathrm{d}x\,\mathrm{d}t &\le C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_1}},\\ \iint \rho_k\beta_k v\cdot\mathbb{P}_{\mathrm{curl}}(\rho_k^\sharp\varpi\times w)\,\mathrm{d}x\,\mathrm{d}t &\le C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_3}},\\ \iint \rho_k\beta_k v\cdot\mathbb{P}_{\nabla}(\rho_k^\sharp\varpi\times v)\,\mathrm{d}x\,\mathrm{d}t &\le C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_2}}.\end{aligned}$$

So the sum is bounded in

$$\iint \rho_k\beta_k v\cdot(\mathbf{W}+\mathbf{W}_2)\,\mathrm{d}x\,\mathrm{d}t = \iint \rho_k\beta_k v\cdot(\mathbf{W}_{\mathrm{int}}+\mathbf{W}_{\mathrm{ext}})\,\mathrm{d}x\,\mathrm{d}t \le C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_3}} \tag{4.46}$$

provided $U_{k-1}<1$.

4.5.4 Proof of Proposition 4.18

Proof of Proposition 4.18. Coming back to (4.41), by estimates (4.42) on the first term, (4.43) on the trilinear terms, (4.44), (4.45) on the $\mathbf{B}, \mathbf{L}$ terms and (4.46) on the $\mathbf{W}$ terms, we conclude that

$$U_k \leq C^k U_{k-1}^{\min\{\frac{5}{3}-\frac{2}{3p_3},\frac{4}{3}\}}$$

provided $U_{k-1} < 1$. Here $p_3 > 1$ ensures the index is strictly greater than 1. Since

$$\begin{aligned} U_0 &= \sup_{t\in(-1,0)} \int |v_0|^2 \,\mathrm{d}x + \int_{-1}^{0}\int_{B_1} d_0^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \sup_{t\in(-1,0)} \int |v|^2 \,\mathrm{d}x + \int_{-1}^{0}\int_{B_1} |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \eta \end{aligned}$$

by Proposition 4.14, we know that if η is small enough, $U_k \to 0$ as $k \to \infty$. So in $Q_{\frac{1}{2}}$, $|v| \leq 1$ a.e.. This finishes the proof of Proposition 4.18. □

4.6 Local Study: Part Three, More Regularity

In this section, we will show that the vorticity ω is smooth in space. We will only work with the vorticity equation from now on. After the previous two steps, in $B_{\frac{1}{2}}$ we should always decompose $u = v + w$, because v is bounded and w is harmonic.

For convenience, given a vector ω, we denote

$$\omega^0 := \frac{\omega}{|\omega|}, \qquad \omega^\alpha := |\omega|^\alpha \omega^0, \alpha \in \mathbb{R}.$$

Let $\partial_\bullet$ be the partial derivative in any space direction or time, then we have

$$\begin{aligned}
\partial_\bullet(|\omega|^\alpha) &= \alpha\omega^{\alpha-1}\cdot\partial_\bullet\omega,\\
\partial_\bullet(\omega^\alpha) &= |\omega|^{\alpha-1}\partial_\bullet\omega + (\alpha-1)(\omega^{\alpha-2}\cdot\partial_\bullet\omega)\omega,\\
\frac{1}{\alpha}\partial_\bullet\partial_\bullet(|\omega|^\alpha) &= |\omega|^{\alpha-2}|\partial_\bullet\omega|^2 + (\alpha-2)(\omega^{\frac{\alpha}{2}-1}\cdot\partial_\bullet\omega)^2 + \omega^{\alpha-1}\cdot\partial_\bullet\partial_\bullet\omega\\
&\geq (\alpha-1)(\omega^{\frac{\alpha}{2}-1}\cdot\partial_\bullet\omega)^2 + \omega^{\alpha-1}\cdot\partial_\bullet\partial_\bullet\omega\\
&= \frac{4(\alpha-1)}{\alpha^2}\left|\partial_\bullet\omega^{\frac{\alpha}{2}}\right|^2 + \omega^{\alpha-1}\cdot\partial_\bullet\partial_\bullet\omega.
\end{aligned}$$

4.6.1 Bound Vorticity in the Energy Space

We will first show ω is bounded in the energy space.

Proposition 4.21. *If $u = v + w$ in $Q_{\frac{1}{2}}$, where v, w are bounded in*

$$\|v\|_{L^\infty(Q_{\frac{1}{2}})} + \|\nabla v\|_{L^2(Q_{\frac{1}{2}})} \leq 2, \tag{4.47}$$

$$\|\operatorname{curl} w\|_{L^2_t L^{\frac{3}{2}}_x(Q_{\frac{1}{2}})} + \|w\|_{L^{\frac{4}{3}}_t \operatorname{Lip}_x(Q_{\frac{1}{2}})} \leq 2, \tag{4.48}$$

$\omega = \operatorname{curl} u$ solves the vorticity equation (4.6), then

(a) $\|\omega^{\frac{3}{4}}\|_{\mathcal{E}(Q_{\frac{1}{4}})} \leq C$,

(b) $\|\omega\|_{\mathcal{E}(Q_{\frac{1}{8}})} \leq C$,

Proof of Proposition 4.21 (a). We fix a pair of smooth space-time cut-off functions ϱ and ς which satisfy

$$\mathbf{1}_{Q_{\frac{1}{8}}} \leq \varsigma \leq \mathbf{1}_{Q_{\frac{1}{4}}} \leq \varrho \leq \mathbf{1}_{Q_{\frac{1}{2}}}.$$

Take the dot product of the vorticity equation (4.6) with $\frac{3}{2}\omega^{\frac{1}{2}}$:

$$\frac{3}{2}\omega^{\frac{1}{2}} \cdot \partial_t \omega = \partial_t(|\omega|^{\frac{3}{2}}),$$
$$\frac{3}{2}\omega^{\frac{1}{2}} \cdot (u \cdot \nabla)\omega = (u \cdot \nabla)(|\omega|^{\frac{3}{2}}),$$
$$\frac{3}{2}\omega^{\frac{1}{2}} \cdot \Delta\omega \le \Delta(|\omega|^{\frac{3}{2}}) - \frac{4}{3}|\nabla\omega^{\frac{3}{4}}|^2.$$

Therefore,

$$(\partial_t + u \cdot \nabla - \Delta)(|\omega|^{\frac{3}{2}}) + \frac{3}{2}\omega \cdot \nabla u \cdot \omega^{\frac{1}{2}} + \frac{4}{3}|\nabla\omega^{\frac{3}{4}}|^2 \le 0.$$

Multiply by ϱ^6 then integrate over space,

$$\int \varrho^6(\partial_t + u \cdot \nabla - \Delta)(|\omega|^{\frac{3}{2}})\,\mathrm{d}x + \frac{4}{3}\int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2\,\mathrm{d}x \le -\frac{3}{2}\int \varrho^6\omega \cdot \nabla u \cdot \omega^{\frac{1}{2}}\,\mathrm{d}x. \tag{4.49}$$

For the left hand side, we can integrate by part,

$$\begin{aligned} &\int \varrho^6(\partial_t + u \cdot \nabla - \Delta)(|\omega|^{\frac{3}{2}})\,\mathrm{d}x \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int \varrho^6|\omega|^{\frac{3}{2}}\,\mathrm{d}x - \int \left((\partial_t + u \cdot \nabla + \Delta)\varrho^6\right)|\omega|^{\frac{3}{2}}\,\mathrm{d}x, \end{aligned} \tag{4.50}$$

where the latter can be controlled by

$$\int \left((\partial_t + u \cdot \nabla + \Delta)\varrho^6\right)|\omega|^{\frac{3}{2}}\,\mathrm{d}x \le C\left(1 + \|u\|_{L^\infty(B_{\frac{1}{2}})}\right)\int \varrho^4|\omega|^{\frac{3}{2}}\,\mathrm{d}x. \tag{4.51}$$

For the right hand side, using $u = v + w$ over the support of ϱ we can separate

$$\int \varrho^6\omega \cdot \nabla u \cdot \omega^{\frac{1}{2}}\,\mathrm{d}x = \int \varrho^6\omega \cdot \nabla v \cdot \omega^{\frac{1}{2}}\,\mathrm{d}x + \int \varrho^6\omega \cdot \nabla w \cdot \omega^{\frac{1}{2}}\,\mathrm{d}x, \tag{4.52}$$

The ∇v term can be controlled by

$$\int \varrho^6 \omega \cdot \nabla v \cdot \omega^{\frac{1}{2}} \, \mathrm{d}x = -\int \omega \cdot \nabla(\varrho^6 \omega^{\frac{1}{2}}) \cdot v \, \mathrm{d}x \qquad (4.53)$$
$$= -\int \varrho^6 \omega \cdot \nabla(\omega^{\frac{1}{2}}) \cdot v \, \mathrm{d}x - \int \omega \cdot (\omega^{\frac{1}{2}} \otimes \nabla \varrho^6) \cdot v \, \mathrm{d}x,$$

where

$$\omega \cdot \nabla(\omega^{\frac{1}{2}}) = |\omega|^{-\frac{1}{2}} \omega \cdot \nabla \omega - \frac{1}{2}(\omega \cdot \nabla \omega \cdot \omega^{-\frac{3}{2}})\omega = \omega^{\frac{1}{2}} \cdot \nabla \omega - \frac{1}{2}(\omega^{\frac{1}{2}} \cdot \nabla \omega \cdot \omega^0)\omega^0$$
$$\Rightarrow |\omega \cdot \nabla(\omega^{\frac{1}{2}})| \leq \left| \frac{3}{2} \omega^{\frac{1}{2}} \cdot \nabla \omega \right| = 2|\omega|^{\frac{3}{4}} \left| \frac{3}{4} \omega^{-\frac{1}{4}} \cdot \nabla \omega \right| = 2|\omega|^{\frac{3}{4}} \left| \nabla |\omega|^{\frac{3}{4}} \right|$$
$$\leq 2|\omega|^{\frac{3}{4}} |\nabla \omega^{\frac{3}{4}}| \leq |\omega|^{\frac{3}{2}} + |\nabla \omega^{\frac{4}{3}}|^2.$$

Here the second to the last inequality is due to $\partial_i |\omega|^{\frac{3}{4}} = \partial_i \omega^{\frac{3}{4}} \cdot \omega^0$. Since $|v| \leq 1$ over the support of ϱ,

$$\int \varrho^6 \omega \cdot \nabla(\omega^{\frac{1}{2}}) \cdot v \, \mathrm{d}x \leq \int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x + \int \varrho^6 |\nabla \omega^{\frac{3}{4}}|^2 \, \mathrm{d}x. \qquad (4.54)$$

By using (4.50)-(4.54) in (4.49), we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x + \frac{4}{3} \int \varrho^6 |\nabla \omega^{\frac{3}{4}}|^2 \, \mathrm{d}x$$
$$\leq \int \left[(\partial_t + u \cdot \nabla + \Delta) \varrho^6 \right] |\omega|^{\frac{3}{2}} \, \mathrm{d}x$$
$$+ \int \varrho^6 \omega \cdot \nabla w \cdot \omega^{\frac{1}{2}} \, \mathrm{d}x$$
$$+ \int \omega \cdot (\omega^{\frac{1}{2}} \otimes \nabla \varrho^6) \cdot v \, \mathrm{d}x$$
$$+ \int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x + \int \varrho^6 |\nabla \omega^{\frac{3}{4}}|^2 \, \mathrm{d}x$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x + \frac{1}{3} \int \varrho^6 |\nabla \omega^{\frac{3}{4}}|^2 \, \mathrm{d}x$$
$$\leq C \left(1 + \|u(t)\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{2}})} \right) \int \varrho^4 |\omega|^{\frac{3}{2}} \, \mathrm{d}x.$$

By Hölder's inequality,

$$\int \varrho^4 |\omega|^{\frac{3}{2}} \, \mathrm{d}x \leq \|\omega(t)\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(B_{\frac{1}{2}})} \left(\int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x \right)^{\frac{2}{3}}.$$

Therefore we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x + \frac{1}{3} \int \varrho^6 |\nabla \omega^{\frac{3}{4}}|^2 \, \mathrm{d}x \leq C \Phi(t) \left(1 + \int \varrho^6 |\omega|^{\frac{3}{2}} \, \mathrm{d}x \right),$$

where

$$\begin{aligned} \Phi(t) &= \left(1 + \|u(t)\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{2}})} \right) \|\omega(t)\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(B_{\frac{1}{2}})} \\ &\leq \left(2 + \|w(t)\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{2}})} \right) \\ &\quad \times \left(\|\operatorname{curl} v(t)\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(B_{\frac{1}{2}})} + \|\operatorname{curl} w(t)\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(B_{\frac{1}{2}})} \right) \end{aligned}$$

since $u = w + v$, and $|v| \leq 1$ inside $B_{\frac{1}{2}}$. By (4.47),

$$\int_{-\frac{1}{4}}^{0} \Phi(t) \, \mathrm{d}t \lesssim \left(1 + \|w\|_{L_t^{\frac{4}{3}} \mathrm{Lip}_x(Q_{\frac{1}{2}})} \right) \left(\|\nabla v\|^{\frac{1}{2}}_{L^2(Q_{\frac{1}{2}})} + \|\operatorname{curl} w(t)\|^{\frac{1}{2}}_{L_t^2 L_x^{\frac{3}{2}}(Q_{\frac{1}{2}})} \right) \leq C.$$

So by Grönwall's inequality,

$$\|\omega^{\frac{3}{4}}\|^2_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{\frac{1}{4}})} \leq e^C - 1.$$

□

Proof of Proposition 4.21 (b). From Proposition 4.21 (a) and Sobolev embedding,

$$\|\omega\|_{L_t^\infty L_x^{\frac{3}{2}} \cap L_t^{\frac{3}{2}} L_x^{\frac{9}{2}}(Q_{\frac{1}{4}})} \leq C,$$

this interpolates the space

$$\|\omega\|_{L_t^4 L_x^2(Q_{\frac{1}{4}})} \leq C.$$

Multiply the vorticity equation (4.6) by $\varsigma^2\omega$ then integrate over $\mathbb{R}^3$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \varsigma^2 \frac{|\omega|^2}{2}\,\mathrm{d}x + \int \varsigma^2 |\nabla\omega|^2\,\mathrm{d}x = &\int (\partial_t \varsigma^2 + \Delta\varsigma^2)\frac{|\omega|^2}{2}\,\mathrm{d}x \\ &- \int (u\cdot\nabla\omega)\cdot\varsigma^2\omega\,\mathrm{d}x \\ &+ \int (\omega\cdot\nabla u)\cdot\varsigma^2\omega\,\mathrm{d}x.\end{aligned}$$

The first integral is L^1 in time because $\omega \in L_t^4 L_x^2$. For the second,

$$\begin{aligned}\int (u\cdot\nabla\omega)\cdot\varsigma^2\omega\,\mathrm{d}x &= \int \varsigma^2 u\cdot\nabla\frac{|\omega|^2}{2}\,\mathrm{d}x \\ &= -\int \frac{|\omega|^2}{2} u\cdot\nabla\varsigma^2\,\mathrm{d}x \\ &= -\int \varsigma|\omega|^2 u\cdot\nabla\varsigma \\ &\leq \|\varsigma\omega\|_{L^2}\|u\cdot\nabla\varsigma|\omega|\|_{L^2},\end{aligned}$$

the latter is bounded L^1 in time, by $u \in L_t^{\frac{4}{3}} L_x^\infty$ and $\omega \in L_t^4 L_x^2$. For the third integral,

$$\int (\omega\cdot\nabla u)\cdot\varsigma^2\omega\,\mathrm{d}x = \int (\omega\cdot\nabla v)\cdot\varsigma^2\omega\,\mathrm{d}x + \int (\omega\cdot\nabla w)\cdot\varsigma^2\omega\,\mathrm{d}x.$$

w is bounded in $L_t^{\frac{4}{3}}\,\mathrm{Lip}_x$, and for v,

$$\begin{aligned}\int (\omega\cdot\nabla v)\cdot\varsigma^2\omega\,\mathrm{d}x &= \int v\cdot(\omega\cdot\nabla)(\varsigma^2\omega)\,\mathrm{d}x \\ &= \int v\cdot\omega(\omega\cdot\nabla\varsigma^2)\,\mathrm{d}x + \int v\cdot(\varsigma^2\omega\cdot\nabla\omega)\,\mathrm{d}x.\end{aligned}$$

The former is L^1 in time, while the latter can be bounded by Cauchy-Schwartz,

$$\int v \cdot (\varsigma^2 \omega \cdot \nabla \omega)\, \mathrm{d}x \le \frac{1}{2} \int |v \otimes \varsigma \omega|^2 \, \mathrm{d}x + \frac{1}{2} \int \varsigma^2 |\nabla \omega|^2 \, \mathrm{d}x.$$

In conclusion,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int \varsigma^2 \frac{|\omega|^2}{2} \, \mathrm{d}x + \frac{1}{2} \int \varsigma^2 |\nabla \omega|^2 \, \mathrm{d}x \\ &\quad \le C \|\omega(t)\|^2_{L^2(B_{\frac14})} + C \|u(t)\|_{L^\infty(B_{\frac14})} \|\omega(t)\|_{L^2(B_{\frac14})} \|\varsigma \omega(t)\|_{L^2} \\ &\qquad + C \|\nabla w\|_{L^\infty(B_{\frac14})} \|\varsigma \omega(t)\|^2_{L^2} \\ &\quad \le C \Phi(t) \left(1 + \int \varsigma^2 \frac{|\omega|^2}{2} \, \mathrm{d}x \right) \end{aligned}$$

where

$$\Phi(t) = \|\omega(t)\|^2_{L^2(B_{\frac14})} + \|u(t)\|_{L^\infty(B_{\frac14})} \|\omega(t)\|_{L^2(B_{\frac14})} + \|\nabla w(t)\|_{L^\infty(B_{\frac14})},$$

whose integral is bounded using (4.47),

$$\int_{-\frac{1}{16}}^0 \Phi(t) \, \mathrm{d}t \le \|\omega\|^2_{L^2(Q_{\frac14})} + \|u\|_{L_t^{\frac43} L_x^\infty(Q_{\frac14})} \|\omega\|_{L_t^4 L_x^2(Q_{\frac14})} + \|\nabla w\|_{L_t^{\frac43} L_x^\infty(Q_{\frac14})} \le C.$$

By a Grönwall argument, we have

$$\|\omega\|^2_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{\frac18})} \le e^C - 1.$$

□

4.6.2 Bound Higher Derivatives in the Energy Space

Now we bootstrap to higher regularity of ω using similar ideas as in the proof of Proposition 4.21.

Proposition 4.22. *For any $n \geq 1$, if $u = v + w$ in $Q_{8^{-n}}$, where v, w are bounded in*

$$\|v\|_{L^\infty(Q_{8^{-n/2}})} + \|v\|_{L^2_t H^{n+1}_x(Q_{8^{-n/2}})} \leq c_n, \tag{4.55}$$

$$\|w\|_{L^{\frac{4}{3}}_t C^{n+1}_x(Q_{8^{-n/2}})} \leq c_n, \tag{4.56}$$

for some constant c_n, $\omega = \operatorname{curl} u$ solves the vorticity equation (4.6), and is bounded in

$$\|\omega\|_{L^\infty_t H^{n-1}_x \cap L^2_t H^n_x(Q_{8^{-n/2}})} \leq c_n, \tag{4.57}$$

then for any multiindex α with $|\alpha| = n$,

(a) $\|\nabla^\alpha \omega^{\frac{3}{4}}\|_{\mathcal{E}(Q_{8^{-n/4}})} \leq C_n$

(b) $\|\nabla^\alpha \omega\|_{\mathcal{E}(Q_{8^{-n-1}})} \leq C_n$

for some C_n depending on c_n and n.

Proof of Proposition 4.22 (a). Similarly we fix smooth cut-off functions ϱ_n and ς_n which satisfy

$$\mathbf{1}_{Q_{8^{-n-1}}} \leq \varsigma_n \leq \mathbf{1}_{Q_{8^{-n/4}}} \leq \varrho_n \leq \mathbf{1}_{Q_{8^{-n/2}}}.$$

Differentiate (4.6) by ∇^α,

$$\partial_t \nabla^\alpha \omega + u \cdot \nabla \nabla^\alpha \omega - \nabla^\alpha \omega \cdot \nabla u + \mathbf{P}_\alpha = \Delta \nabla^\alpha \omega, \tag{4.58}$$

where

$$\mathbf{P}_\alpha = \sum_{\beta < \alpha} \binom{\alpha}{\beta} \operatorname{curl}\left(\nabla^\beta \omega \times \nabla^{\alpha-\beta} u\right).$$

Multiply (4.58) by $\frac{3}{2}\varrho_n^6(\nabla^\alpha\omega)^{\frac{1}{2}}$ then integrate in space,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho_n^6|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x + \frac{4}{3}\int \varrho_n^6|\nabla\nabla^\alpha\omega^{\frac{3}{4}}|^2\,\mathrm{d}x \\
&\quad\le \int \left[(\partial_t + u\cdot\nabla + \Delta)\varrho_n^6\right]|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x \\
&\qquad + \int \varrho_n^6\nabla^\alpha\omega\cdot\nabla w\cdot(\nabla^\alpha\omega)^{\frac{1}{2}}\,\mathrm{d}x \\
&\qquad + \int \nabla^\alpha\omega\cdot((\nabla^\alpha\omega)^{\frac{1}{2}}\otimes\nabla\varrho_n^6)\cdot v\,\mathrm{d}x \\
&\qquad + \|v\|_{L^\infty(Q_{8^{-n}})}^2\int \varrho_n^6|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x + \int \varrho_n^6|\nabla\nabla^\alpha\omega^{\frac{3}{4}}|^2\,\mathrm{d}x \\
&\qquad + \frac{3}{2}\int \varrho_n^6(\nabla^\alpha\omega)^{\frac{1}{2}}\cdot\mathbf{P}_\alpha\,\mathrm{d}x
\end{aligned}$$

same as in the proof of Proposition 4.21 (a). So

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho_n^6|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x + \frac{1}{3}\int \varrho_n^6|\nabla\nabla^\alpha\omega^{\frac{3}{4}}|^2\,\mathrm{d}x \\
&\quad\le C\left(1 + \|u(t)\|_{L^\infty(B_{8^{-n}})} + \|\nabla w(t)\|_{L^\infty(B_{8^{-n}})}\right)\int \varrho_n^4|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x \\
&\qquad + \frac{3}{2}\int \varrho_n^6(\nabla^\alpha\omega)^{\frac{1}{2}}\cdot\mathbf{P}_\alpha\,\mathrm{d}x.
\end{aligned}$$

Terms other than $\mathbf{P}_\alpha$ are dealt with by the same way as in Proposition 4.21:

$$\int \varrho_n^4|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x \le \|\nabla^\alpha\omega(t)\|_{L^{\frac{3}{2}}(B_{8^{-n}})}^{\frac{1}{2}}\left(\int \varrho_n^6|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x\right)^{\frac{2}{3}}.$$

The induction condition (4.57) ensures that $\|\nabla^\alpha\omega\|_{L^2(Q_{8^{-n}})} \le c_n$. Therefore

$$\begin{aligned}
&\int_{-8^{-2n}}^0 \left(1 + \|u(t)\|_{L^\infty(B_{8^{-n}})} + \|\nabla w(t)\|_{L^\infty(B_{8^{-n}})}\right)\|\nabla^\alpha\omega(t)\|_{L^{\frac{3}{2}}(B_{8^{-n}})}^{\frac{1}{2}}\,\mathrm{d}t \\
&\quad\lesssim \left(1 + \|v\|_{L^\infty(B_{8^{-n}})} + \|w\|_{L_t^{\frac{4}{3}}C_x^1(B_{8^{-n}})}\right)\|\nabla^\alpha\omega\|_{L^2(Q_{8^{-n}})}^{\frac{1}{2}} \le C_n.
\end{aligned}$$

Now let's focus on $\mathbf{P}_\alpha$.

$$|\mathbf{P}_\alpha| \lesssim \sum_{k=0}^{n} |\nabla^k \omega||\nabla^{n-k+1} u| \leq \sum_{k=0}^{n} |\nabla^k \omega||\nabla^{n-k+1} v| + \sum_{k=0}^{n} |\nabla^k \omega||\nabla^{n-k+1} w|.$$

We denote

$$\mathbf{P}_{v,k} = |\nabla^k \omega||\nabla^{n-k+1} v|, \qquad \mathbf{P}_{w,k} = |\nabla^k \omega||\nabla^{n-k+1} w|.$$

First we estimate $\mathbf{P}_{v,k}$. By (4.55) and (4.57), when $k = 0$,

$$\|\mathbf{P}_{v,0}\|_{L_t^1 L_x^{\frac{3}{2}}(Q_{8-n})} \leq \|\omega\|_{L_t^2 L_x^6(Q_{8-n})} \|\nabla^{n+1} v\|_{L_t^2 L_x^2(Q_{8-n})} \leq C_n,$$

and when $0 < k \leq n$,

$$\|\mathbf{P}_{v,k}\|_{L_t^1 L_x^{\frac{3}{2}}(Q_{8-n})} \leq \|\nabla^k \omega\|_{L_t^2 L_x^2(Q_{8-n})} \|\nabla^{n+1-k} v\|_{L_t^2 L_x^6(Q_{8-n})} \leq C_n.$$

Next we estimate $\mathbf{P}_{w,k}$. When $0 \leq k < n$,

$$\|\mathbf{P}_{w,k}\|_{L_t^1 L_x^{\frac{3}{2}}(Q_{8-n})} \leq \|\nabla^k \omega\|_{L_t^\infty L_x^2(Q_{8-n})} \|\nabla^{n+1-k} w\|_{L_t^{\frac{4}{3}} L_x^\infty(Q_{8-n})} \leq C_n.$$

Finally, when $k = n$,

$$|\mathbf{P}_{w,n}|_{L_x^{\frac{3}{2}}(B_{8-n})} \leq |\nabla^n \omega||\nabla w|.$$

Therefore,

$$\begin{aligned}\int \varrho_n^6 (\nabla^\alpha \omega)^{\frac{1}{2}} \cdot \mathbf{P}_\alpha \, \mathrm{d}x \leq &\left(1 + \int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} \, \mathrm{d}x\right) \\ &\times \left(\sum_{k=0}^{n} \|\mathbf{P}_{v,k}\|_{L_x^{\frac{3}{2}}(B_{8-n})} + \sum_{k=0}^{n-1} \|\mathbf{P}_{w,k}\|_{L_x^{\frac{3}{2}}(B_{8-n})} + \|\nabla w\|_{L_x^\infty(B_{8-n})}\right)\end{aligned}$$

In conclusion, we have shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho_n^6|\nabla^\alpha\omega|^{\frac{3}{2}}\,\mathrm{d}x+\frac{1}{3}\int \varrho_n^6|\nabla\nabla^\alpha\omega^{\frac{3}{4}}|^2\,\mathrm{d}x\le C\Phi(t)\left(1+\int \varrho_n^6|\nabla^n\omega|^{\frac{3}{2}}\,\mathrm{d}x\right),$$

where

$$\begin{aligned}\Phi(t)=&\left(1+\|u(t)\|_{L^\infty(B_{8^{-n}})}+\|\nabla w(t)\|_{L^\infty(B_{8^{-n}})}\right)\|\nabla^\alpha\omega(t)\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(B_{8^{-n}})}\\&+\sum_{k=0}^{n}\|\mathbf{P}_{v,k}\|_{L_x^{\frac{3}{2}}(B_{8^{-n}})}+\sum_{k=0}^{n-1}\|\mathbf{P}_{w,k}\|_{L_x^{\frac{3}{2}}(B_{8^{-n}})}+\|\nabla w\|_{L_x^\infty(B_{8^{-n}})}\end{aligned}$$

with integral

$$\int_{-8^{-2n}/4}^{0}\Phi(t)\,\mathrm{d}t\le C_n.$$

Taking the sum over all multi-index α with size $|\alpha|=n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho_n^6|\nabla^n\omega|^{\frac{3}{2}}\,\mathrm{d}x+\frac{1}{3}\int \varrho_n^6|\nabla^{n+1}\omega^{\frac{3}{4}}|^2\,\mathrm{d}x\le C\Phi(t)\left(1+\int \varrho_n^6|\nabla^{n+1}\omega|^{\frac{3}{2}}\,\mathrm{d}x\right),$$

Finally, Grönwall inequality gives

$$\||\nabla^{n+1}\omega|^{\frac{3}{4}}\|_{L_t^\infty L_x^2\cap L_t^2\dot{H}_x^1(Q_{8^{-n}/4})}\le C_n.$$

□

Proof of Proposition 4.22 (b). Now we multiply (4.58) by $\varsigma_n^2\nabla^\alpha\omega$ then integrate over

$\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2}\,\mathrm{d}x + \int \varsigma_n^2 |\nabla\nabla^\alpha\omega|^2\,\mathrm{d}x = \int (\partial_t \varsigma_n^2 + \Delta\varsigma_n^2)\frac{|\nabla^\alpha\omega|^2}{2}\,\mathrm{d}x \\ - \int (u\cdot\nabla\nabla^\alpha\omega)\cdot\varsigma_n^2\nabla^\alpha\omega\,\mathrm{d}x \\ + \int (\nabla^\alpha\omega\cdot\nabla u)\cdot\varsigma_n^2\nabla^\alpha\omega\,\mathrm{d}x \\ + \int \varsigma_n^2\nabla^\alpha\omega\cdot\mathbf{P}_\alpha\,\mathrm{d}x$$

For the same reason, the only term that we need to take care of is $\mathbf{P}_\alpha$ term, and the others are dealt the same as in Proposition 4.21 (b):

$$\int (\partial_t\varsigma_n^2 + \Delta\varsigma_n^2)\frac{|\nabla^\alpha\omega|^2}{2}\,\mathrm{d}x - \int (u\cdot\nabla\nabla^\alpha\omega)\cdot\varsigma_n^2\nabla^\alpha\omega\,\mathrm{d}x + \int (\nabla^\alpha\omega\cdot\nabla u)\cdot\varsigma_n^2\nabla^\alpha\omega\,\mathrm{d}x \\ \lesssim_n \|\nabla^\alpha\omega\|^2_{L^2(Q_{8-n/4})} + \|u\|_{L^\infty(Q_{8-n/4})}\|\nabla^\alpha\omega\|_{L^2(Q_{8-n/4})}\left(\int \varsigma_n^2\frac{|\nabla^\alpha\omega|^2}{2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\ + \|\nabla w\|_{L^\infty(Q_{8-n/4})}\int \varsigma_n^2\frac{|\nabla^\alpha\omega|^2}{2}\,\mathrm{d}x + \|v\|_{L^\infty(Q_{8-n/4})}\|\nabla^\alpha\omega\|^2_{L^2(Q_{8-n/4})} \\ + \frac{1}{\varepsilon}\|v\|^2_{L^\infty(Q_{8-n/4})}\int \varsigma_n^2\frac{|\nabla^\alpha\omega|^2}{2}\,\mathrm{d}x + \varepsilon\int \varsigma_n^2|\nabla\nabla^\alpha\omega|^2\,\mathrm{d}x.$$

The last term can be absorbed into the left, and we will use Grönwall on the remaining terms.

Now we shall focus on the $\mathbf{P}_\alpha$ term. From Proposition 4.22 (a), we have

$$\|\nabla^n\omega\|_{L_t^\infty L_x^{\frac{3}{2}}\cap L_x^{\frac{3}{2}}L_t^{\frac{9}{2}}(Q_{8-n/4})} \le C_n. \tag{4.59}$$

Again by interpolation,

$$\|\nabla^n\omega\|_{L_t^4L_x^2(Q_{8-n/4})} \le C_n, \qquad \|\nabla^n\omega\|_{L_t^2L_x^3(Q_{8-n/4})} \le C_n,$$

First we estimate $\mathbf{P}_{w,k}$. In this case, for any $0 \le k \le n$,

$$\|\mathbf{P}_{w,k}\|_{L^1_t L^2_x(Q_{8-n/4})} \le \|\nabla^k \omega\|_{L^4_t L^2_x(Q_{8-n/4})} \|\nabla^{n+1-k} w\|_{L^{\frac{4}{3}}_t L^\infty_x(Q_{8-n})} \le C_n.$$

Then we estimate $\mathbf{P}_{v,k}$. When $0 < k \le n$,

$$\|\mathbf{P}_{v,k}\|_{L^1_t L^2_x(Q_{8-n})} \le \|\nabla^k \omega\|_{L^2_t L^3_x(Q_{8-n})} \|\nabla^{n+1-k} v\|_{L^2_t L^6_x(Q_{8-n})} \le C_n.$$

For the case $k = 0$ of the v term, we put the curl on $\nabla^\alpha \omega$,

$$\begin{aligned}
&\int \varsigma_n^2 \nabla^\alpha \omega \cdot \operatorname{curl}(\omega \times \nabla^\alpha v) \, \mathrm{d}x \\
&\quad = \int (\omega \times \nabla^\alpha v) \cdot \operatorname{curl}(\varsigma_n^2 \nabla^\alpha \omega) \, \mathrm{d}x \\
&\quad \le \int \varsigma_n^2 |\omega| |\nabla^\alpha v| |\nabla \nabla^\alpha \omega| + \varsigma_n |\nabla \varsigma_n| |\omega| |\nabla^\alpha v| |\nabla^\alpha \omega| \, \mathrm{d}x \\
&\quad \le \int \varsigma_n^2 |\omega|^2 |\nabla^\alpha v|^2 \, \mathrm{d}x + \varepsilon \int \varsigma_n^2 |\nabla \nabla^\alpha \omega|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int |\nabla \varsigma_n|^2 |\nabla^\alpha \omega|^2 \, \mathrm{d}x.
\end{aligned}$$

where $|\nabla \nabla^\alpha \omega|$ term can be absorbed to the left. By (4.59) and Sobolev embedding,

$$\|\omega\|_{L^\infty_t L^3_x(Q_{8-n/4})} \le C_n.$$

Therefore

$$\iint \varsigma_n^2 |\omega|^2 |\nabla^\alpha v|^2 \, \mathrm{d}x \, \mathrm{d}t \le \|\omega\|^2_{L^\infty_t L^3_x(Q_{8-n/4})} \|\nabla^\alpha v\|^2_{L^2_t L^6_x(Q_{8-n/4})} \le C_n.$$

In conclusion,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \, \mathrm{d}x + \int \varsigma_n^2 |\nabla \nabla^\alpha \omega|^2 \, \mathrm{d}x \le C\Phi(t) \left(1 + \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \, \mathrm{d}x\right),$$

where

$$\begin{aligned}\Phi(t) = &\|\nabla^\alpha \omega(t)\|^2_{L^2(B_{8^{-n}/4})} + \|u\|_{L^\infty(B_{8^{-n}/4})}\|\nabla^\alpha\omega\|_{L^2(B_{8^{-n}/4})}\\&+\|\nabla w\|_{L^\infty(B_{8^{-n}/4})} + \|v\|_{L^\infty(B_{8^{-n}/4})}\|\nabla^\alpha\omega\|^2_{L^2(B_{8^{-n}/4})}\\&+\frac{1}{\varepsilon}\|v\|^2_{L^\infty(B_{8^{-n}/4})} + \sum_{k=0}^{n}\|\mathbf{P}_{w,k}\|_{L^2(B_{8^{-n}/4})} + \sum_{k=0}^{n-1}\|\mathbf{P}_{v,k}\|_{L^2(B_{8^{-n}/4})}\\&+\|\omega\|^2_{L^3(B_{8^{-n}/4})}\|\nabla^\alpha v\|^2_{L^6(B_{8^{-n}/4})} + \frac{1}{\varepsilon}\|\nabla^\alpha\omega(t)\|^2_{L^2(B_{8^{-n}/4})}\end{aligned}$$

has integral $\int_{-8^{-2n}/16}^{0}\Phi(t)\,\mathrm{d}t \le C_n$. Finally Grönwall inequality gives

$$\|\nabla^\alpha\omega\|_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q_{8^{-n-1}})} \le C_{n+1}.$$

□

4.6.3 Proof of the Local Theorem

Proof of the Local Theorem 4.3. First, Proposition 4.14 gives

$$\|v\|_{\mathcal{E}(Q_1)} \le \eta$$

where η can be chosen arbitrarily small if we pick η_1 small. Next, by Proposition 4.18, we know

$$\|v\|_{L^\infty(Q_{\frac{1}{2}})} \le 1.$$

These two steps implies (4.47). As for (4.48), $\operatorname{curl} w = \varpi$ in B_1, so we use interpolation in (4.37):

$$\|\operatorname{curl} w\|_{L^2_t L^{\frac{3}{2}}_x(Q_{\frac{1}{2}})} \le \|\varpi\|_{L^2_t L^{\frac{12}{7}}_x} \le \|\varpi\|^{\frac{1}{2}}_{L^{p_1}_t L^{q_3}_x}\|\varpi\|^{\frac{1}{2}}_{L^{p_2}_t L^{q_4}_x} \le \eta$$

w is harmonic inside B_1, therefore

$$\|w\|_{L_t^{\frac{4}{3}} C_x^n(Q_{\frac{1}{2}})} \lesssim_n \|w\|_{L_t^{\frac{4}{3}} L_x^1(Q_1)} \leq \eta$$

due to (4.36) and $p_1 \geq \frac{4}{3}$. Therefore, we can use Proposition 4.21 to obtain

$$\|\omega\|_{\mathcal{E}(Q_{\frac{1}{8}})} \leq C.$$

The next step is to use Proposition 4.22 iteratively. Suppose for $n \geq 1$ we know that

$$\|\nabla^{n-1}\omega\|_{\mathcal{E}(Q_{8^{-n}})} \leq c_n$$

which is equivalent to (4.57). Let φ_n and $\varphi_n^\sharp$ be a pair of smooth spatial cut-off functions, with

$$\mathbf{1}_{B_{\frac{1}{8^n+4}}} \leq \varphi_n \leq \mathbf{1}_{B_{\frac{1}{8^n+3}}}, \qquad \mathbf{1}_{B_{\frac{1}{8^n+2}}} \leq \varphi_n^\sharp \leq \mathbf{1}_{B_{\frac{1}{8^n+1}}},$$

and set

$$v_n := -\operatorname{curl} \varphi_n^\sharp \Delta^{-1} \varphi_n \omega, \qquad w_n = \varphi_n u - v_n.$$

On the one hand, ∇v_n is a Riesz transform of $\varphi_n \omega$ up to lower order terms, so by the boundedness of Riesz transform we know

$$\|\nabla^{n+1} v_n\|_{L^2(Q_{8-n/2})} \leq \|\nabla^n \omega\|_{L^2(Q_{8-n})} \leq c_{n-1}.$$

On the other hand, we have similar boundedness estimates following Proposition

4.18 as before,

$$\|v_n\|_{L^\infty(Q_{8^{-n}/2})} \le 1.$$

w_n is harmonic in $B_{\frac{1}{8^n+4}}$, so we also have

$$\|w_n\|_{L_t^{\frac{4}{3}} C_x^{n+1}(Q_{8^{-n}/2})} \lesssim_n \|w_n\|_{L_t^{\frac{4}{3}} L_x^1(Q_{\frac{1}{8^n+4}})} \le \eta.$$

Therefore, by Proposition 4.22

$$\|\nabla^n \omega\|_{\mathcal{E}(Q_{8^{-n-1}})} \le C_n.$$

By induction, we have

$$\|\nabla^n \omega\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{8^{-n-1}})} \le C_n$$

for any n. By Sobolev embedding, this implies for any n,

$$\|\nabla^n \omega\|_{L^\infty(Q_{8^{-n-3}})} \lesssim \|\nabla^n \omega\|_{L_t^\infty L_x^2(Q_{8^{-n-3}})} + \|\nabla^{n+2} \omega\|_{L_t^\infty L_x^2(Q_{8^{-n-3}})} \le C_n.$$

□

4.7 Appendix: Suitability of Solutions

Theorem 4.23. *Let u be a suitable weak solution to the Navier–Stokes equation in $\mathbb{R}^3$. That is, $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ solves the following equation in the sense of distribution,*

$$\partial_t u + u \cdot \nabla u + \nabla P = \Delta u, \qquad \operatorname{div} u = 0 \tag{4.60}$$

where P is the pressure, and u satisfies the following local energy inequality in the sense of distribution,

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + P\right)\right) + |\nabla u|^2 \leq \Delta \frac{|u|^2}{2}. \tag{4.61}$$

Suppose $v \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ is compactly supported in space and solves the following equation,

$$\partial_t v + \omega \times v + \nabla \mathbf{R}(u \otimes v) = \Delta v + \mathbf{C}_v, \qquad \operatorname{div} v = 0 \tag{4.62}$$

where $\omega = \operatorname{curl} u$ is the vorticity, $\mathbf{C}_v \in L_t^1 L_{\mathrm{loc},x}^2 + L_t^2 L_{\mathrm{loc},x}^{\frac{6}{5}}$ is a force term, and

$$\mathbf{R} = \frac{1}{2}\operatorname{tr} - \Delta^{-1} \operatorname{div}\operatorname{div}$$

is a symmetric Riesz operator. Moreover, suppose v differs from φu by

$$\varphi u - v = w \in L_t^\infty H_x^1 \cap L_t^2 H_x^2$$

for some fixed $\varphi \in C_c^\infty(\mathbb{R}^3)$. Then v satisfies the following local energy inequality,

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}(v\mathbf{R}(u \otimes v)) + |\nabla v|^2 \leq \Delta \frac{|v|^2}{2} + v \cdot \mathbf{C}_v. \tag{4.63}$$

Proof. It is well-known that the pressure P can be recovered from u by

$$P = -\Delta^{-1} \operatorname{div}\operatorname{div}(u \otimes u).$$

Since

$$u \cdot \nabla u + \nabla P = \nabla \frac{|u|^2}{2} + \omega \times u - \nabla \Delta^{-1} \operatorname{div} \operatorname{div}(u \otimes u)$$
$$= \omega \times u + \nabla \mathbf{R}(u \otimes u),$$

The Navier–Stokes equation (4.60) can be rewritten as

$$\partial_t u + \omega \times u + \nabla \mathbf{R}(u \otimes u) = \Delta u, \tag{4.64}$$

and local energy inequality (4.61) can be rewritten as

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u \mathbf{R}(u \otimes u)\right) + |\nabla u|^2 \leq \Delta \frac{|u|^2}{2}, \tag{4.65}$$

First, multiply (4.64) by φ,

$$\partial_t \varphi u + \omega \times \varphi u + \nabla \mathbf{R}(u \otimes \varphi u) = \Delta(\varphi u) + [\nabla \mathbf{R}, \varphi](u \otimes u) + [\varphi, \Delta] u.$$

Denote

$$\mathbf{C}_u = [\nabla \mathbf{R}, \varphi](u \otimes u) + [\varphi, \Delta] u$$

for these commutator terms. Subtracting the equation of v from this equation of φu, we will have the equation for w. In summary,

$$\partial_t \varphi u + \omega \times \varphi u + \nabla \mathbf{R}(u \otimes \varphi u) = \Delta(\varphi u) + \mathbf{C}_u, \tag{4.66}$$
$$\partial_t v + \omega \times v + \nabla \mathbf{R}(u \otimes v) = \Delta v + \mathbf{C}_v, \tag{4.67}$$
$$\partial_t w + \omega \times w + \nabla \mathbf{R}(u \otimes w) = \Delta w + \mathbf{C}_u - \mathbf{C}_v. \tag{4.68}$$

Recall from [Vas10] that $\Delta u \in L^{\frac{4}{3}-\varepsilon}_{\mathrm{loc}(t,x)}$. Since $\Delta w \in L^2_{t,x}$, we have $\Delta v \in L^{\frac{4}{3}-\varepsilon}_{\mathrm{loc}(t,x)}$.

Moreover, $\mathbf{C}_u, \mathbf{C}_v \in L^1_t L^2_{\text{loc},x} + L^2_t L^{\frac{6}{5}}_{\text{loc},x}$, and $\varphi u, v \in \mathcal{E}$ are compactly supported. Therefore, we can multiply (4.66) and (4.67) by w, and (4.68) by φu and v,

$$w \cdot \partial_t(\varphi u) + w \cdot \omega \times \varphi u + w \cdot \nabla \mathbf{R}(u \otimes \varphi u) = w \cdot \Delta(\varphi u) + w \cdot \mathbf{C}_u, \tag{4.69}$$

$$w \cdot \partial_t v + w \cdot \omega \times v + w \cdot \nabla \mathbf{R}(u \otimes v) = w \cdot \Delta v + w \cdot \mathbf{C}_v \tag{4.70}$$

$$\varphi u \cdot \partial_t w + \varphi u \cdot \omega \times w + \varphi u \cdot \nabla \mathbf{R}(u \otimes w) = \varphi u \cdot \Delta w + \varphi u \cdot (\mathbf{C}_u - \mathbf{C}_v). \tag{4.71}$$

$$v \cdot \partial_t w + v \cdot \omega \times w + v \cdot \nabla \mathbf{R}(u \otimes w) = v \cdot \Delta w + v \cdot (\mathbf{C}_u - \mathbf{C}_v). \tag{4.72}$$

Now take the sum of (4.69)-(4.72). ∂_t terms are

$$\begin{aligned} &\varphi u \cdot \partial_t w + w \cdot \partial_t(\varphi u) + v \cdot \partial_t w + w \cdot \partial_t v \\ &= \partial_t(\varphi u \cdot w) + \partial_t(w \cdot v) \\ &= \partial_t(|\varphi u|^2 - |v|^2). \end{aligned}$$

$\omega \times$ terms are

$$w \cdot \omega \times \varphi u + \varphi u \cdot \omega \times w + w \cdot \omega \times v + v \cdot \omega \times w = 0.$$

$\nabla\mathbf{R}$ terms are

$$\begin{aligned}
& w\cdot\nabla\mathbf{R}(u\otimes\varphi u)+v\cdot\nabla\mathbf{R}(u\otimes w)\\
&\qquad+\varphi u\cdot\nabla\mathbf{R}(u\otimes w)+w\cdot\nabla\mathbf{R}(u\otimes v)\\
&=\operatorname{div}(w\mathbf{R}(u\otimes\varphi u))+\operatorname{div}(v\mathbf{R}(u\otimes w))\\
&\qquad+\operatorname{div}(\varphi u\mathbf{R}(u\otimes w))+\operatorname{div}(w\mathbf{R}(u\otimes v))\\
&\qquad-\operatorname{div}(w)\nabla\mathbf{R}(u\otimes\varphi u)-\operatorname{div}(v)\nabla\mathbf{R}(u\otimes w)\\
&\qquad-\operatorname{div}(\varphi u)\nabla\mathbf{R}(u\otimes w)-\operatorname{div}(\varphi)\nabla\mathbf{R}(u\otimes v)\\
&=2\operatorname{div}(\varphi u\mathbf{R}(u\otimes\varphi u)-v\mathbf{R}(u\otimes v))\\
&\qquad-(u\cdot\nabla\varphi)\left(\nabla\mathbf{R}(u\otimes\varphi u)+\nabla\mathbf{R}(u\otimes w)+\nabla\mathbf{R}(u\otimes v)\right)\\
&=2\operatorname{div}(\varphi u\mathbf{R}(u\otimes\varphi u)-v\mathbf{R}(u\otimes v))-2(u\cdot\nabla\varphi)\mathbf{R}(u\otimes\varphi u).
\end{aligned}$$

Here we use $\operatorname{div} v=0,\operatorname{div}(\varphi u)=\operatorname{div} w=u\cdot\nabla\varphi$. Δ terms are

$$\begin{aligned}
&\varphi u\cdot\Delta w+w\cdot\Delta(\varphi u)+v\cdot\Delta w+w\cdot\Delta v\\
&\qquad=\Delta(u\cdot w)-2\nabla(\varphi u):\nabla w+\Delta(v\cdot w)-2\nabla v:\nabla w\\
&\qquad=\Delta(|\varphi u|^2-|v|^2)-2(|\nabla(\varphi u)|^2-|\nabla v|^2).
\end{aligned}$$

Commutator terms are

$$w\cdot\mathbf{C}_u+\varphi u\cdot(\mathbf{C}_u-\mathbf{C}_v)+w\cdot\mathbf{C}_v+v\cdot(\mathbf{C}_u-\mathbf{C}_v)=2\varphi u\cdot\mathbf{C}_u-2v\cdot\mathbf{C}_v.$$

In summary, half the sum of these four identities (4.69)-(4.72) gives

$$\begin{aligned}
&\partial_t\frac{|\varphi u|^2-|v|^2}{2}+\operatorname{div}(\varphi u\mathbf{R}(u\otimes\varphi u)-v\mathbf{R}(u\otimes v))+|\nabla(\varphi u)|^2-|\nabla v|^2\\
&\qquad=\Delta\frac{|\varphi u|^2-|v|^2}{2}+\varphi u\cdot\mathbf{C}_u-v\cdot\mathbf{C}_v+(u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u).
\end{aligned}\tag{4.73}$$

Next, multiply local energy inequality of u (4.65) by φ^2,

$$\begin{aligned}
&\partial_t \frac{|\varphi u|^2}{2} + |\varphi \nabla u|^2 + \operatorname{div}\left(\varphi^2 u \mathbf{R}(u \otimes u)\right) \\
&\qquad \leq \Delta \frac{|\varphi u|^2}{2} + [\varphi^2, \Delta] \frac{|u|^2}{2} + [\operatorname{div}, \varphi^2]\left(u \mathbf{R}(u \otimes u)\right), \\
&\partial_t \frac{|\varphi u|^2}{2} + |\nabla(\varphi u)|^2 + \operatorname{div}\left(\varphi u \mathbf{R}(u \otimes \varphi u)\right) \\
&\qquad \leq \Delta \frac{|\varphi u|^2}{2} + [\varphi^2, \Delta] \frac{|u|^2}{2} + |u \otimes \nabla \varphi|^2 + 2(u \otimes \nabla \varphi) : (\varphi \nabla u) \\
&\qquad\qquad + [\operatorname{div}, \varphi^2]\left(u \mathbf{R}(u \otimes u)\right) + \operatorname{div}(\varphi u [\mathbf{R}, \varphi](u \otimes u)).
\end{aligned} \tag{4.74}$$

The quadratic commutator terms in (4.74) are

$$\begin{aligned}
&[\varphi^2, \Delta] \frac{|u|^2}{2} + |u \otimes \nabla \varphi|^2 + 2(u \otimes \nabla \varphi) : (\varphi \nabla u) \\
&\qquad = [\varphi^2, \Delta] \frac{|u|^2}{2} + |u|^2 |\nabla \varphi|^2 + 2 \nabla \varphi \cdot \varphi \nabla u \cdot u \\
&\qquad = -2 \nabla(\varphi^2) \cdot \nabla \frac{|u|^2}{2} - \Delta(\varphi^2) \frac{|u|^2}{2} + |u|^2 |\nabla \varphi|^2 + 2 \nabla \varphi \cdot \nabla u \cdot \varphi u \\
&\qquad = -4 \varphi \nabla \varphi \cdot \nabla \frac{|u|^2}{2} - \frac{1}{2} \Delta(\varphi^2) |u|^2 + |u|^2 |\nabla \varphi|^2 + 2 \nabla \varphi \cdot \nabla u \cdot \varphi u \\
&\qquad = -2 \varphi \nabla \varphi \cdot \nabla u \cdot u - \varphi \Delta \varphi |u|^2 \\
&\qquad = \varphi u \cdot (-2 \nabla \varphi \cdot \nabla u - (\Delta \varphi) u) \\
&\qquad = \varphi u \cdot [\varphi, \Delta] u.
\end{aligned}$$

The cubic commutator terms in (4.74) are

$$
\begin{aligned}
&[\mathrm{div}, \varphi^2]\,(u\mathbf{R}(u\otimes u)) + \mathrm{div}(\varphi u[\mathbf{R},\varphi](u\otimes u))\\
&\qquad = 2\varphi\nabla\varphi\cdot u\mathbf{R}(u\otimes u) + \varphi u\cdot\nabla[\mathbf{R},\varphi](u\otimes u) + \mathrm{div}(\varphi u)[\mathbf{R},\varphi](u\otimes u)\\
&\qquad = 2\varphi(u\cdot\nabla\varphi)\mathbf{R}(u\otimes u) + \varphi u\cdot\nabla[\mathbf{R},\varphi](u\otimes u) + (u\cdot\nabla\varphi)[\mathbf{R},\varphi](u\otimes u)\\
&\qquad = 2\varphi(u\cdot\nabla\varphi)\mathbf{R}(u\otimes u) + \varphi u\cdot\nabla[\mathbf{R},\varphi](u\otimes u)\\
&\qquad\qquad + (u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u) - (u\cdot\nabla\varphi)\varphi\mathbf{R}(u\otimes u)\\
&\qquad = \varphi u\cdot\nabla\varphi\mathbf{R}(u\otimes u) + \varphi u\cdot\nabla[\mathbf{R},\varphi](u\otimes u) + (u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u)\\
&\qquad = \varphi u\cdot[\nabla,\varphi]\mathbf{R}(u\otimes u) + \varphi u\cdot\nabla[\mathbf{R},\varphi](u\otimes u) + (u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u)\\
&\qquad = \varphi u\cdot\left([\nabla,\varphi]\mathbf{R} - \nabla[\varphi,\mathbf{R}]\right)(u\otimes u) + (u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u)\\
&\qquad = \varphi u\cdot[\nabla\mathbf{R},\varphi](u\otimes u) + (u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u).
\end{aligned}
$$

Therefore, local energy inequality for φu can be simplified as

$$
\begin{aligned}
&\partial_t\frac{|\varphi u|^2}{2} + |\nabla(\varphi u)|^2 + \mathrm{div}\left(\varphi u\mathbf{R}(u\otimes\varphi u)\right)\\
&\qquad \le \Delta\frac{|\varphi u|^2}{2} + \varphi u\cdot\mathbf{C}_u + (u\cdot\nabla\varphi)\mathbf{R}(\varphi u\otimes u).
\end{aligned}
$$

Subtracting (4.73) from this, we obtain (4.63). $\square$

Chapter 5

Boundary Vorticity and Inviscid Limit

5.1 Introduction

For dimension $d = 2, 3$, we consider the periodic channel with physical boundary at $x_d = 0$ and $x_d = 1$: $\Omega = \mathbb{T}^{d-1} \times (0, 1)$, where $\mathbb{T} = [0, 1]_{\text{per}}$ denotes the unit periodic domain. For any kinematic viscosity $\nu > 0$, we denote $u^\nu : (0, T) \times \Omega \to \mathbb{R}^d$ the velocity field of an incompressible fluid confined in Ω, subject to no-slip boundary conditions, and $P^\nu : (0, T) \times \Omega \to \mathbb{R}$ the associated pressure field. The dynamic of the flow is described by the following Navier–Stokes Equation:

$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla P^\nu = \nu \Delta u^\nu & \text{in } (0, T) \times \Omega \\ \operatorname{div} u^\nu = 0 & \text{in } (0, T) \times \Omega \\ u^\nu = 0 & \text{for } x_d = 0, \text{ and } x_d = 1. \end{cases} \tag{NSE$_\nu$}$$

For any $A > 0$, we investigate the inviscid asymptotic behavior of u^ν when ν converges to 0, under the condition that the initial values converge to a shear flow of strength A:

$$\lim_{\nu\to 0}\|u^\nu(0) - Ae_1\|_{L^2(\Omega)} = 0. \tag{5.1}$$

Note that the steady shear flow $\bar{u}(t,x) = Ae_1$ is solution to the Euler equation with impermeability boundary condition:

$$\begin{cases} \partial_t \bar{u} + \bar{u}\cdot\nabla\bar{u} + \nabla\bar{P} = 0 & \text{in } (0,T)\times\Omega \\ \operatorname{div}\bar{u} = 0 & \text{in } (0,T)\times\Omega \\ \bar{u}\cdot n = 0 & \text{for } x_d = 0, \text{ and } x_d = 1, \end{cases} \tag{EE}$$

where n is the outer normal as shown in Figure 5.1. However, it is an outstanding open question (even in dimension 2) whether, in the double limit (5.1) and $\nu \to 0$, the solution u^ν of (NSE_ν) converges to this shear flow Ae_1. The difficulty of this problem stems from the discrepancy between the no-slip boundary condition for the Navier–Stokes equation and the impermeable boundary condition of the Euler equation. Kato [Kat84] showed in 1984 a conditional result ensuring this convergence under the a priori assumption that the energy dissipation rate in a very thin boundary layer Γ_ν of width proportional to ν vanishes:

$$\lim_{\nu\to 0}\int_0^T \int_{\Gamma_\nu} \nu|\nabla u^\nu|^2 \,\mathrm{d}x\,\mathrm{d}t = 0.$$

This condition has been sharpened in a variety of ways (see, for instance [TW97, Wan01, Kel07, Kel08] and Kelliher [Kel17], for a general review), and similar other conditional results have been derived (see for instance [BTW12, CKV15, CEIV17, CV18]). Non-conditional results of strong inviscid limits have been obtained only for real analytic initial data [SC98], vanishing vorticity near the boundary [Mae14, FTZ18], or symmetries [LFMNLT08, MT08]. Since [Pra04], it is expected that in favorable cases, the Prandtl boundary layer describes the behavior of the solution u^ν up to a distance proportional to $\sqrt{\nu}$. However, even in the simple shear flow case, it is possible to engineer families of initial values $u^\nu(0)$ converging to the shear flow, but associated to Prandtl boundary layers which are either strongly unstable [Gre00], blow up in finite time [E00], or even ill-posed in the Sobolev framework [GVD10, GVN12].

It is actually believed that the inviscid asymptotic limit may fail due to turbulence (See Bardos and Titi [BT13]). This scenario is consistent with the non-uniqueness pathology of the shear flow solution for the Euler system (EE). Indeed, an adaptation to the boundary value problem (EE) of the construction based on convex integration of Szekelyhidi in [Szé11] provides infinitely many solutions to (EE) with initial value Ae_1 (see also Bardos, Titi, Wiedemann [BTW12] for a different boundary geometry). More precisely, the following estimate can be proved on this construction (see appendix 5.5).

Proposition 5.1. *For any* $0 < C < 2$*, there exists a solution* v *to* (EE) *with initial value* Ae_1 *such that for any time* $T < 1/(2A)$*:*

$$\|v(T) - Ae_1\|^2_{L^2(\Omega)} = CA^3T.$$

The convex integration is a powerful tool introduced by De Lellis and Szekelyhidi [DLS09] to construct spurious solutions to the Euler equation. It proved itself to be a powerful tool to model turbulence. For instance, the technique was successfully applied by Isett [Ise18] to prove the Onsager theorem (see also [BDLSV19] for the construction of admissible solutions, and [CET94] for the proof of the other direction). It shows that turbulent flows can have regularity C^α for any α up to $1/3$, a property conjectured by Onsager [Ons49]. Proposition 5.1 predicts the possible deviation from the initial shear flow Ae_1 due to turbulence, a phenomenon called layer separation. Moreover, it provides an explicit value for the L^2 norm of this layer separation.

This article aims to provide an upper bound on the L^2 norm of possible layer separations through the double limit inviscid asymptotic. In our channel framework, the Reynolds number is given by $\mathsf{Re} = A/\nu$. Our main theorem is the following.

Theorem 5.2. *Let Ω be a unit periodic channel in $\mathbb{R}^d$ of dimension $d = 2, 3$. There exists $C > 0$ depending on d only, such that the following is true. Let $\bar{u} = Ae_1$ be a constant shear flow for some $A > 0$, and let u^ν be a Leray–Hopf solution to* (NSE$_\nu$) *with kinematic viscosity $\nu > 0$. For any $T > 0$, we have*

$$\begin{aligned}\|u^\nu(T) - \bar{u}\|^2_{L^2(\Omega)} &+ \frac{\nu}{2}\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \\ &\leq 4\|u^\nu(0) - \bar{u}\|^2_{L^2(\Omega)} + CA^3T + CA^2\mathsf{Re}^{-1}\log(2 + \mathsf{Re}).\end{aligned}$$

This theorem is the special case of a more general result given in Theorem 5.5 at the end of this section. By Leray–Hopf solution, we mean any weak solutions to (NSE$_\nu$) which in addition verifies the energy inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu\|^2_{L^2(\Omega)} \leq -\nu\|\nabla u^\nu\|^2_{L^2(\Omega)}.$$

We have the following corollary on any weak inviscid limit, which corresponds to the layer separation predicted by Proposition 5.1.

Corollary 5.3. *There exists a universal constant $C > 0$ such that the following is true. Consider any family u^ν of a Leray–Hopf solutions to* (NSE$_\nu$) *such that u_0^ν converges strongly in $L^2(\Omega)$ to Ae_1. Then, for any weak limit u^∞ of weakly convergent subsequences of u^ν, we have for almost every $T > 0$ that*

$$\|u^\infty(T) - Ae_1\|^2_{L^2(\Omega)} \leq CA^3T.$$

Note that the solutions u^ν are uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\Omega))$. Therefore they converge weakly up to a subsequence in $L^2_{t,x}$.

This result bets on the fact that the double limit to Ae_1 in the inviscid asymptotic may fail, which is related to the physical relevance of the solutions constructed by convex integration. An interesting question is whether such solutions can be themselves obtained via double limit in the inviscid asymptotic. A first result in this direction was provided by Buckmaster and Vicol [BV19] where they constructed via convex integration, in the case without boundary, spurious solutions at the level of Navier–Stokes. They show that the inviscid limit of this family of Navier–Stokes solutions can converge to spurious solutions of Euler. However, these spurious solutions constructed at the level of Navier–Stokes do not have enough regularity to be Leray–Hopf solutions, and therefore do not fit in the framework of Corollary 5.3.

Non-uniqueness and pattern predictability. The non-uniqueness of solutions to the Euler equation, as proved by convex integration, puts under question the ability of the model itself to predict the future. Theorem 5.2 provides a first example of how non-uniqueness and pattern predictability can be reconciled. The energy of the shear flow is A^2, while the maximum energy of the layer separation is bounded above by CA^3T. This predicts pattern visibility on a lapse of time $1/A$. On this lapse of time, the layer separation stays negligible compared to the shear flow pattern. Especially, the smaller the pattern is (small A), the longer the prediction stays accurate.

Inviscid limit and boundary vorticity. It is well known that the possible growth of the layer separation is closely related to the creation of boundary vorticity (see Kelliher [Kel07] for instance). To see this, we formally compute the evolution of the L^2 distance between u^ν and $\bar{u}$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu-\bar{u}\|_{L^2}^2 &= (u^\nu-\bar{u},\partial_t u^\nu)\\ &= -(u^\nu-\bar{u},u^\nu\cdot\nabla u^\nu)-(u^\nu-\bar{u},\nabla P^\nu)+\nu(u^\nu-\bar{u},\Delta u^\nu)\\ &= \nu(u^\nu,\Delta u^\nu)-\nu(\bar{u},\Delta u^\nu)\\ &= -\nu\|\nabla u^\nu\|_{L^2}^2-\int_{\partial\Omega} J[\bar{u}]\cdot(\nu\omega^\nu)\,\mathrm{d}x'\end{aligned} \tag{5.2}$$

where $J[\bar{u}]=n^\perp\cdot\bar{u}$ when $d=2$ and $J[\bar{u}]=n\times\bar{u}$ when $d=3$, and ω^ν is the vorticity of u^ν. Since $\bar{u}$ is a constant on the boundaries, it is crucial to estimate the mean boundary vorticity. If the convergence $\nu\omega^\nu\big|_{\partial\Omega}\to 0$ holds in the average sense, then the inviscid limit would be valid. For a general static smooth solution to Euler's equation $\bar{u}$ in a general domain Ω, we only need $\nu\omega^\nu\big|_{\partial\Omega}\to 0$ in distribution. This convergence may fail and we could lose uniqueness, but we can still control the size of the impact from this boundary vorticity using Theorem 5.4 below.

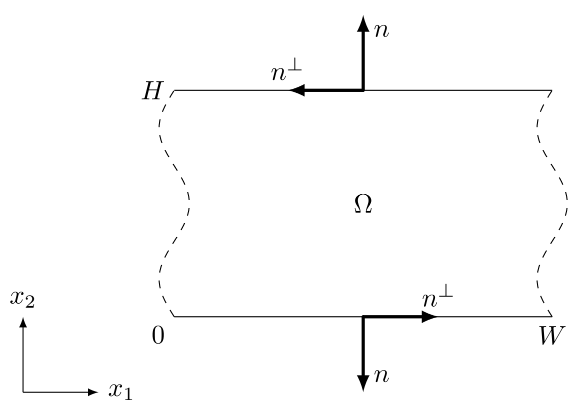

Figure 5.1: 2D Periodic Channel

Before showing the theorem, we first illustrate which estimates we may expect and how they prove Theorem 5.2. Denote the energy dissipation by

$$D:=\nu\|\nabla u^\nu\|_{L^2((0,T)\times\Omega)}^2.$$

If we take the curl of (NSE_ν), we have the vorticity equation,

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u.$$

The main difficulties are due to the transport term $u \cdot \nabla \omega$, and the boundary. Let us put aside those two difficulties for now, and focus on the other terms. Then the regularity we could expect for ω is at best

$$\nu^2 \|\nabla^2 \omega\|_{L^1((0,T)\times\Omega)} \lesssim_d \nu \|\omega \cdot \nabla u\|_{L^1((0,T)\times\Omega)} \leq D.$$

Here $A \lesssim_d B$ means $A \leq C(d) B$ for some constant $C(d)$ depending in dimension d only. This is not rigorous because the parabolic regularization is false in L^1, but let us also ignore this issue for the moment. By interpolation, we have

$$\nu^{\frac{3}{2}} \left\|\nabla^{\frac{2}{3}} \omega\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}((0,T)\times\Omega)} \lesssim_d \left(\nu^2 \|\nabla^2 \omega\|_{L^1((0,T)\times\Omega)}\right)^{\frac{1}{2}} \left(\nu \|\omega\|^2_{L^2((0,T)\times\Omega)}\right)^{\frac{1}{2}} \lesssim_d D.$$

Finally the trace theorem suggests that (again, this is the borderline case for the trace theorem, so in no way a rigorous proof)

$$\|\nu\omega\|^{\frac{3}{2}}_{L^{\frac{3}{2}}((0,T)\times\partial\Omega)} \lesssim_d D. \tag{5.3}$$

Using this $L^{\frac{3}{2}}$ estimate, if we integrate (5.2) from 0 to T, we have

$$\begin{aligned}
&\frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(T) + D \\
&\qquad \leq \frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(0) + \|J[\bar{u}] \cdot \nu\omega^\nu\|_{L^1((0,T)\times\partial\Omega)} \\
&\qquad \leq \frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(0) + \|\nu\omega^\nu\|_{L^{\frac{3}{2}}((0,T)\times\partial\Omega)} \|\bar{u}\|_{L^3((0,T)\times\partial\Omega)} \\
&\qquad \leq \frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(0) + \frac{1}{2}D + CA^3 T|\partial\Omega|
\end{aligned}$$

for some constant C depending on d only. By absorbing $\frac{1}{2}D$ to the left we finish the proof of Theorem 5.2. Note however, that this direct proof collapses due to the transport term. In dimension three, u can be controlled at best in $L^{10/3}_{t,x}$ while the best control of $\nabla\omega$ is in the Lorentz spaces $L^{4/3,q}_{t,x}$ for any $q > 4/3$ (see [VY21b]). But this is far from enough to bound the transport term $u\nabla\omega$ in $L^1_{t,x}$. In dimension 2, the transport term can almost be controlled in L^1. But the bound is in negative power of ν and so is useless for the asymptotic limit. However, we can use blow-up techniques inspired by [Vas10] (see also [CV14, VY21b]) which naturally deplete the strength of the transport term.

Boundary vorticity control for the unscaled Navier–Stokes equation. In the review paper [MM18], Maekawa and Mazzucato summarized the difficulties of considering inviscid limit with boundary:

> Mathematically, the main difficulty in the case of the no-slip boundary condition is the lack of a priori estimates on strong enough norms to pass to the limit, which in turn is due to the lack of a useful boundary condition for vorticity or pressure.

Following this remark, our proof relies on a new boundary vorticity control. This is a regularization result for the unscaled Navier–Stokes equation. However, it is remarkable that this estimate is rescalable through the inviscid limit $\nu \to 0$. The strategy of looking for uniform estimates with respect to the inviscid scaling was first introduced for 1D conservation laws in [KV21a]. It was successfully applied to obtain the unconditional double limit inviscid asymptotic in the case of a single shock [KV21b]. Note that if (u^ν, P^ν) is a solution to (NSE$_\nu$), then $u(t,x) = u^\nu(\nu t, \nu x)$, $P(t,x) = P^\nu(\nu t, \nu x)$ solves the Navier–Stokes equation with unit viscosity coefficient in $(0,T/\nu)\times(\Omega/\nu)$:

$$\partial_t u + u\cdot\nabla u + \nabla P = \Delta u, \qquad \operatorname{div} u = 0. \tag{NSE}$$

The regularization result on the vorticity at the boundary is as follows.

Theorem 5.4 (Boundary Regularity)**.** *There exists a universal constant $C>0$ such that the following holds. Let Ω be a periodic channel of period W and height H of dimension $d=2$ or 3. For any Leray–Hopf solution u to* (NSE$_1$) *in $(0,T)\times\Omega$, there exists a parabolic dyadic decomposition* [1]

$$\text{closure}\big\{(0,T)\times\partial\Omega\big\} = \text{closure}\Big\{\bigcup_i (s^i,t^i)\times\bar{B}_{r_i}(x^i)\Big\},$$

where $0\le s^i<t^i\le T$, $0<r^i<\frac{W}{2}$, $x^i\in\partial\Omega$, and

$$\bar{B}_r(y) = \big\{(x',x_d)\in\partial\Omega : \big\|x'-y'\big\|_{\ell^\infty}<r, x_d=y_d\big\}$$

is a box of dimension $d-1$ in $\partial\Omega$, such that the following is true. Define a piecewise constant function $\tilde{\omega}:(0,T)\times\partial\Omega\to\mathbb{R}$ by taking averages

$$\tilde{\omega}(t,x) = \frac{1}{\big|\bar{B}_{r^i}\big|}\int_{\bar{B}_{r^i}(x^i)}\left|\frac{1}{t^i-s^i}\int_{s^i}^{t^i}\omega\,\mathrm{d}t\right|\mathrm{d}x', \qquad \text{for } t\in(s^i,t^i), x\in\bar{B}_{r^i}(x^i).$$

Then

$$\left\|\tilde{\omega}\mathbf{1}_{\left\{\tilde{\omega}>\max\left\{\frac{1}{t},\frac{1}{W^2},\frac{1}{H^2}\right\}\right\}}\right\|_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)}^{\frac{3}{2}} \le C\|\nabla u\|^2_{L^2((0,T)\times\Omega)}.$$

This theorem provides a "scaling invariant" nonlinear estimate, that is, both sides of the estimate have the same scaling under the canonical scaling of the Navier–Stokes equation $(t,x)\mapsto\varepsilon u(\varepsilon^2 t,\varepsilon x)$. The bounds in the theorem do not depend on

[1] A dyadic decomposition into cubes of parabolic scaling. See Definition 5.14.

the size of Ω or the terminal time T, and we do not require any smallness for the initial energy.

The conclusion of this theorem is slightly different from what we hope in (5.3), due to some difficulties that we overlooked in the formal argument. To begin with, the higher regularity $\nabla^2\omega \in L^1$ is not known. As mentioned before, one reason is the transport term $u\cdot\nabla\omega$ is indeed hard to control. Using blow-up techniques along the trajectories of the flow first introduced in [Vas10], it was proved in [VY21b] that without boundary in $\Omega = \mathbb{R}^3$, $\nabla^2\omega \in L^{1,q}$ locally for $q > 1$ but miss the endpoint L^1. The bounded domain is even more complicated because of the lack of convenient global control on the pressure. In turn, it means that no control on the pressure can be brought locally through the blow-up process. This poses problems when applying the boundary regularity theory for the linear evolutionary Stokes equation. Indeed, a counterexample constructed in [Ser14] shows that we cannot control that way oscillations in time. The idea which remedies this problem consists in smoothing locally in time to gain some integrability. We can then apply the boundary Stokes estimate for $\int u\,\mathrm{d}t$ instead of u. This justifies the construction of $\tilde{\omega}$ via local smoothing in Theorem 5.4. Lastly, because the maximal function is not a bounded operator in L^1, we only obtained weak $L^{\frac{3}{2}}$ norm instead of $L^{\frac{3}{2}}$ norm.

Note that because $J[\bar{u}]$ is constant on the boundary $\partial\Omega$, and because $\tilde{\omega}$ is constructed via local smoothing on disjoint domains, we have

$$\left|\int_0^T\int_{\partial\Omega} J[\bar{u}]\cdot\omega^\nu\,\mathrm{d}x'\,\mathrm{d}t\right| \le \left|\int_0^T\int_{\partial\Omega} J[\bar{u}]\cdot\tilde{\omega}^\nu\,\mathrm{d}x'\,\mathrm{d}t\right|.$$

We can then apply Theorem 5.4, and proceed as in the formal computation. One last difficulty is that Theorem 5.4 is a regularization result, and so the estimate weakens when t goes to 0. Indeed, it controls only $\tilde{\omega} > \max\{\frac{1}{t}, \frac{1}{W^2}, \frac{1}{H^2}\}$. If we integrate the remainder, there will be a logarithmic singularity at $t = 0$. To avoid this, we apply the vorticity bound only in the time interval $t \in (T_\nu, T)$ for some small time $T_\nu \approx \nu^3$, and for $t \in (0, T_\nu)$ we use a very short time stability of a stable Prandtl layer to bridge the gap.

General case. We actually do the proof in a slightly more general setting. We will consider a periodic channel with width W and height H, where the physical boundary are localized at $x_d = 0$ and $x_d = H$ (see Figure 5.1):

$$\Omega = \Big\{(x', x_d) : 0 \le x_d \le H, x' \in [0, W]^{d-1}_{\text{per}}\Big\}.$$

The following theorem estimates the layer separation for a more general shear flow $\bar{u}$ of the following form:

$$\bar{u}(x) = \begin{cases} \bar{U}(x_2)e_1 & d = 2 \\ \bar{U}_1(x_3)e_1 + \bar{U}_2(x_3)e_2 & d = 3 \end{cases}.$$

In this configuration, we define the Reynolds number as

$$\mathsf{Re} = \frac{AH}{\nu}$$

where $A = \|\bar{u}\|_{L^\infty(\partial\Omega)}$ is the boundary shear.

Theorem 5.5 (General Shear Flow)**.** *There exists a universal constant $C > 0$ such that the following holds. Let Ω be a bounded periodic channel with period W and height H in $\mathbb{R}^d$ with $d = 2$ or 3. Let $\bar{u}$ be a static shear flow in Ω with bounded vorticity, and let u^ν be a Leray–Hopf solution to* (NSE$_\nu$)*. For a given $\bar{u}$ defined as above, denote the maximum shear, boundary velocity, and kinetic energy of $\bar{u}$ by*

$$G := \|\nabla \bar{u}\|_{L^\infty(\Omega)}, \qquad A := \|\bar{u}\|_{L^\infty(\partial\Omega)}, \qquad E := \|\bar{u}\|^2_{L^2(\Omega)}.$$

For any $T > 0$, we have

$$\sup_{0\le t\le T}\Big\{\|u^\nu-\bar{u}\|^2_{L^2(\Omega)}(t)+\frac{\nu}{2}\|\nabla u^\nu\|^2_{L^2((0,t)\times\Omega)}\Big\}$$
$$\le \exp(2GT)\Bigg\{4\|u^\nu(0)-\bar{u}\|^2_{L^2(\Omega)}+2\nu G^2T|\Omega|+CA^2|\Omega|\mathsf{Re}^{-1}\log(2+\mathsf{Re})$$
$$+2\mathsf{Re}^{-1}E+CA^3T|\partial\Omega|\max\{H/W,1\}^2\Bigg\}.$$

Note that Theorem 5.2 is a direct consequence of Theorem 5.5 with $H = W = 1$, $\bar{U} = A$ for $d = 2$, and $\bar{U}_1 = A, \bar{U}_2 = 0$ for $d = 3$.

This paper is organized as follows. We first introduce necessary tools in Section 5.2. The boundary vorticity estimate and the proof of Theorem 5.4 is shown in Section 5.3. In Section 5.4 we finish the proof of the main result, which are Theorem 5.2 and Theorem 5.5. Finally, we prove Proposition 5.1 in the appendix.

5.2 Notations and Preliminary

We begin with some notations. We will be working with boxes more often than balls. For this reason, let us denote the spatial box and the space-time cube of radius r by

$$B_r := \Big\{x\in\mathbb{R}^d : \|x\|_{\ell^\infty} < r\Big\}, \qquad Q_r := (-r^2,0)\times B_r.$$

We denote the same box and cube centered at x and (t,x) by $B_r(x)$ and $Q_r(t,x)$ respectively. Near the boundary $\{x_d = 0\}$, we denote the half-box and its boundary part by

$$B_r^+ := \big\{(x',x_d) : \|x'\|_{\ell^\infty} < r, 0 < x_d < r\big\}, \qquad \bar{B}_r := \big\{(x',0) : \|x'\|_{\ell^\infty} < r\big\},$$

and denote their space-time version by

$$Q_r^+ = (-r^2, 0) \times B_r^+, \qquad\qquad \bar{Q}_r = (-r^2, 0) \times \bar{B}_r.$$

Finally, for a bounded set Ω and $f \in L^2(\Omega)$, we denote the average of f in Ω as

$$\fint_\Omega f \,\mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega f \,\mathrm{d}x.$$

In this section, we provide some useful preliminary results and some corollaries, which will be used later in the paper. Most are widely known, and we do not claim any originality in the proof, but we include them here for completeness.

5.2.1 Evolutionary Stokes Equation

Let (u, P) be the solution to the following Stokes equation.

$$\begin{cases} \partial_t u + \nabla P = \Delta u + f & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \end{cases}. \tag{SE}$$

Recall the following estimates on Stokes equations, which can be found in the book of Seregin [Ser14].

Theorem 5.6 (Cauchy Problem, Section 4.4 Theorem 4.5)**.** *Let Ω be a bounded domain with smooth boundary. Let $1 < p, q < \infty$, and $f \in L^p(0, T; L^q(\Omega))$. There exists a unique solution (u, P) to* (SE) *such that*

(1) *u satisfies the zero initial-boundary condition:*

$$u = 0 \text{ at } t = 0,$$
$$u = 0 \text{ on } (0, T) \times \partial\Omega.$$

(2) *P satisfies the zero mean condition:*

$$\int_\Omega P(t,x)\,\mathrm{d}x = 0 \text{ at any } t \in (0,T).$$

Moreover, we have the coercive estimate

$$\big\||\partial_t u| + |\nabla^2 u| + |\nabla P|\big\|_{L^p(0,T;L^q(\Omega))} \le C(\Omega,p,q)\|f\|_{L^p(0,T;L^q(\Omega))}.$$

Theorem 5.7 (Local Boundary Regularity, Section 7.10 Proposition 7.10)**.** *Let $1 < p < \infty$, $1 < q \le q' < \infty$. Assume $u, \nabla u, P \in L^p_t L^q_x(Q_2^+)$, $f \in L^p_t L^{q'}_x(Q_2^+)$ and (u,P) satisfy* (SE) *in $\Omega = Q_2^+$. Moreover, assume*

$$u = 0 \text{ on } \{x_d = 0\}. \tag{5.4}$$

Then we have the local boundary estimate

$$\begin{aligned}&\big\||\partial_t u| + |\nabla^2 u| + |\nabla P|\big\|_{L^p_t L^{q'}_x(Q_1^+)} \\ &\quad \le C(p,q,q')\Big(\||u| + |\nabla u| + |P|\|_{L^p_t L^q_x(Q_2^+)} + \|f\|_{L^p_t L^{q'}_x(Q_1^+)}\Big).\end{aligned}$$

Combining these two estimates, we derive the following mixed case.

Corollary 5.8. *Let $1 < p_2 < p_1 < \infty$, $1 < q_1, q_2 < \infty$, $f \in L^{p_1}_t L^{q_1}_x(Q_2^+)$, $u, \nabla u, P \in L^{p_2}_t L^{q_2}_x(Q_2^+)$. If (u,P) satisfies* (SE) *in Q_2^+ and u satisfies* (5.4)*, then $u = u_1 + u_2$ satisfying for any $q' < \infty$, there exists a constant $C = C(p_1,p_2,q_1,q_2,q')$ such that*

$$\begin{aligned}&\big\||\partial_t u_1| + |\nabla^2 u_1|\big\|_{L^{p_1}_t L^{q_1}_x(Q_1^+)} + \big\||\partial_t u_2| + |\nabla^2 u_2|\big\|_{L^{p_2}_t L^{q'}_x(Q_1^+)} \\ &\quad \le C\Big(\|f\|_{L^{p_1}_t L^{q_1}_x(Q_2^+)} + \||u| + |\nabla u| + |P|\|_{L^{p_2}_t L^{q_2}_x(Q_2^+)}\Big).\end{aligned}$$

Proof. Let Ω' be a smooth domain such that $B^+_{\frac{3}{2}} \subset \Omega' \subset B^+_2$. Define u_1 to be the solution to the Cauchy problem in Ω' with force f. By Theorem 5.6, we obtain

$$\big\||\partial_t u_1| + |\nabla^2 u_1| + |\nabla P_1|\big\|_{L^{p_1}(-4,0;L^{q_1}(\Omega'))} \le C\|f\|_{L_t^{p_1}L_x^{q_1}(Q_2^+)}.$$

Since u_1 has trace zero, P_1 has mean zero, we have

$$\||u_1| + |\nabla u_1| + |P_1|\|_{L^{p_1}(-4,0;L^{q_1}(\Omega'))} \le C\|f\|_{L_t^{p_1}L_x^{q_1}(Q_2^+)}.$$

Now we define $u_2 = u - u_1, P_2 = P - P_1$. Since $p_1 > p_2$, we have

$$\begin{aligned}&\||u_2| + |\nabla u_2| + |P_2|\|_{L_t^{p_2}L_x^{\min\{q_1,q_2\}}(Q_{3/2}^+)} \\ &\qquad \le C\Big(\|f\|_{L_t^{p_1}L_x^{q_1}(Q_2^+)} + \||u| + |\nabla u| + |P|\|_{L_t^{p_2}L_x^{q_2}(Q_2^+)}\Big).\end{aligned}$$

Note that u_2 solves (SE) with zero force term in $Q_{\frac{3}{2}}^+$, so the desired result follows by applying Theorem 5.7. □

5.2.2 Inhomogeneous Sobolev Embedding

We show that given partial derivatives bounded in inhomogeneous Lebesgue spaces, a binary function is bounded in L^∞.

Lemma 5.9 (Inhomogeneous Supercritical Sobolev Embedding). *Let $\alpha \in (0,1)$, and $\Omega = \{(t,z) : t \in [-1,0], z \in [0,1]\}$. Let $u \in L^1(\Omega)$ with weak partial derivatives bounded in inhomogeneous spaces*

$$\partial_t u \in L_t^1 L_z^\infty(\Omega) + L_t^q L_z^1(\Omega), \qquad \partial_z u \in L_t^p L_z^\infty(\Omega) + L_t^\infty L_z^r(\Omega),$$

with $p > \frac{1}{\alpha}, q > \frac{1}{1-\alpha}, r > 1$, then $u \in C(\Omega)$ is continuous with oscillation bounded *by*

$$\sup_\Omega u - \inf_\Omega u = \|u\|_{\mathrm{osc}(\Omega)} \le C\Big(\|\partial_t u\|_{L^1_t L^\infty_z + L^q_t L^1_z} + \|\partial_z u\|_{L^p_t L^\infty_z + L^\infty_t L^r_z}\Big)$$

where $C = C(p,q,r)$ *depends on* p, q, r.

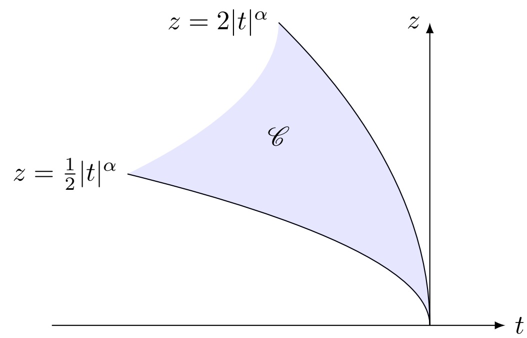

Figure 5.2: Inhomogeneous Sobolev Embedding

Proof. Up to cutoff and mollification, we may assume $u \in C^\infty((-\infty,0]\times[0,\infty))$ with compact support in $2\Omega = (-2,0]\times[0,2)$. Up to translation, we show $u(0,0)$ is bounded. By the fundamental theorem of calculus, for any $\lambda > 0$, we have

$$0 = u(0,0) + \int_0^\infty \frac{\partial}{\partial s} u(-s, \lambda s^\alpha)\,\mathrm{d}s.$$

Taking average for $\lambda \in \left(\frac{1}{2}, 2\right)$ yields

$$|u(0,0)| \le \int_{\frac{1}{2}}^2 \int_0^\infty |\partial_t u| + \lambda\alpha s^{\alpha-1}|\partial_z u|\,\mathrm{d}s\,\mathrm{d}\lambda.$$

The Jacobian of $(t,z) = (s, \lambda s^\alpha)$ is

$$\frac{D(t,z)}{D(s,\lambda)} = \det\begin{bmatrix} -1 & 0 \\ \lambda\alpha s^{\alpha-1} & s^\alpha \end{bmatrix} = s^\alpha = |t|^\alpha \sim z,$$

thus we can bound $u(0,0)$ via a change of variable by

$$\begin{aligned}|u(0,0)| &\leq \int_{\mathscr{C}} \Big(|\partial_t u| + \alpha z |t|^{-1} |\partial_z u|\Big) |t|^{-\alpha} \, \mathrm{d}z \, \mathrm{d}t \\ &= \int_{\mathscr{C}} |t|^{-\alpha} |\partial_t u| + \alpha\lambda |t|^{-1} |\partial_z u| \, \mathrm{d}z \, \mathrm{d}t.\end{aligned}$$

where $\mathscr{C}$ is the region illustrated in Figure 5.2.

Now we compute inhomogeneous norms of $|t|^{-1}$ and $|t|^{-\alpha}$ in $\mathscr{C}$:

$$\begin{aligned}\int_{\frac{1}{2}|t|^\alpha}^{2|t|^\alpha} |t|^{-\alpha} \, \mathrm{d}z &= \frac{3}{2} \in L_t^\infty(-2,0), \\ \big\| |t|^{-\alpha} \big\|_{L_z^\infty(\frac{1}{2}|t|^\alpha, 2|t|^\alpha)} &= |t|^{-\alpha} \in L_t^{q'}(-2,0), \\ \int_{\frac{1}{2}|t|^\alpha}^{2|t|^\alpha} |t|^{-1} \, \mathrm{d}z &= \frac{3}{2} |t|^{\alpha-1} \in L_t^{p'}(-2,0), \\ \|1/t\|_{L_z^{r'}(\frac{1}{2}|t|^\alpha, 2|t|^\alpha)} &= |t|^{-1} \left(\frac{3}{2}|t|^\alpha\right)^{\frac{1}{r'}} \lesssim |t|^{\frac{\alpha}{r'}-1} \in L_t^1(-2,0).\end{aligned}$$

Here $p' < \frac{1}{\alpha}, q' < \frac{1}{1-\alpha}, r' < \infty$ are the Hölder conjugate of p, q, r respectively. In conclusion, $|t|^{-1}$ and $|t|^{-\alpha}$ are bounded in spaces

$$|t|^{-\alpha} \in L_t^\infty L_z^1 \cap L_t^{q'} L_z^\infty, \qquad\qquad |t|^{-1} \in L_t^{p'} L_z^1 \cap L_t^1 L_z^{r'},$$

which completes the proof of this lemma by Hölder inequality. □

5.2.3 Parabolic Maximal Function

Let us introduce the following notion of maximal function adapted to the parabolic scaling.

Definition 5.10 (Parabolic Maximal Function)**.** For $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$, we define the parabolic maximal function by taking the greatest mean values

$$\mathcal{M}f(t,x) := \sup_{r>0} \fint_{t-r^2}^{t+r^2} \fint_{B_r(x)} |f(s,y)| \,\mathrm{d}y \,\mathrm{d}s.$$

For $f \in L^1((0,T)\times\Omega)$ where $\Omega \subset \mathbb{R}^d$ is a bounded set, we can define $\mathcal{M}f$ by applying the previous definition on the zero extension of f in $\mathbb{R}\times\mathbb{R}^d$.

Recall the classical weak type $(1,1)$ bound on the maximal function $\mathcal{M}$:

$$\|\mathcal{M}f\|_{L^{1,\infty}} \le C_d \|f\|_{L^1}.$$

5.2.4 Lipschitz Decay of 1D Heat Equation

We end this section by reminding the readers that solutions to the 1D heat equation have a decay rate of $t^{-\frac{3}{4}}$ in the Lipschitz norm. It will be useful to control the Prandtl layer in a small initial time of order $O(\nu^3)$. This result is very elementary. We give the proof for the sake of completeness.

Lemma 5.11. *For $z > 0$ we have*

$$\sum_{n=1}^{\infty} n^2 e^{-n^2 z} < z^{-\frac{3}{2}}.$$

Proof. We can approximate this infinite series by

$$\begin{aligned}\sum_{n=1}^{\infty} n^2 e^{-n^2 z} &= z^{-\frac{3}{2}} \sum_{n=1}^{\infty} (\sqrt{z}n)^2 e^{-(\sqrt{z}n)^2} \sqrt{z} \\ &= z^{-\frac{3}{2}} \left(\int_0^{\infty} x^2 e^{-x^2} \,\mathrm{d}x + O(\sqrt{z}) \right) \\ &= \frac{\sqrt{\pi}}{4} z^{-\frac{3}{2}} + O(z^{-1}),\end{aligned}$$

when $z \to 0$ is small, and

$$\sum_{n=1}^{\infty} n^2 e^{-n^2 z} \le \sum_{n=1}^{\infty} n^2 e^{-nz} = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\sum_{n=1}^{\infty} e^{-nz}\right) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{e^z - 1}\right) = \frac{(e^z+1)e^z}{(e^z-1)^3} \approx e^{-z}$$

when $z \to \infty$ is large. This proves that the left hand side is bounded by $Cz^{-\frac{3}{2}}$ for some constant C, which can be easily determined by carefully examine the estimates. □

Using this lemma, we can compute the decay rate.

Lemma 5.12. *Let $\nu > 0$, $H > 0$, and suppose $v(t, x_d)$ solves the following 1D heat equation in $[0, H]$:*

$$\begin{cases} \partial_t v = \nu v_{xx} & \text{in } (0,\infty) \times (0, H) \\ v = 0 & \text{on } (0,\infty) \times \{0, H\} \\ v = v_0 & \text{at } t = 0 \end{cases}$$

with $v_0 \in L^2(0, H)$. Then

$$\|\nabla v(t)\|_{L^\infty} \le \frac{1}{2}(\nu t)^{-\frac{3}{4}} \|v_0\|_{L^2}.$$

Proof. We can write the solutions explicitly in terms of Fourier series. We expand v_0 by sine series as

$$v_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{H}\right),$$

with

$$\sum_{n=1}^{\infty} b_n^2 = \frac{2}{H}\|v_0\|_{L^2}^2 < \infty.$$

The solution can be explicitly written as

$$v(t,x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{H}\Big) e^{-\nu \frac{n^2\pi^2}{H^2} t},$$

so the derivative is bounded by

$$\begin{aligned}
|\partial_x v(t,x)| &\le \left| \sum_{n=1}^{\infty} b_n \cos\Big(\frac{n\pi x}{H}\Big)\Big(\frac{n\pi}{H}\Big) e^{-\nu \frac{n^2\pi^2}{H^2} t} \right| \\
&\le \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \Big(\frac{n\pi}{H}\Big)^2 e^{-2\nu \frac{n^2\pi^2}{H^2} t}, \right)^{\frac{1}{2}} \\
&\le \left(\frac{2}{H} \|v_0\|_{L^2}^2 \right)^{\frac{1}{2}} \Big(\frac{\pi}{H}\Big) \left(\frac{2\nu\pi^2 t}{H^2} \right)^{-\frac{3}{4}} \\
&\le \frac{1}{2} (\nu t)^{-\frac{3}{4}} \|v_0\|_{L^2}
\end{aligned}$$

using the previous lemma. □

5.3 Boundary Regularity for the Navier–Stokes Equation

The goal of this section is to prove the boundary regularity for the Navier–Stokes equation with unit viscosity constant: Theorem 5.4. This relies on the following local estimate.

Proposition 5.13. *Suppose* (u, P) *is a weak solution to the Navier–Stokes equation* (NSE) *with forcing term* $f \in L^1(-4,0;L^2(B_2^+))$*, such that* $u \in L^\infty(-4,0;L^2(B_2^+))$*,*

$\nabla u \in L^2(Q_2^+)$, *and in distribution they satisfy*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u + f & \text{in } Q_2^+ \\ \operatorname{div} u = 0 & \text{in } Q_2^+ \\ u = 0 & \text{on } \bar{Q}_2 \end{cases}.$$

If we denote

$$c_0 := \int_{-4}^{0} \|\nabla u(t)\|_{L^2(B_2^+)}^2 + \|f\|_{L^2(B_2^+)} \,\mathrm{d}t,$$

then we can bound the average-in-time vorticity on the boundary by

$$\int_{\bar{B}_1} \left| \int_{-1}^{0} \omega(t, x', 0) \,\mathrm{d}t \right| \mathrm{d}x' \le C(c_0 + c_0^{\frac{1}{2}}).$$

Proof. For $t \in (-3, 0)$, we define

$$U(t, x) = \int_{t-1}^{t} u(s, x) \,\mathrm{d}s.$$

As explained in the introduction, this is needed to tame the time oscillation of the local pressure, which comes from $\partial_t u$. This allows us to apply the local Stokes estimate at the boundary. Denote $\rho(t) = \mathbf{1}_{[0,1]}(t)$, then $U = u *_t \rho$, where $*_t$ stands for convolution in t variable only. If we denote $Q = P *_t \rho$, and $F = (f - u \cdot \nabla u) *_t \rho$, then U satisfies the following system:

$$\begin{cases} \partial_t U + \nabla Q = \Delta U + F & \text{in } (-3, 0) \times B_2^+ \\ U = 0 & \text{on } \{x_d = 0\} \end{cases}.$$

The proof of this theorem can be divided into three steps: the first two estimate terms in this system, and the last step uses the Stokes estimate and the Sobolev embedding.

Step 1. Estimates on $u, U, \partial_t U, \Delta U$. We have via Sobolev embedding and using that $u = 0$ on $\bar{Q}_2$ that

$$\|u\|_{L_t^2 L_x^6(Q_2^+)} \leq C c_0^{\frac{1}{2}} \tag{5.5}$$

for both dimension 2 and 3. Since $\partial_t U(t,x) = u(t,x) - u(t-1,x)$, we have

$$\|\partial_t U\|_{L_t^2 L_x^6((-3,0)\times B_2^+)} \leq C c_0^{\frac{1}{2}},$$

On the other hand, the Laplacian of U is bounded by

$$\|\Delta U\|_{L_t^\infty H_x^{-1}((-3,0)\times B_2^+)} \leq C\|\Delta u\|_{L_t^2 H_x^{-1}(Q_2^+)} \leq C\|\nabla u\|_{L^2(Q_2^+)} \leq C c_0^{\frac{1}{2}}.$$

Step 2. Estimates on F and Q. Applying Hölder's inequality, by (5.5) we have

$$\|u\cdot\nabla u\|_{L_t^1 L_x^{\frac{3}{2}}(Q_2^+)} \leq C c_0.$$

Also by (5.5) we have by embedding that

$$\|\mathrm{div}(u\otimes u)\|_{L_t^1 W_x^{-1,3}(Q_2^+)} \leq C c_0.$$

for both dimension 2 and 3. By convolution, we bound F by

$$\|F\|_{L_t^\infty L_x^{\frac{3}{2}}((-3,0)\times B_2^+)}, \|F\|_{L_t^\infty W_x^{-1,3}((-3,0)\times B_2^+)} \leq C c_0.$$

Next we estimate Q. Using $\nabla Q = \Delta U + F - \partial_t U$ we have

$$\|\nabla Q\|_{L_t^2 H_x^{-1}} \leq C c_0 + C c_0 + C c_0^{\frac{1}{2}} \leq C(c_0 + c_0^{\frac{1}{2}}).$$

Without loss of generality we assume that the average of Q is zero at every t. Then by Nečas theorem (see [Ser14], Section 1.4),

$$\|Q\|_{L^2_{t,x}} \le C(c_0 + c_0^{\frac{1}{2}}).$$

Step 3. Stokes estimates and Trace theorem. By Corollary 5.8, we can split $U = U_1 + U_2$, where for any $p < \infty$, we have

$$\big\||\partial_t U_1| + |\nabla^2 U_1|\big\|_{L^p_t L^{\frac{3}{2}}_x (Q_1^+)} + \big\||\partial_t U_2| + |\nabla^2 U_2|\big\|_{L^2_t L^p_x (Q_1^+)} \le C(c_0 + c_0^{\frac{1}{2}}).$$

Denote $\Omega(t, x_d) := \int_{\bar{B}_1} |\nabla U(t, x', x_d)| \, \mathrm{d}x'$, then $\partial_{x_d}\Omega$ is bounded in

$$\partial_{x_d}\Omega \in L^2_t L^p_{x_d} + L^p_t L^{\frac{3}{2}}_{x_d}((-1,0)\times(0,1)).$$

for any $p < \infty$. Note that

$$\partial_t \Omega = \int |\nabla u| \, \mathrm{d}x' \in L^2_{t,x_d}((-1,0)\times(0,1)).$$

Since by interpolation, $L^1_t L^\infty_{x_d} \cap L^\infty_t L^1_{x_d} \subset L^2_{t,x_d}$, by duality $\partial_t\Omega$ is bounded in $L^2_{t,x_d} \subset L^1_t L^\infty_{x_d} + L^\infty_t L^1_{x_d}$. Similarly, $\partial_{x_d}\Omega$ is bounded in

$$\partial_{x_d}\Omega \in L^2_t L^p_{x_d} + L^p_t L^{\frac{3}{2}}_{x_d}((-1,0)\times(0,1)) \subset L^r_t L^\infty_{x_d} + L^\infty_t L^r_{x_d}((-1,0)\times(0,1))$$

for any $p > 3$ and $r > 1$ sufficiently small. Now we can use Lemma 5.9 to show Ω is continuous up to the boundary with oscillation bounded by

$$\|\Omega\|_{\mathrm{osc}((-1,0)\times(0,1))} \le C(c_0 + c_0^{\frac{1}{2}}).$$

Since the average of Ω is also bounded as

$$\int \Omega \,\mathrm{d}x_d \,\mathrm{d}t = \int_{Q_1^+} |\nabla u| \,\mathrm{d}x \,\mathrm{d}t \leq C c_0^{\frac{1}{2}},$$

we have Ω is bounded in L^∞, in particular

$$\int_{\bar{B}_1} \left| \int_{-1}^{0} \nabla u(t, x', 0) \,\mathrm{d}t \right| \mathrm{d}x' = \Omega(0,0) \leq C_0(c_0 + c_0^{\frac{1}{2}}).$$

This concludes the proof of this proposition. □

The proof of Theorem 5.4 relies on a domain decomposition inspired by the Calderón–Zygmund decomposition introduced for the study of singular integrals (see [Ste93]). We first define the parabolic dyadic decomposition.

Definition 5.14 (Parabolic Dyadic Decomposition)**.** Let $L > 0$, and let Ω be a periodic channel of period W and height H. We define the parabolic dyadic decomposition of $(0, L) \times \Omega$ as below. Denote

$$R_0 = \min\left\{ \sqrt{L}, \frac{W}{2}, \frac{H}{2} \right\}. \tag{5.6}$$

Then we can find positive integer k_L, k_W, k_H, such that

$$L = 4^{k_L} L_0, \qquad W = 2 \cdot 2^{k_W} W_0, \qquad H = 2 \cdot 2^{k_H} H_0,$$

where L_0, W_0, H_0 satisfy

$$R_0 \leq \sqrt{L_0}, W_0, H_0 \leq 2R_0.$$

First, we evenly divide $(0, L) \times \Omega$ into $4^{k_L} \cdot 2^{k_W+1} \cdot 2^{k_H+1}$ cubes of length L_0, width W_0 and height H_0, and denote $\mathcal{Q}_0$ to be this set of cubes. For each $Q \in \mathcal{Q}_0$, we can divide Q into 4×2^d subcubes with length $L_0/4$, width $W_0/2$, and height $H_0/2$. This set is denoted by $\mathcal{Q}_1$. For each cube in $\mathcal{Q}_1$, we can continue to dissect it into 4×2^d smaller cubes with a quarter the length, half the width, and half the height. We denote the resulted family by $\mathcal{Q}_2$. We proceed indefinitely and define $\mathcal{Q} = \cup_{k\in\mathbb{N}}\mathcal{Q}_k$ to be the parabolic dyadic decomposition of $(0, L) \times \Omega$.

Proof of Theorem 5.4. The partition of $(0, T) \times \Omega$ is constructed as follows. Among the parabolic dyadic decomposition of $(0, T) \times \Omega$, we first select a family of disjoint cubes, denoted by $\mathcal{Q}^\circ$, according to the following rule:

a) For any integer $k \geq 1$, in $\left\{4^{-k}L_0 \leq t \leq 4^{-k+1}L_0\right\}$, we pick every parabolic cube in $\mathcal{Q}_k$, which are cubes of size $4^{-k}L_0 \times 2^{-k}W_0 \times 2^{-k}H_0$.

b) In $\{t \geq L_0\}$, we pick every parabolic cube in $\mathcal{Q}_0$.

The selection of these cubes ensures enough gap from the initial time $t = 0$, which allows the local parabolic regularization to apply around these cubes. As shown in Figure 5.3 and Figure 5.4, they form a partition of $(0, T)\times\Omega$. Figure 5.3 corresponds to when $R_0 = \min\left\{\frac{W}{2}, \frac{H}{2}\right\} < \sqrt{L_0}$, and figure 5.4 corresponds to when $R_0 = \sqrt{L_0} = \sqrt{T}$, in which case **b)** does not happen.

We are interested in cubes that touch the boundary, i.e., having zero distance from $\partial\Omega$. We call these cubes the "boundary cubes". Given a boundary cube $Q \in \mathcal{Q}_k$ that meets the boundary $\{x_d = 0\}$, we denote its length as $l = 4^{-k}L_0$, width as $w = 2^{-k}W_0$, and height as $h = 2^{-k}H_0$. Thus for some $(t, x', 0) \in (0, T) \times \partial\Omega$, Q can be expressed as

$$Q = (t - l, t) \times \bar{B}_{w/2}(x') \times (0, h), \qquad \bar{B}_{w/2}(x') = \left\{y' : \left\|x' - y'\right\|_{\ell^\infty} < w/2\right\}$$

Let us denote

$$2Q = (t - 2l, t) \times \bar{B}_w(x') \times (0, 2h).$$

Similar definition applies to boundary cubes that touch $\{x_d = H\}$. A boundary

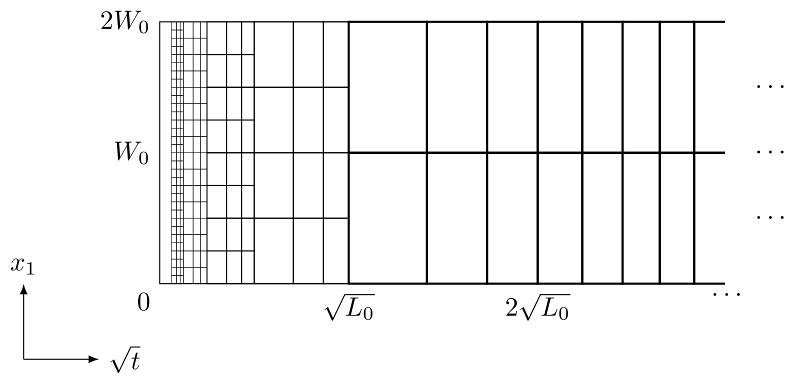

Figure 5.3: Initial Partition $\mathcal{Q}^\circ$ of a Long Channel $(0, L) \times \Omega$

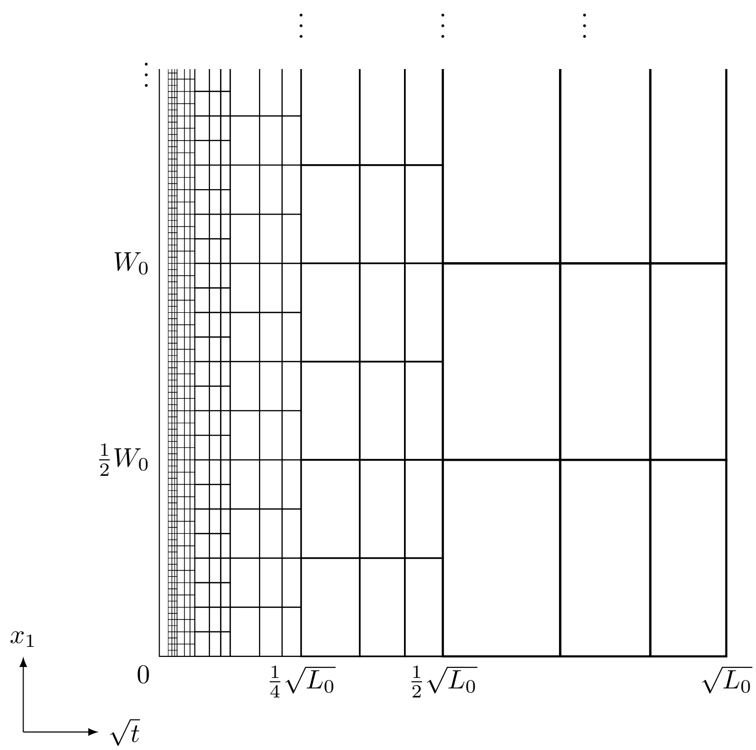

Figure 5.4: Initial Partition $\mathcal{Q}^\circ$ of a Wide Channel $(0, L) \times \Omega$

cube $Q \in \mathcal{Q}_k$ is said to be suitable if it satisfies

$$\fint_{2Q} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le c_0 (2^{-k} R_0)^{-4} \tag{S}$$

for some c_0 to be determined.

Starting from $\mathcal{Q}^\circ$, we decompose the boundary cubes based on the following rules. For each boundary cube in the initial partition $\mathcal{Q}^\circ$ that is not suitable, we dyadically dissect it into 4×2^d smaller parabolic cubes. For each smaller boundary cube, we continue to dissect it until the suitability condition (S) is satisfied. This process will finish in finitely many steps almost everywhere because ∇u is bounded in L^2 for any Leray–Hopf solutions, so all sufficiently small cubes are suitable.

The final partition will contain a subcollection of dyadic boundary cubes $\{Q^i\}_{i\in\Lambda} \subset \mathcal{Q}$ that are suitable, mutually disjoint, and verify closure $\{(0,T)\times\partial\Omega\} =$ closure $\{\bigcup_i \bar{Q}_i\}$. For each boundary cube $Q^i \in \mathcal{Q}_k$ centered at $(t^{(i)}, x^{(i)})$, we denote its length as $l_i = 4^{-k} L_0$, width as $w_i = 2^{-k} W_0$, and height as $h_i = 2^{-k} H_0$. Thus Q^i can be expressed as

$$Q^i = (t^{(i)} - l_i, t^{(i)}) \times \bar{B}^i \times (0, h_i), \qquad \bar{B}^i = \bar{B}_{w_i/2}(x^{(i)}).$$

It is easy to see from our construction that $2Q^i \subset (0,T) \times \Omega$. Denote $r_i = 2^{-k} R_0$, then from Definition 5.14 we have

$$r_i \le \sqrt{l_i}, w_i, h_i \le 2r_i.$$

Suitability (S) of Q^i implies

$$\fint_{2Q^i} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le c_0 r_i^{-4}.$$

Using the canonical scaling of the Navier–Stokes equation $u_r(t,x) := r u(r^2 t, rx)$, Proposition 5.13 implies that

$$\tilde{\omega}|_{\bar{Q}^i} = \fint_{\bar{B}^i} \left| \fint_{t^{(i)}-l_i}^{t^{(i)}} \omega(t,x',0)\,\mathrm{d}x' \right| \mathrm{d}t \le C(c_0 + c_0^{\frac{1}{2}}) r_i^{-2} =: c_1 r_i^{-2}.$$

We can use this Proposition because Q^i is comparable to a parabolic cube.

Now we separate three cases:

1. If $Q^i \in \mathcal{Q}^\circ \cap \mathcal{Q}_k$ with $k \ge 1$, then by condition **a)**, any $(t,x) \in Q^i$ satisfies $t < 4l_i \le 16r_i^2$, thus in $\bar{Q}^i$ we have

$$\tilde{\omega} \le \frac{16c_1}{t}.$$

   We can select c_0 small enough such that $16c_1 = 1$.

2. If $Q^i \in \mathcal{Q}^\circ \cap \mathcal{Q}_0$, then by condition **b)**, any $(t,x) \in Q^i$ satisfies $L_0 = l_i < t < T$, $r_i = R_0$, thus in $\bar{Q}^i$ we have

$$\tilde{\omega} \le c_1 R_0^{-2} = \frac{1}{16} R_0^{-2},$$

   Note that this case only happen when $T > L_0 \ge R_0^2$, so in fact we know $R_0 = \min\{W,H\}/2$, thus $\tilde{\omega} \le \min\{W,H\}^{-2}$.

3. If $Q^i \notin \mathcal{Q}^\circ$ is not one of the initial cubes in the grid, then its antecedent cube $\tilde{Q}^i$ is also a boundary cube and is not suitable, so

$$\fint_{2\tilde{Q}^i} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t > c_0 (2r_i)^{-4},$$

   By the definition of the maximal function $\mathcal{M}$ (recall Definition 5.10), this implies

$$\min_{Q^i} \mathcal{M}(|\nabla u|^2) \geq c_2 r_i^{-4}.$$

for some c_2 comparable to c_0.

Combining these three cases, for any $r_\star = 2^l R_0$ with $l \in \mathbb{Z}$, we have

$$\Big\{(t,x') \in (0,T) \times \partial\Omega : \tilde{\omega} > \max\big\{c_1 r_\star^{-2}, t^{-1}, W^{-2}, H^{-2}\big\}\Big\}$$
$$\subset \bigcup_i \{\bar{Q}^i : r_i < r_\star\} \subset \bigcup_i \bigcup_{k=1}^{\infty} \Big\{\bar{Q}^i : r_i = 2^{-k} r_\star\Big\}.$$

Therefore the measure of the upper level set is controlled by the total measure of these suitable boundary cubes, that is

$$\begin{aligned} \Big|\Big\{\tilde{\omega} > \max\big\{c_1 r_\star^{-2}, t^{-1}, W^{-2}, H^{-2}\big\}\Big\}\Big| &\leq \sum_{k=1}^{\infty} \sum_{r_i = 2^{-k} r_\star} |\bar{Q}^i| \\ &\leq \sum_{k=1}^{\infty} \frac{2^k}{r_\star} \sum_{r_i = 2^{-k} r_\star} |Q^i|. \end{aligned}$$

Note that

$$\bigcup_i \Big\{Q^i : r_i = 2^{-k} r_\star\Big\} \subset \Big\{\mathcal{M}(|\nabla u|^2) \geq c_2 (2^{-k} r_\star)^{-4}\Big\},$$

which implies that

$$
\begin{aligned}
&\Big|\Big\{\tilde{\omega} > \max\{c_1 r_\star^{-2}, t^{-1}, W^{-2}, H^{-2}\}\Big\}\Big| \\
&\quad \le \sum_{k=1}^{\infty} \frac{2^k}{r_\star}\Big|\Big\{\mathcal{M}(|\nabla u|^2) \ge c_2(2^{-k}r_\star)^{-4}\Big\}\Big| \\
&\quad \lesssim \sum_{k=1}^{\infty} \frac{2^k}{r_\star}\Big\|\mathcal{M}(|\nabla u|^2)\Big\|_{L^{1,\infty}_{\mathrm{loc}}((0,T)\times\Omega)}(2^{-k}r_\star)^4 \\
&\quad \lesssim \Big\||\nabla u|^2\Big\|_{L^1((0,T)\times\Omega)} r_\star^3.
\end{aligned}
$$

By the definition of Lorentz space, this shows

$$
\Big\|\tilde{\omega}\mathbf{1}_{\left\{\tilde{\omega}>\max\left\{\frac{1}{t},\frac{1}{W^2},\frac{1}{H^2}\right\}\right\}}\Big\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)} \lesssim \|\nabla u\|^2_{L^2((0,T)\times\Omega)}.
$$

This completes the proof of the theorem. □

5.4 Proof of the Main Result

This section is dedicated to the proof of Theorem 5.5. Theorem 5.4 provides a control on the large part of $\tilde{\omega}$, but it leaves a remainder in the region $\tilde{\omega} < \frac{1}{t}$, whose integral has a logarithmic singularity at $t = 0$. To avoid this singularity, we should apply Theorem 5.4 only away from $t = 0$, and near $t = 0$ we should adopt a different strategy.

Let u^ν_{Pr} be a shear solution to (NSE$_\nu$) with initial value $u^\nu_{\mathrm{Pr}}\big|_{t=0} = \bar{u}$ (the pressure term is 0). Then u^ν_{Pr} can be written as

$$
u^\nu_{\mathrm{Pr}}(t,x) = \begin{cases} U^\nu_{\mathrm{Pr}}(t,x_2)e_1 & d=2 \\ U^\nu_{\mathrm{Pr}1}(t,x_3)e_1 + U^\nu_{\mathrm{Pr}2}(t,x_3)e_2 & d=3 \end{cases},
$$

where U^ν_{Pr} solves the Prandtl layer equation,

$$\begin{cases} \partial_t U^\nu_{\mathrm{Pr}} = \nu \partial_{x_d x_d} U^\nu & \text{in } (0,T)\times(0,H) \\ U^\nu_{\mathrm{Pr}} = 0 & \text{on } (0,T)\times\{0,H\} \cdot \\ U^\nu_{\mathrm{Pr}} = \bar{U} & \text{at } t=0 \end{cases} \tag{Pr_ν}$$

We choose a small positive number $T_\nu < T$ to be determined later, and separate the evolution into two parts: in a short period $(0, T_\nu)$, we compare u^ν and $\bar{u}$ with the Prandtl layer u^ν_{Pr}, while in the remaining time (T_ν, T), we compare u^ν and $\bar{u}$ using the boundary vorticity.

Before we proceed, let us remark on a few useful computations and estimates that will be used repeatedly in this section. If v, w are two divergence-free vector fields in $(0,T)\times\Omega$ satisfying the no-slip boundary condition $v = 0$ and the no-flux boundary condition $w \cdot n = 0$ on $\partial\Omega$ respectively, then we have the following three estimates:

$$\begin{aligned} (v-w, v\cdot\nabla v - w\cdot\nabla w) &= (v-w, v\cdot\nabla(v-w)) + (v-w, (v-w)\cdot\nabla w) \\ &\le \|\nabla w\|_{L^\infty}\|v-w\|^2_{L^2}, \end{aligned} \tag{5.7}$$

$$(v-w, \nabla P) = \int_{\partial\Omega} P(v-w)\cdot n \,\mathrm{d}S = 0, \tag{5.8}$$

$$\begin{aligned} (v-w, \Delta v) &= -\|\nabla v\|^2_{L^2(\Omega)} + (\nabla w, \nabla v) - \int_{\partial\Omega} w\cdot\partial_n v \,\mathrm{d}S \\ &\le -\frac{1}{2}\|\nabla v\|^2_{L^2} + \frac{1}{2}\|\nabla w\|_{L^2} - \int_{\partial\Omega} J[w]\cdot \operatorname{curl} v \,\mathrm{d}S. \end{aligned} \tag{5.9}$$

Here $J[w]$ is a rotation of w and $\operatorname{curl} v$ is the vorticity of v defined by

$$J[w] := \begin{cases} n^\perp \cdot w & d=2 \\ n\times w & d=3 \end{cases}, \qquad \operatorname{curl} v := \begin{cases} \nabla^\perp \cdot v & d=2 \\ \nabla\times v & d=3 \end{cases},$$

where $n^\perp$ is the rotation of the normal vector counterclockwise by a right angle, and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. Moreover, note that $w \cdot \nabla w = 0$ in (5.7) when w is a shear flow.

5.4.1 Prandtl Timespan

To compute the evolution of $u^\nu - u^\nu_{\mathrm{Pr}}$, first we subtract their equations and obtain

$$\partial_t(u^\nu - u^\nu_{\mathrm{Pr}}) + u^\nu \cdot \nabla u^\nu + \nabla P^\nu = \nu\Delta(u^\nu - u^\nu_{\mathrm{Pr}}).$$

The evolution of $u^\nu - u^\nu_{\mathrm{Pr}}$ can be computed using (5.7)–(5.9) as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu - u^\nu_{\mathrm{Pr}}\|^2_{L^2} + \nu\|\nabla(u^\nu - u^\nu_{\mathrm{Pr}})\|^2_{L^2} &\le -(u^\nu - u^\nu_{\mathrm{Pr}}, u^\nu \cdot \nabla u^\nu) \\ &\le \|\nabla u^\nu_{\mathrm{Pr}}\|_{L^\infty}\|u^\nu - u^\nu_{\mathrm{Pr}}\|^2_{L^2}.\end{aligned}$$

By Lemma 5.12, the Lipschitz norm of the Prandtl layer at time t is

$$\|\nabla u^\nu_{\mathrm{Pr}}\|_{L^\infty}(t) = \|\nabla U^\nu_{\mathrm{Pr}}\|_{L^\infty} \le \frac{1}{2}(\nu t)^{-\frac{3}{4}}\left(\frac{E}{|\partial\Omega|}\right)^{\frac{1}{2}}.$$

Integrating in time, we have

$$2\|\nabla u^\nu_{\mathrm{Pr}}\|_{L^1(0,T_\nu;L^\infty(\Omega))} \le \int_0^{T_\nu} (\nu t)^{-\frac{3}{4}}\left(\frac{E}{|\partial\Omega|}\right)^{\frac{1}{2}} \mathrm{d}t \le \log 2 \tag{5.10}$$

if we choose T_ν small enough such that

$$T_\nu \le T_* := \left(\frac{\log 2}{4}\right)^4 E^{-2}|\partial\Omega|^2\nu^3. \tag{5.11}$$

By Grönwall's inequality, we have for any $0 < t < T_\nu$,

$$\frac{1}{2}\|u^\nu - u^\nu_{\mathrm{Pr}}\|^2_{L^2(\Omega)}(t) + \nu\|\nabla(u^\nu - u^\nu_{\mathrm{Pr}})\|^2_{L^2((0,t)\times\Omega)} \le \|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(0). \tag{5.12}$$

The evolution of $u^\nu_{\mathrm{Pr}} - \bar{u}$ can be computed using (5.9) as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu_{\mathrm{Pr}} - \bar{u}\|^2_{L^2(\Omega)} &= (u^\nu_{\mathrm{Pr}} - \bar{u}, \partial_t u^\nu_{\mathrm{Pr}}) = (u^\nu_{\mathrm{Pr}} - \bar{u}, \nu\Delta u^\nu_{\mathrm{Pr}}) \\ &\le -\frac{\nu}{2}\|\nabla u^\nu_{\mathrm{Pr}}\|^2_{L^2} + \frac{\nu}{2}\|\nabla\bar{u}\|^2_{L^2} - \nu\int_{\partial\Omega}\bar{u}\cdot\partial_n u^\nu_{\mathrm{Pr}}\,\mathrm{d}x'\end{aligned}$$

where $\|\nabla\bar{u}\|^2_{L^2} \le G^2|\Omega|$ and

$$\left|\int_{\partial\Omega}\bar{u}\cdot\partial_n u^\nu_{\mathrm{Pr}}\,\mathrm{d}x'\right| \le \|\nabla u^\nu_{\mathrm{Pr}}\|_{L^\infty(\partial\Omega)}\|\bar{u}\|_{L^\infty(\partial\Omega)}|\partial\Omega|.$$

Integration in time gives for any $0 < t < T_\nu$, we have

$$\begin{aligned}&\frac{1}{2}\|u^\nu_{\mathrm{Pr}} - \bar{u}\|^2_{L^2(\Omega)}(t) + \frac{\nu}{2}\|\nabla u^\nu_{\mathrm{Pr}}\|^2_{L^2((0,t)\times\Omega)} \\ &\qquad \le \frac{\nu}{2}G^2|\Omega|t + A\nu|\partial\Omega|\|\nabla u^\nu_{\mathrm{Pr}}\|_{L^1(0,T_\nu;L^\infty(\Omega))} \\ &\qquad \le \frac{\nu}{2}G^2|\Omega|t + \frac{1}{2}A^2|\Omega|\mathsf{Re}^{-1}\end{aligned}$$

where the last inequality used (5.10).

Combined with (5.12), we have for any $0 < t \le T_\nu$,

$$\begin{aligned}&\frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(t) + \frac{\nu}{2}\|\nabla u^\nu\|^2_{L^2((0,t)\times\Omega)} \\ &\qquad \le 2\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(0) + \nu G^2|\Omega|t + A^2|\Omega|\mathsf{Re}^{-1}.\end{aligned} \tag{5.13}$$

5.4.2 Main Timespan

The evolution of $u^\nu - \bar u$ can be computed using (5.7)–(5.9) as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu - \bar u\|_{L^2}^2 &= (u^\nu - \bar u, \partial_t u^\nu)\\ &\le -(u^\nu - \bar u, u^\nu\cdot\nabla u^\nu) - (u^\nu - \bar u, \nabla P^\nu) + \nu(u^\nu - \bar u, \Delta u^\nu)\\ &\le \|\nabla \bar u\|_{L^\infty}\|u^\nu - \bar u\|_{L^2}^2 - \frac{1}{2}\nu\|\nabla u^\nu\|_{L^2}^2 + \frac{1}{2}\nu\|\nabla \bar u\|_{L^2}^2\\ &\quad - \int_{\partial\Omega} J[\bar u]\cdot(\nu\omega^\nu)\,\mathrm{d}x'.\end{aligned}$$

Since $\bar u$ is a constant on each connecting component of $\partial\Omega$, by integrating from T_ν to T, we have

$$\begin{aligned}&\frac{1}{2}\|u^\nu - \bar u\|_{L^2(\Omega)}^2(T) + \frac{\nu}{2}\|\nabla u^\nu\|_{L^2((T_\nu,T)\times\Omega)}^2\\ &\quad\le \frac{1}{2}\|u^\nu - \bar u\|_{L^2(\Omega)}^2(T_\nu) + G\int_{T_\nu}^T \|u^\nu - \bar u\|_{L^2(\Omega)}^2(t)\,\mathrm{d}t + \frac{\nu}{2}G^2(T-T_\nu)|\Omega|\\ &\qquad + A\left(\left|\int_{T_\nu}^T\int_{\{x_d=0\}}\nu\omega^\nu\,\mathrm{d}x'\,\mathrm{d}t\right| + \left|\int_{T_\nu}^T\int_{\{x_d=H\}}\nu\omega^\nu\,\mathrm{d}x'\,\mathrm{d}t\right|\right).\end{aligned}$$

Adding (5.13) at $t = T_\nu$, we have for any $T > T_\nu$ that

$$\begin{aligned}&\frac{1}{2}\|u^\nu - \bar u\|_{L^2(\Omega)}^2(T) + \frac{\nu}{2}\|\nabla u^\nu\|_{L^2((0,T)\times\Omega)}^2\\ &\le 2\|u^\nu - \bar u\|_{L^2(\Omega)}^2(0) + G\int_{T_\nu}^T \|u^\nu - \bar u\|_{L^2(\Omega)}^2(t)\,\mathrm{d}t + \nu G^2 T|\Omega| + A^2|\Omega|\mathsf{Re}^{-1}\\ &\quad + A\left(\left|\int_{T_\nu}^T\int_{\{x_d=0\}}\nu\omega^\nu\,\mathrm{d}x'\,\mathrm{d}t\right| + \left|\int_{T_\nu}^T\int_{\{x_d=H\}}\nu\omega^\nu\,\mathrm{d}x'\,\mathrm{d}t\right|\right).\end{aligned} \tag{5.14}$$

5.4.3 Proof of Theorem 5.5

We first note that Theorem 5.5 is only interesting when the initial kinetic energy $\|u^\nu(0)\|_{L^2}$ and $\|\bar u\|_{L^2}$ are comparable.

Lemma 5.15. *Let $\bar u \in L^2(\Omega)$, and let u^ν be a Leray–Hopf solution to* (NSE$_\nu$), *so* *the energy inequality holds:*

$$\frac{1}{2}\|u^\nu(T)\|^2_{L^2(\Omega)} + \nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \le \frac{1}{2}\|u^\nu(0)\|_{L^2(\Omega)}.$$

For any $C' > 1$, there exists $C > 0$ such that if $\|u^\nu(0)\|_{L^2(\Omega)} > C\|\bar{u}\|_{L^2(\Omega)}$ or $\|\bar{u}\|_{L^2(\Omega)} > C\|u^\nu(0)\|_{L^2(\Omega)}$, then

$$\|u^\nu(T) - \bar{u}\|^2_{L^2(\Omega)} + 2\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \le C'\|u^\nu(0) - \bar{u}\|^2_{L^2(\Omega)}.$$

Proof. If $\|u^\nu(0)\|_{L^2} > C\|\bar{u}\|_{L^2(\Omega)}$, by energy inequality we can bound

$$\begin{aligned}
\|u^\nu(T) - \bar{u}\|^2_{L^2(\Omega)} &\le \left(1 + \frac{1}{C}\right)\left(\|u^\nu(T)\|^2_{L^2(\Omega)} + C\|\bar{u}\|^2_{L^2(\Omega)}\right) \\
&= \left(1 + \frac{1}{C}\right)\|u^\nu(0)\|^2_{L^2(\Omega)} - 2\left(1 + \frac{1}{C}\right)\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \\
&\quad + (C+1)\|\bar{u}\|^2_{L^2(\Omega)} \\
&\le \left(1 + \frac{1}{C}\right)^2\left(\|u^\nu(0) - \bar{u}\|^2_{L^2(\Omega)} + C\|\bar{u}\|^2_{L^2(\Omega)}\right) \\
&\quad - 2\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} + (C+1)\|\bar{u}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Since $\|u^\nu(0)\|_{L^2} > C\|\bar{u}\|_{L^2}$ implies $\|\bar{u}\|_{L^2} < \frac{1}{C-1}\|u^\nu(0) - \bar{u}\|_{L^2}$, we conclude

$$\|u^\nu(T) - \bar{u}\|^2_{L^2(\Omega)} + 2\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \le C'\|u^\nu(0) - \bar{u}\|^2_{L^2(\Omega)}$$

for some $C' \to 1^+$ as $C \to \infty$. If $\|u^\nu(0)\|_{L^2} < \frac{1}{4}\|\bar{u}\|_{L^2}$, then by the energy inequality we can estimate

$$\begin{aligned}\|u^\nu(T)-\bar u\|^2_{L^2(\Omega)} &\le \left(1+\frac{1}{C}\right)\left(C\|u^\nu(T)\|^2_{L^2(\Omega)}+\|\bar u\|^2_{L^2(\Omega)}\right)\\ &\le (1+C)\|u^\nu(0)\|^2_{L^2(\Omega)} - 2(1+C)\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)}\\ &\quad + \left(1+\frac{1}{C}\right)\|\bar u\|^2_{L^2(\Omega)}\\ &\le \left(1+\frac{1}{C}\right)^2\left(\|u^\nu(0)-\bar u\|^2_{L^2(\Omega)}+C\|u^\nu(0)\|^2_{L^2(\Omega)}\right)\\ &\quad - 2\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} + (1+C)\|u^\nu(0)\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $\|\bar u\|_{L^2} > C\|u^\nu(0)\|_{L^2}$ implies $\|u^\nu(0)\|_{L^2} < \frac{1}{C-1}\|u^\nu(0)-\bar u\|_{L^2}$, we again have

$$\|u^\nu(T)-\bar u\|^2_{L^2(\Omega)} + 2\nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \le C'\|u^\nu(0)-\bar u\|^2_{L^2(\Omega)}$$

and the result also follows. □

Because of this lemma, from here we assume

$$\frac{E}{C} \le \|u^\nu(0)\|^2_{L^2(\Omega)} \le CE$$

for some universal constant C. Under this assumption, we see there is a trivial upper bound on layer separation as

$$\frac{1}{2}\|u^\nu(T)-\bar u\|^2_{L^2(\Omega)} + \nu\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \le CE \tag{5.15}$$

again using the energy inequality.

Next we study the rescaled boundary vorticity. Since u^ν solve (NSE$_\nu$) in $(0,T)\times\Omega$, its rescale $u(t,x) = u^\nu(\nu t, \nu x)$ solves (NSE) in $(0,T/\nu)\times(\Omega/\nu)$. Moreover,

$$\nabla u(t,x) = \nu\nabla u^\nu(\nu t,\nu x), \qquad \omega(t,x) = \nu\omega^\nu(\nu t,\nu x).$$

Now we apply Theorem 5.4 on u, and we have a rescaled estimate on u^ν as

$$\left\| \nu\tilde{\omega}^\nu \mathbf{1}_{\left\{\nu\tilde{\omega}^\nu > \max\left\{\frac{\nu}{t}, \frac{\nu^2}{W^2}, \frac{\nu^2}{H^2}\right\}\right\}} \right\|_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)}^{\frac{3}{2}} \leq C\nu \|\nabla u^\nu\|_{L^2((0,T)\times\Omega)}^2. \tag{5.16}$$

Proof of Theorem 5.5. We choose $T_\nu = 4^{-K}T$ for some integer K such that

$$\frac{1}{4}T_* \leq T_\nu \leq T_*$$

where T_* is defined in (5.11). The average of ω^ν in (T_ν, T) is thus bounded by the average of $\tilde{\omega}^\nu$. To estimate the boundary vorticity in (5.14), we split it as

$$\begin{aligned} \left| \int_{T_\nu}^T \int_{\{x_d=0\}} \nu\omega^\nu \,\mathrm{d}x' \,\mathrm{d}t \right| &\leq \int_{T_\nu}^T \int_{\{x_d=0\}} \nu\tilde{\omega}^\nu \,\mathrm{d}x' \,\mathrm{d}t \\ &\leq \int_{T_\nu}^T \int_{\{x_d=0\}} \nu\tilde{\omega}^\nu \mathbf{1}_{\left\{\nu\tilde{\omega}^\nu > \max\left\{\frac{\nu}{t}, \frac{\nu^2}{W^2}, \frac{\nu^2}{H^2}\right\}\right\}} \,\mathrm{d}x' \,\mathrm{d}t \\ &\quad + \int_{T_\nu}^T \int_{\{x_d=0\}} \max\left\{\frac{\nu}{t}, \frac{\nu^2}{W^2}, \frac{\nu^2}{H^2}\right\} \mathrm{d}x' \,\mathrm{d}t. \end{aligned} \tag{5.17}$$

For the first term in (5.17), we apply (5.16) and obtain

$$\begin{aligned} &\int_{T_\nu}^T \int_{\{x_d=0\}} A\nu\tilde{\omega}^\nu \mathbf{1}_{\left\{\nu\tilde{\omega}^\nu > \max\left\{\frac{\nu}{t}, \frac{\nu^2}{W^2}, \frac{\nu^2}{H^2}\right\}\right\}} \,\mathrm{d}x' \,\mathrm{d}t \\ &\quad \leq \left\| \nu\tilde{\omega}^\nu \mathbf{1}_{\left\{\nu\tilde{\omega}^\nu > \max\left\{\frac{\nu}{t}, \frac{\nu^2}{W^2}, \frac{\nu^2}{H^2}\right\}\right\}} \right\|_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)} \|A\|_{L^{3,1}((0,T)\times\partial\Omega)} \\ &\quad \leq \frac{1}{8}\nu \|\nabla u^\nu\|_{L^2((0,T)\times\Omega)}^2 + CA^3 T|\partial\Omega|. \end{aligned} \tag{5.18}$$

For the second term in (5.17), it is bounded by

$$\begin{aligned}
&A\int_{T_\nu}^{T}\int_{\{x_d=0\}}\max\left\{\frac{\nu}{t},\frac{\nu^2}{W^2},\frac{\nu^2}{H^2}\right\}\mathrm{d}x'\,\mathrm{d}t\\
&\quad\le A\int_{T_\nu}^{T}\int_{\{x_d=0\}}\frac{\nu}{t}\,\mathrm{d}x'\,\mathrm{d}t + A\int_{T_\nu}^{T}\int_{\{x_d=0\}}\max\left\{\frac{\nu^2}{W^2},\frac{\nu^2}{H^2}\right\}\mathrm{d}x'\,\mathrm{d}t\\
&\quad\le A\nu\log\left(\frac{T}{T_\nu}\right)|\partial\Omega| + A\nu^2\min\{W,H\}^{-2}T|\partial\Omega|\\
&\quad\le A^2|\Omega|\mathsf{Re}^{-1}\log\left(\frac{4T}{T_*}\right) + A^3T|\partial\Omega|\mathsf{Re}^{-2}\max\{H/W,1\}^2.
\end{aligned}$$

Since $\frac{1}{T_*} = CE^2|\partial\Omega|^{-2}\nu^{-3} = C\left(\frac{E}{A^2|\Omega|}\right)^2\mathsf{Re}^3\frac{A}{H}$, we separate the log as

$$\log\left(\frac{4T}{T_*}\right) \le 3\log\mathsf{Re} + 2\left(\frac{E}{A^2|\Omega|}\right) + \frac{AT}{H} + C.$$

Thus the second term in (5.17) is bounded by

$$\begin{aligned}
&A\int_{T_\nu}^{T}\int_{\{x_d=0\}}\max\left\{\frac{\nu}{t},\frac{\nu^2}{W^2},\frac{\nu^2}{H^2}\right\}\mathrm{d}x'\,\mathrm{d}t\\
&\quad\le A^2|\Omega|\mathsf{Re}^{-1}\log(\mathsf{Re}+C) + 2\mathsf{Re}^{-1}E\\
&\qquad + A^3T|\partial\Omega|\left(\mathsf{Re}^{-1} + \mathsf{Re}^{-2}\max\{H/W,1\}^2\right).
\end{aligned}\tag{5.19}$$

Plugging (5.18)-(5.19) into (5.17) and applying to (5.14) (naturally for the other boundary $\{x_d = H\}$ the same estimate), we conclude for every $T > T_\nu$ that

$$\begin{aligned}
&\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(T) + \frac{\nu}{2}\|\nabla u^\nu\|^2_{L^2((T_\nu,T)\times\Omega)}\\
&\quad\le 4\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(0) + 2G\int_{T_\nu}^{T}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(t)\,\mathrm{d}t\\
&\qquad + 2\nu G^2T|\Omega| + A^2|\Omega|\mathsf{Re}^{-1}\log(\mathsf{Re}+C) + 2\mathsf{Re}^{-1}E\\
&\qquad + CA^3T|\partial\Omega|\left(1 + \mathsf{Re}^{-2}\max\{H/W,1\}^2\right).
\end{aligned}$$

Combined with (5.13) we see indeed that the above inequality is true for any $T > 0$, so applying Grönwall's inequality yields

$$\begin{aligned}
&\sup_{0\le t\le T}\left\{\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}(t) + \frac{\nu}{2}\|\nabla u^\nu\|^2_{L^2((0,t)\times\Omega)}\right\} \\
&\le \exp(2GT)\left\{4\|u^\nu(0) - \bar{u}\|^2_{L^2(\Omega)} + CA^3T|\partial\Omega|\left(1 + \mathsf{Re}^{-2}\max\{H/W, 1\}^2\right) + R_\nu\right\},
\end{aligned}$$

where the remainder terms R_ν is defined as

$$R_\nu = 2\nu G^2 T|\Omega| + A^2|\Omega|\mathsf{Re}^{-1}\log(\mathsf{Re} + C) + 2\mathsf{Re}^{-1}E.$$

Finally, if Re is sufficiently small, then the estimate holds true automatically by $\mathsf{Re}^{-1}E$ term according the trivial bound (5.15). Otherwise, by $\mathsf{Re}^{-2} \le C$ and $\mathsf{Re}^{-1}\log(\mathsf{Re} + C) \le C\log(2 + \mathsf{Re})$ we complete the proof. □

Proof of Theorem 5.2. In this particular setting, $G = 0$, $E = A^2|\Omega|$, $W/H = 1$. Therefore we can bound

$$R_\nu \le CA^2|\Omega|\mathsf{Re}^{-1}\log(2 + \mathsf{Re}) + 2\mathsf{Re}^{-1}E \le CA^2|\Omega|\mathsf{Re}^{-1}\log(2 + \mathsf{Re})$$

which finishes the proof of the theorem. □

5.5 Appendix: Construction of Weak Solutions to the Euler Equation with Layer Separation

This appendix is dedicated to the proof of Proposition 5.1. In [Szé11], Székelyhidi constructed weak solutions to (EE) with strictly decreasing energy profile with vortex sheet initial data in a unit torus $\Omega = \mathbb{T}^d$, by means of convex integration introduced in [DLS10].

We will first construct a weak (distributional) solution (v, P) to (EE) in a two-dimensional set $\mathbb{T} \times (0,1)$, such that $v = e_1$ at $t = 0$ and $\frac{1}{2}\|v\|_{L^2}^2(t) = \frac{1}{2} - rt$ at a constant rate $r > 0$ for small t. To achieve this, we follow the ideas of [Szé11]. However, we first construct a subsolution $\bar{v}$ on a bigger domain $\tilde{\Omega} = \mathbb{T} \times [-1, 2]$, that we will convex integrate only on $\mathbb{T} \times (0,1)$. The result function v is a solution to (EE) only inside $\mathbb{T} \times (0,1)$, but it keeps the global incompressibility $\operatorname{div} v = 0$ in $\mathbb{T} \times [-1,2]$, together with $v = 0$ on $\mathbb{T} \times (-1,0) \cup (1,2)$. This provides the impermeability condition needed at the boundary. More precisely, consider $(\bar{v}, \bar{u}, \bar{q}) : (0,T) \times \tilde{\Omega} \to \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \times \mathbb{R}$ with respect to some $\bar{e} : (0,T) \times \tilde{\Omega} \to [0, \infty)$, satisfying $\bar{v} \in L^2_{\text{loc}}$, $\bar{u} \in L^1_{\text{loc}}$, $\bar{q} \in \mathcal{D}'$, and in the distribution sense

$$\begin{cases} \partial_t \bar{v} + \operatorname{div} \bar{u} + \nabla \bar{q} = 0 \\ \operatorname{div} \bar{v} = 0 \end{cases} \tag{5.20}$$

and almost everywhere

$$\bar{v} \otimes \bar{v} - \bar{u} \leq \bar{e}\, \mathrm{Id}.$$

Here $\mathcal{S}_0^{2\times 2}$ is the space of trace-free two-by-two matrices.

To achieve this, we set

$$\bar{v} = (\alpha, 0), \qquad \bar{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \qquad \bar{q} = \beta$$

for some $\alpha(t, x_2), \beta(t, x_2), \gamma(t, x_2)$ to be fixed. With this choice, we need

$$\partial_t \alpha + \partial_{x_2} \gamma = 0, \qquad \begin{pmatrix} \bar{e} - \alpha^2 + \beta & \gamma \\ \gamma & \bar{e} - \beta \end{pmatrix} \geq 0.$$

The second constraint can be simplified to

$$2\bar{e} - \alpha^2 \geq 0, \qquad\qquad (\bar{e} - \alpha^2 + \beta)(\bar{e} - \beta) \geq \gamma^2.$$

Denote $\bar{f} = \bar{e} - \frac{1}{2}\alpha^2$, $\delta = \beta - \frac{1}{2}\alpha^2$, then

$$\begin{cases} \bar{f} \geq 0 \\ (\bar{f} + \delta)(\bar{f} - \delta) \geq \gamma^2 \end{cases} \Rightarrow \bar{f} \geq \sqrt{\gamma^2 + \delta^2} \Rightarrow \bar{e} \geq \frac{1}{2}\alpha^2 + \sqrt{\gamma^2 + \delta^2} \geq \frac{1}{2}\alpha^2 + |\gamma|,$$

which will be the only constraint by setting $\beta = \frac{1}{2}\alpha^2$ thus $\delta = 0$. It suffices to find (α, γ) that solves $\partial_t \alpha + \partial_{x_2} \gamma = 0$, i.e. we require the conservation of momentum and need

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \alpha \, \mathrm{d}x_2 = 0, \qquad \gamma = \int_{0.5}^{x_2} \partial_t \alpha \, \mathrm{d}x_2, \qquad \bar{e} \geq \frac{1}{2}\alpha^2 + |\gamma|.$$

Let us mimic the strategy in [Szé11] and work with a different vortex-sheet initial data:

$$\alpha(0, x_2) = \begin{cases} 1 & 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $\alpha(t, x_2)$ be the piecewise linear function interpolating $(-1, 0)$, $(0, 0)$, $(\lambda t, 1)$, $(1 - \lambda t, 1)$, $(1, 0)$, $(2, 0)$ for some fixed $\lambda > 0$ to be determined as in Figure 5.5.

Under this setup, it is simple to see that

$$\partial_{x_2} \gamma = -\partial_t \alpha = \lambda \alpha |\partial_{x_2} \alpha|,$$

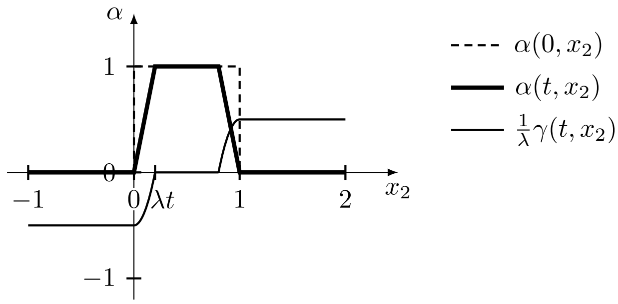

Figure 5.5: The graph of $\alpha(t,x_2)$, $\frac{1}{\lambda}\gamma(t,x_2)$ for a fixed $0 \le t < T = \frac{1}{2\lambda}$

from which we can recover

$$\gamma(t,x_2) = \begin{cases} -\frac{\lambda}{2}\big(1-\alpha^2(t,x_2)\big) & -1 \le x_2 \le \frac{1}{2} \\ \frac{\lambda}{2}\big(1-\alpha^2(t,x_2)\big) & \text{otherwise} \end{cases}$$

and as a consequence, we need

$$\bar{e} \ge \frac{1}{2}\alpha^2 + |\gamma| = \frac{1}{2}\alpha^2 + \frac{\lambda}{2}(1-\alpha^2) = \frac{1}{2} - \frac{1}{2}(1-\lambda)(1-\alpha^2).$$

Let us fix $\lambda, \varepsilon \in (0,1)$, and set

$$\bar{e} = \frac{1}{2} - \frac{\varepsilon}{2}(1-\lambda)(1-\alpha^2). \tag{5.21}$$

Then $\bar{e} > \frac{1}{2}\alpha^2 + |\gamma|$ in the space-time region $\mathcal{U} := (0,T)\times\mathbb{T}\times(0,1)\cap\{\alpha<1\}$.

We are now ready to apply Theorem 1.3 of [Szé11] when convex integrating in $(0,T)\times\mathbb{T}\times(0,1)$ only. This provides infinitely many $(\tilde{v},\tilde{u}) \in L^\infty_{loc}((0,T)\times\tilde{\Omega})$ with $\tilde{v} \in C(0,T;L^2_{\text{weak}}(\tilde{\Omega})$ such that $(\tilde{v},\tilde{u},0)$ satisfies (5.20), $(\tilde{v},\tilde{u}) = 0$ a.e. in $\mathcal{U}^c = (0,T)\times\mathbb{T}\times((-1,0)\cup(1,2))\cup\{\alpha=1\}$, and $v := \bar{v}+\tilde{v}$, $u := \bar{u}+\tilde{u}$ satisfy

$$v\otimes v - u = \bar{e}\,\mathrm{Id} \qquad \text{a.e. in } (0,T)\times\mathbb{T}\times(0,1).$$

From the second equation of (5.20), $\partial_{x_2} v_2 = -\partial_{x_1} v_1$, and $v_2 \in C_{x_2}(W^{-1,\infty}_{x_1})$. But since we didn't convex integrate on $(0,T)\times\mathbb{T}\times((-1,0)\cup(0,1))$, we still have $v_2 = 0$ at $x_2 = 0$ and $x_2 = 1$. This provides the impermeability boundary conditions at these points.

Then (v, P) satisfies (EE) with the impermeability conditions in $(0,T)\times\mathbb{T}\times(0,1)$ in the distributional sense for $P = \bar{q} - \bar{e}$, and $\frac{1}{2}|v|^2 = \bar{e}$ matches the energy density profile given in (5.21) (note that the constructed solution is not solution to (EE) in the domain $(0,T)\times\mathbb{T}\times(-1,2)$). Now, we have on $(0,T)\times\mathbb{T}\times(0,1)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|v|^2}{2}\,\mathrm{d}x = \varepsilon(1-\lambda)\int \alpha\partial_t\alpha\,\mathrm{d}x_2 = -\varepsilon\lambda(1-\lambda)\int \alpha^2|\partial_{x_2}\alpha|\,\mathrm{d}x_2 = -\frac{2}{3}\varepsilon\lambda(1-\lambda),$$

i.e. $\frac{1}{2}\|v\|^2_{L^2}$ decreases linearly at rate $r := \frac{2}{3}\varepsilon\lambda(1-\lambda)$.

We consider the deviation from initial value. Since $\tilde{v} = 0$ a.e. at $t = 0$, we know $v(0) = \bar{v}(0) = \pm e_1$, and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |v(t)-v(0)|^2\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|v(t)|^2}{2}\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int v(t)\cdot v(0)\,\mathrm{d}x \\ &= -r - \int \partial_t v(t)\cdot v(0)\,\mathrm{d}x \\ &= -r + \int \operatorname{div} u(t)\cdot v(0)\,\mathrm{d}x.\end{aligned}$$

The quantity $\bar{e}$ and $\bar{q}$ depend only on t, x_2, so the equation on v_1 from (5.20) has no pressure and verify:

$$\partial_{x_2} u_{12} = -\partial_t v_1 - \partial_{x_1} u_{11}.$$

Especially, $u_{12} \in C_{x_2}(W^{-1,\infty}_{t,x_1})$. Therefore,

$$\begin{aligned}\int \operatorname{div} u(t) \cdot v(0)\,\mathrm{d}x &= \int_{\mathbb{T}} -u_{12}(t,x_1,0) + u_{12}(t,x_1,1)\,\mathrm{d}x_1 \\ &= \int_{\mathbb{T}} -\bar{u}_{12}(t,x_1,0) + \bar{u}_{12}(t,x_1,1)\,\mathrm{d}x_1 \\ &= -\gamma(t,0) + \gamma(t,1) = \lambda.\end{aligned}$$

This gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |v(t) - v(0)|^2\,\mathrm{d}x = \lambda - r = \lambda - \frac{2}{3}\varepsilon\lambda(1-\lambda).$$

This rate converges to 1 by setting $\lambda \to 1$ and $\varepsilon \to 0$, thus

$$\frac{1}{2}\|v(t) - e_1\|^2_{L^2(\mathbb{T}\times[0,1])} = Ct, \qquad \forall t \in \left(0, \frac{1}{2\lambda}\right).$$

Moreover, $v = 0$ on $\{x_2 = 0, 1\}$.

Now for some $A > 0$, define $(v^*, P^*) : (0, \frac{1}{2\lambda A}) \times \Omega \to \mathbb{R}^2 \times \mathbb{R}$ by time rescaling $v^*(t,x) = Av(At,x)$, $P^*(t,x) = A^2 P(At,x)$, where $\Omega = \mathbb{T}\times[0,1]$ is the unit channel. Then $v^*(0) = Ae_1$ in Ω, $v^*(t) = 0$ on $\partial\Omega$ and

$$\frac{1}{2}\|v(t) - Ae_1\|^2_{L^2(\mathbb{T}\times[0,1])} = CA^3 t, \qquad \forall t \in \left(0, \frac{1}{2\lambda A}\right)$$

for some C, λ satisfying $0 < C < \lambda < 1$.

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