The Cannon-Thurston map

Ino Loukidou

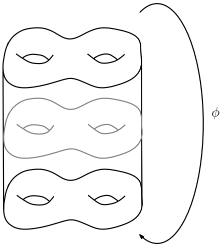
April 2026

A Cannon Thurston map

3-manifolds that fiber over S^1

Mapping tori

Σ surface, $\phi : \Sigma \to \Sigma$. Define $M_\phi := {}^{\Sigma \times [0,1]}\!\big/_{(x,1) \sim (\phi(x),0)}$.



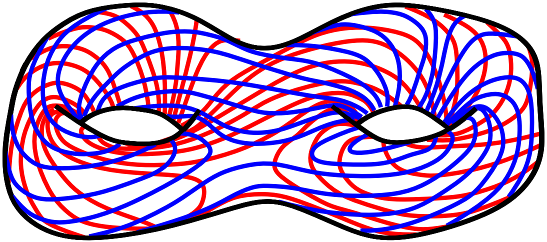
Short exact sequence:

$$1 \to \pi_1(\Sigma) \to \pi_1(M_\phi) \to \mathbb{Z} \to 1$$

Thus, $\pi_1(M_\phi)$ generated by $\pi_1(\Sigma)$ and t.

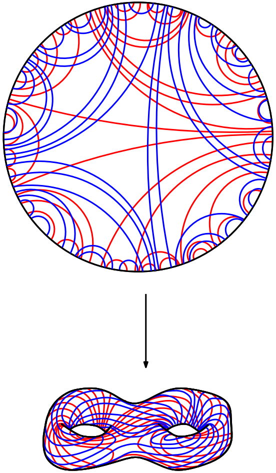
pseudo-Anosov homeomorphisms

$\phi : \Sigma \to \Sigma$ pseudo-Anosov $\iff$ preserves pair of geodesic laminations $L^{\pm}$ of Σ (stable, unstable).



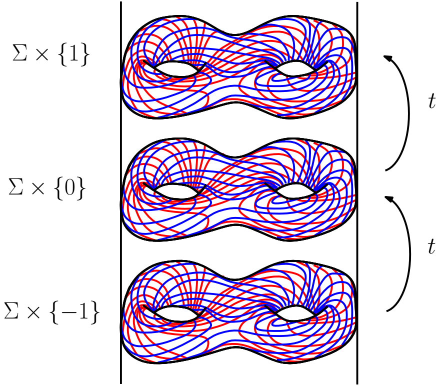
Universal cover of Σ

$\tilde{\Sigma} \simeq \mathbb{H}^2$. Laminations $L^{\pm}$ lift to geodesic laminations $\Lambda^{\pm}$ of $\mathbb{H}^2$.



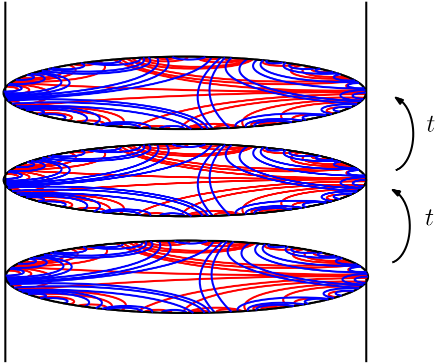
A cover of M_ϕ

Normal cyclic cover for $\pi_1(\Sigma) \triangleleft \pi_1(M_\phi) \simeq \Sigma \times \mathbb{R}$. Deck transformations $\langle t \rangle$.



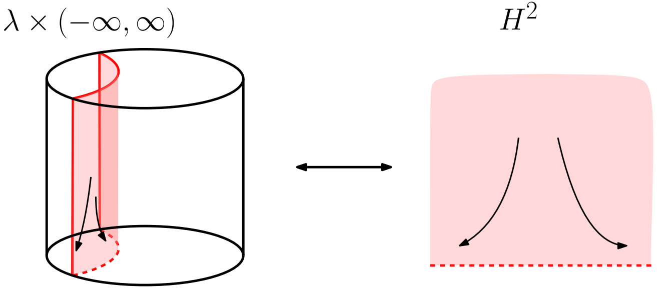
The universal cover of M_ϕ

$\tilde{M}_\phi \simeq \tilde{\Sigma} \times \mathbb{R} \simeq \mathbb{H}^2 \times \mathbb{R}$. Has (pseudo) metric ρ: as t goes up contracts red leaves and expands blue leaves.



Hyperbolic planes

In this metric, $\lambda \times \mathbb{R} \simeq \mathbb{H}^2$.



Hyperbolic structures

Thurston's theorem

Theorem

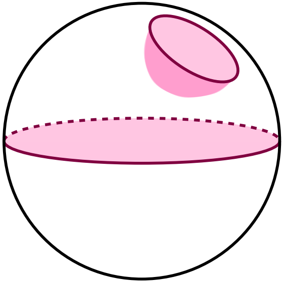
(Thurston): Σ *hyperbolic,* $\phi : \Sigma \to \Sigma$ *pseudo-Anosov* $\implies M_\phi$ *hyperbolic.*

Theorem

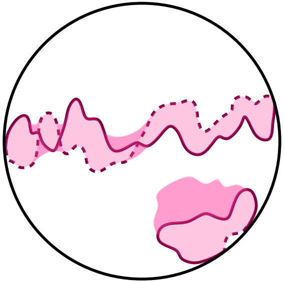
ρ *and the hyperbolic metric are quasi-comparable.*

Planes in $\mathbb{H}^3$

Fuchsian

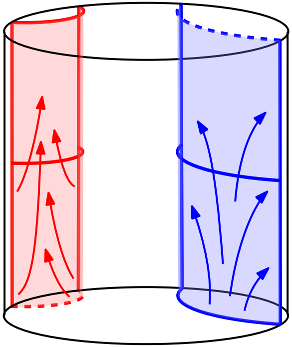


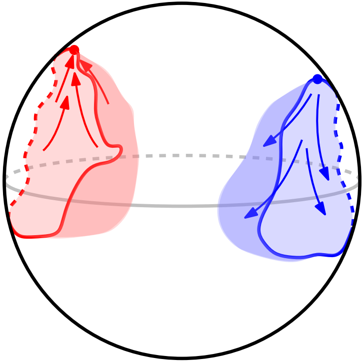
Quasi-fuchsian



An example

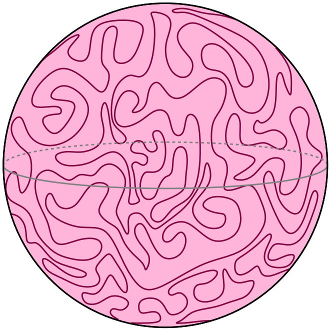
$\lambda \times \mathbb{R}$ quasi-plane in $\mathbb{H}^3$.





What about $\tilde{\Sigma} \times \{0\}$?

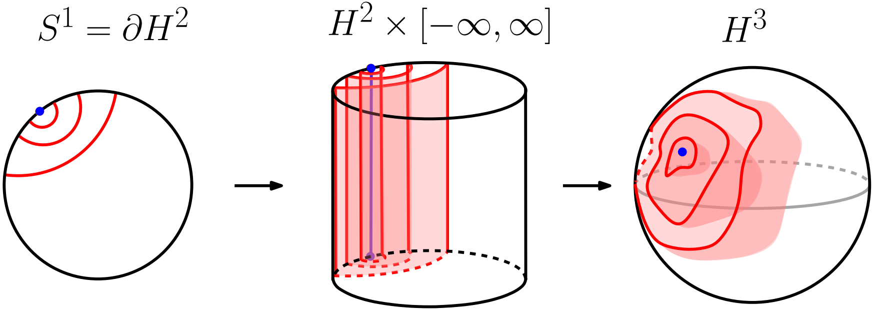
$\pi_1(\Sigma) \lhd \pi_1(M) \implies \tilde{\Sigma}$ accumulates to all of S^2_∞.



Thus, if we show the inclusion extends, the boundary map is sphere-filling.

The inclusion extends

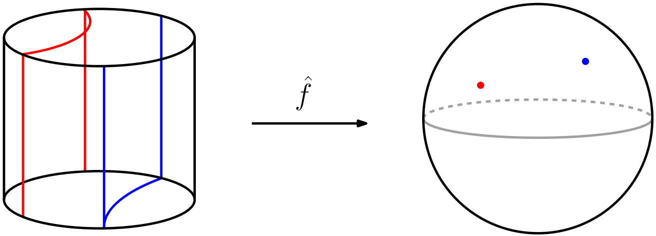
The inclusion extends



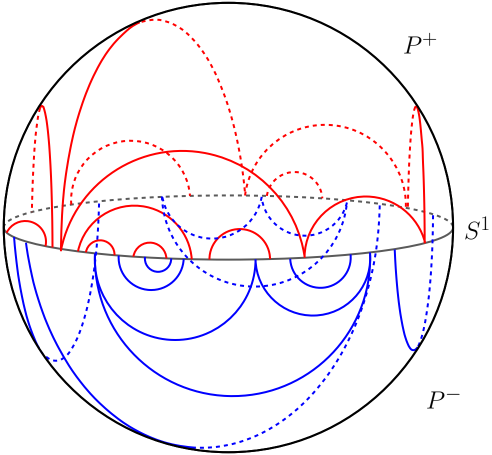
$p \in S^1_\infty$ has well-defined image in $S^2 \implies f : S^1_\infty \to S^2_\infty$.

How does the map look like?

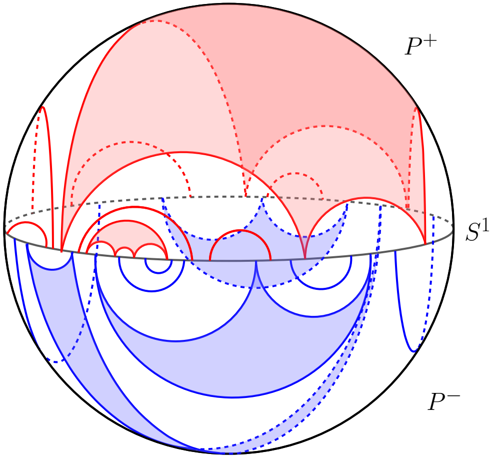
Actually the whole red leaf (up) and the blue leaf (down) maps to a single point since it is quasi-isometric to a hyperbolic plane



Collapsing laminations



Collapsing laminations

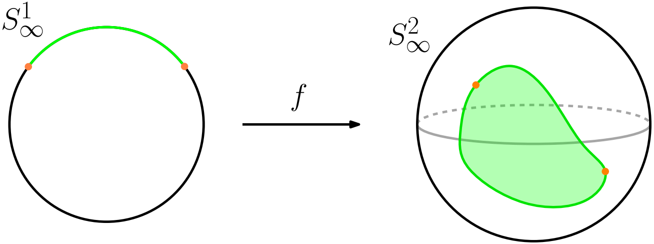


properties of the map

Properties of $f : S^1 \to S^2$

Moore for surfaces with boundary $\implies$ For every interval $I \subseteq S^1$

- $f(I) \simeq D^2$
- $f(\partial I) \subseteq \partial f(I)$.



Zippers

The images $Z^{\pm} = f(P^{\pm})$. They are G-invariant, dense and path connected.

Thank you!