Exceptional Lie groups and some related geometry

Outline of the talk

- Compact simple Lie groups – Maximal tori, Weyl groups and root systems
- Crash course in exceptional groups
- Applications
- Exceptional groups occur in many contexts of mathematics and physics, and the main interest in this talk is
- Some instances of geometry arising from arrangements defined by Weyl groups of exceptional groups
Examples

Let $T \subset G$ be a maximal torus in a compact simple Lie group.
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**Example**

- Let $V$ be a real vector space of dimension $n$ with an inner product; then the group of linear automorphisms of $V$ *preserving* the inner product is a compact group denoted $O(n)$ (the *orthogonal* group).

  \[ s(x, y) = ^t x S y \text{ means } s(x, y) = s(gx, gy) = ^t(Mx)S(My) \Rightarrow M = (^tM)^{-1}. \]
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- Let $V$ be a complex vector space of dimension $n$ with an inner product; then the group of (complex) linear automorphisms of $V$ which preserve the inner product is a compact Lie group denoted $U(n)$ (the unitary group). Then $M = (M^*)^{-1}$. 

In each case, the determinant gives rise to an exact sequence

$$1 \rightarrow SG \rightarrow G \rightarrow \text{det} \rightarrow K \rightarrow 1$$

where $K$ is the field $\mathbb{R}$ or $\mathbb{C}$ and $G$ is the group above. Then:

$SG$ is a simple compact Lie group $SO(n)$ resp. $SU(n)$. 

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Maximal tori

For the two groups considered, the maximal tori are easily described.

Example

- For $U(n)$, it is the group $T = \{ \text{diag}(t_1, \ldots, t_n) \}$ of diagonal matrices with complex entries $t_i = e^{i \theta_i}$. For $SU(n)$ one has in addition $\prod_{i} t_i = 1$. 
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- Let $B(\theta) := \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}$ be the $2 \times 2$ real matrix representing a rotation of $\theta$ degrees in $\mathbb{R}^2$. For the group $O(n)$ $T$ is the group of diagonal block matrices $T = \{\text{diag}(B_1(\theta_1), \ldots, B_{\lfloor \frac{n}{2} \rfloor}(\theta_{\lfloor \frac{n}{2} \rfloor}))\}$ when $n = 2m$ is even and $T = \{B_1(\theta_1), \ldots, B_{\lfloor \frac{n}{2} \rfloor}(\theta_{\lfloor \frac{n}{2} \rfloor}), 1\}$ when $n = 2m + 1$ is odd.
Let $N(T) \subset G$ denote the normalizer of $T$; the group $W(G) = N(T)/T$ is called the Weyl group of $G$.

**Example**

- For $U(n)$, in addition to multiplication on itself, permutations of the factors $t_1, \ldots, t_n$ preserve (normalize) the torus $T$. The relation defining $SU(n)$ makes $t_n$ dependent on the others, so the group of permutations is $\Sigma_{n-1}$.
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**Example**

- For \( U(n) \), in addition to multiplication on itself, permutations of the factors \( t_1, \ldots, t_n \) preserve (normalize) the torus \( T \). The relation defining \( SU(n) \) makes \( t_n \) dependent on the others, so the group of permutations is \( \Sigma_{n-1} \).

- For \( SO(n) \), one has the permutations of the \textit{blocks}; but in addition, in each block \( \theta_i \mapsto -\theta_i \) preserves the torus. The Weyl group is \( W(SO(2m+1)) = \Sigma_m \rtimes \mathbb{Z}_2^m \), for \( SO(2m) \) the number of sign changes needs to be even, \( W(SO(2m)) = \Sigma_m \rtimes \mathbb{Z}_2^{m-1} \).
Using quaternions

$\mathbb{H}$ is the $\mathbb{R}$-algebra generated by 1 and elements $i, j, k$ such that

\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji, ik = -ki, jk = -kj. \]

$\mathbb{H}$ forms a *division algebra* over $\mathbb{R}$, meaning it is a multiplicative algebra over $\mathbb{R}$ without zero-divisors. $\mathbb{H}$ satisfies all the axioms of a field except commutativity, hence it can be used as the scalar field for a vector space $V$ of dimension $n$ over $\mathbb{H}$ (dimension 4 over $\mathbb{R}$).

**Convention**

One needs to fix the scalar multiplication; standard is: $V$ is a *right* $\mathbb{H}$-vector space, then matrices operate from the left.

Using a $\mathbb{H}$-hermitian form, what was done above for the inner products can be extended to the case of *quaternionic matrices*. This leads to compact simple Lie groups called the *symplectic groups* $Sp(n)$. The Weyl group is $\Sigma_n \rtimes \mathbb{Z}_2^n$. 
A representation of $G$ is a group homomorphism of $G$ in the automorphism group of a vector space. Conjugation in $G$, $x \mapsto c_s(x) := sx s^{-1}$, defines a map

$$\text{Inn} : G \longrightarrow \text{Aut}(G), s \mapsto c_s : G \longrightarrow G.$$ 

This leads to the *adjoint representation* of $G$, defined as follows:

$$\text{Ad} : G \longrightarrow GL(T_e(G)),
\quad s \mapsto T(c_s) : T_e(G) \longrightarrow T_e(G).$$ (1)

Representations are studied by restricting to $T$: any representation of $T$ is multiplication by $(t_1^{a_1}, \ldots, t_n^{a_n})$ with integer $a_i$. The torus itself is $T = \mathbb{R}^n / \mathbb{Z}^n$, and the exponents correspond to elements in a *lattice*. The *roots of $G$* are the eigenvalues of the adjoint representation. In this way a *root system* is defined, and leads to the classification

$$A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8.$$
Classical and exceptional groups

In the notation, the subscript denotes the *dimension of the maximal torus*

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Exceptional groups

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The classification is called the Cartan-Killing classification and goes back to independent determinations by those two mathematicians. Any idea what these exceptional groups represent?
Note that $\mathbb{C} = \mathbb{R} + \text{i}\mathbb{R}, \, i^2 = -1, \, \mathbb{H} = \mathbb{C} + \text{j}\mathbb{C}, \, j^2 = -1$. Such algebras are called composition algebras. The construction can be extended one more time, $\mathbb{O} = \mathbb{H} + e_3\mathbb{H}$ to give a division algebra. This algebra is given by the relations:

The division algebra of octonions (Cayley-Graves algebra)

\begin{align*}
e_j^2 &= -e_0 = -1 \quad j = 1, \ldots, 7 \\
e_j e_k &= -e_k e_j \quad j \neq k, \quad j, k = 1, \ldots, 7 \\
e_i \cdot e_{i+1} &= e_{i+3}, \quad e_{i+1} \cdot e_{i+3} = e_i, \quad e_{i+3} \cdot e_i = e_{i+1}, \quad \text{indices taken mod 7} \\
\end{align*}

in which $e_0, e_i, e_{i+1}, e_{i+3}$ form a subalgebra isomorphic to $\mathbb{H}$. 
Multiplication in $\mathbb{O}$

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Multiplication in $\mathfrak{g}$
Each colored triangle is a subalgebra $\cong \mathbb{H}$. For example, this shows $e_3 \cdot e_5 = e_2$ and $e_5 \cdot e_1 = -e_6$. 
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Jordan algebras were studied at the beginning of the twentieth century in a quest to find interesting algebras for quantum physics. Again there is a classification, with a *unique exceptional* one, a 27-dimensional algebra (the $3 \times 3$ algebra above).
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There is an exotic way to define, starting with a Jordan algebra and a $\mathbb{R}$-division algebra, a Lie algebra, the Tits-Vinberg construction.
The Tits-Vinberg construction

Let $\mathcal{A}$ be a composition algebra (one of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) and $\mathcal{J}$ a Jordan algebra. Define the algebra

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\mathcal{T}(\mathcal{A}, \mathcal{J}) = \text{Der}(\mathcal{A}) \oplus \text{Der}(\mathcal{J}) + \mathcal{A}_0 \otimes \mathcal{J}_0.
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Provided with a rather complicated bracket, this can be made to a Lie algebra. The algebra $\mathbf{H}_3(\mathcal{A})$ of “hermitian” $3 \times 3$-matrices for a composition algebra $\mathcal{A}$ is a Jordan algebra; taking for $\mathcal{A}, \mathcal{B}$ the various composition algebras with $\mathcal{T} = \mathcal{T}(\mathcal{A}, \mathbf{H}_3(\mathcal{B}))$ leads to the *magic square*:
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- Dynkin diagrams of equally laced groups arise as resolution graphs of rational singularities and universal deformations as singular orbits in the adjoint representations.
- The exceptional groups give rise to a series of *generalized projective planes*. The book “Locally mixed symmetric spaces”, just appeared at Springer, details this as an example of *symmetric spaces*. 

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- The exceptional groups, in fact the entire magic square, arises in context of compactification of *supergravity in dimension 11*.
- The Weyl groups and root systems define arrangements which arise in various fascinating situations in algebraic geometry.
Main topic of this talk

Arrangements defined by Weyl groups

The Weyl group $W(G)$ can be defined as a reflection group, generated in $GL_n(\mathbb{R})$ ($n$ the rank of $G$) by the reflections on the roots. Considering the corresponding projective arrangement one dimension lower defines a hyperplane arrangement in $\mathbb{P}^n(\mathbb{R})$ (or in complex projective space). A particular case is considered in “The geometry of some special arithmetic quotients”. More details will be given in the upcoming monograph by the author.
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Weyl group actions and arrangements

The group $G_2$, being only of rank 2, defines only a set of points on $\mathbb{P}^1(\mathbb{C})$, but the others give rise to interesting arrangements.

- $F_4$ defines an arrangement of planes in $\mathbb{P}^3(\mathbb{C})$: the 24 planes form an arrangement which has two subsets of 12 roots, each corresponding to a subroot system of type $D_4$, called a triple of desmic tetrahedra, one arising from the long roots, one from the short ones.
Arrangements defined by Weyl groups

Weyl group actions and arrangements

- $E_6$ defines an arrangement of 36 hyperplanes ($\mathbb{P}^4$) in $\mathbb{P}^5(\mathbb{C})$. The group $W(E_6)$ is isomorphic to the group of incidence-preserving permutations of the 27 lines on a smooth cubic surface, and all the geometry has wonderful interpretations of geometric configurations. This geometry has been studied in detail in “The geometry of some special arithmetic quotients”.

- $E_7$ defines an arrangement of 63 hyperplanes ($\mathbb{P}^5$) in $\mathbb{P}^6(\mathbb{C})$. The group $W(E_7)$ is similarly related to the 28 bitangents of a quartic curve in $\mathbb{P}^2(\mathbb{C})$, and a similar correspondence to geometric properties is mirrored in the arrangement.

- $E_8$ defines an arrangement of 120 hyperplanes ($\mathbb{P}^6$) in $\mathbb{P}^7(\mathbb{C})$. This arrangement is very “universal”: all the above can be derived as subarrangements of this arrangement. The lattice in $\mathbb{R}^8$ defined by the roots can be identified with a very special order in $O$, that is a sublattice which is also a subring. Many other connections exist.
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The Weyl group $W(E_6)$ is isomorphic to the group of permutations of the 27 lines on a cubic surface. This isomorphism has a beautiful geometric expression in the properties of the arrangement defined.

Properties of the 27 lines on a smooth cubic surface

- There are 45 special hyperplane sections, each of which intersects the cubic surface in 3 of the 27 lines; this plane is tangent at the 3 intersection points of the lines, hence the name *tritangent*.
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- There are 36 *double sixes* which are sets of 12 of the lines, arranged in a $2 \times 6$ matrix, such that two lines meet if and only if they are different rows and columns.

- There are 120 *trihedral pairs*, sets of nine of the lines, cut out by 6 of the tritangents, which determine the equation of the cubic surface.

- There are 40 *triads of trihedral pairs*, such that the triple contains all 27 lines.
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Subloci of the arrangement defined by $W(E_6)$

As a Weyl group, $W(E_6)$ is generated by reflections on the roots. Since the rank is 6, the projective arrangement is defined in $\mathbb{P}^5(\mathbb{C})$.

### 36 hyperplanes

- Since $E_6$ has 36 positive roots, this defines 36 hyperplanes (each a $\mathbb{P}^4(\mathbb{C})$), one corresponding to each double-six.
- Corresponding to the 120 triples of azygetic double-sixes, there are 120 $\mathbb{P}^3(\mathbb{C})$’s which are the intersection of 3 of the 36.
- Corresponding to the 270 pairs of syzegetic pairs of double sixes, there are 270 $\mathbb{P}^3(\mathbb{C})$’s which are the intersection of two of the 36.
- Corresponding to the 120 trihedral pairs, there are 120 $\mathbb{P}^1(\mathbb{C})$’s (lines) which are the intersection of 6 of the 36.

Where are the 27 lines?
The 27 weights

The group $E_6$ has two complex-conjugate 27-dimensional representations, in the exceptional Jordan algebra (the highest weight is the fundamental weight corresponding to the first (resp. last) root $\alpha_1$ (resp. $\alpha_6$)). This defines 27 weights, which are accordingly an orbit under the Weyl group. It is these weights which correspond to the 27 lines. (Note: everything taking place in the universal cover of the maximal torus identified with $H^1(T, \mathbb{Z})$).
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The arrangement defined by the 27 hyperplanes

This defines 27 hyperplanes ($\mathbb{P}^4$). Some of the interesting loci:

- Corresponding to the 45 tritangent planes, there are 45 $\mathbb{P}^3(\mathbb{C})$’s which are the intersection of 3 of the 27.
- Corresponding to the 216 pairs of skew lines there are 216 $\mathbb{P}^3(\mathbb{C})$’s which are the intersection of two of the 27.
- Recalling that each of the 120 trihedral pairs defines a set of 9 lines, there are 120 lines which are the intersection of 9 of the 27.
- etc...
The invariants of the classical groups described above are polynomials invariant under permutations of the coordinates, or under permutations of the squares of the coordinates. Here it is a more challenging matter. The degrees of the invariants are: 2, 5, 6, 8, 9, 12. Of these, only the quintic is unique. As a general definition, for invariants of even degree one can take the sum of the powers of the roots. For odd degrees, this sum is always 0 because with any root also its negative is a root.

The $k^{th}$ powers of the 27 fundamental weights

Using the 27 weights (with equations $a_i = 0, b_i = 0, c_{ij} = 0$), one gets for any degree $k$ an invariant of the corresponding degree.

$$I_k := \sum_{i,j} \{ a_i^k + b_i^k + c_{ij}^k \}.$$
The invariant quintic

A whole chapter in “The geometry of some special arithmetic quotients” is devoted to the quintic \( I_5 = 0 \), a 4-dimensional variety in \( \mathbb{P}^5(\mathbb{C}) \). Some of the beautiful properties are just listed here.

### Loci on the quintic

- The *singular locus* of the quintic consists of 120 lines which intersect in 36 points.
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**Loci on the quintic**

- The *singular locus* of the quintic consists of 120 lines which intersect in 36 points.
- The 45 $\mathbb{P}^3(\mathbb{C})$’s mentioned above are all contained in the quintic; in fact the intersection of $I_5$ with any of the 27 hyperplanes, a quintic, splits into 5 of these 45.
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The invariant quintic

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- Both the Nieto quintic and the invariant quintic 4-fold are compactifications of *ball quotients*. This was conjectured in the lecture notes mentioned above and has been verified in the mean time. In fact, also the 45 $\mathbb{P}^3(\mathbb{C})$’s above are ball quotients.
This year's discovery: the invariant nontic $I_9$

The only invariants of odd degree are $I_5$ and $I_9$, but while the quintic is unique, $I_9$ is only unique up to addition of terms $I_2^2 I_5$. Nevertheless it seemed natural to consider the invariant $I_9$, which had not been done back in the 1990's when $I_5$ was studied. An initial study already showed the following.

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  - The 120 lines are contained *simply*, leaving a curve of degree 1080 to explain.
  - The 120 lines are arranged in $\mathbb{P}^5(\mathbb{C})$ such that 4 of them lie in certain planes.
More geometry of $I_9$

- Restricted to these planes, 5 of the octics vanish and the remaining one splits off these 4 lines, leaving a quartic curve in each of the planes. These planes are the so-call $c$-planes, and the 4 lines intersect at 6 of the 36 singular points of intersection.
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- The intersection of $I_9$ with the symmetric hyperplanes $P^4(\mathbb{C})$ is a nontic threefold $\mathcal{H}_9$. It intersects the Nieto quintic (the intersection of $I_5$ with the same hyperplane) in the union of 15 + 15 planes, the $c$-planes mentioned above, and the so-called Segre planes, each containing 3 of the nodes and 4 of the 10 isolated double points of $\mathcal{N}_5$, the $c$-planes being double in the intersection.
Part of a bigger story?

The proof that $\mathcal{N}_5$ is (birational to) a ball quotient requires manipulations of Chern numbers on $\mathcal{N}_5$ and a branched cover $W \to \mathcal{N}_5$ of $\mathcal{N}_5$. The branch locus of that cover is contained in the union of the $c$-planes and the Segre planes, i.e., in the intersection $\mathcal{N}_5 \cap \mathcal{K}_9$. This means that the computation done already for $\mathcal{N}_5$ applies immediately to $\mathcal{K}_9$! Apart from the singular quartics of $\mathcal{K}_9$ in each of the $c$-planes, ...
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A conjecture on $\mathcal{H}_9$

There is a cover $Y \to \mathbb{P}^4(\mathbb{C})$ which restricted to $N_5$ is the cover mentioned above. Then the cover $Z \to \mathcal{H}_9$, branched at the union of the $c$-planes and the Segre planes, is (birational to) a compactification of a ball quotient.
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A conjecture on $K_9$

There is a cover $Y \to \mathbb{P}^4(\mathbb{C})$ which restricted to $N_5$ is the cover mentioned above. Then the cover $Z \to K_9$, branched at the union of the $c$-planes and the Segre planes, is (birational to) a compactification of a ball quotient.

A more daring conjecture

Let $\mathcal{V} \subset \mathbb{P}^4(\mathbb{C})$ be a symmetric variety which contains the Segre planes and/or the $c$-planes. Then the restriction of $Y$ to $\mathcal{V}$ is (birational to) a ball quotient. $Y$ is itself a 4-dimensional ball quotient.
A conjecture on the nontic $I_9$ ...

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The big conjecture and beyond

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- The 36 exceptional divisors of the 36 nodes, each a copy of the Segre cubic.
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The compactification locus is the image of the 120 singular lines in the birational model.

… and beyond

Assuming the above conjectures are correct, we would have the following situation: in $\mathbb{P}^5(\mathbb{C})$ there are two hypersurfaces $I_5$ and $I_9$ which are (birational to) compactifications of ball quotients; these meet in 45 linear spaces $\mathbb{P}^3(\mathbb{C})$ which are 3-dimensional subball quotients.
Bigger and bigger...

On the ambient space $\mathbb{P}(L(T_{E_6}))$

It is natural then to wonder whether $\mathbb{P}^5(\mathbb{C})$ is a ball quotient such that $I_5$ and $I_9$ are codimension 1 subballs?
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Move up to $\mathbb{P}^6(\mathbb{C})$, which is the ambient space for the $W(E_7)$-arrangement. Each of the 28 hyperplanes arising from the 28 weights contains exactly the configuration just considered. Is $\mathbb{P}^6(\mathbb{C})$ a ball quotient with 28 subball quotients $\mathbb{P}^5(\mathbb{C})$ as above?

The biggest challenge to proving the above

Currently our understanding of branched covers and birational geometry in dimension $> 3$ is not quite complete enough to describe explicitly what happens and what the birational models should precisely look like. Some progress here will be necessary.
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