The Pigeonhole Principle

"If you shove 8 pigeons into 7 holes, then there is a hole with at least 2 pigeons."

Warm-up

1. Ten people are swimming in the lake. Prove that at least two of them were born on the same day of the week.

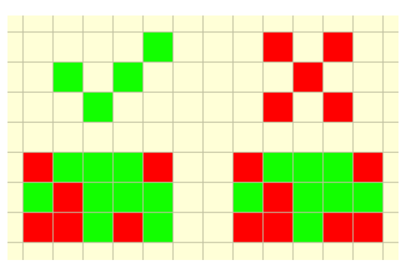
2. Seventeen children are in an elevator. Prove that at least three of them were born on the same day of the week.

3. Briar the cat likes to wear socks on all four of its feet. Briar's sock drawer is filled with yellow, cyan, and pink socks. Every morning Briar pulls socks out of the drawer one at a time until four matching socks are found. What is the largest number of socks Briar may pull from the drawer before finding a complete set?

4. Sarah writes down random positive integers when she gets bored. Prove that if Sarah writes 1001 numbers, then there must be at least 2 with the same last three digits.

Workout

1. Simone is coloring in the squares on a (really really big) sheet of graph paper with red and green pencils. Her goal is to color all the squares on the page so that there is no rectangle all of whose corners are the same color (Simone calls such rectangles *unichrome* and she hates them.) For example, the picture below shows a successful start on the left and a failure on the right. This is a failure since the 4 boxes in the corner of this rectangle are all red.



(a) Prove that it is impossible for Simone to successfully color the entire sheet of graph paper without any unichrome rectangles.

(b) What is the largest $3 \times n$ box Simone can color without making a unichrome rectangle?

(c) Prove that using 3 colors instead of 2 will not help Simone avoid the dreaded unichrome rectangles.

2. Devon picks 7 numbers from $\{1, 2, 3, \ldots, 10, 11\}$. Prove he has a pair that add up to 12.

3. Tammy notices that whenever she selects 7 whole numbers, there is always a triple a, b, c of numbers in her collection that all differ from each other by a multiple of 3. Tammy conjectures this will always be the case. Prove Tammy's conjecture.

4. The Queen has a garden in the shape of an equilateral triangle with each side measuring 2 kilometers. The 5 royal children like hide in the garden as far away from each other as possible. Prove that no matter how hard they try to get away from each other, there are still always two royal siblings within 1 kilometer of each other.

Challenge

1. Given any two people, they have either high fived each other at least once in their lives or they have not. Prove that if there are 6 people in a library, then there are 3 of them who have either all high fived one another or who have never high fived one another.

2. A *closed cap* on a sphere is a hemisphere including the circle on its boundary. Prove that for any 5 points on a sphere, there is some closed cap containing at least 4 of them.

3. Suppose that 101 positive integers are arranged in a circle. The sum of all the numbers is 300. Prove that you can always choose a consecutive sequence of numbers which sum to 200.

4. If α is a real number and $n \geq 1$ is a whole number, show there is a rational number p/q so that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}.$$

Find solutions at
`http://www-personal.umich.edu/~tghyde/PHP_sols.pdf`