Polynomial Factorization Statistics and Point Configurations in $\mathbb{R}^3$

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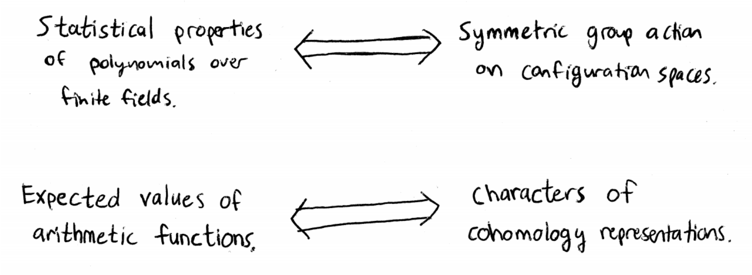
Overview



Ex. The following facts are equivalent.

- Average number of $\mathbb{F}_q$-roots of a degree d polynomial in $\mathbb{F}_q[x]$ is

$$1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{d-1}}$$

- Let ψ_d^k be the S_d-character of $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$, then

$$\langle 1_d,\ \psi_d^0 \rangle = 1 \qquad \langle \mathrm{Std}_d,\ \psi_d^k \rangle = 1 \quad 1 \leq k \leq d-1,$$

Polynomials

- $\mathrm{Poly}_d(\mathbb{F}_q) := \{\text{monic deg } d \text{ polynomials } f(x) \in \mathbb{F}_q[x]\}$
- The **factorization type** of $f(x) \in \mathrm{Poly}_d(\mathbb{F}_q)$ is the partition $\lambda_f \vdash d$ given by the degrees of the irred. factors of $f(x)$.
- **Ex.** The factorization type of

$$f(x) = x(x+1)^2(x^2+1) \in \mathrm{Poly}_5(\mathbb{F}_3)$$

is $\lambda_f = (1^3\, 2^1)$.

- **Note:** factorization type is insensitive to factor multiplicity,

$$g(x) = x^2 \qquad\qquad h(x) = x(x+1)$$

both have factorization type (1^2).

Factorization Statistics

A **factorization statistic** $P : \mathrm{Poly}_d(\mathbb{F}_q) \to \mathbb{Q}$ is a function where $P(f)$ depends only on the factorization type λ_f.

Examples:

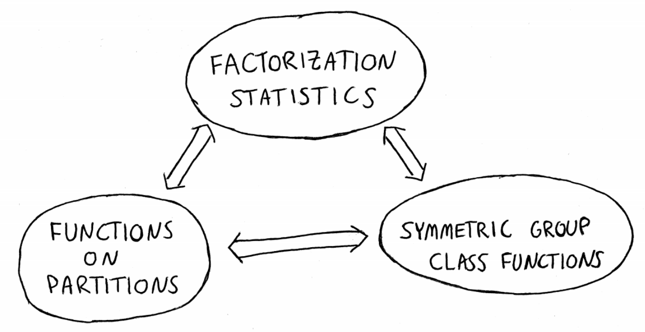
- $R(f) = \#\, \mathbb{F}_q$-roots of f with multiplicity
- $L(f) = (-1)^{\#\text{ irred factors of } f}$ (Liouville function)
- $Q(f) = \#$ red. quad. factors $-$ # irred. quad. factors

If $f(x) = x(x+1)^2(x^2+1) \in \mathrm{Poly}_5(\mathbb{F}_3)$, then

- $R(f) = 3$
- $L(f) = (-1)^4 = 1$
- $Q(f) = 3 - 1 = 2$

Factorization Statistics

A **factorization statistic** $P : \mathrm{Poly}_d(\mathbb{F}_q) \to \mathbb{Q}$ is a function where $P(f)$ depends only on the factorization type λ_f.



Expected Values

If P is a factorization statistic, let $E_d(P)$ denote the expected value of P on $\mathrm{Poly}_d(\mathbb{F}_q)$

$$E_d(P) = \frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d(\mathbb{F}_q)} P(f).$$

Ex. (Quadratic excess)
$Q(f) = \#$ red. quad. factors $-$ # irred. quad. factors

d	$E_d(Q)$
3	$\frac{2}{q} + \frac{1}{q^2}$
4	$\frac{2}{q} + \frac{2}{q^2} + \frac{2}{q^3}$
5	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{2}{q^4}$
6	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{3}{q^5}$
10	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{6}{q^5} + \frac{6}{q^6} + \frac{8}{q^7} + \frac{8}{q^8} + \frac{5}{q^9}$

Expected Values

$Q(f) = \#$ red. quad. factors $-$ $\#$ irred. quad. factors

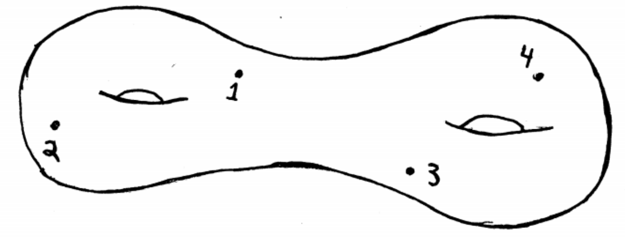
d	$E_d(Q)$	$E_d(Q)_{q=1}$
3	$\frac{2}{q} + \frac{1}{q^2}$	3
4	$\frac{2}{q} + \frac{2}{q^2} + \frac{2}{q^3}$	6
5	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{2}{q^4}$	10
6	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{3}{q^5}$	15
10	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{6}{q^5} + \frac{6}{q^6} + \frac{8}{q^7} + \frac{8}{q^8} + \frac{5}{q^9}$	45

- degree $d - 1$
- positive integer coefficients
- coefficients sum to $\binom{d}{2}$
- coefficientwise convergence as $d \to \infty$

Configuration Space

Let X be a topological space

- $\mathrm{PConf}_d(X) := \{(x_1, x_2, \ldots, x_d) \in X^d : x_i \neq x_j\}$
- Symmetric group S_d acts freely on $\mathrm{PConf}_d(X)$ by permuting coordinates.
- $H^k(\mathrm{PConf}_d(X), \mathbb{Q})$ is an S_d-representation for each k.



Twisted Grothendieck-Lefschetz

- Let $\mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q) \subseteq \mathrm{Poly}_d(\mathbb{F}_q)$ denote the subset of *squarefree polynomials*.
- Let ϕ_d^k be the S_d-character of $H^k(\mathrm{PConf}_d(\mathbb{C}), \mathbb{Q})$.

Theorem (Church, Ellenberg, Farb, 2014)

If P is a factorization statistic, then

$$\frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} (-1)^k \frac{\langle P, \phi_d^k \rangle}{q^k}.$$

Geometric Idea

Theorem (Church, Ellenberg, Farb, 2014)

If P is a factorization statistic, then

$$\frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} (-1)^k \frac{\langle P, \phi_d^k \rangle}{q^k}.$$

- $\mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{C}) \cong \mathrm{Conf}_d(\mathbb{C}) := \mathrm{PConf}_d(\mathbb{C})/S_d$

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_d) \iff \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$$

- Grothendieck-Lefschetz trace formula with "twisted coefficients".

Main Result

Let ψ_d^k be the S_d-character of $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$.

Theorem (H. 2017)

If P is a factorization statistic, then

$$E_d(P) := \frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} \frac{\langle P, \psi_d^k \rangle}{q^k}.$$

- Geometric idea = ??? (in progress)
- Proof uses cycle index series for cohomology reps derived from Orlik-Solomon algebras.
  (Lehrer-Solomon, Hanlon, Sundaram-Welker.)
- Same technique proves the Church, Ellenberg, Farb result.

Quadratic Excess Revisited

$Q(f) = \#$ red. quad. factors $-$ $\#$ irred. quad. factors

$$E_d(Q) = \sum_{k=0}^{d-1} \frac{\langle Q, \psi_d^k \rangle}{q^k}.$$

d	$E_d(Q)$
3	$\frac{2}{q} + \frac{1}{q^2}$
4	$\frac{2}{q} + \frac{2}{q^2} + \frac{2}{q^3}$
5	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{2}{q^4}$
6	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{3}{q^5}$
10	$\frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{6}{q^5} + \frac{6}{q^6} + \frac{8}{q^7} + \frac{8}{q^8} + \frac{5}{q^9}$

Quadratic Excess is a Character

$Q(f) = \#$ red. quad. factors $-$ $\#$ irred. quad. factors

- $\mathbb{Q}[d]$ is the permutation rep. of S_d with basis $e_1, e_2, \ldots, e_d$.
- $\bigwedge^2 \mathbb{Q}[d]$ is an S_d-rep. with basis $e_i \wedge e_j$ such that $i < j$.
- If $\sigma \in S_d$, then

$$\begin{aligned} \mathrm{Tr}_{\bigwedge^2 \mathbb{Q}[d]}(\sigma) &= \#\{i < j : \sigma \text{ fixes } i, j\} - \#\{i < j : \sigma \text{ transposes } i, j\} \\ &= \binom{x_1(\sigma)}{2} - \binom{x_2(\sigma)}{1} \\ &= Q(\sigma) \end{aligned}$$

- Therefore Q, viewed as an S_d-class function, is a character!

Coefficientwise Convergence

Let x_j for $j \geq 1$ be the class function

$$\begin{aligned} x_j(\sigma) &= \#\, j\text{-cycles of } \sigma, \\ x_j(f) &= \#\text{ deg. } j \text{ irreducible factors of } f. \end{aligned}$$

$P \in \mathbb{Q}[x_1, x_2, \ldots]$ are called **character polynomials**.

Theorem (H. 2017)

If P is a character polynomial, then the expected values $E_d(P)$ of P on $\mathrm{Poly}_d(\mathbb{F}_q)$ converge coefficientwise as $d \to \infty$ to

$$\lim_{d\to\infty} E_d(P) = \sum_{k=0}^{\infty} \frac{\langle P, \psi^k \rangle}{q^k},$$

where $\langle P, \psi^k \rangle := \lim_{d\to\infty} \langle P, \psi_d^k \rangle$.

Key fact: $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ exhibits **representation stability**.

Quadratic Excess is a Character Polynomial

$$Q = \binom{x_1}{2} - \binom{x_2}{1} \implies Q \text{ is a char. poly.}$$

Therefore $E_d(Q)$ converge coefficientwise as $d \to \infty$

d	$E_d(Q)$
3	$\frac{2}{q} + \frac{1}{q^2}$
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$$\lim_{d\to\infty} E_d(Q) = \frac{2}{q} + \frac{2}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + \frac{6}{q^5} + \frac{6}{q^6} + \frac{8}{q^7} + \frac{8}{q^8} + \ldots$$

Regular Representation

Theorem

The total cohomology of $\mathrm{PConf}_d(\mathbb{R}^3)$ is isomorphic to the regular representation.

$$\bigoplus_{k=0}^{d-1} H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}) \cong \mathbb{Q}[S_d].$$

In other words,

$$\sum_{k=0}^{d-1} \psi_d^k = \rho.$$

where ρ is the character of the regular representation $\mathbb{Q}[S_d]$.

Regular Representation

Theorem

The total cohomology of $\mathrm{PConf}_d(\mathbb{R}^3)$ *is isomorphic to the regular representation.*

$$\bigoplus_{k=0}^{d-1} H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}) \cong \mathbb{Q}[S_d].$$

Coefficient Sum

Sum of $E_d(P)$ coeffs. is the same as the evaluation $E_d(P)_{q=1}$.

Corollary (H. 2017)

If P is a factorization statistic, then

- $E_d(P)_{q=1} = P(1^d)$.
- *If P is the character of an S_d-rep. V, then*

$$E_d(P)_{q=1} = \dim V$$

Ex. $\dim \bigwedge^2 \mathbb{Q}[d] = \binom{d}{2}$, hence

$$E_d(Q)_{q=1} = \binom{d}{2}.$$

Coefficient Sum

Corollary (H. 2017)

If P is a factorization statistic, then

- $E_d(P)_{q=1} = P(1^d)$.
- *If P is the character of an S_d-rep. V, then*

$$E_d(P)_{q=1} = \dim V$$

Proof.

$$E_d(P)_{q=1} = \sum_{k=0}^{d-1} \langle P, \psi_d^k \rangle = \langle P, \sum_{k=0}^{d-1} \psi_d^k \rangle = \langle P, \mathbb{Q}[S_d] \rangle = P(1^d).$$

□

Global Constraint

$$\bigoplus_{k=0}^{d-1} H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}) \cong \mathbb{Q}[S_d].$$

Distribution of the irred. components of $\mathbb{Q}[S_d]$ among $H^*(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ determines the expected values of polynomial factorization statistics!

$$E_d(P) = \sum_{k=0}^{d-1} \frac{\langle P, \psi_d^k \rangle}{q^k}.$$

Trivial Representation

- The trivial rep. $\mathbf{1}$ of S_d is one dimensional, hence there is exactly one k for which $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ has a trivial component.
- Trivial character is the constant $= 1$ factorization statistic.
- Therefore
$$1 = E_d(1) = \sum_{k=0}^{d-1} \frac{\langle 1, \psi_d^k \rangle}{q^k},$$
hence $H^0(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ has the trivial component.
- $\mathrm{PConf}_d(\mathbb{R}^3)$ is path connected, so $H^0(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}) \cong \mathbf{1}$.

Trivial Representation

$$H^0(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q}) \cong \mathbf{1} \Longrightarrow \psi_d^0 = 1$$

The large q limit of expected values is completely determined by the trivial multiplicity in a factorization statistic.

Corollary (H. 2017)

Suppose P is a factorization statistic, then

$$\lim_{q \to \infty} E_d(P) = \langle P, 1 \rangle.$$

Pf:

$$E_d(P) = \sum_{k=0}^{d-1} \frac{\langle P, \psi_d^k \rangle}{q^k} \Longrightarrow \lim_{q \to \infty} E_d(P) = \langle P, \psi_d^0 \rangle = \langle P, 1 \rangle.$$

Sign Representation

Where is the sign representation **Sgn**?

Theorem

For all $d \geq 1$*,*

$$\langle \mathrm{sgn}, \psi_d^k \rangle = \begin{cases} 1 & k = \lfloor d/2 \rfloor \\ 0 & \textit{otherwise.} \end{cases}$$

That is, $H^{2\lfloor d/2 \rfloor}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ *is the only cohomological degree with a sign component.*

Also computed by Carlitz (1932), Hanlon (1990), Lehrer (1999).

Even Type

- Let $\mathcal{E}$ be the factorization statistic

$$\mathcal{E}(f) = \begin{cases} 1 & \lambda_f \text{ is even } (f \text{ has } \textbf{even type}) \\ 0 & \text{otherwise.} \end{cases}$$

- $E_d(\mathcal{E})$ is the prob. of a random $f \in \mathrm{Poly}_d(\mathbb{F}_q)$ having an even type.
- First guess $E_d(\mathcal{E}) \approx \frac{1}{2}$.
-
$$\mathcal{E} = \frac{1}{2}(1 + \mathrm{sgn}) \Longrightarrow E_d(\mathcal{E}) = \frac{1}{2} + \frac{1}{2q^{\lfloor d/2 \rfloor}}.$$

- Thus there is a slight bias toward an even type!

Expected Roots

- Let R be the "number of roots" factorization statistic.
- R is the character of the permutation rep. $\mathbb{Q}[d]$.

$$R = 1 + \chi_{\mathrm{Std}}$$

- Expect one $\mathbb{F}_q$-root for random deg. d poly. when q is large.

Expected Roots

$$E_d(R) = 1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{d-1}} \qquad \text{(generating functions)}$$

$$= \sum_{k=0}^{d-1} \frac{\langle R, \psi_d^k \rangle}{q^k}$$

$$= \langle 1, \psi_d^0 \rangle + \frac{\langle \chi_{\mathrm{Std}}, \psi_d^1 \rangle}{q} + \frac{\langle \chi_{\mathrm{Std}}, \psi_d^2 \rangle}{q^2} + \ldots + \frac{\langle \chi_{\mathrm{Std}}, \psi_d^{d-1} \rangle}{q^{d-1}}$$

$$\langle 1,\ \psi_d^0 \rangle = 1 \qquad \langle \chi_{\mathrm{Std}},\ \psi_d^k \rangle = 1 \quad 1 \leq k \leq d-1,$$

Thank you!