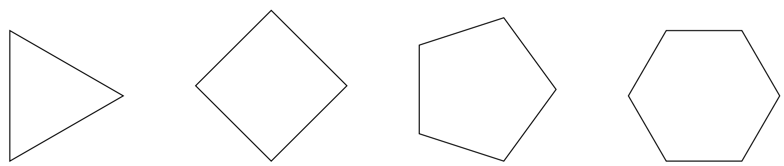
REGULAR POLYTOPES REALIZED OVER $\mathbb{Q}$

TREVOR HYDE

A *regular polytope* is a d-dimensional generalization of a regular polygon and a Platonic solid. Roughly, they are convex geometric objects with maximal rotational symmetry. To avoid stating a precise definition we appeal to a known complete classification of all regular polytopes. The codimension 1 parts of a polytope are called *facets*. For example, the facets of a cube are squares.

- **Dimension 2:** In two dimensions, the regular polytopes are the familiar *regular polygons*. There is a regular n-gon for every $n \geq 3$.



- **Dimension 3:** There are five regular polytopes in three dimensions, these are the *Platonic solids*:
  - Tetrahedron, or 3-simplex
  - Cube, or 3-cube
  - Octahedron, or 3-orthoplex
  - Dodecahedron
  - Icosahedron

  Each regular polytope P has a *dual*—also a regular polytope—formed by placing vertices at the center of each facet of P and then taking the convex hull of these vertices. Taking the dual of the resulting polytope gives us a copy of the original polytope, hence the terminology. Each regular n-gon is its own dual. The tetrahedron is its own dual. The cube and octahedron are dual, as are the dodecahedron and icosahedron.

- **Dimension 4:** We have six regular polytopes in four dimensions described below by their three dimensional facets:
  - 4-simplex (3-simplex facets)
  - 4-cube (3-cube facets)
  - 4-orthoplex (3-simplex facets)
  - 24-cell (octahedral facets)
  - 120-cell (dodecahedral facets)
  - 600-cell (3-simplex facets)

  The 4-simplex and the 24-cell are both self-dual. The 4-cube and 4-orthoplex are dual, as are the 120-cell and 600-cell. Four dimensions is, sadly, the end of interesting new regular polytopes.

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- **Dimension $d \geq 5$:** Only three families of regular polytope persist beyond dimension four.
  - d-simplex $\big((d-1)$-simplex facets$\big)$
  - d-cube $\big((d-1)$-cube facets$\big)$
  - d-orthoplex $\big((d-1)$-simplex facets$\big)$

  The d-simplex is self-dual in all dimensions. The d-cube is dual to the d-orthoplex. Note that the 3-orthoplex is the octahedron, so that one may think of the d-orthoplex as a d-dimensional analog of the octahedron.

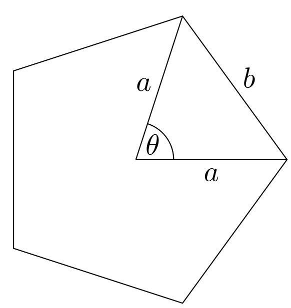
When we encounter a regular polytope P in the real (mathematical) world, it is given by some collection of points in a d-dimensional Euclidean space. The coordinates of these points belong to some field k, and we call such a collection of vertices a *realization of P over k*. Every d-dimensional regular polytope can be realized in d-dimensions over $\mathbb{R}$. Generic real numbers are, however, annoying to write down. The easiest numbers to write down are rationals. This motivates our main question:

Main Question: Which regular polytopes can be realized in d-dimensions over $\mathbb{Q}$?

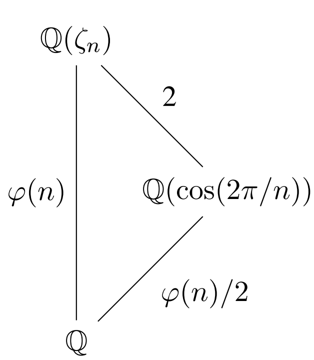
In this note we'll completely answer this question for all regular polytopes in every dimension. From now, when we talk about *realizing* a polytope, we mean *realizing over* $\mathbb{Q}$ unless otherwise specified.

1. Regular Polygons

Suppose we have a regular n-gon realized in d-dimensional Euclidean space for some $d \geq 2$. The center of the polygon is the average of the vertices. As the vertices have rational coordinates, so will the center. Consider the triangle formed by connecting two adjacent vertices to the center. The lengths of the edges of this triangle need not be rational $\big($consider the hypotenuse of the triangle with vertices $(0,0), (1,0), (1,1),\big)$ but their *squares* must be. Suppose the edge lengths are a and b as in the figure below, with internal angle θ.



The Law of Cosines gives us $b^2 = 2a^2\big(1 - \cos(\theta)\big)$. Since $a^2, b^2 \in \mathbb{Q}^\times$, it follows that $\cos(\theta) \in \mathbb{Q}$. This is a serious obstruction for the realization of regular polygons—notice that it is independent of the dimension of the ambient space. In a regular n-gon, we note that $\theta = 2\pi/n$, so that a necessary condition for a regular n-gon to be realized in any dimension $d \geq 2$ is whether $\cos(2\pi/n) \in \mathbb{Q}$. We answer this question using the theory of cyclotomic fields.



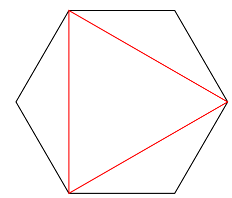
If ζ_n is a primitive nth root of unity, then $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$, where φ is Euler's totient function. The real subfield of $\mathbb{Q}(\zeta_n)$ is precisely $\mathbb{Q}(\cos(2\pi/n))$ from which it follows that $[\mathbb{Q}(\cos(2\pi/n)) : \mathbb{Q}] = \varphi(n)/2$. In other words, $\cos(2\pi/n) \in \mathbb{Q}$ if and only if $\varphi(n) = 2$. One checks that the only solutions to this equation are $n = 3, 4, 6$. Therefore, if $n \neq 3, 4, 6$, then we cannot realize a regular n-gon in any d-dimensional space.

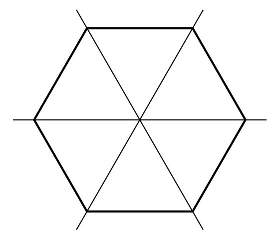
What about $n = 3, 4, 6$? The case $n = 4$ is the easiest: the four points $\{(0,0), (1,0), (0,1), (1,1)\}$ realize a square in the plane.

Note that if a regular polytope can be realized over $\mathbb{Q}$ in dimension d, then it may be realized in all higher dimensions. The regular triangle ($n = 3$) and the regular hexagon ($n = 6$) have a special relationship which manifests in this context as *corealizability*: If we can realize the regular hexagon, then any alternating collection of vertices gives a realization of the regular triangle. On the other hand, if we can realize the regular triangle, then by reflecting vertices across opposite edges we realize a regular tiling of the entire plane in which the triangle resides. Given a point in this tiling, the six nearest points provide a realization of the regular hexagon.

In three dimensions we have the simple realization

$$\{(1,0,0),(0,1,0),(0,0,1)\}$$





of the regular triangle. Hence, $n = 3, 6$ may be realized in all dimensions $d \geq 3$. What about in the plane ($d = 2$)? The area of a regular triangle in the plane may be computed from the vertices using determinants, for example, hence must be rational if the vertices are. However, if the side length of the triangle is a one readily computes the area to be $\frac{\sqrt{3}}{4}a^2$ which is not in $\mathbb{Q}$. Here we see another arithmetic obstruction to realizability from which we deduce the much stronger conclusion: *The regular triangle and hexagon are realizable in the plane over a field k iff k contains a square root of three.*

Thus, we have answered our question completely for all regular polygons. The Law of Cosines prevents all but three regular polygons from being realized in any dimension over $\mathbb{Q}$ and $\sqrt{3}$ prevents the regular triangle and hexagon from being realized in the plane over $\mathbb{Q}$.

2. Higher Dimensional Polytopes

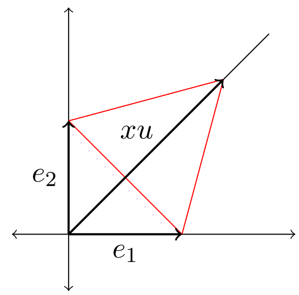
Using the non-realizability of the regular pentagon in any dimension we conclude that the dodecahedron and the 120-cell, hence their duals the icosahedron and 600-cell, are not realizable in any dimension. All the other regular polytopes have two-dimensional facets which are either regular triangles or squares. In particular three families remain: the d-simplex, the d-cube, and the d-orthoplex; and one exceptional polytope in four dimensions: the 24-cell.

Cubes and Orthoplexes: The d-cube is easily realized in d-dimensions (hence all higher) using all sums of subsets of the standard basis vectors. Duality implies the d-orthoplex may also be realized in d-dimensions. However, there is a nicer realization of the d-orthoplex given by $\{\pm e_i : 1 \leq i \leq d\}$. One advantage the latter realization has over the dual construction is that it is integral (even defined over $\mathbb{F}_{1^2}$) while computing the dual vertices of the cube involves either multiplying or dividing by 4. This detail could be important to a similar investigation of realizability over fields of positive characteristic.

Simplices: This is the most interesting case! The d-simplex is d-dimensional but not always realizable in d-dimensions as witnessed above by the 2-simplex in two dimensions. However, there is a simple and well-known realization of the d-simplex in $(d+1)$-dimensions defined over $\mathbb{Q}$ (even better, over $\mathbb{F}_1$) given by $\{e_i : 1 \leq i \leq d+1\}$ the standard basis vectors. Therefore we are reduced to the question of realizability of the d-simplex in d-dimensions. The facets of a d-simplex are $(d-1)$-simplices, which by the above observation can be realized via the standard basis vectors in d-space. Furthermore, any realization is equivalent via a change in coordinates to one where all but one vertex is a standard basis vector.[1] Let $u = \sum_i e_i$ be the vector of all 1's. Then the final vertex is on the line spanned by u, say it is xu for some scalar x.

The defining property of x is that this vertex is distance $\sqrt{2}$ from all other vertices. This gives us a quadratic relation on x:

$$2 = \langle xu - e_i, xu - e_i \rangle^2$$
$$2 = (x-1)^2 + \sum_{i=1}^{d-1} x^2$$
$$0 = dx^2 - 2x - 1.$$



Thus, $x = \frac{1 \pm \sqrt{d+1}}{d}$. We conclude that in dimension d, the d-simplex is first realized over $\mathbb{Q}(\sqrt{d+1})$, hence over $\mathbb{Q}$ precisely when $d = m^2 - 1$ for some m, and in that case $x = \frac{1}{m-1}$. Below is a short list of the first few such d.

$$d = 3, 8, 15, 24, 35, 48, 63, \ldots$$

We see that the tetrahedron is the first example, with realization

$$\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}.$$

A nice coincidence, these are non-adjacent vertices of the standard cube.

24-Cell: The one remaining case is the 24-cell in four dimensions. *Wikipedia* maintains pages on all the regular polytopes in small dimensions, and a quick check gives us the realization with the following 24 vertices of the 24-cell [2]:

$$\{\pm e_i + \pm e_j : 1 \leq i < j \leq 4\},$$

answering our question in the affirmative for all dimensions $d \geq 4$.

We have thus completely classified which regular polytopes are realizable over $\mathbb{Q}$ in every dimension!

3. Further Questions

For those regular polytopes not realizable over $\mathbb{Q}$ one natural question is **over what field *is* each polytope realizable? Is there a smallest *field of definition*?** For the pentagon, dodecahedron, icosahedron, 120-cell, and 600-cell the answer appears to be $\mathbb{Q}(\sqrt{5})$ (over which all are defined in their native dimension) and although I believe this to be **the** field of definition, I have yet to verify this.

One may pose the same questions of realizability over a field of positive characteristic. Does the same classification of regular polytopes hold in positive characteristic? Realizability becomes more subtle, since computations of barycenters may not be defined (hence the corealizability of a polytope and its dual could potentially fail). Grothendieck makes the following (somewhat cryptic) remark in his *Sketch of a Programme* [1, Pg. 19] which first motivated me to think about realizability, though I still do not fully understand his insight.

> Thus, examining the Pythagorean polyhedra one after the other, I saw the same small miracle was repeated each time, which I called the combinatorial paradigm of the polyhedra under consideration. Roughly speaking, it can be described by saying that when we consider the specialization of the polyhedra in the, or

[1] Changing coordinates can be subtle for some questions of realizability, but in this case there is no issue as you should convince yourself.

> one of the, most singular characteristic(s) (namely characteristics 2 and 5 for the icosahedron, characteristic 2 for the octahedron), we read off from the geometric regular polyhedron over the finite field ($\mathbb{F}_4$ and $\mathbb{F}_5$ for the icosahedron, $\mathbb{F}_2$ for the octahedron) a particularly elegant (and unexpected) description of the combinatorics of the polyhedron. It seems to me that I perceived there a principle of great generality, which I believed I found again for example in a later reflection on the combinatorics of the system of 27 lines on a cubic surface, and its relations with the root system E_7. Whether it happens that such a principle really exists, and even that we succeed in uncovering it from its cloak of fog, or that it recedes as we pursue it and ends up vanishing like a Fata Morgana, I find in it for my part a force of motivation, a rare fascination, perhaps similar to that of dreams. No doubt that following such an unformulated call, the unformulated seeking form, from an elusive glimpse which seems to take pleasure in sumultaneously hiding and revealing itself—can only lead far, although no one could predict where…

There are other families of polytopes of interest besides the regular ones. I have not begun to consider their realizability over $\mathbb{Q}$, although I would start with their two dimensional facets for quick obstructions.

Finally, note that a realization of a regular polytope over a field k gives a representation of its rotational symmetry group defined over k—a finite subgroup of the orthogonal group. These groups may be studied with the general tools of arithmetic geometry. It would be nice to see if either that theory can completely answer realizability questions over any field for any kind of polytope, or if the hands-on analysis as done here gives complete classifications of finite subgroups of $\mathrm{O}_d(k)$.

References

[1] Alexander Grothendieck. Sketch of a Programme. http://www.landsburg.com/grothendieck/EsquisseEng.pdf, 1997.

[2] Wikipedia. 24-cell — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=24-cell&oldid=704849122, 2016.

DEPT. OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043,
E-mail address: tghyde@umich.edu