Reduction Stability and Iterate Decomposition Stability

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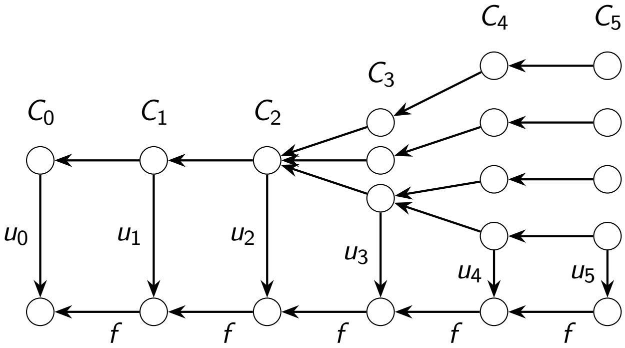
$f^n(x) = u(y)$

- Let $K = \overline{K}$, $\mathrm{char}(K) = 0$.
- Let f, u be non-constant rational functions defined over K with $\deg f \geq 2$.

$$C_n : f^n(x) = u(y)$$

- C_n arise in the study of the dynamical Mordell-Lang problem.
- Is C_n irreducible? What can we say about the components of C_n?
- For each n we have a finite map

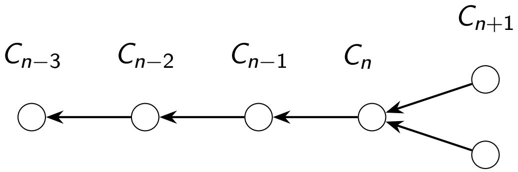
$$C_{n+1} \to C_n$$
$$(x, y) \mapsto (f(x), y)$$



- $u_n : C_n \to \mathbb{P}^1$ defined by $u_n(x, y) = x$.
- Total degree of the projection u_n is $\deg u$.
- Restricting u_n to irreducible components gives a partition of $\deg u$.
- Hence the branching must eventually stabilize.

Question

- How long does it take for the C_n to stabilize?
- Can we have a situation like this for large n?



Reduction Stability and Iterate Decomposition Stability

Theorem (H, Zieve) Let $K = \overline{K}$, $\mathrm{char}(K) = 0$. Suppose f, u are non-constant rational functions defined over K such that $\deg f \geq 2$.

- (RS) There exists a constant $b = b(\deg u)$ such that if $C_b : f^b(x) = u(y)$ is irreducible, then C_n is irreducible for all $n \geq 0$.
- (RS') There exists a constant $b' = b'(\deg u)$ such that for all $n \geq b'$, C_n has the same number of irreducible components as $C_{b'}$.
- (IDS) There exists a constant $b'' = b''(\deg u)$ such that if $f^n = u \circ v$ for some $n \geq 1$ and rational function v, then $f^{b''} = u \circ w$ for some rational function w.

RS' follows from RS by induction.

RS $\Rightarrow$ IDS

- $f^n = u \circ v$ iff $C_n : f^n(x) = u(y)$ has a genus 0 component of the form $y = v(x)$ iff C_n has a component D for which the x-coordinate projection $u_n : D \to \mathbb{P}^1$ has degree 1.
- RS' provides b' so that $C_{b'} : f^{b'}(x) = u(y)$ must have genus 0 component for which the x-coordinate projection has degree 1.

IDS $\Rightarrow$ RS

Theorem (Fried) Let g, h be non-constant rational functions defined over a field K. If $g(x) = h(y)$ is reducible, then we have

$$g = g_0 \circ g_1$$
$$h = h_0 \circ h_1$$

such that g_0, h_0 have the same Galois closure and $g_0(x) = h_0(y)$ is reducible.

- Suppose $C_n : f^n(x) = u(y)$ were reducible. Let $u = u_0 \circ u_1$ and $f^n = f_0 \circ f_1$ be the decompositions given by Fried's theorem.
- u_0 and f_0 having same Galois closure implies $\deg f_0 \leq \deg u_0! \leq \deg u!$.
- IDS provides b'' so that $f^{b''} = f_0 \circ f_2$ for some f_2.
- Then $f_0(x) = u_0(y)$ reducible implies $C_{b''} : f^{b''}(x) = f_0(f_2(x)) = u_0(u_1(y)) = u(y)$ reducible.

RS Proof Outline

- Using Fried's theorem we reduce to the case where $C_b : f^b(x) = u(y)$ is irreducible of genus 0.
- Riemann-Hurwitz argument to show that if $b \geq \log\left((2 + 1/42)\deg u\right)/\log(2)$, then the x-projections $u_i : C_i \to \mathbb{P}^1$ have Galois closure of genus at most 1 for $i \leq b/2$ and $\#\{p : p \text{ is a critical value of } u_i \text{ for some } i \leq b/2\} \leq 4$.
- Rational functions $u(y)$ with Galois closure of genus at most 1 can be classified up to change of coordinates.

RS Proof Outline

- $u(y)$ is, after a change of coordinates, either $y^m, y^m + y^{-m}, \pm T_m(y)$, or one of finitely many functions with Galois group A_4, S_4, or A_5; or comes from an isogeny of elliptic curves (for example, Lattès maps.)
- In each case, knowing the ramification of u and assuming C_b is irreducible of genus 0, R-H limits the possible ramification of f over the critical values of u.
- If b is sufficiently large, the ramification of f is constrained enough that we can classify all possibilities.
- But then we conclude in each case that C_n is always irreducible.

Reduction Stability and Iterate Decomposition Stability

Theorem (H, Zieve) Let B, C be projective curves defined over an algebraically closed field K of characteristic 0. Suppose

$$u : C \to B$$
$$f : B \to B$$

are finite morphisms defined over K such that $\deg f \geq 2$.

- (RS) There exists a constant $b = b(\deg u)$ such that if the fiber product C_b of f^b and u is irreducible, then C_n is irreducible for all $n \geq 0$.
- (RS') There exists a constant $b' = b'(\deg u)$ such that for all $n \geq b'$, the fiber product C_n of f^n and u has the same number of irreducible components as $C_{b'}$.
- (IDS) There exists a constant $b'' = b''(\deg u)$ such that if $f^n = u \circ v$ for some $n \geq 1$ and $v : B \to C$, then $f^{b''} = u \circ w$ for some $w : B \to C$.

Thank you!

These slides may be found on my website:
www-personal.umich.edu/~tghyde/