Factorization Statistics, Representation Stability, and the Growing Gaps Principle

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Factorization Statistics

▷ $\mathrm{Poly}_d(\mathbb{F}_q)$ = the set of monic degree d polynomials in $\mathbb{F}_q[x]$.

▷ The **factorization type** of $f(x) \in \mathrm{Poly}_d(\mathbb{F}_q)$ is the partition of d given by the degrees of the irreducible factors of $f(x)$.

Ex.

$$x^2(x+1)(x^2+1)^3 \in \mathrm{Poly}_9(\mathbb{F}_3)$$

has factorization type $\lambda = (1^3\, 2^3)$.

▷ A **factorization statistic** is a function $P : \mathrm{Poly}_d(\mathbb{F}_q) \to \mathbb{Q}$ such that $P(f)$ depends only on the factorization type of $f(x)$.

Ex. R = total number of $\mathbb{F}_q$-roots with multiplicity.

Ex. F = total number of irreducible factors.

Squarefree Polynomial Statistics

- Let ϕ_d^k be the S_d-character of $H^k(\mathrm{PConf}_d(\mathbb{R}^2), \mathbb{Q})$ and let $\langle \cdot, \cdot \rangle$ be the standard inner product of S_d-class functions.

- $\mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q)$ is the set of **squarefree** polynomials.

Theorem (Church, Ellenberg, Farb 2014)

Let P be a factorization statistic, then

$$\frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} (-1)^k \frac{\langle P, \phi_d^k \rangle}{q^k}.$$

Idea: $\mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{C}) \cong \mathrm{PConf}_d(\mathbb{C})/S_d \cong \mathrm{PConf}_d(\mathbb{R}^2)/S_d$, apply Grothendieck-Lefschetz trace formula with twisted coefficients.

Unrestricted Polynomial Statistics

▷ Let ψ_d^k be the S_d-character of $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$.

Theorem (H. 2017)

Let P be a factorization statistic, then

$$\frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d(\mathbb{F}_q)} P(f) = \sum_{k=0}^{d-1} \frac{\langle P, \psi_d^k \rangle}{q^k}.$$

Algebraic geometry does not (apparently) help.

▷ **Need a different approach!**

Reduction to Interpretation of Measures

▷ Let $\nu(\lambda)$ denote the probability of an $f(x) \in \mathrm{Poly}_d(\mathbb{F}_q)$ having factorization type $\lambda \vdash d$. We call ν the **splitting measure**.

$$\frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d(\mathbb{F}_q)} P(f) = \sum_{\lambda \vdash d} P(\lambda)\nu(\lambda)$$

▷ Theorem is equivalent to showing

$$\nu(\lambda) = \frac{1}{z_\lambda} \sum_{k=0}^{d-1} \frac{\psi_d^k(\lambda)}{q^k},$$

for all partitions $\lambda = 1^{m_1} 2^{m_2} \cdots$ where $z_\lambda = \prod_{j \geq 1} j^{m_j} m_j!$.

(Recall: ψ_d^k is the S_d-character of $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$.)

Generating Functions and Euler Products

Idea: Combine all $\nu(\lambda)$ into one generating/symmetric function.

▷ Unique factorization translates into an "Euler product",

$$\sum_{d\geq 0}\sum_{\lambda\vdash d}\nu(\lambda)p_\lambda = \prod_{j\geq 1}\left(\frac{1}{1-\frac{p_j}{q^j}}\right)^{M_j(q)},$$

where $M_d(q) = \frac{1}{d}\sum_{e|d}\mu(e)q^{d/e}$ is the d**th necklace polynomial**.

▷ Sundaram, Hanlon, and others used the plethystic description of $H^*(\mathrm{PConf}_*(\mathbb{R}^3), \mathbb{Q})$ as $\mathrm{Sym}(\mathrm{Lie})$ to compute its Frobenius characteristic:

$$\sum_{d\geq 0}\sum_{\lambda\vdash d}\left(\frac{1}{z_\lambda}\sum_{k=0}^{d-1}\frac{\psi_d^k(\lambda)}{q^k}\right)p_\lambda = \prod_{j\geq 1}\left(\frac{1}{1-\frac{p_j}{q^j}}\right)^{M_j(q)}.$$

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Generating Functions and Euler Products

▷ Same strategy works in the squarefree case to give another proof of CEF result,

$$\sum_{d\geq 0}\sum_{\lambda\vdash d}\left(\frac{1}{z_\lambda}\sum_{k=0}^{d-1}(-1)^k\frac{\phi_d^k(\lambda)}{q^k}\right)p_\lambda=\prod_{j\geq 1}\left(1+\frac{p_j}{q^j}\right)^{M_j(q)},$$

(Recall: ϕ_d^k is the S_d-character of $H^k(\mathrm{PConf}_d(\mathbb{R}^2),\mathbb{Q})$.)

Bonus: Splitting measure interpretation gives us an efficient, direct way to compute the characters ψ_d^k and ϕ_d^k.

Representation Stability $\Longrightarrow$ Asymptotic Stability

▷ For each $k \geq 0$, the sequences $H^k(\mathrm{PConf}_d(\mathbb{R}^2), \mathbb{Q})$ and $H^{2k}(\mathrm{PConf}_d(\mathbb{R}^3), \mathbb{Q})$ are representation stable.

▷ CEF showed rep. stability translates into asymptotic stability for first moments of factorization stats given by character polynomials P,

$$\lim_{d \to \infty} \frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q)} P(f) = \sum_{k \geq 0} (-1)^k \frac{\langle P, \phi^k \rangle}{q^k}$$

$$\lim_{d \to \infty} \frac{1}{q^d} \sum_{f \in \mathrm{Poly}_d(\mathbb{F}_q)} P(f) = \sum_{k \geq 0} \frac{\langle P, \psi^k \rangle}{q^k}.$$

Representation Stability from Growing Gaps

▷ Rep. stability and asymp. stability follow directly from Euler products (observed by Fulman, Chen, Hersh-Reiner, and others.)
▷ Connection to configuration spaces only needed to get Schur positivity.

$$\sum_{d\geq 0}\sum_{\lambda\vdash d}\left(\frac{1}{z_\lambda}\sum_{k=0}^{d-1}(-1)^k\frac{\phi_d^k(\lambda)}{q^k}\right)p_\lambda=\prod_{j\geq 1}\left(1+\frac{p_j}{q^j}\right)^{M_j(q)},$$

$$\sum_{d\geq 0}\sum_{\lambda\vdash d}\left(\frac{1}{z_\lambda}\sum_{k=0}^{d-1}\frac{\psi_d^k(\lambda)}{q^k}\right)p_\lambda=\prod_{j\geq 1}\left(\frac{1}{1-\frac{p_j}{q^j}}\right)^{M_j(q)}.$$

Key: $M_d(q)=\frac{1}{d}q^d+O(q^{d/2})$
▷ Gaps between leading and subsequent term grow with d.
▷ Growing gaps imply values of ϕ_d^k and ψ_d^k are given by character polynomials independent of d.

Representation Stability from Growing Gaps

Theorem (Growing Gaps Principle)

Let $F_d(q)$ for $d \geq 1$ be a sequence of polynomials with $\deg F_d(q) = d$ such that for each $g \geq 1$, $F_d(q) = \frac{1}{d}q^d + O(q^{d-g})$ for all but finitely many $d \geq 1$. Define symmetric group class functions χ_d^k by an Euler product,

$$\sum_{d \geq 0} \sum_{\lambda \vdash d} \left(\frac{1}{z_\lambda} \sum_{k=0}^{d} \frac{\chi_d^k(\lambda)}{q^k} \right) p_\lambda := \prod_{j \geq 1} \left(\frac{1}{1 \pm \frac{p_j}{q^j}} \right)^{\pm F_j(q)} .$$

Then for each $k \geq 0$, the sequence χ_d^k exhibits representation stability.

▷ This is a preliminary version of a general principle.

Bounded Multiplicity Polynomial Statistics

▷ Let $m \geq 1$ and let $\mathrm{Poly}_d^m(\mathbb{F}_q)$ be the subset of polynomials in $\mathrm{Poly}_d(\mathbb{F}_q)$ with max factor multiplicity $\leq m$.

Ex. $\mathrm{Poly}_d^{\mathrm{sf}}(\mathbb{F}_q) = \mathrm{Poly}_d^1(\mathbb{F}_q)$.

▷ Let $\nu^m(\lambda) := \frac{1}{q^d}|\{f \in \mathrm{Poly}_d^m(\mathbb{F}_q) : \text{ fact. type of } f = \lambda\}|$ for $\lambda \vdash d$, then

$$\sum_{d \geq 0} \sum_{\lambda \vdash d} \nu^m(\lambda) p_\lambda = \prod_{j \geq 1} \left(\frac{1 - \frac{p_j^{m+1}}{q^j}}{1 - \frac{p_j}{q^j}} \right)^{M_j(q)}$$

▷ Growing gap principle implies coefficients of ν^m satisfy rep. stability and thus asymp. stability.

▷ Coefficients of ν^m are typically virtual characters.

Sundaram's Lie Variants

▷ Let $g : \mathbb{N} \to \mathbb{R}$ be a function and consider

$$F_d(q) := \frac{1}{d} \sum_{e|d} g(e) q^{d/e}.$$

▷ In recent work Sundaram uses the symmetric functions defined by the coefficients of the Euler products

$$\prod_{j \geq 1} \left(\frac{1}{1 \pm p_j t^j} \right)^{\pm F_j(\pm q)}$$

to study variations of the Lie and Foulkes representations, Schur positivity of sums of power sums, and positivity of restricted row sums in symmetric group character tables.

▷ Growing gaps principle implies these symmetric functions exhibit rep. stability.

Divisor Statistics on Varieties over $\mathbb{F}_q$

- Let V be a variety defined over $\mathbb{F}_q$.

$$\mathrm{Conf}_d(V)(\mathbb{F}_q) := \{\text{Subsets } C \subseteq V(\overline{\mathbb{F}}_q) : |C| = d,\ \mathrm{Frob}_q(C) = C\}$$
$$\mathrm{Sym}_d(V)(\mathbb{F}_q) := \{\text{Multisubsets } C \subseteq V(\overline{\mathbb{F}}_q) : |C| = d,\ \mathrm{Frob}_q(C) = C\}$$

- Elements of $\mathrm{Conf}_d(V)(\mathbb{F}_q)$ and $\mathrm{Sym}_d(V)(\mathbb{F}_q)$ have "factorization types" given by Frobenius orbits.

$$\sum_{d\geq 0}\sum_{\lambda\vdash d} |\mathrm{Conf}_\lambda(V)(\mathbb{F}_q)| p_\lambda = \prod_{j\geq 1}(1+p_j)^{M_j(V)}$$
$$\sum_{d\geq 0}\sum_{\lambda\vdash d} |\mathrm{Sym}_\lambda(V)(\mathbb{F}_q)| p_\lambda = \prod_{j\geq 1}\left(\frac{1}{1-p_j}\right)^{M_j(V)},$$

where

$$M_d(V) := \frac{1}{d}\sum_{e|d}\mu(e)|V(\mathbb{F}_{q^{d/e}})|$$

counts the number of length d Frobenius orbits in $V(\overline{\mathbb{F}}_q)$.

Divisor Statistics on Varieties over $\mathbb{F}_q$

$$M_d(V) := \frac{1}{d} \sum_{e|d} \mu(e) |V(\mathbb{F}_{q^{d/e}})|$$

- Weil conjecture imply that $|V(\mathbb{F}_{q^m})|$ is a polynomial in q and finitely many other parameters.

- Chen used equivalent generating functions to show asymp. stability for fac. statistics on these spaces.

- Weil conjectures imply $M_d(V)$ has growing gaps, hence we get rep. stability (for essentially any way we choose to define our class functions.)

Thank you!