A DYNAMICAL PROOF OF THE CANTOR-BERNSTEIN THEOREM

TREVOR HYDE

This note presents a dynamical proof of the Cantor-Bernstein theorem.

Theorem 1 (Cantor-Bernstein). *Let X and Y be sets. If there exist injections $f : X \to Y$ and $g : Y \to X$, then there exists a bijection $h : X \to Y$.*

If X and Y are finite sets, then the result follows from the simple fact that no finite set admits a bijection with a proper subset. So the interesting case is when X and Y are infinite. Then, for example, $x \mapsto x + 1$ defines a bijection between $\mathbb{N}$ and a proper subset, so the same logic does not apply. This allows neither f nor g to be surjective, forcing us to construct by some means an auxiliary bijection $h : X \to Y$.

There are many proofs of Theorem 1—enough to fill a book [1]. The proof presented below was found by the author, but the fundamental idea is the same as other known proofs. The only possibly original contribution of this note is framing the proof in dynamical language. Our perspective paints a different conceptual picture than classic proofs; this picture seems to be the source of clarity. A similar picture appears in [2], but the phrasing does not fit nicely into a larger, more familiar context.

Dynamical Proof. Before beginning the proof, we introduce the requisite terminology from dynamics. Let $u : X \to X$ be a function. For any non-negative integer m we write u^m for the mth composite of u with itself. Given $x \in X$ we let $\mathcal{O}_u(x)$ denote the *grand orbit* of x under u, which is the union of the forward and backward orbits of u:

$$\mathcal{O}_u(x) = \{u^m(x) : m \geq 0\} \cup \{w \in X : u^n(w) = x \text{ for some } n \geq 0\}.$$

The grand orbits under u partition the set X. Furthermore, if $\mathcal{O}$ is a grand orbit under u, then $\mathcal{O}$ is *totally invariant* under u in the sense that $u(\mathcal{O}) \subseteq \mathcal{O}$ and $u^{-1}(\mathcal{O}) = \mathcal{O}$. The grand orbits of an injective function are especially simple: we classify them into two types. Let $\mathcal{O}$ be a grand orbit under an injective function $u : X \to X$. If u is surjective on $\mathcal{O}$, which is to say $u(\mathcal{O}) = \mathcal{O}$, then it is bijective, and we call $\mathcal{O}$ a $\mathbb{Z}$-orbit. If u is not surjective on $\mathcal{O}$ we call $\mathcal{O}$ an $\mathbb{N}$-orbit. This terminology is justified by the following picture: if $\mathcal{O}$ is a $\mathbb{Z}$-orbit, then for any $x \in \mathcal{O}$ we have $\mathcal{O} = \{u^m(x) : m \in \mathbb{Z}\}$ and we visualize $\mathcal{O}$ as

$$\cdots \xrightarrow{u} u^{-2}(x) \xrightarrow{u} u^{-1}(x) \xrightarrow{u} u^0(x) \xrightarrow{u} u^1(x) \xrightarrow{u} u^2(x) \xrightarrow{u} \cdots$$

If $\mathcal{O}$ is an $\mathbb{N}$-orbit, then there exists some $x \in \mathcal{O}$ such that $\mathcal{O} = \{u^m(x) : m \in \mathbb{N}\}$ which looks like

$$u^0(x) \xrightarrow{u} u^1(x) \xrightarrow{u} u^2(x) \xrightarrow{u} u^3(x) \xrightarrow{u} \cdots$$

Note that $\mathbb{Z}$-orbits may be finite, however $\mathbb{N}$-orbits must be infinite when u is injective.

Date: August 14th, 2016.

Proof. Consider the compositions $u = gf : X \to X$ and $v = fg : Y \to Y$. Since f and g are injective, so are u and v. Therefore the grand orbits of u and v may be classified as $\mathbb{N}$- and $\mathbb{Z}$-orbits. We have commuting squares

$$\begin{CD} X @>u>> X \\ @VfVV @VVfV \\ Y @>v>> Y \end{CD} \qquad \begin{CD} Y @>v>> Y \\ @VgVV @VVgV \\ X @>u>> X \end{CD} \tag{1}$$

which follow from the associativity of composition,

$$fu = f(gf) = (fg)f = vf$$
$$gv = g(fg) = (gf)g = ug.$$

These identities give us a correspondence between the grand orbits under u and v: f maps grand orbits of X under u into grand orbits of Y under v, similarly for g. We show there are bijections between each pair of corresponding grand orbits, and since the grand orbits partition X and Y, we will be done.

Suppose $\mathcal{O}_u(x)$ is a $\mathbb{Z}$-orbit. If we let $y = f(x)$, then (1) implies $f\big(\mathcal{O}_u(x)\big) = \mathcal{O}_v(y)$, so f maps $\mathcal{O}_u(x)$ bijectively onto $\mathcal{O}_v(y)$.

$$\begin{CD} \cdots @>u>> u^{-2}(x) @>u>> u^{-1}(x) @>u>> x @>u>> u^{1}(x) @>u>> u^{2}(x) @>u>> \cdots \\ @. @VVfV @VVfV @VVfV @VVfV @VVfV \\ \cdots @>v>> v^{-2}(y) @>v>> v^{-1}(y) @>v>> y @>v>> v^{1}(y) @>v>> v^{2}(y) @>v>> \cdots \end{CD}$$

FIGURE 1. f restricts to a bijection on any $\mathbb{Z}$-orbit.

Next suppose $\mathcal{O}^1$ is an $\mathbb{N}$-orbit. Then the corresponding orbit $\mathcal{O}^2$ under v is also an $\mathbb{N}$-orbit. At least one of $\mathcal{O}^1$, $\mathcal{O}^2$ maps surjectively, hence bijectively, onto the other under f, g respectively.

$$\begin{CD} x @>u>> u^{1}(x) @>u>> u^{2}(x) @>u>> u^{3}(x) @>u>> \cdots \\ @VVfV @VVfV @VVfV @VVfV \\ y @>v>> v^{1}(y) @>v>> v^{2}(y) @>v>> v^{3}(y) @>v>> \cdots \end{CD}$$

FIGURE 2. If $\mathcal{O}^1 = \mathcal{O}_u(x)$ is an $\mathbb{N}$-orbit and $y = f(x)$, then f maps $\mathcal{O}^1$ onto $\mathcal{O}^2 = \mathcal{O}_v(y)$.

This concludes the proof. □

References

[1] Arie Hinkis. *Proofs of the Cantor-Bernstein Theorem.* Springer, 2013.
[2] Berthold Schweizer. Cantor, schroder, and bernstein in orbit. *Mathematics Magazine*, 73(4):311, 2000.

DEPT. OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043,
E-mail address: tghyde@umich.edu