Quantitative algebraic topology and Lipschitz homotopy

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We consider when it is possible to bound the Lipschitz constant a priori in a homotopy between Lipschitz maps. If one wants uniform bounds, this is essentially a finiteness condition on homotopy. This contrasts strongly with the question of whether one can homotop the maps through Lipschitz maps. We also give an application to cobordism and discuss analogous isotopy questions.

Lipschitz homotopy | amenable group | uniformly finite homology

Introduction

The classical paradigm of geometric topology, exemplified by, at least, immersion theory, cobordism, smoothing and triangulation, surgery, and embedding theory is that of reduction to algebraic topology (and perhaps some additional pure algebra). A geometric problem gives rise to a map between spaces, and solving the original problem is equivalent to finding a nullhomotopy or a lift of the map. Finally, this homotopical problem is solved, typically by the completely nongeometric methods of algebraic topology, e.g. localization theory, rational homotopy theory, spectral sequences etc.

While this has had enormous successes in answering classical qualitative questions, it is extremely difficult (as has been emphasized by Gromov [11]) to understand the answers quantitatively. One general type of question that tests one's understanding of the solution of a problem goes like this: introduce a notion of complexity, and then ask about the complexity of the solution to the problem in terms of the complexity of the original problem. Other possibilities can involve understanding typical behavior or the implications of making variations of the problem.

In this task, often the complexity of the problem is reflected, somewhat imperfectly, in the Lipschitz constant of the map. Indeed, one can often view the Lipschitz constant of the map as a measure of the complexity of the geometric problem.

For concreteness, let us quickly review the classical case of cobordism, following Thom [17]. Let M be a compact smooth manifold. The problem is: When is M^n the boundary of some other compact smooth W^{n+1}?

There are many possible choices of manifolds in this construction, such as oriented manifolds, manifolds with some structure on their (stabilized) tangent bundles, PL and topological versions and so on. But, for now, we will confine our attention to this simplest version.

Thom embeds M^m in a high dimensional euclidean space $M \subset \mathbb{S}^{m+N-1} \subset \mathbb{R}^{m+N}$, and then classifies the normal bundle by a map $\nu_M : \nu M \to E(\xi^N \downarrow Gr(N, m+N))$ from the normal bundle to the universal bundle of N-planes in $m+N$ space. Including $\mathbb{R}^{m+N}$ into S^{m+N} via one-point compactification, we can think of νM as being a neighborhood of M in this sphere (via the tubular neighborhood theorem), and extend this map to S^{m+N} if we include $E(\xi^N \downarrow Gr(N, m+N))$ into its one point compactification $E(\xi^N \downarrow Gr(N, m+N))\hat{}$, the Thom space of the universal bundle. Let us call this map

$$\Phi_M : S^{m+N} \to E(\xi^N \downarrow Gr(N, m+N))\hat{}$$

Thom shows, among other things, that:

1. M bounds iff Φ_M is homotopic to a constant map. If M is the boundary of W, one embeds W in D^{m+N+1}, extending the embedding of M into S^{m+N}. Extending Thom's construction over this disk gives a nullhomotopy of Φ_M. Conversely, one uses the nullhomotopy and takes the transverse inverse of Gr(N, m+N) under a good smooth approximation to the homotopy to the constant map ∞ to produce the nullcobordism.

2. Φ_M is homotopic to a constant map iff $\nu_{M*}([M]) = 0 \in H_m(Gr(N, m+N); Z_2)$. This condition is often reformulated in terms of the vanishing of Stiefel-Whitney numbers.

It is now reasonable to define the complexity of M in terms of the volume of a Riemannian metric on M whose local structure is constrained, e.g., by having curvature and injectivity radius bounded appropriately, $|K| \le 1$, inj ≥ 1, or, alternatively, by counting the number of simplices in a triangulation (again whose local structure is bounded). In that case, supposing M bounds, i.e., the conclusion of Thom's theorem holds, then can we bound the complexity of the manifold that M bounds?

Note that the Lipschitz constant of Φ_M is related to the complexity that we have chosen in that the curvature controls the local Lipschitz constant of ν, but that there is an additional global aspect that comes from the embedding. It is understood that, because of expander graphs, for example, one might have to have extremely thin tubular neighborhoods when one embeds a manifold in a high dimensional Euclidean space. This increases the Lipschitz constant accordingly when extending ν_M to Φ_M. We shall refer to the Lipschitz constant of a Thom map for M as the *Thom complexity of* M.

Of course, we can separate the problems and deal with the problem of understanding the complexity of embedded coboundaries based on a complexity involving $|K|$ and τ, which is by definition, the *feature size* of computational topol-

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ogy, the smallest size at which normal exponentials to M in a high-dimensional sphere collide. Doing this, studying the Lipschitz constant of Φ_M essentially is the same as considering $\sup(|K|, 1/\tau)$. In this paper, that is the approach we take, but in a sequel, we plan to use the method of [3] to avoid embeddings for application to unoriented cobordism.

The second issue, then, is item (2). How does one get information about the size of the nullhomotopy? Algebraic topology does not directly help us because it reasons algebraically involving many formally defined groups and their structures, constantly identifying objects with equivalence classes. Despite this, Gromov has suggested the following:

Optimistic possibility [11]: If X and Y are finite complexes and Y is simply connected, then there is a constant K, such that if $f, g : X \to Y$ are homotopic Lipschitz maps with Lipschitz constant L, then they are homotopic through KL-Lipschitz maps.

The rest of this paper makes some initial comments regarding this problem. We shall first discuss a stronger problem, that of constructing a KL-Lipschitz homotopy. We give necessary and sufficient conditions for there to be a KL-Lipschitz homotopy between homotopic L-Lipschitz maps with constant K only depending on $\dim(X)$. The hypotheses of this situation are sufficient for unoriented cobordism and give a linear increase of Thom complexity for the problem of unoriented smooth embedded cobordism because in that case the Thom space is a finite complex with finite homotopy groups. However, we leave open natural extensions even to unoriented PL manifolds or oriented smooth manifolds, as well as the more natural volume measures of complexity, which we hope to discuss in a future paper.

We also give some contrasting results where, essentially for homological reasons, one cannot find Lipschitz homotopies, but homotopies through Lipschitz maps are possible.

Finally, we make some comments and conjectures about the related problem of isotopy of embeddings in both the Lipschitz setting and in the C^k setting. Both of these contrast with the C^1 setting considered in [10].

Constructing Lipschitz homotopies

Theorem 1. *Let Y be a finite complex with finite homotopy groups in dimensions $\leq d$. Then there exists $C(d)$ so that for all simplicial path metric spaces X with dimension $X \leq d$ such that the restriction of the metric to each simplex of X is standard, if $f, g : X \to Y$ are homotopic L-Lipschitz maps with $L \geq 1$, then there is a $C(d)L$-Lipschitz homotopy F from f to g.*

Conversely, if a Lipschitz homotopy always exists, then the homotopy groups of Y are finite in dimensions $\leq d$.

Proof: We begin with the positive direction; the argument is a slight adaptation of that in [15] which can be referred to for more detail and for a generalization. We may assume that Y is a compact smooth manifold embedded in $\mathbb{R}^k_+$ for some k, with $\partial Y \subset \mathbb{R}^{k-1}$. The normal bundle projection from a tubular neighborhood N of Y to Y retracts N to Y by a Lipschitz map which we may assume to have Lipschitz constant less than 2. Choose an $\epsilon > 0$ so that the straight line connecting points y' and y'' in Y is contained in N whenever $d(y', y'') < \epsilon$.

The space of μ-Lipschitz maps $\partial\Delta^\ell \to Y$ is compact for each ℓ and $\mu > 0$, so for each $\ell \leq d+1$ and $\mu > 0$, we can choose a finite collection $\{\phi_{i,\ell,\mu} : \partial\Delta^\ell \to Y\}$ of μ-Lipschitz maps which is ϵ-dense in the space of all μ-Lipschitz maps $\partial\Delta^\ell \to Y$. For each such map that extends to Δ^ℓ, choose Lipschitz extensions, $\bar{\phi}_{i,\ell,\mu,j} : \Delta^\ell \to Y$ in each homotopy class of extensions of $\phi_{i,\ell,\mu}$. Since the homotopy groups of Y are finite, there are only finitely many such extensions for each $\phi_{i,\ell,\mu}$. If $f : \partial\Delta^\ell \to Y$ is μ-Lipschitz, f is homotopic to some $\phi_{i_0,\ell,\mu}$ by a linear homotopy in $\mathbb{R}^k_+$. Retracting this into Y along the normal bundle gives a 2μ-Lipschitz homotopy from f to $\phi_{i_0,\ell,\mu}$. If f extends over Δ^ℓ, then some $\bar{\phi}_{i_0,\ell,\mu,j}$ can be pieced together with the Lipschitz homotopy from f to $\phi_{i_0,\ell,\mu}$ to give a Lipschitz extension of $f|\partial\Delta^\ell$.

The result of this is that for every ℓ, μ, there is a ν so that if $f : \Delta^\ell \to Y$ is a map with $f|\partial\Delta^\ell$ μ-Lipschitz, then $f|\partial\Delta^\ell$ has a ν-Lipschitz extension to Δ^ℓ which is homotopic to f. It is now an induction on the skeleta of X to show that there is a ν so that if f and g are homotopic 1-Lipschitz maps from X^d to Y, then f and g are ν-Lipschitz homotopic. The rest of the argument in the forward direction is a rescaling and subdivision argument. If f and g are homotopic L-Lipschitz maps, $L \geq 1$, as in the statement of the theorem, we can rescale the metric and subdivide to obtain maps with Lipschitz constant 1. After extending, we rescale back to the original metric, obtaining the desired result.

For the rescaling argument, it is helpful to work with cubes, rather than standard simplices because subdivisions are easier to handle. We consider X as a subcomplex of the standard simplex, embedded as a subcomplex of the standard N-simplex, thought of as the convex hull of unit vectors in $\mathbb{R}^{N+1}$. There is a map sending the barycenter of each simplex $< e_{i_0}, ..., e_{i_k} >$ to $e_{i_0} + ... + e_{i_k}$, which is a vertex of the unit cube in $\mathbb{R}^{N+1}$. Extending linearly throws our complex onto a subcomplex of the unit cube via a homeomorphism whose bi-Lipschitz constant is controlled by d. It is now an easy matter to subdivide the cube into smaller congruent pieces. Here, by the "bi-Lipschitz constant," we mean the sup of the Lipschitz constants of the embedding and its inverse.

We now proceed to the converse. Assuming that Y is a finite complex with at least one non-finite homotopy group, we will show that there is a nullhomotopic Lipschitz map $\mathbb{R}^n \to Y$ for some n which is not Lipschitz nullhomotopic. The argument shows that there is no constant $C(n)$ that works for all subcubes of $\mathbb{R}^n$.

Suppose that $\pi_1(Y)$ is infinite. We give $\widetilde{Y}$ the path metric obtained by pulling the metric in Y up locally. Then by König's Lemma [12], the 1-skeleton of $\widetilde{Y}$ contains an infinite path R that has infinite diameter in $\widetilde{Y}$. Let $X = [0, \infty)$ with the usual simplicial structure. The map $f : X \to R$ is 1-Lipschitz and nullhomotopic. If there were a Lipschitz homotopy F from f to a constant, then the length of each path $F|\{x\} \times I$ would be bounded independently of x. Lifting to $\widetilde{Y}$ would give uniformly bounded paths from $f(x)$ to the basepoint for all x, contradicting the fact that R has infinite diameter.

Assume that $\pi_k(Y)$ is finite for $k \leq n-1$, $n \geq 2$, and assume that $\pi_n(Y)$ is infinite. Let us first consider the case in which

Y is simply connected. By Serre's extension of the Hurewicz theorem, the first infinite homotopy group of Y is isomorphic to the corresponding homology group modulo torsion, so we have maps $S^n \to Y \to K(\mathbb{Z}, n)$ such that the generator of $H^n(K(\mathbb{Z}, n)) \cong \mathbb{Z}$ pulls back to a generator of $H^n(S^n)$. The image of Y is compact, so the image of Y lies in a finite skeleton of $K(\mathbb{Z}, n)$ which, for definiteness, we could take to be a finite symmetric product of S^n's. The composition is homotopic to a map into $S^n \subset K(\mathbb{Z}, n)$ and this composition $S^n \to S^n$ has degree one.

Consider the map $\mathbb{R}^n \to T^n \to S^n$, where the last map is the degree one map obtained by squeezing the complement of the top cell to a point. The n-form representing the generator of $H^n(K(\mathbb{Z}, n))$ pulls back to a form cohomologous to the volume form on S^n and, therefore to a closed form on T^n cohomologous to a positive multiple of the volume form. This pulls back to a closed form on $\mathbb{R}^n$ boundedly cohomologous to a multiple of the volume form.

Suppose that the map $\mathbb{R}^n \to Y$ is Lipschitz nullhomotopic. The Lipschitz nullhomotopy can be approximated by a smooth Lipschitz nullhomotopy. See, for example, [2]. By the proof of the Poincaré lemma, this shows that the volume form on $\mathbb{R}^n$ is $d\alpha$ for some bounded form α. This is easily seen to be impossible by Stokes' theorem, since the integral of the volume form over an $m \times m \times \cdots \times m$ cube grows like m^n in m, while the integral of α over the boundary grows like m^{n-1}. This contradiction shows that the composition $\mathbb{R}^n \to T^n \to S^n \to Y$ is not Lipschitz nullhomotopic.

In case Y has nontrivial finite fundamental group, the universal cover $\widetilde{Y}$ is compact and simply connected. As above, we assume that our map $\mathbb{R}^n \to Y$ is Lipschitz nullhomotopic. The Lipschitz nullhomotopy lifts to $\widetilde{Y}$. Applying the argument above in $\widetilde{Y}$ produces a contradiction.

In [5], Block and Weinberger discuss uniformly finite cohomology theory for manifolds of bounded geometry. This theory uses cochains that are uniformly bounded on simplices in a triangulation of finite complexity or, in the de Rham version, uses k-forms that are uniformly bounded on k-tuples of unit vectors. The argument above, then, shows that there is an obstruction in this theory which must vanish in order for a nullhomotopic map to be Lipschitz nullhomotopic. We will develop this theory explicitly in a future paper.

Under favorable circumstances, it is also possible to use this theory to construct Lipschitz nullhomotopies.

Theorem 2. *Let M be a closed orientable manifold with non-amenable fundamental group. Then the composition*

$$\widetilde{M} \to M \to S^n$$

is Lipschitz nullhomotopic. Here, $M \to S^n$ is the degree one map obtained by crushing out the $n-1$-skeleton of M and sending the interior of one n-simplex homeomorphically onto $S^n - \{*\}$, as in the proof of the first theorem. Everything else goes to $*$. Conversely, if the fundamental group is amenable, then this composite is not Lipschitz nullhomotopic.

Proof: Triangulate $\widetilde{M}$ by pulling up a triangulation of M. The composition in the statement of the theorem gives an element of the n^{th} uniformly finite cohomology of $\widetilde{M}$ with coefficients in $\pi_n(S^n) = \mathbb{Z}$. There is a duality theorem stated in [5] to the effect that the n^{th} uniformly finite cohomology of $\widetilde{M}$ is isomorphic to its 0^{th} uniformly finite homology. One of the main theorems of [4] says that the 0^{th} uniformly finite homology of the universal cover of a manifold is trivial if and only if the fundamental group is non-amenable.

Since the proof of the duality theorem quoted above is previously undocumented, we outline a proof in the case we used. Orient the cells of M, and therefore of $\widetilde{M}$ in such a way that the sum of the positively oriented top dimensional cells is a locally finite cycle. Consider the dual cell decomposition on $\widetilde{M}$. Each vertex in the dual cell complex is the barycenter of a top dimensional cell, so we can assign an element of $\pi_n(S^n)$ to each vertex in the dual complex. By the theorem from [4] quoted above, this chain is the boundary of a uniformly bounded 1-chain in the dual skeleton. Assigning the coefficient of each 1-simplex to the $(n-1)$-cell it pierces, gives a uniformly bounded cochain whose coboundary is the original cocycle, up to sign, exactly as in the classical PL proof of Poincaré duality.

It follows, then, that there is a uniformly bounded $(n-1)$-cochain whose coboundary is equal to the obstruction. One uses this cochain exactly as in ordinary obstruction theory to build a homotopy from the given composition to a constant map. Since the maps used in the construction were chosen from a finite collection, this nullhomotopy can be taken to be globally Lipschitz.

The converse follows from the de Rham argument used in the proof of theorem 1 applied to a Følner sequence.

We note that the theory of homotopies h_t that are Lipschitz for every t is quite different from the theory of Lipschitz homotopies. For instance, contracting the domain in itself shows that every Lipschitz map $\mathbb{R}^n \to S^n$ is nullhomotopic through Lipschitz maps, while the construction in Theorem 1 shows that such a map need not be Lipschitz nullhomotopic. In fact, we have

Theorem 3. *If M is a closed connected manifold with infinite fundamental group, the map described in Theorem 2 is nullhomotopic through Lipschitz maps.*

Proof: We may assume that M has dimension ≥ 2, since the circle is the only one-dimensional example and the theorem is clearly true in this case. We may also assume that M is 1- or 2-ended, since Stallings' structure theorem for ends of groups implies that otherwise $\pi_1(M)$ is nonamenable, [8]. We will begin by assuming that M is 1-ended, which means that for any compact $C \subset M$, there is a compact $C \subset D \subset M$ so that any two points in $M - D$ are connected by an arc in $M - C$. By induction, we can write M as a nested union of compact sets $C_i \subset C_{i+1}$, so that any two points in $M - C_{i+1}$ are connected by an arc in $M - C_i$. See figure 1.

Shrink the support of the map $M \to S^n$ to lie in a small ball in the interior of a top-dimensional simplex. Here, by support, we mean the inverse image of $S^n - \{*\}$. Run a proper ray out to infinity in the 1-skeleton of $\widetilde{M}$, as in the proof of Theorem 1 and connect the ray by a geodesic segment to the

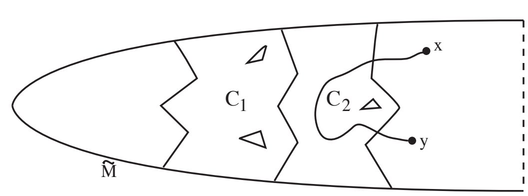

Fig. 1: A one-ended manifold

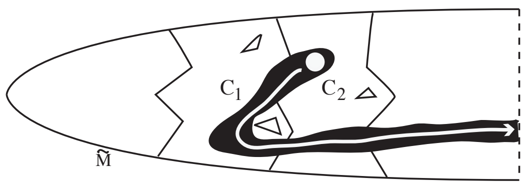

Fig. 2: The support of the degree one map is pushed to infinity

center of a lift of the support ball. Smooth the ray by rounding angles in the 1-skeleton and thicken the ray to a map of $[0, \infty) \times D^{n-1} \to \widetilde{M}$. Since there are only finitely many different angles in the 1-skeleton of $\widetilde{M}$, we can assume that this thickened ray has constant thickness larger than the diameter of the lifted support ball.

Homotop the map $\widetilde{M} \to S^n$ to a map which is constant on this tube by pushing the support out to infinity during the interval $t \in [0, 1/2]$. This homotopy is Lipschitz for every t because it is smooth and agrees with the original Lipschitz map outside a compact set for every t. Repeat this construction for every lift of the support ball, being careful to choose each ray so that if the support ball lies in $\widetilde{M} - C_{i+1}$, the ray lies in $\widetilde{M} - C_i$ and parameterizing these pushes to infinity to occur on intervals $[1/2, 3/4]$, $[3/4, 7/8]$, etc. The resulting homotopies are constant on larger and larger compact sets and converge to the constant map when $t \to 1$. Note that the Lipschitz constants of these Lipschitz maps are globally bounded. See figure 2.

The argument for the two-ended case is similar, except that the complements of the C_i's will have two unbounded components. One must be careful that for $i > 1$ a ball which lies in an unbounded component of $M - C_{i+1}$ should be pushed to infinity in that component after dropping back no further than into $C_i - C_{i-1}$.

Remark 1. *Calder and Siegel [6] have shown that if Y is a finite complex with finite fundamental group, then for each n there is a b so that if X is n-dimensional and $f, g : X \to Y$ are homotopic maps, then there is a homotopy h_t from f to g so that the path $\{h_t(x) \mid 0 \leq t \leq 1\}$ is b-Lipschitz for every $x \in X$. In the case of our map $\mathbb{R}^2 \to S^2$, such a homotopy can be obtained by lifting to S^3 via the Hopf map and contracting the image along geodesics emanating from a point not in the image of $\mathbb{R}^2$. One way to prove the general case uses a construction from [9]. If Y is a finite simplicial complex with finite fundamental group, given $n > 0$, Theorem 2′ of [9] produces a PL map q from a contractible finite polyhedron to Y that has the approximate lifting property for n-dimensional spaces. Taking a regular neighborhood in some high-dimensional euclidean space gives a space PL homeomorphic to a standard ball. Composing q with this PL homeomorphism and the regular neighborhood collapse gives a PL map from a standard ball PL ball, B, to Y which has the approximate lifting property for n-dimensional spaces. If $\dim X \leq n$ and $f : X \to Y$ is a nullhomotopic map, there is a map $\bar{f} : X \to B$ so that $p \circ \bar{f}$ is ϵ-close to f, with ϵ as small as we like. Coning off in B gives a nullhomotopy in B with lengths of paths bounded by the diameter of B. Composing with p gives a nullhomotopy of f where the lengths of the tracks of the homotopy are bounded by the diameter of B times the Lipschitz constant of p.*

Thus, in our construction $\widetilde{M} \to M \to S^n$, we can achieve Lipschitz nullhomotopies in the t-direction whenever $n \geq 2$ and Lipschitz nullhomotopies in the $\widetilde{M}$-direction whenever the fundamental group of M is infinite, but we can achieve both simultaneously if and only if the fundamental group of M is nonamenable.

Remark 2. *As mentioned in the introduction, for geometric applications it is much more useful to have Lipschitz homotopies than homotopies through Lipschitz maps. However, as pointed out in [19], pp. 102-104, it is possible to turn the latter into the former at the cost of increasing the length of time the homotopy takes. More precisely, if one is a situation where any two ϵ-close maps are homotopic, then the length of the homotopy does not have to be any larger than the number of ϵ-balls it takes to cover the space of maps with Lipschitz constant at most CL. In our situation, if X is compact and d-dimensional, then this observation would allow a Lipschitz homotopy that is roughly of size e^{L^d}. Our examples of homotopic Lipschitz maps that are not at all Lipschitz homotopic thus, of course, require the noncompactness of the domain.*

Some remarks on Isotopy classes.

If X and Y are manifolds, then we can consider analogous problems for embeddings rather than just maps. In his seminal paper [10], Gromov used Haefliger's reduction of metastable embedding theory to homotopy theory (i.e. the theory of embeddings $M \to N$ when the dimensions satisfy $2n > 3m + 2$) to show that a bound on the bi-Lipschitz constant of an embedding cuts the possible number of isotopy classes down to a finite (polynomial) number when the target is simply connected.

In general he pointed out that because of the existence of Haefliger knots, that is infinite families of smooth embeddings of $S^{4k-1} \subset S^{6k}$, there are infinite smooth families with a bound on the bi-Lipschitz constant.

It is possible to continue this line of thought into much lower codimension with the following theorems, by changing the categories. See figure 3. The picture represents a sequence of Haefliger-knotted spheres with bi-Lipschitz convergence to an embedding which is bi-Lipschitz but not C^1. If the Haefliger knots are replaced by ordinary codimension two knots, this becomes an example showing that Theorem 4 is false in codimension 2.

Theorem 4. *The number of topological isotopy classes of embeddings of $M \to N$ represented by locally flat embeddings with a given bi-Lipschitz constant is finite whenever $m - n \neq 2$.*

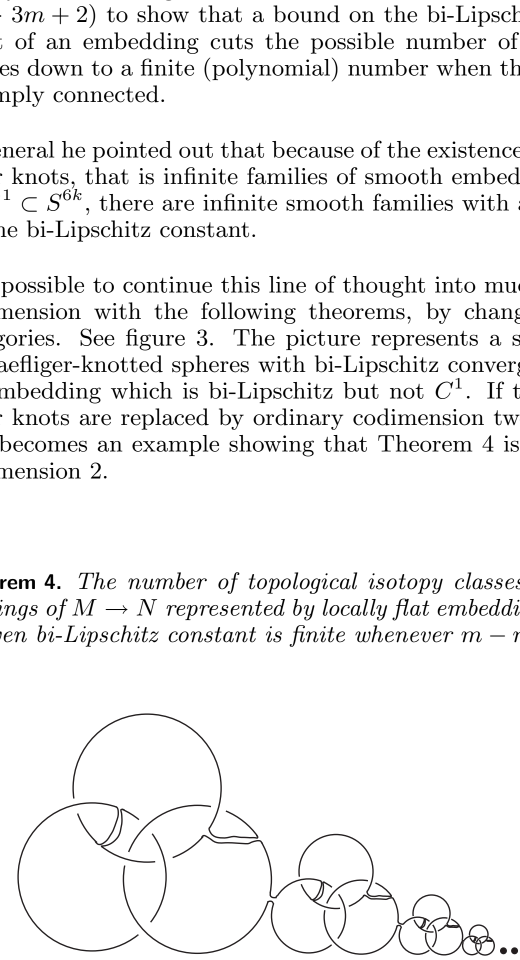

Fig. 3: The Lipschitz norm is uniformly bounded, yet the C^2 norm necessarily grows

This is proved in codimension > 2 in J. Maher's University of Chicago thesis [13] by using the Browder-Casson-Haefliger-Sullivan-Wall analysis of topological embeddings in terms of Poincaré embeddings (see [18]) together with the observation that any such embedding will be topologically locally flat using the 1-LC flattening theorem, [7], corollary 5.7.3, p. 261, and the fact that the image has Hausdorff codimension at least 3. More precisely, let $\{f_i\}$ be a sequence of K-bi-Lipschitz embeddings. Using Arzela-Ascoli we can extract a convergent subsequence that remains K-bi-Lipschitz. Since the codimension is at least three, this image with have Hausdorff codimension 3, and therefore will be 1-LC, as observed by Siebenmann and Sullivan in [16]. All the f_i sufficiently C^0 close to this limit will be topologically isotopic to it because they induce the same Poincaré embedding. In codimension 1, a similar argument works, except that the limit cannot be assumed to be locally flat. However, by theorem 7.3.1 of Davermann-Venema [7], any two locally flat embeddings C^0 close enough to the limit must be isotopic, so the conclusion follows in this case, as well.

Theorem 5. *The same is true in the smooth category (in all codimensions) if one bounds the C^2 norms of the embeddings.*

A linear homotopy between sufficiently C^1-close C^1 embeddings gives an isotopy of embeddings which extends to an ambient isotopy. Thus, a sequence of C^2 embeddings which C^1 converges contains only finitely many topological isotopy classes of embeddings.

Both of these theorems then give rise to interesting quantitative questions, both in terms of bounding the number of embeddings and also in terms of understanding how large the Lipschitz constants must grow during the course of an isotopy.

At the moment, unlike the situation for maps, we do not even see any effective bounds on the number of isotopy classes. Nevertheless, based on Gromov's ideas, the following conjecture seems plausible:

Conjecture 6. *If N is simply connected, and $\dim M < \dim N$-2, then the number of L-bi-Lipschitz isotopy classes of embeddings of M in N grows like a polynomial in L. Furthermore, there is a C so that any two such embeddings that are isotopic are isotopic by an isotopic through CL-bi-Lipschitz embeddings.*

A very interesting case suggested by the techniques of this paper is the following:

Proposition 7. *Let Σ^n be a rational homology sphere. Then the number of isotopy classes of topological bi-Lipschitz embeddings of Σ^n in S^{n+k} is finite for $k > 2$.*

The finiteness follows from considerations about Poincaré embeddings. Standard facts about spherical fibrations give finiteness of the normal data, and Alexander duality enormously restricts the homology of the complement. The number of homotopy types is readily bounded via analysis of k-invariants to be at most a tower of exponentials where the critical parameter is the size of the torsion homology, and then obstruction theory allows only finitely many possibilities for each of these (when taking into account that the total space of this Poincaré embedding is a homotopy sphere).

We note that the homotopy theory of this situation can be studied one prime at a time using a suitable pullback diagram. For large enough primes, the issues involved resemble rational homotopy theory – the core homotopical underpinning of Gromov's paper [10]. This gives us some hope that this special case of Conjecture 5 might be accessible.

We now turn to the case of hypersurfaces: $\dim M = \dim N - 1$.

Remarks

1. For codimension two embeddings of the sphere, [14] shows that one cannot tell whether a C^2 knot is isotopic to the unknot. Therefore there is no computable function that can bound the size of an isotopy.
2. By taking tubular neighborhoods, this gives an analogous result for $S^1 \times S^{n-2}$ in S^n, for $n > 4$.
3. Perhaps more interesting from the point of view of this paper is that the same holds true in the smooth setting even if the hypersurface has nontrivial finite fundamental group by [17], pp. 83-85, so that finiteness of a homotopy group is not enough to give a quantitative estimate on the size of an isotopy. It still seems possible that a version of conjecture 5 can saved for embeddings in codimension 1 that are "incompressible", i.e. that induce injections on their fundamental group.

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