

AN INTRODUCTION TO K3 SURFACES AND THEIR DYNAMICS

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ABSTRACT. These notes provide an introduction to the geometry of K3 surfaces and the dynamics of their automorphisms. The notes are based on lectures delivered in Grenoble in July 2018, and in Beijing in July 2019.

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1. Introduction

The study of dynamical systems is full of very challenging problems. One reason is that it is easy to write down a transformation, but it is difficult to predict the behavior of its iterates. It is therefore valuable to have some property of the dynamical system preserved by the transformation.

Complex dynamics is the study of transformations that preserve a complex structure on the underlying space. The tools of complex analysis in one and several variables can then be used to understand the behavior of iterates. In one complex variable these methods led to rich developments, see e.g. [Mil06] for an introduction, and higher dimensions are also actively explored, see e.g. [DS10, Lyu14, Can18].

In these notes we concentrate on K3 surface automorphisms. This puts us in complex dimension 2, but in addition ensures that our automorphisms preserve a smooth volume form. K3 surfaces have several additional features that become handy when looking at their automorphisms.

The first feature is that K3s have good moduli spaces, which turn out to be homogeneous spaces for appropriate Lie groups. This is related to the Hodge structure on the cohomology of a K3 surface and quite a bit about an automorphism can be understood already by looking at its action on cohomology. In particular, one can construct interesting automorphisms by specifying linear-algebraic data, instead of giving say algebraic equations defining the K3 surface. An example of this is McMullen's construction of a K3 surface with a Siegel disc (see [McM02] and §6.1).

A second feature is the existence of special Riemannian metrics on the K3 surface, compatible with the complex structure and volume form. The metrics turn out to have vanishing Ricci curvature, although their sectional curvature (generically) does not vanish. An application of these metrics to the dynamics of holomorphic automorphisms is included in [Theorem 7.2.2](#) below.

The above features are specific to K3 surfaces, but many of the ideas and tools developed in these notes also appear in other situations in

complex dynamics. I hope that the reader will get a sense for this active field of investigation. At times I have included brief remarks on directions not immediately connected to dynamics or K3 surfaces, but which I believe help situate the discussion in the general landscape.

Overview of contents. Section 2 starts with the place of K3 surfaces in the general classification of compact complex surfaces. The basic definitions and examples follow, as well as a discussion of the topology and Hodge theory of K3 surfaces.

Section 3 takes the point of view of Kähler and Riemannian geometry. A discussion of Ricci-flat metrics and Monge–Ampère equations is followed by a description of holonomy groups and hyperkähler metrics.

Section 4 contains a brief discussion of the Torelli theorems. It starts with a general discussion of complex deformation theory, followed by an application to period mappings of K3 surfaces. Several versions of the Torelli theorem are then stated.

Section 5 is the start of the dynamical part. After introducing some examples of K3 automorphisms, entropy is discussed in the context of the Gromov–Yomdin theorem. The section ends with a reminder on Salem numbers and a few other properties of K3 automorphisms.

Section 6 contains two results that are related to non-hyperbolic dynamics on K3s. The first is McMullen’s [McM02] construction of K3 automorphisms that have invariant open sets on which the action is conjugated to a rigid rotation. The second is Cantat’s [Can01b] classification of invariant measures, and orbit closures, for sufficiently large automorphism groups of K3s.

Section 7 contains a discussion of hyperbolic aspects of K3 dynamics. First we present Cantat’s [Can01a] construction of invariant currents and the measure of maximal entropy. Then we present a proof of a result of Cantat & Dupont [CD20b], following [FT19], that the measure of maximal entropy is equal to the volume form only in Kummer examples.

Some familiarity with complex geometry is assumed, as contained for example in the first chapters of Griffiths & Harris [GH78]. The textbook of Huybrechts [Huy05] has an emphasis on topics closer to the needs of these notes.

Analogies. For readers familiar with Teichmüller theory and the geometry of Riemann surfaces, Weil’s brief report [Wei09, pg. 390] can provide a motivation for the study of K3 surfaces. The table of analogies included below can also serve as a dictionary for many of the structures in the present text.

Riemann surfaces	K3 surfaces
Mapping classes of diffeomorphisms: pseudo-Anosov, reducible, periodic	Holomorphic automorphisms: hyperbolic, parabolic, elliptic
Stable and unstable foliations	Stable and unstable currents
Entropy, action on curves	Entropy, action on H^2
Hodge theory $H^1 = H^{1,0} \oplus H^{0,1}$	Hodge theory $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$
Teichmüller space	Period Domain(s)
Flat metrics	Ricci-flat (hyperkähler) metrics
Holomorphic 1-form	Holomorphic 2-form
Straight lines for the flat metric	Special Lagrangians
Periodic trajectories	Special Lagrangian tori
Completely periodic foliations	Torus fibrations
\mathbb{S}^1 : directions for straight lines	\mathbb{S}^2 : twistor (hyperkähler) rotation
square-tiled surfaces	Kummer surfaces
Lyapunov exponents for families	

Some omitted topics. This text is mainly concerned with infinite order automorphisms of complex K3 surfaces. One can also consider finite order automorphisms, or K3s in positive characteristic. Both topics have been extensively studied but are not mentioned further in this text. One can also consider automorphisms over other ground fields, e.g. non-archimedean ones such as $\mathbb{C}((t))$, leading to a “tropicalization” of the discussion (see [Fil19b]). Let us also note that the action of algebraic automorphisms on triangulated categories associated to algebraic manifolds promises to be another direction of fruitful investigation, see e.g. [FFH⁺21]. Additionally, questions about arithmetic properties of points on K3 surfaces can be investigated using dynamical ideas, see e.g. [FT21].

More recently, the joint dynamics of several automorphisms became a topic of active investigation. Many of the problems that for now appear intractable for a single automorphism, such as the properties of Lebesgue measure (e.g. positivity of entropy, ergodicity) can be studied for groups generated by several automorphisms. See [CD20a, FT21] for more in this direction.

Finally, one can associate Lyapunov exponents to families of K3 surfaces – these measure the non-triviality of the family. For families of Riemann surfaces, these considerations started with Kontsevich’s article [Kon97], which also connected the subject to Teichmüller dynamics. A version for K3 surfaces is discussed in [Fil18].

Further reading. There are many excellent sources that present in greater depth some of the material in these notes. Our hope is that the brief overview presented here will entice the reader to learn more about the subject.

An excellent introduction for the nonspecialist is contained in the seminar notes [K3-85]. A modern introduction, with a stronger algebraic flavor than [K3-85], is the monograph of Huybrechts [Huy16]. Differential-geometric aspects are treated in the collection of notes [GHJ03]. Yau’s solution of the Calabi conjecture, essential to much of the geometry of K3 surfaces, is in [Yau78].

Further reading in dynamics. The initial impetus for studying automorphisms of K3 surfaces came from Mazur’s [Maz92] questions about rational points on them. Cantat’s paper [Can01a] introduced complex-analytic tools to the subject and constructed the measure of maximal entropy. McMullen [McM02] constructed the first examples of positive entropy K3 surface automorphisms which admit a Siegel disc, i.e. an open domain on which the dynamics is conjugated to a rotation on a polydisc. Further examples of automorphisms with small but positive entropy were constructed in [McM11].

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2. Basic structures

Outline of section. In §2.1 we set the context for K3 surfaces by recalling the classification of compact Riemann surfaces and compact complex surfaces. The three broad classes – elliptic, parabolic, and hyperbolic – divide the landscape according to their geometric and algebraic properties. An important theme in this setting is that of n -forms on n -dimensional manifolds. Whether they have poles, zeros, or trivialize the canonical bundle, determines to a large extent the geometry.

In §2.2 we define K3s and give some examples. This is followed by a brief discussion of their topology in §2.3 and a recollection of essential Hodge-theoretic facts in §2.4.

2.1. Classification of surfaces

2.1.1. The case of Riemann surfaces. Compact Riemann surfaces are divided into three fundamentally different classes:

genus 0: The only possibility is $\mathbb{P}^1(\mathbb{C})$; it carries a metric of constant positive curvature and has no holomorphic 1-forms.

genus 1: Elliptic curves have a flat metric and exactly one holomorphic 1-form.

genus ≥ 2 : Higher genus surfaces have *canonical* constant negative curvature metrics and plenty of holomorphic 1-forms.

Only the genus 0 and genus 1 Riemann surfaces admit infinite-order endomorphisms with non-trivial dynamics. Indeed, any holomorphic self-map of a higher genus surface must act as a semi-contraction for the hyperbolic metric (by the Schwarz lemma). The map then must either be an isometry, hence finite order, or a uniform contraction because the surface is compact (and due to the equality case in the Schwarz lemma). If the map is a uniform contraction, then a sufficiently high iterate will take everything to a neighborhood of the fixed point, but the map is proper, and unless the image is a single point the map is also open, which leads to a contradiction. In general, the Schwarz lemma is a powerful tool that's used to study endomorphisms of $\mathbb{P}^1(\mathbb{C})$. In higher dimensions its analogues, such as the notion of Kobayashi hyperbolicity, also proved very effective.

Alternatively¹ one can argue that a holomorphic endomorphism of a genus $g \geq 2$ Riemann surface must be an automorphism since by the

¹I am grateful to the referee for this suggestion.

Riemann–Hurwitz formula its critical set is empty. Its set of holomorphic automorphisms is then finite by a theorem of Hurwitz.

2.1.2. Classification of compact complex surfaces. Compact complex surfaces also admit a similar classification, due to Enriques for the algebraic case and to Kodaira in general. There are also two distinct geometric conditions on a compact complex surface: being algebraic, and being Kähler. A comprehensive introduction to compact complex surfaces is [BHPVdV04], and Friedman’s [Fri98] and Beauville’s [Bea96] textbooks also provide instructive treatments.

2.1.3. Kodaira dimension. The key invariant distinguishing complex surfaces is the number of holomorphic differentials. Namely, let K_X denote the canonical bundle of a compact complex surface X ; its sections are given in local coordinates by $f(z_1, z_2)dz_1 \wedge dz_2$ with f holomorphic. Define the *Kodaira dimension* by

$$\kappa(X) := \limsup_n \frac{\log h^0(K_X^{\otimes n})}{\log n}$$

where $h^0(L)$ denotes the dimension of $H^0(L)$ – the space of sections of a line bundle L . It is known that $h^0(K_X^{\otimes n})$ grows polynomially in n , of degree at most 2. When X is algebraic, the above lim sup can be replaced by a genuine limit.

2.1.4. Enriques–Kodaira classification. Since blowing up a point does not change the birational isomorphism class of a surface, assume that the surface is *minimal* [BHPVdV04, VI.1]. The possibilities are then:

- $\kappa = -\infty$: *Rational surfaces*, i.e. ones bimeromorphic to \mathbb{P}^2 .
Ruled surfaces, i.e. \mathbb{P}^1 -bundles over curves (equivalently: projectivizations of 2-dimensional vector bundles over curves).
Class VII surfaces, they are not algebraic.
- $\kappa = 0$: *Tori*, *K3 surfaces*, *Enriques surfaces*, *bielliptic surfaces*
Kodaira surfaces, they are not Kähler.
- $\kappa = 1$: Properly elliptic surfaces, of the form $X \rightarrow C$ with general fiber an elliptic curve and with C a curve.
- $\kappa = 2$: General type surfaces.

Note that an Enriques surface is double-covered by a K3 surface, and bielliptic surfaces are isogenous to locally trivial bundles of elliptic curves over elliptic curves.

Only surfaces with $\kappa \leq 0$ admit nontrivial *endomorphisms*. Dynamically interesting endomorphisms on blowups of \mathbb{P}^2 at finitely many points have been constructed by Bedford and Kim [BK06] and McMullen [McM07].

For dynamically interesting *automorphisms*, on minimal surfaces one has to restrict to $\kappa = 0$, see [Can14, §2.5], and this case will be developed in the remainder of these notes with the study of K3 surfaces. Their study was initiated by Cantat [Can01a].

2.2. Definition and examples of K3s

2.2.1. Definition. A compact complex surface X is called² a *K3 surface* if it satisfies both of the following:

- (i) The canonical bundle K_X is holomorphically trivial, i.e. there exists a nowhere vanishing holomorphic 2-form Ω .
- (ii) It is simply connected.

The conditions can be succinctly expressed as $\pi_1(X) = 0$ and $K_X = 0$. The simple connectivity condition can be weakened but leads to the same surfaces: it suffices to assume that $H_1(X) = 0$, or a condition that also makes sense in characteristic p is $H^1(X, \mathcal{O}_X) = 0$.

2.2.2. Quartics. Consider smooth degree 4 surfaces in \mathbb{P}^3 . They are simply connected by the Lefschetz hyperplane theorem, and admit a nowhere vanishing holomorphic 2-form by the residue construction (see §2.2.3). Alternatively, using the adjunction formula one checks that K_X is trivial. Recall that $K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$ and for a quartic $X \subset \mathbb{P}^3$ we have:

$$K_X \cong K_{\mathbb{P}^3}(X)|_X \cong \mathcal{O}_{\mathbb{P}^3}(-4 + 4)|_X \cong \mathcal{O}_X$$

One can phrase the calculation differently as follows. The statement $K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$ says that a 3-form on \mathbb{P}^3 must have a pole along a surface of degree 4 (this can be seen by working in local coordinates and writing an explicit differential form). The subsequent application of the adjunction formula is a rephrasing of the residue construction, which we now discuss.

2.2.3. Residues. Suppose that M is a complex n -dimensional manifold and $S \subset M$ is complex $(n - 1)$ -dimensional. Assume that Ω is a meromorphic n -form on M with poles only along S , i.e. in local coordinates where $S = \{z_1 = 0\}$ we have

$$\Omega = \frac{f(z_1, \dots, z_n)}{z_1^k} dz_1 \wedge \dots \wedge dz_n$$

²“ainsi nommées en l’honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire” see comments to [Wei09], which also give information on the origin of the name “Teichmüller spaces”.

with f holomorphic. Write $f = \sum_{i \geq 0} z_1^i f_i(z_2, \dots, z_n)$ and define

$$\text{Res}_S \Omega := f_{k-1}(z_2, \dots, z_n) dz_2 \wedge \cdots \wedge dz_n.$$

2.2.4. Exercise (Residues are well-defined). Show that the residue $\text{Res}_S \Omega$ is a well-defined holomorphic $(n-1)$ -form on S , independent of the choice of coordinate system. *Hint: The head-on approach leads to complicated calculations. Use instead that in dimension 1, $\text{Res}(dg) = 0$ for any meromorphic function g and write any meromorphic 1-form in 1 variable as $\Omega = r \frac{dz_1}{z_1} + dg$.*

Using residues, build a nowhere vanishing holomorphic 2-form on a quartic surface in \mathbb{P}^3 .

2.2.5. Exercise (Uniqueness of holomorphic form). Show that if a line bundle L over a compact complex manifold has a nowhere vanishing holomorphic section Ω , then any other holomorphic section of L is a scalar multiple of Ω .

2.2.6. Kummer examples. Let $T := \mathbb{C}^2/\Lambda$ be a complex torus and set $Q := T/\pm 1$ to be the quotient by the involution $x \mapsto -x$ on the torus. Then Q has 16 singular double points and blowing them up gives a K3 surface X . Indeed the standard symplectic form on T survives the construction and vanishes nowhere on X , and one can check that $H_1(X) = 0$ using that the involution of T acts as -1 on $H_1(T)$.

2.2.7. Blowups. Recall that in local coordinates on \mathbb{A}^2 , the blowup at the origin is described as

$$\text{Bl}_0 \mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1 \text{ with coordinates } (x, y) \times (s : t)$$

using the equation $xt = ys$. The reader can gain familiarity with blowups using the following calculations.

2.2.8. Exercise (Log-canonical thresholds). For a function $f: \mathbb{C}^2 \rightarrow \mathbb{C}$, set

$$\text{lct}(f) := \sup \left\{ s : \int_{B_\varepsilon} |f(x, y)|^{-s} d\text{Vol} < +\infty \right\}$$

where B_ε denotes the ball of radius ε at the origin, for some sufficiently small $\varepsilon > 0$. Compute $\text{lct}(f)$ for $f(x, y) = x^a y^b$ and $f(x, y) = y^2 - x^3$. *Hint: For the second example, blow up the origin until it looks like the first example.*

2.2.9. Exercise (Volume form in Kummer construction). Verify that the holomorphic volume form on a complex torus descends, via the construction in §2.2.6, to a nowhere vanishing volume form on the associated Kummer K3. *Hint: To do so, it suffices to consider the quotient map $\mathbb{A}^2 \rightarrow Q$ given by $(x, y) \mapsto (-x, -y)$, compute Q explicitly,*

and then blow up its singular point. Equivalently, blow up \mathbb{A}^2 first to get $Q' \rightarrow \mathbb{A}^2$ and then lift the involution of \mathbb{A}^2 to Q' . Then compute what happens to $dx \wedge dy$ along these maps.

2.3. Topology of K3 surfaces

All K3 surfaces are diffeomorphic and so have the same topology. Indeed, a deformation theory argument (see §4.1) shows that any K3 surface can be put in a holomorphic family containing a Kummer example (§2.2.6). By Poincaré duality $H^3(X) = H^1(X) = 0$ so the only non-trivial homology group is H^2 . Cup product gives it a symmetric non-degenerate bilinear form and we first recall some relevant structures.

2.3.1. Lattices. A lattice is a finite rank free \mathbb{Z} -module Λ equipped with a non-degenerate symmetric bilinear form $\Lambda \times \Lambda \rightarrow \mathbb{Z}$, with the pairing of two elements denoted $v \cdot w$. A lattice is *unimodular* if the map induced by the bilinear form $\Lambda \rightarrow \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z})$ is an isomorphism. A lattice is *even* if v^2 is even for all $v \in \Lambda$.

Denote the extension of scalars to the reals by $\Lambda_{\mathbb{R}}$. The *signature* of Λ is the signature of the bilinear form on $\Lambda_{\mathbb{R}}$. Say that Λ is *indefinite* if the signature is indefinite, i.e. has both positive and negative directions.

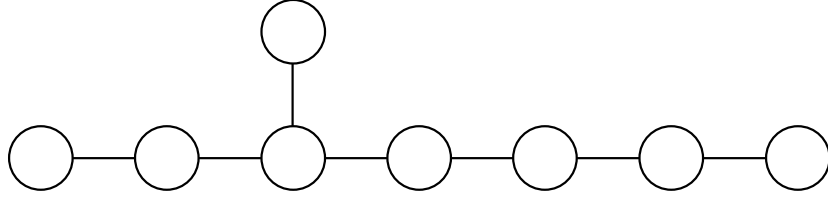
It is a fundamental theorem that if Λ is an even, unimodular, indefinite lattice, then it is unique up to isomorphism. Moreover if the signature of the pairing on $\Lambda_{\mathbb{R}}$ is (m, n) then $m \equiv n \pmod{8}$. See [Ser73, Ch. V] for a concise introduction to these questions.

2.3.2. Examples of even, unimodular lattices. The matrix $U := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ determines on \mathbb{Z}^2 an even unimodular lattice structure, of signature $(1, 1)$. It is sometimes called (confusingly) the hyperbolic plane, and we will denote by \mathbb{U} the corresponding even unimodular lattice.

The lattice \mathbb{E}_8 is determined from the E_8 Dynkin diagram as follows. The symmetric matrix determining the bilinear form on \mathbb{Z}^8 has 2 on the diagonal, -1 in the (i, j) entry if the vertices i, j are adjacent in the diagram, and 0 otherwise.

Together, \mathbb{E}_8 and \mathbb{U} serve as the building blocks of all even, unimodular, indefinite lattices.

2.3.4. The K3 lattice. The rank of $H^2(X, \mathbb{Z})$ for a K3 surface X is 22 (compute the Euler characteristic of the quartic in \mathbb{P}^3) and cup product makes it a unimodular lattice (by Poincaré duality). A calculation with Stiefel–Whitney classes implies that the lattice is even and it has signature $(3, 19)$ (via Hodge theory, see §2.4). It follows that there is

FIGURE 2.3.3. The E_8 Dynkin diagram.

a (non-unique) isomorphism $H^2(X, \mathbb{Z}) \rightarrow \mathbb{I}_{3,19} := \mathbb{U}^{\oplus 3} \oplus (-\mathbb{E}_8)^{\oplus 2}$ with the fixed lattice constructed from the basic building blocks.

2.3.5. Cohomology of the Kummer surface. One can alternatively see that for a K3 surface the rank of $H^2(X)$ is 22 by looking at Kummer examples from §2.2.6. Indeed for a torus T we have that $H^2(T; \mathbb{Z}) \cong \mathbb{Z}^6$, since it is the second exterior power $\Lambda^2 H^1(T; \mathbb{Z})$ of the first cohomology group. This also gives that as a lattice $H^2(T; \mathbb{Z}) \cong \mathbb{U}^3$ and so has signature $(3, 3)$. Note that the involution $x \mapsto -x$ on the torus acts trivially on H^2 , since it acts by (-1) on H^1 and hence by $(-1)^2 = 1$ on H^2 .

Let us denote by $\tilde{T} \rightarrow T$ the blowup of T at the 16 fixed points of the involution. Then $H^2(\tilde{T}; \mathbb{Z}) \cong \mathbb{Z}^{22}$ (where $22 = 6 + 16$) and note that the self-intersection of an exceptional curve of the blowup is (-1) . Now the involution of T lifts to \tilde{T} and still acts trivially on $H^2(\tilde{T})$, so at least the rank over \mathbb{Q} of $H^2(X)$ doesn't change. Similarly the signature $(3, 19)$ follows from this calculation, but note that the descent of integral lattice structure on H^2 from \tilde{T} to X requires more care. See [K3-85, Exposé VIII] for a careful treatment of this and more related to Kummer surfaces.

2.4. Hodge theory on K3 surfaces

We introduce the basic notions of Kähler geometry below in §3.1, but for now it suffices to know that every K3 surface is Kähler ([Siu83]) and thus admits a Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

with $H^{p,q} = \overline{H^{q,p}}$. The space $H^{2,0}$ is spanned by the holomorphic 2-form Ω and the intersection pairing determines a positive-definite hermitian metric on $H^{2,0} \oplus H^{0,2}$, while on $H^{1,1}$ the signature is $(1, 19)$. Denote by $H_{\mathbb{R}}^{1,1}$ the real space whose complexification is $H^{1,1}$.

2.4.1. Néron–Severi group. Define

$$\mathrm{NS}(X) := H^{1,1} \cap H^2(X, \mathbb{Z})$$

which is isomorphic, by the Lefschetz (1, 1)-theorem and the vanishing of $H^1(X, \mathbb{Z})$, with the group of holomorphic line bundles on X . A line bundle \mathcal{L} is identified with its first Chern class denoted $[\mathcal{L}]$. Moreover, the fundamental class of a complex curve $C \subset X$, denoted $[C]$, will also be in $\mathrm{NS}(X)$.

Denote by $\rho := \mathrm{rk}_{\mathbb{Z}} \mathrm{NS}(X)$; the signature of cup product on $\mathrm{NS}(X)$ can be $(1, \rho - 1)$, $(0, \rho)$ or $(0, \rho - 1)$. The K3 is algebraic if and only if the signature is $(1, \rho - 1)$, by the Kodaira embedding theorem [GH78, §1.4].

2.4.2. Transcendental lattice. Denote by $\mathcal{T}(X)$ the smallest subspace of $H^2(X; \mathbb{C})$ that is defined over \mathbb{Q} and contains $H^{2,0}$. It is clear that $\mathcal{T}(X) \subseteq \mathrm{NS}(X)^\perp$, and when cup product on $\mathrm{NS}(X)$ is non-degenerate (e.g. if X is algebraic) we have in fact equality.

2.4.3. Riemann–Roch and Serre duality. Because the canonical bundle of X is trivial, Serre duality implies that $h^i(\mathcal{L}) = h^{2-i}(\mathcal{L}^\vee)$, where \mathcal{L}^\vee denotes the dual line bundle of \mathcal{L} . The Riemann–Roch formula then becomes

$$(2.4.4) \quad h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^0(\mathcal{L}^\vee) = \frac{1}{2}[\mathcal{L}]^2 + 2$$

and it implies the existence of holomorphic sections of either \mathcal{L} or \mathcal{L}^\vee as soon as $[\mathcal{L}]^2 \geq -2$.

2.4.5. The (-2) curves. Given a $\delta \in \mathrm{NS}(X)$ with $\delta^2 = -2$, there exists a compact curve $C \subset X$ such that $[C] = \pm\delta$, and C is a union $C = \cup C_i$ with each $C_i \cong \mathbb{P}^1$ and $[C_i]^2 = -2$. This follows from an application of Eqn. (2.4.4) and an analysis of the possibilities, see [Huy16, 2.1.4].

2.4.6. The Weyl group. Denote by $\Delta_X \subset \mathrm{NS}(X)$ the set of all classes δ with $\delta^2 = -2$, and by Δ_X^\pm those which are represented by classes $[C]$ of (-2) curves. Consider the reflection

$$s_\delta(x) := x + (x \cdot \delta)\delta$$

and the group of orthogonal transformations generated by the transformations s_δ , called the *Weyl group* $W_X \subset \mathbf{O}(H^2)$. Because we have $\delta \in \mathrm{NS}(X)$, the action of each s_δ and hence all of W_X preserves the Hodge decomposition and the integral structure.

2.4.7. The Kähler chamber. Consider the action of W_X on $H^{1,1}$, where the transformation s_δ fixes the hyperplane $H_\delta \subset H^{1,1}(X)$ of classes orthogonal to δ . The cohomology classes $\alpha \in H^{1,1}(X)$ with $\alpha^2 > 0$ form two cones (exchanged by $\alpha \mapsto -\alpha$) and the set of classes outside all the hyperplanes H_δ form chambers; the action of W_X on the chambers is transitive (within a fixed cone).

Any Kähler metric (see §3.1) gives a cohomology class $[\omega] \in H_{\mathbb{R}}^{1,1}$. Denote by \mathcal{K}_X the set of all cohomology classes represented by a Kähler metric, and called the *Kähler cone*. It is a convex subset of $H^{1,1}$ invariant under positive scaling.

A Kähler metric ω picks out a chamber for W_X from the geometric condition

$$\int_C \omega > 0 \text{ for any compact curve } C.$$

Indeed, this condition applies to the (-2) curves $[C] = \delta$ and so $[\omega] \cdot \delta > 0$.

It is more difficult, but true, that \mathcal{K}_X coincides with the distinguished chamber of W_X . In other words, any cohomology class $[\omega] \in H_{\mathbb{R}}^{1,1}$ which pairs strictly positively with all (-2) curves, and satisfies $[\omega]^2 > 0$, can be represented by a Kähler metric ([K3-85, XIII, Prop. 4]).

3. Differential Geometry

Outline of section. In §3.1 we introduce the basic notions of Kähler geometry. These are followed by a discussion of Monge–Ampère equations and their connection to the space of Kähler metrics.

The Riemannian geometry point of view is taken up in §3.2, through the concept of holonomy. We introduce hyperkähler manifolds, of which K3 surfaces are fundamental examples.

3.1. Kähler geometry

For the following discussion, it is convenient to assume that X is a general compact complex n -dimensional manifold, with integrable complex structure $I: T_{\mathbb{R}}X \rightarrow T_{\mathbb{R}}X$ such that $I^2 = -\mathbf{1}$. Here $T_{\mathbb{R}}X$ denotes the real tangent bundle and its complexification $T_{\mathbb{C}}X := T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ splits as $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ according to the eigenvalues $\pm\sqrt{-1}$ of I .

3.1.1. Definition (Kähler metric). A Kähler form is a differential 2-form ω with $d\omega = 0$ and such that the symmetric bilinear form $g(-, -) := \omega(I-, -)$ is a Riemannian metric, which is called a Kähler metric.

3.1.2. Bundles and curvature. The canonical bundle $K_X := \Lambda^n(T_{\mathbb{C}}^{1,0}X)$ is a holomorphic line bundle and a Kähler form ω induces a metric ω^n on K_X . The Ricci curvature of the Kähler metric associated to ω is equal to the curvature of the holomorphic line bundle K_X equipped with the metric ω^n . In coordinates, if

$$\omega = \sqrt{-1} \sum g_{i,\bar{j}} dz_i \wedge \bar{d}z_j$$

then

$$(3.1.3) \quad \rho = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{i,\bar{j}})$$

is the alternating form giving the Ricci curvature.

The metric is called *Kähler–Einstein* if

$$(3.1.4) \quad \rho = k \cdot \omega \quad \text{with } k \in \mathbb{R}.$$

Recall that on a compact Riemann surface there always exists a Kähler–Einstein metric. For $\mathbb{P}^1(\mathbb{C})$ we have $k = 1$ (positive curvature), for elliptic curves $k = 0$ (zero curvature) and for higher genus $k = -1$ (negative curvature). In this particular case we can even give explicit formulas for ω :

$$\frac{2|dz|^2}{(1+|z|^2)^2} \quad |dz|^2 \quad \frac{2|dz|^2}{(1-|z|^2)^2}$$

The first expression gives the constant curvature +1 metric on $\mathbb{P}^1(\mathbb{C})$ in a chart on \mathbb{C} . The second expression gives a flat metric on \mathbb{C} which descends to any elliptic curve exhibited as a quotient of \mathbb{C} by a lattice. The last expression gives the constant curvature -1 metric on $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and it descends to any compact genus $g \geq 2$ Riemann surface exhibited as a quotient of Δ by a lattice in $\mathbf{PSL}_2(\mathbb{R})$.

Let us note that the group of holomorphic automorphisms of $\mathbb{P}^1(\mathbb{C})$ is $\mathbf{PSL}_2(\mathbb{C})$ and, since it is not compact, this group does not preserve any metric on $\mathbb{P}^1(\mathbb{C})$. On the other hand, any holomorphic automorphism of a compact Riemann surface of genus $g \geq 1$ will preserve a given constant curvature metric on that surface. This is reflected also in substantial analytic challenges when constructing Kähler–Einstein metrics in the case $k > 0$.

3.1.5. Yau’s theorems. Returning to the general setting of a compact complex n -manifold X from §3.1.2, the cohomology class $[\rho]$ of ρ as defined in Eqn. (3.1.3) is expressed in terms of the first Chern class of the tangent bundle and equal to $(2\pi)c_1(T_{\mathbb{C}}^{1,0}X)$. Therefore, since Eqn. (3.1.4) must also hold in cohomology, we must first look for a $k \in \mathbb{R}$ and Kähler class $[\omega]$ such that

$$[\rho] = k[\omega].$$

The existence of a Kähler–Einstein metric when $-\rho$ is a Kähler class (so $k < 0$) is due independently to Aubin and Yau, and the case $\rho = 0$ (so $k = 0$) is due to Yau [Yau78]. The case $k > 0$ is significantly more difficult and the subject of more recent activity by Chen, Donaldson, and Sun [CDS14].

For K3 surfaces we have $c_1(T_{\mathbb{C}}^{1,0}X) = 0$ and $k = 0$, so Yau’s theorem says that in any Kähler class $[\omega']$ there exists a unique Ricci-flat Kähler form ω . Because of the dd^c (or $\partial\bar{\partial}$)-lemma, see [Huy05, 3.A.22] we know that two cohomologous Kähler metrics are related by

$$\omega = \omega' + \sqrt{-1}\partial\bar{\partial}\phi$$

where ϕ is called a *potential*.

If Ω denotes the holomorphic 2-form on a K3 surface, then $\Omega \wedge \bar{\Omega}$ induces a flat metric on K_X , so the existence of a Ricci-flat metric is equivalent to solving the equation

$$(\omega' + \sqrt{-1}\partial\bar{\partial}\phi)^2 = \Omega \wedge \bar{\Omega}$$

provided that $\int_X (\omega')^2 = \int_X \Omega \wedge \bar{\Omega}$. This is called a *Monge–Ampère equation*. The result proved by Yau is more general.

3.1.6. Theorem ([Yau78]). *On an n -dimensional Kähler manifold (X, ω) let $f \in C^\infty(X)$ be a function such that $\int_X \omega^n = \int_X e^f \omega^n$. Then there exists $\phi \in C^\infty(X)$ (unique up to a constant) such that*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^f \omega^n$$

with $\omega + \sqrt{-1}\partial\bar{\partial}\phi$ also a Kähler metric.

One way to solve the Monge–Ampère equation in [Theorem 3.1.6](#) is to use the continuity method. Concretely, consider a 1-parameter family of functions f_s such that $f_0 \equiv 0$ and $f_1 = f$, for instance by scaling f linearly and adjusting the constants appropriately. One shows that the values of s for which a solution ϕ_t exists is both open and closed, and since it contains $s = 0$ by construction, it follows that a solution exists for $s = 1$ as well. Openness follows from an elementary application of the inverse function theorem in appropriate function spaces. Closedness is the heart of the problem and requires a priori estimates for the solutions of Monge–Ampère equations.

3.1.7. The space of Kähler metrics. A different approach to solving Monge–Ampère equations is based on a variational technique, i.e. solutions are characterized as extremizers of functionals. While showing the regularity (e.g. smoothness) of extremizers is as difficult as showing closedness in the continuity method, the formal aspects of the functionals that appear reveal more about the structure of the space

of all Kähler metrics. The presentation below is heuristic and will not introduce the necessary function spaces, working formally instead. A rigorous presentation is in the recent monograph of Guedj and Zeriahi [GZ17].

For a Kähler metric ω , the space of all cohomologous Kähler metrics is parametrized by the space of functions $\phi \in C^\infty(X)$, called *potentials*, subject to the requirement

$$\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi \text{ is a Kähler metric}$$

and modulo constant functions; denote the space by \mathcal{K}_ω . View the Monge–Ampère operator as a map from functions to measures³:

$$MA(\phi) := \omega_\phi^n$$

Since functions and measures are dual one can view MA as a differential 1-form on the space of functions. It is closed, and in fact has the following explicit primitive:

$$E(\phi) := \frac{1}{n+1} \sum_{i=0}^n \int_X \phi \omega_\phi^i \wedge \omega^{n-i}$$

called the energy of ϕ . This is justified by the following calculations, which show that moreover E is a concave functional on the space of functions.

3.1.8. Proposition. *Suppose that ϕ_t is a 1-parameter family of potentials. Then*

$$\frac{d}{dt} E(\phi_t) = \int_X \dot{\phi}_t MA(\phi_t)$$

which shows that formally $dE = MA$ on the space of Kähler potentials.

Suppose additionally that $\phi_t = \phi_0 + tv$ where $v \in C^\infty(X)$. Then

$$\frac{d^2}{dt^2} E(\phi_t) \leq 0$$

Note that formulating the concavity of E (the second statement above) uses the affine structure of the space of functions.

³Probability measures, if $\int_X \omega^n = 1$

Proof. We calculate directly:

$$\begin{aligned}
\frac{d}{dt}E(\phi_t) &= \frac{1}{n+1} \frac{d}{dt} \sum_{j=0}^n \int_X \phi_t \omega_{\phi_t}^j \wedge \omega^{n-j} \\
&= \frac{1}{n+1} \sum_{j=0}^n \int_X \left(\dot{\phi}_t \omega_{\phi_t}^j + j \cdot \phi_t (\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t) \omega_{\phi_t}^{j-1} \right) \wedge \omega^{n-j} \\
&= \frac{1}{n+1} \sum_{j=0}^n \int_X \left(\dot{\phi}_t \omega_{\phi_t}^j + j \cdot (\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t) \dot{\phi}_t \omega_{\phi_t}^{j-1} \right) \wedge \omega^{n-j} \\
&= \frac{1}{n+1} \sum_{j=0}^n \int_X \left(\dot{\phi}_t \omega_{\phi_t}^j + j \cdot (\omega_{\phi_t} - \omega) \dot{\phi}_t \omega_{\phi_t}^{j-1} \right) \wedge \omega^{n-j} \\
&= \int_X \dot{\phi}_t \omega_{\phi_t}^n = \int_X \dot{\phi}_t MA(\phi_t)
\end{aligned}$$

where we used: the Leibniz rule, integration by parts, the expression $\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t = \omega_{\phi_t} - \omega$, and finally a telescoping sum.

To compute the second derivative, recall that now ϕ_t varies affinely, so $\ddot{\phi}_t = 0$. Compute directly again, using the previous expression as a starting point:

$$\begin{aligned}
\frac{d^2}{dt^2}E(\phi_t) &= \int_X \ddot{\phi}_t \omega_{\phi_t}^n + \int_X \dot{\phi}_t n (\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t) \wedge \omega_{\phi_t}^{n-1} \\
&= \int_X \dot{\phi}_t n (\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t) \wedge \omega_{\phi_t}^{n-1} \\
&= -n \int_X \sqrt{-1} (\partial \dot{\phi}_t) \wedge (\bar{\partial} \dot{\phi}_t) \wedge \omega_{\phi_t}^{n-1} \leq 0
\end{aligned}$$

since $\sqrt{-1} \partial \xi \wedge \bar{\partial} \xi \geq 0$ for any $\xi \in C^\infty(X)$. \square

3.1.9. The variational approach. In Proposition 3.1.8 we established that $E: \mathcal{K}_\omega \rightarrow \mathbb{R}$ is a concave function, with $dE = MA$ formally. Moreover E is “increasing” in the sense that if $\dot{\phi}_t \geq 0$ then $\frac{d}{dt}E(\phi_t) \geq 0$. In order to solve the equation $MA(\phi) = \mu_0$ for a fixed measure μ_0 , consider the functional

$$\mathcal{F}_{\mu_0}(\phi) := E(\phi) - \int_X \phi d\mu_0$$

on the space \mathcal{K}_ω . One expects a maximum of \mathcal{F}_{μ_0} , achieved at ϕ_0 , to solve $dE(\phi_0) = \mu_0$. For further information in this direction, see also Demailly’s survey [Dem17].

3.2. Holonomy point of view

For a more detailed exposition of the concepts in this section, see [K3-85, Exp. XV]. On a Riemannian manifold (X, g) the Levi-Civita connection defines parallel transport along paths connecting $x, y \in X$ inducing maps between tangent spaces $T_x X \rightarrow T_y X$.

3.2.1. Definition (Holonomy of a Riemannian metric). The set of all maps in $\mathbf{GL}(T_x X)$ obtained as parallel transport along loops based at $x \in X$ is called *the holonomy group of the metric g , at the point x* . Restricting to loops based at $x \in X$ that are null homotopic defines the *restricted holonomy group*.

3.2.2. Remark.

- (i) The holonomy group is contained in the orthogonal group $\mathbf{O}(T_x X)$ determined by the metric, since the Levi-Civita connection preserves the metric. Since the group is also closed, it is a compact Lie group.
- (ii) A smooth path connecting $x, y \in X$ induces by parallel transport a map $T_x X \rightarrow T_y X$ which identifies the holonomy groups. Hence the conjugacy class of the holonomy group is independent of the basepoint and we can speak of “the” holonomy group (assuming X is connected).
- (iii) The Lie algebra of the holonomy group can be computed in terms of the curvature tensors at all the points.

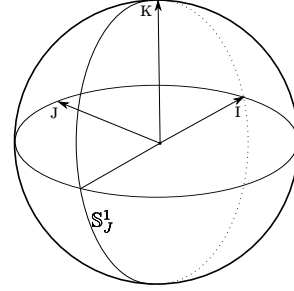
3.2.3. Example.

- (i) The holonomy group of a Kähler manifold is contained in $\mathbf{U}(n)$. Indeed, the condition $d\omega = 0$ is equivalent to the complex structure $I: TX \rightarrow TX$ being preserved by parallel transport.
- (ii) A Kähler manifold is Ricci-flat if and only if the *restricted* holonomy group is contained in $\mathbf{SU}(n)$. On the other hand, the existence of a holomorphic nowhere vanishing volume form is equivalent to the holonomy group being contained in $\mathbf{SU}(n)$.
- (iii) The manifold is called *hyperkähler* if the holonomy is contained in $\mathbf{Sp}(n)$, the group of $n \times n$ quaternion matrices which are unitary for an appropriate metric. In this case, parallel transport preserves three complex structures I, J, K with the usual relations, and in fact any complex structure of the form $xI + yJ + zK$ with $x^2 + y^2 + z^2 = 1$.
- (iv) For a symmetric space G/K with G a semisimple Lie group (with finite center), K a maximal compact, and the metric given by the Killing form, the holonomy group is the connected

component of the identity of K . The same holds for G compact and $K \subset G$ a compact subgroup.

3.2.4. K3s as hyperkähler manifolds. For an in depth treatment of these concepts, see the collection of notes [GHJ03].

The exceptional isomorphism of compact Lie groups $\mathbf{SU}(2) \cong \mathbf{Sp}(1)$ implies that on a Ricci-flat K3 surface, there exists besides the complex structure I another one J , with $IJ = -JI = K$. In fact there is a whole sphere \mathbb{S}^2 of complex structures, as per Example 3.2.3(iii), called the *twistor sphere*.



There is a relationship between the 2-form ω defining the Ricci-flat metric, the holomorphic 2-form Ω , and the complex structures I, J, K . After rescaling Ω by an appropriate complex number, we have

$$\begin{aligned} \omega &= \omega_I = g(I-, -) \\ \operatorname{Re} \Omega &= \omega_J = g(J-, -) \\ \operatorname{Im} \Omega &= \omega_K = g(K-, -) \end{aligned}$$

Moreover, for $I_t := xI + yJ + zK$ with $x^2 + y^2 + z^2 = 1$, the 2-form $\omega_{I_t} := g(I_t-, -)$ is also Kähler, i.e. $d\omega_{I_t} = 0$, as can be seen from the above relations.

3.2.5. Special Lagrangians in dimension 1. Consider the complex plane \mathbb{C} equipped with the holomorphic form $\Omega := dz$ and Euclidean flat metric. Then straight lines can be characterized as distance-minimizing curves. Alternatively, if they meet the horizontal at angle θ , straight lines are characterized as curves on which $e^{\sqrt{-1}\theta}\Omega$ restricts to a real-valued 1-form inducing the length element of the ambient flat metric.

3.2.6. Special Lagrangians in general. Suppose now that (X, ω, Ω) is a complex n -dimensional manifold, ω is a Kähler metric, and Ω is a holomorphic n -form. Then a real n -dimensional submanifold $L \subset X$ is *special Lagrangian* (abbreviated: sLag) if:

Lagrangian: The restriction $\omega|_L \equiv 0$, i.e. L is Lagrangian in the symplectic manifold (X, ω) .

special: The restriction $\Omega|_L$ is a real n -form inducing the same volume on L as the Riemannian metric on X determined by ω .

Note that in the definition, we can start with $e^{\sqrt{-1}\theta}\Omega$ to have a “rotated” variant. Special Lagrangians are locally volume minimizing, since they

are calibrated manifold, i.e. their volume can be computed using a closed differential form.

3.2.7. Counting sLags in dimension 1. In complex dimension 1, i.e. on Riemann surfaces, we saw that sLags are the same as straight lines determined by the flat metric induced by a holomorphic 1-form. Counting the number of such curves that are *closed* has been studied extensively and implies also counts for the number of closed billiard trajectories in rational-angled polygons. Veech [Vee89] showed that the number of closed billiard trajectories of length at most L is asymptotic to $c_n L^2$ for an explicit constant c_n . For general rational-angled polygons Masur [Mas88] proved that the number of closed trajectories has quadratic upper and lower bounds, and results of Eskin, Mirzakhani, and Mohammadi [EMM15] imply a quadratic asymptotic in an averaged sense.

Two features are important and recur. First, given one closed billiard trajectory, perturbing it (but keeping the angle fixed) gives another closed billiard trajectory. Second, the angles of the trajectories will equidistribute on the unit circle.

3.2.8. Counting sLags in dimension 2. When (X, ω, Ω) is a K3 surface with a Ricci-flat metric, there is again an abundance of special Lagrangian 2-tori. In dimension 1 the angle was on the unit circle, while on K3s the choice of angle corresponds to equators on the twistor sphere. Again given one sLag torus, one can deform it to obtain a foliation (with closed leaves) on the K3 surface. In fact, while in dimension 1 there can be “barriers” to obtaining a foliation with closed leaves on the entire space, in dimension 2 this barrier can be passed and one gets a *special Lagrangian torus fibration* of the entire K3 surface.

It is shown in [Fil20] that the number of such fibrations, with volume of a fiber bounded by V , is

$$N(V) = C \cdot V^{20} + O(V^{20-\delta})$$

at least when the K3 surface is sufficiently general. The constant C is, up to rational factors, equal to $\frac{1}{\pi^{20}\zeta(11)}$ and arises as a ratio of volumes of homogeneous moduli spaces.

3.2.9. SLags and hyperkähler metrics. Using the hyperkähler structure on a K3 surface, one finds that a special Lagrangian on a K3 surface is, in fact, a holomorphic curve for a *different* complex structure. For example, sLag tori lead to elliptic curves. This connection allows one to reduce the question of counting sLags to counting special vectors in the K3 lattice, and then to a problem in homogeneous dynamics.

4. Torelli theorems

Outline of section. A remarkable feature of K3 surfaces is that their geometry is, to a large extent, determined by the Hodge structure. For example, if two K3 surfaces have abstractly isomorphic Hodge structures, then they are in fact isomorphic. The correspondence is even stronger and allows one to construct automorphisms of K3s using Hodge structures.

This section describes some of these results and the related background. Basic facts from deformation theory are recalled in §4.1. The period domains relevant to K3 surfaces are described in §4.2. Finally, some of the Torelli theorems valid for K3 surfaces are in §4.3.

4.1. Complex deformation theory

4.1.1. Setup. The discussion in this section is quite general and extends beyond K3 surfaces. For a more in depth treatment of the concepts in this section, see [K3-85, Exp. V]. We will consider proper holomorphic submersions $\mathcal{X} \xrightarrow{\pi} B$ between complex manifolds. Assume that B is simply connected, e.g. the unit ball in \mathbb{C}^N , and equipped with a basepoint $b_0 \in B$. For a point $b \in B$ let \mathcal{X}_b denote $\pi^{-1}(b)$. The data of $\mathcal{X} \xrightarrow{\pi} B$ will be called a *deformation* of \mathcal{X}_{b_0} .

4.1.2. Definition (Universal family). The deformation $\mathcal{X} \xrightarrow{\pi} B$ is a *universal deformation* of \mathcal{X}_{b_0} if the following holds. For any other deformation $\pi': \mathcal{X}' \rightarrow B'$ and isomorphism $\chi_0: \mathcal{X}'_{b'_0} \rightarrow \mathcal{X}_{b_0}$ there exists an open $B'' \subset B'$ containing b'_0 and *unique* holomorphic maps χ, β giving a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}'' := \pi'^{-1}(B'') & \xrightarrow{\chi} & \mathcal{X} \\ \pi' \downarrow & & \downarrow \pi \\ B'' & \xrightarrow{\beta} & B \end{array}$$

such that $\chi|_{\mathcal{X}'_{b'_0}} = \chi_0$ and $\beta(b'_0) = b_0$. If the maps are not required to be unique, the deformation is called *versal*.

The following result gives a useful criterion for when a universal deformation exists. For the statement, Θ_X denotes the sheaf of holomorphic vector fields on X and $H^\bullet(X, \Theta_X)$ are the sheaf cohomology groups.

4.1.3. Theorem (Kodaira–Spencer–Nirenberg). *Suppose X is a compact complex manifold with $H^0(X, \Theta_X) = 0$ and $H^2(X, \Theta_X) = 0$.*

Then there exists a universal deformation of X , whose base is an open subset of $H^1(X, \Theta_X)$ containing the origin.

The space $H^0(X, \Theta_X)$ denotes the global holomorphic vector fields on X , which can be viewed as infinitesimal automorphisms. When $H^2(X, \Theta_X) \neq 0$, a versal deformation space still exists, but it can be singular.

4.1.4. Tangent sheaf for K3 surfaces. Suppose now that X is a compact complex manifold with a nowhere vanishing holomorphic 2-form Ω . Then Ω induces a sheaf isomorphism $\Theta_X \rightarrow \Theta_X^\vee$, where Θ_X^\vee is the sheaf of differential 1-forms. When X is Kähler, the dimension of the sheaf cohomology groups of Θ_X^\vee can be computed from the Hodge numbers as $\dim H^p(X, \Theta_X^\vee) = \dim H^{1,p}(X)$ and in fact there are canonical isomorphism between the corresponding vector spaces.

In the case of K3 surfaces we have $H^{p,q}(X) = 0$ unless $p + q$ equals 0, 2, 4, which implies that the conditions of [Theorem 4.1.3](#) are satisfied. Furthermore the dimension of the universal deformation is 20 (which is $\dim H^{1,1}(X)$), and following through the cohomological calculations gives that the deformations are canonically parametrized by an open set in $\text{Hom}(H^{2,0}, H^{1,1})$.

4.2. Period domains

4.2.1. Definition (Marked K3s). Let $\Lambda := \mathbb{I}_{3,19}$ denote the unique even, unimodular lattice of signature $(3, 19)$.

A *marking* of a K3 surface X is an isomorphism of lattices $\iota: \Lambda \rightarrow H^2(X, \mathbb{Z})$. A *marked family* of K3 surfaces $\mathcal{X} \xrightarrow{\pi} B$ is a marking on each fiber $\mathcal{X}_t = \pi^{-1}(t)$, compatible with local identifications of $H^2(\mathcal{X}_t, \mathbb{Z})$.

Let \mathcal{M}_Λ denote the space of marked K3 surfaces, up to marking-preserving isomorphisms. For the lattice Λ , extensions of scalars to a field or ring k are denoted Λ_k .

4.2.2. Period domain. Consider the *period domain*

$$(4.2.3) \quad \mathcal{D}_\Lambda := \{[\alpha] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : \alpha \cdot \alpha = 0, \alpha \cdot \bar{\alpha} > 0\}$$

An element $[\alpha] \in \mathcal{D}_\Lambda$ determines a Hodge decomposition

$$\Lambda_{\mathbb{C}} = [\alpha] \oplus ([\alpha] \oplus [\bar{\alpha}])^\perp \oplus [\bar{\alpha}]$$

which mimics the Hodge decomposition of the second cohomology of a K3 surface.

4.2.4. Associated groups. Consider the orthogonal groups $G := \mathbf{O}(\Lambda_{\mathbb{R}})$, $\Gamma = \mathbf{O}(\Lambda_{\mathbb{Z}})$ and $H = \text{Stab}_G(\alpha) \cong \mathbf{O}_2(\mathbb{R}) \times \mathbf{O}_{1,19}(\mathbb{R})$, for some $[\alpha] \in \mathcal{D}_{\Lambda}$. Therefore we have

$$\mathcal{D}_{\Lambda} \cong G/H$$

in a G -equivariant way, and in particular there is a Γ -action on \mathcal{D}_{Λ} .

4.2.5. Period map. There is a natural map $\mathcal{M}_{\Lambda} \xrightarrow{\text{Per}} \mathcal{D}_{\Lambda}$, called the period map, defined as follows. For a marked K3 surface $(X, \iota) \in \mathcal{M}_{\Lambda}$, set

$$\text{Per}(x, \iota) := \iota^{-1} \left(H^{2,0}(X) \right) \in \mathbb{P}(\Lambda_{\mathbb{C}})$$

That the period map lands in \mathcal{D}_{Λ} follows from the properties of the Hodge structure of a K3 surface. The period map is holomorphic (this holds more generally and follows from basic results in the deformation theory of complex manifolds). Furthermore, it is Γ -equivariant by construction.

4.3. Torelli theorems

The following result is due, in various levels of generality, to Pyatetski-Shapiro–Shafarevich, Looijenga–Peters, Todorov, and Burns–Rapoport.

4.3.1. Theorem (Torelli theorem for K3 families). *The period map $\mathcal{M}_{\Lambda} \rightarrow \mathcal{D}_{\Lambda}$ is a local covering map between complex manifolds.*

The image of the period map is all of \mathcal{D}_{Λ} .

The next construction, due to Atiyah [Ati58], illustrates how \mathcal{M}_{Λ} can fail to be separated.

4.3.2. Example (Flops). There exist two holomorphic families $\mathcal{X}_i \xrightarrow{\pi_i} \Delta = \{|z| < 1\}$ with the following properties. First, the central fibers are biholomorphic: $\mathcal{X}_{1,0} \cong \mathcal{X}_{2,0}$. Second, the families over the punctured disc are isomorphic: there exists an isomorphism

$$\mathcal{X}_1|_{\Delta^{\times}} \xrightarrow{\sim} \mathcal{X}_2|_{\Delta^{\times}}$$

which commutes with projections to $\Delta^{\times} = \{0 < |z| < 1\}$. Nevertheless, there does not exist an isomorphism $\mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ commuting with projection to Δ .

The monodromy of the transformation going around the central fiber squares to the identity in the smooth mapping class group (Kronheimer) but is infinite order in the symplectic mapping class group (Seidel). See [Sei08] for more on this.

The next result refines [Theorem 4.3.1](#) to identify isomorphism classes of K3s and their automorphisms.

4.3.3. Theorem (Torelli theorem for individual K3s). *Suppose that X_1, X_2 are two K3 surfaces. If there exists an isomorphism $f: H^2(X_1) \rightarrow H^2(X_2)$ preserving the \mathbb{Z} -structure, Hodge structure, and cup product, then $X_1 \cong X_2$.*

If moreover f takes the Kähler cone of X_1 to that of X_2 , then there exists a unique isomorphism $F: X_2 \rightarrow X_1$ with $F^ = f$ on cohomology.*

4.3.4. Kummer examples can be characterized cohomologically. [K3-85, Exp. IX, Prop. 2] gives a cohomological characterization of Kummer examples. Specifically, suppose that $[\alpha] \in \mathcal{D}_\Lambda \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ is a point in the period domain. Set $L_{[\alpha]} := ([\alpha] \oplus [\bar{\alpha}]) \cap \Lambda_{\mathbb{R}}$ to be the real 2-dimensional subspace, whose complexification would correspond to $H^{2,0} \oplus H^{0,2}$ in the Hodge decomposition. Then $[\alpha]$ is the period of a marked Kummer surface if and only if $L_{[\alpha], \mathbb{Z}} := L_{[\alpha]} \cap \Lambda_{\mathbb{Z}}$ has rank 2 over \mathbb{Z} , and if $x \in L_{[\alpha], \mathbb{Z}}$ then $x^2 \equiv 0 \pmod{4}$.

Using this cohomological characterization, as well as an analysis of the period map, shows that any K3 surface can be deformed to a Kummer example. To do so, it suffices to show that the period points of Kummer surfaces are dense in the associated period domain.

5. Dynamics on K3s

Outline of section. We can now discuss examples and basic properties of K3 surface automorphisms. After giving some examples in §5.1, we proceed to discuss entropy and the Gromov–Yomdin theorem in §5.2. Finally, some elementary and useful general properties are discussed in §5.3.

5.1. Some basic examples

We begin by describing some concrete examples of K3 surfaces with dynamically interesting automorphisms.

5.1.1. The $(2, 2, 2)$ examples. Consider a smooth surface $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ cut out by a multi-degree $(2, 2, 2)$ polynomial, i.e. if $(X_0 : X_1, Y_0 : Y_1, Z_0 : Z_1)$ are the homogeneous coordinates, then the equation for F has degree 2 in each of the variables. Concretely, in a chart given by \mathbb{A}^3 one can take

$$x^2 + y^2 + z^2 + t(xyz) + 1 = 0$$

with t as a parameter, and compactify to $(\mathbb{P}^1)^3$ by homogenizing each variable individually.

For each of the \mathbb{P}^1 factors, projecting X along it to $\mathbb{P}^1 \times \mathbb{P}^1$ gives a $2 : 1$ map and an involution exchanges the two sheets. Concretely, for the above example we have

$$\sigma_x(x, y, z) = \left(\frac{1 + y^2 + z^2}{x}, y, z \right)$$

The first entry is determined from the formula for the coefficients of a quadratic equation in terms of the roots, and can be alternatively written as $-tyz - x$. Analogously one defines σ_y, σ_z and together these generate a free group, modulo the relations $\sigma_{\bullet}^2 = 1$.

See [Maz92] for further questions about this family.

5.1.2. Kummer examples. Suppose that T is a complex 2-torus with $f_T : T \rightarrow T$ a linear automorphism; for example take $T = E \times E$ with E an elliptic curve, then f_T can be constructed from a matrix in $\mathbf{SL}_2(\mathbb{Z})$ using the group structure on E .

Perform the Kummer construction on T (see §2.2.6) and observe that the linear automorphism f_T extends to $f_X : X \rightarrow X$. The topological entropy (see §5.2) of f_T and f_X is the same, the measure of maximal entropy is given by the holomorphic 2-form, and the invariant currents are smooth (see Theorem 7.1.1 for these concepts).

A more general definition of Kummer examples is introduced in [CD20b, Def. 1.3]. Specifically, X can be any projective surface and f an automorphism, and the requirements are that there exists a birational map $X \rightarrow X'$ to an orbifold, a finite orbifold cover $T \rightarrow X'$, where T is a torus, and corresponding automorphisms $f_{X'}, f_T$, with the natural commutation relations between them and to f . In dynamical systems terminology, one would say that (X, f) and (T, f_T) admit a common (finite) factor.

5.1.3. Automorphisms and Hodge theory. Although the existence of a Hodge decomposition requires a Kähler metric, the decomposition itself only depends on the complex structure. Therefore, any holomorphic automorphism preserves the Hodge decomposition of a complex manifold. Furthermore, the automorphism preserves the \mathbb{Z} -structure and cup product in cohomology.

In the case of K3 surfaces, any holomorphic automorphism preserves the decomposition $H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. In particular, it preserves $H_{\mathbb{R}}^{1,1}$ and hence gives an element in $\mathbf{O}(H_{\mathbb{R}}^{1,1})$, which is an orthogonal group of signature $(1, 19)$.

5.1.4. Types of automorphisms. Elements of $\mathbf{O}(\mathbb{R}^{1,19})$ can be classified (after perhaps raising to a power) according to their action on $\mathbb{R}^{1,19}$ as follows:

- *hyperbolic* elements have an eigenvector v with eigenvalue λ , with $|\lambda| > 1$ and $v \cdot v = 0$.
- *parabolic* elements are unipotent (with non-trivial Jordan block) and fix a vector with $v \cdot v = 0$.
- *elliptic* elements fix a vector with $v \cdot v = 1$.

Hyperbolic elements are also frequently called “loxodromic”. We will use the same adjectives for automorphisms of K3 surfaces, according to their action on $H_{\mathbb{R}}^{1,1}$.

Note that because automorphisms preserve an integral structure in cohomology, elliptic ones will necessarily be of finite order. Parabolic automorphisms will preserve the fibers of a map $X \rightarrow \mathbb{P}^1$, which will be elliptic curves. From the dynamical point of view, the most interesting ones are the hyperbolic automorphisms. They have positive entropy by Gromov–Yomdin’s [Theorem 5.2.3](#) below.

5.2. Entropy

5.2.1. Coverings and Nets. Let (X, d) be a metric space and $\varepsilon > 0$. A subset $S \subset X$ is ε -*separated* if $\forall s_1, s_2 \in S$ we have $d(s_1, s_2) \geq \varepsilon$ when $s_1 \neq s_2$. A subset $S \subset X$ is an ε -*covering* if for all $x \in X$ there is $s \in S$ with $d(x, s) < \varepsilon$.

Observe that a maximal ε -separated set is also an ε -covering. Conversely, given an ε -covering $C \subset X$ and a (2ε) -separated set $S \subset X$, there is an injection $S \hookrightarrow C$ by assigning to each element of S one of the elements in C that is at distance less than ε from it.

As a consequence, for many purposes it is equivalent to work with maximal ε -separated sets or minimal ε -coverings, where maximal and minimal are taken according to the cardinality. For convenience, we will work with maximal ε -separated sets and denote by $\mathcal{S}(X, d, \varepsilon)$ their cardinality.

5.2.2. Topological entropy. Let $f: X \rightarrow X$ be a continuous map of a metric space and define the new distances

$$d_n(x, y) := \max_{i=0 \dots n} d(f^i x, f^i y)$$

which measures the maximal distance at which the two points diverge after n iterates of the dynamics. Define

$$h(f, \varepsilon) := \limsup_{n \rightarrow \infty} \frac{\log \mathcal{S}(X, d_n, \varepsilon)}{n}$$

$$h_{top}(f) := \lim_{\varepsilon \rightarrow 0} h(f, \varepsilon)$$

Provided we can only make ε -accurate measurements, the first quantity measure the exponential growth rate of distinct trajectories, as we observe the dynamics up to time n .

5.2.3. Theorem (Gromov–Yomdin). *Let $f: X \rightarrow X$ be a holomorphic endomorphism of a compact Kähler manifold. Then*

$$h_{\text{top}}(f) = \log \rho(f)$$

where $\rho(f)$ is the spectral radius of f^* acting on the cohomology $H^\bullet(X)$.

5.2.4. Remark.

- (i) Yomdin [Yom87] proved that for any smooth map of a compact manifold one has the inequality

$$h_{\text{top}}(f) \geq \log \rho(f)$$

while Gromov [Gro03] proved, for compact Kähler manifolds, the reverse inequality $h_{\text{top}}(f) \leq \log \rho(f)$. Thus for a K3 surface automorphism, f^* is hyperbolic on H^2 if and only if $h_{\text{top}}(f) > 0$.

Below, we will present the proof of Gromov’s half of [Theorem 5.2.3](#).

- (ii) Gromov’s theorem fails for non-Kähler complex manifolds. For example take a cocompact lattice $\Gamma \subset \mathbf{SL}_2(\mathbb{C})$ coming from a compact hyperbolic 3-manifold. Then the time-one map of the geodesic flow on $\Gamma \backslash \mathbf{SL}_2(\mathbb{C})$ has positive entropy but is homotopic to the identity.

5.2.5. Preparations for Gromov’s theorem. Consider the embedding

$$\begin{aligned} \Delta_{f,n}: X &\rightarrow X \times \cdots \times X \\ x &\mapsto (x, fx, \dots, f^n x) \end{aligned}$$

and denote by $\Gamma_{f,n} \subset X^{n+1}$ the image. Fix a Kähler metric ω on X and endow X^{n+1} with the induced Kähler metric $\omega_{[n]} := \omega \boxtimes \cdots \boxtimes \omega$. Let $d_{[n]}$ denote the induced distance on X^{n+1} . The estimate between the dynamically defined distances d_n and $d_{[n]}$:

$$d_n(x_1, x_2) \leq d_{[n]}(\Delta_{f,n}(x_1), \Delta_{f,n}(x_2)) \leq n \cdot d_n(x_1, x_2)$$

allows us to use $d_{[n]}$ in the definition of entropy from [§5.2.2](#), since the factor of n disappears after taking log and dividing by n .

Gromov’s theorem will follow from the next two results, which we prove below.

5.2.6. Proposition (Volume growth). *For the Kähler metric $\omega_{[n]}$ on X^{n+1} we have*

$$\lim_{n \rightarrow \infty} \frac{\log \text{Vol}(\Gamma_{f,n})}{n} = \log \rho(f)$$

where $\rho(f)$ is the spectral radius of f^* acting on the cohomology of X .

5.2.7. Proposition (Lower bounds on volume). *Let $V \subset M$ be complex manifolds with a Kähler metric on M , with a uniform bound K on the sectional curvatures. Given $\varepsilon > 0$ there exists $\delta = \delta(\dim V, \varepsilon, K) > 0$ (but independent of $\dim M$) such that if $x \in V$ then*

$$\text{Vol}(B(x, \varepsilon) \cap V) \geq \delta$$

where $B(x, \varepsilon)$ denotes the ball of radius ε at x .

This last result is valid more generally for minimal surfaces in Riemannian manifolds, of which complex submanifolds of Kähler manifolds are examples.

5.2.8. Proof of Gromov's theorem. By Proposition 5.2.7 applied to $\Gamma_{f,n} \subset X^{n+1}$ we have that

$$\mathcal{S}(\Gamma_{f,n}, d_{[n]}, \varepsilon) \cdot \delta \leq \text{Vol}(\Gamma_{f,n})$$

since the ε -separated set gives disjoint balls with lower bounds on volume. Combined with Proposition 5.2.6 and the definition of entropy, the result follows.

5.2.9. Proof of Proposition 5.2.6. It is convenient to introduce the quantities

$$(5.2.10) \quad \delta_p(f) := \lim_{n \rightarrow \infty} \left(\int_X \omega^q \wedge (f^*)^n \omega^p \right)^{\frac{1}{n}} \quad \text{where } p + q = \dim_{\mathbb{C}} X.$$

We'll check in a moment that the limit indeed exists, and that $\delta_p(f)$ is the spectral radius of f^* on $H^{p,p}(X)$; assume this for now.

To proceed, note that the volume of $\Gamma_{f,n}$ is computed using the formula

$$\text{Vol}(\Gamma_{f,n}) = \int_X \left(\omega + f^* \omega + \cdots + (f^*)^n \omega \right)^{\dim X}.$$

Using Eqn. (5.2.10), one checks directly that

$$\lim_{n \rightarrow \infty} \frac{\log \text{Vol}(\Gamma_{f,n})}{n} = \max_p \log \delta_p(f)$$

so it remains to establish the existence of the limit defining $\delta_p(X)$ and its equality to the spectral radius of f^* on $H^{p,p}(X)$.

To do so, consider inside $H^{p,p}(X)$ the open cone of classes representable by smooth strongly positive (p, p) -forms (see §7.1.9 for strong positivity). This cone is clearly preserved by f^* and contains $[\omega]^p$, in particular is nonempty. By a generalized version of the Perron–Frobenius theorem, an eigenvector with largest eigenvalue of f^* lies in the closure of the cone. Furthermore, the iterate of any vector in the interior will grow at the maximal rate, given by the largest eigenvalue. Note also that cup-product with $[\omega]^q$ defines a strictly positive function on the boundary of the cone, since it is clearly nonnegative and remains so under any small perturbation of $[\omega]$. This in particular implies that the limit in Eqn. (5.2.10) exists and equals this largest eigenvalue. \square

Let us point out that in the above argument, the largest (in absolute value) eigenvalue need not be unique, and Jordan blocks can occur.

5.2.11. Proof of Proposition 5.2.7. Because the curvature of ω is assumed bounded, we can assume that we work in a fixed neighborhood of $0 \in \mathbb{C}^N$ and ω is Euclidean. The claimed lower bound then follows from the more general result below, which is interesting for both large and small radii:

5.2.12. Theorem (Lelong inequality). *Suppose that $V \subset \mathbb{C}^N$ is a properly embedded complex submanifold, of dimension k and containing $0 \in \mathbb{C}^N$. Setting $B(0, r)$ to be the ball of radius r in \mathbb{C}^N , the function*

$$\frac{\text{Vol}(V \cap B(0, r))}{r^{2k}} \text{ is increasing as } r \text{ increases.}$$

Its limit as $r \rightarrow 0$ is a fixed constant C_k .

Proof. The basic facts are that

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial} \|z\|^2 &\text{ is the Euclidean metric} \\ \sqrt{-1}\partial\bar{\partial} \log \|z\|^2 &\geq 0 \end{aligned}$$

and Stokes theorem (integration by parts) will be used repeatedly. Denote by $V_r = B(0, r) \cap V$ and ∂V_r its boundary (nonempty by properness of the embedding). It will also be more convenient to express the calculation using the real operators

$$d = \partial + \bar{\partial} \quad \text{and} \quad d^c = \sqrt{-1}(\partial - \bar{\partial})$$

so that $\frac{1}{2}dd^c = \sqrt{-1}\partial\bar{\partial}$. Let us first present a heuristic calculation:

$$\begin{aligned}
\frac{1}{r^{2k}} \int_{V_r} (dd^c \|z\|^2)^k &= \frac{1}{r^{2k-2}} \int_{\partial V_r} \left(\frac{d^c \|z\|^2}{\|z\|^2} \right) \wedge (dd^c \|z\|^2)^{k-1} \\
&= \frac{1}{r^{2k-2}} \int_{\partial V_r} (d^c \log \|z\|^2) \wedge (dd^c \|z\|^2)^{k-1} \\
&= \frac{1}{r^{2k-2}} \int_{V_r} (dd^c \log \|z\|^2) \wedge (dd^c \|z\|^2)^{k-1} \\
&= \dots \\
&= \int_{V_r} (dd^c \log \|z\|^2)^k
\end{aligned}$$

Since the integrand is positive, it is clear that the function is increasing in r . Note that the integrand has a singularity at $0 \in \mathbb{C}^N$ and the above integration by parts should be stated using a spherical shell with radii between ε and r . To be more explicit, assume that $k = 1$ for simplicity, i.e. that the manifold V is 1-dimensional but the dimension of the ambient space is arbitrary. Define now:

$$I(r) := \frac{1}{r^2} \int_{V_r} dd^c \|z\|^2$$

Then we can use Stokes's theorem and that $r^2 = \|z\|^2$ on V_r to rewrite it as

$$I(r) = \int_{\partial V_r} d^c \log (\|z\|^2)$$

as we did above in the first step of the calculation. Define now the spherical shell $\mathbb{S}(r, \varepsilon) := \{z \in \mathbb{C}^N : \varepsilon \leq \|z\| \leq r\}$ and compute using Stokes again:

$$\begin{aligned}
I(r) - I(\varepsilon) &= \int_{\partial V_r} d^c \log \|z\|^2 - \int_{\partial V_\varepsilon} d^c \log \|z\|^2 \\
&= \int_{V \cap \mathbb{S}(r, \varepsilon)} dd^c \log (\|z\|^2)
\end{aligned}$$

The integrand is nonnegative so the function is monotonic. \square

5.2.13. Remark (Image in projectivization). The positive form $dd^c \log \|z\|^2$ in \mathbb{C}^N can be identified with $\pi^* \omega_{FS}$ where ω_{FS} is the Fubini–Study form on $\mathbb{P}(\mathbb{C}^N)$ and $\pi: \mathbb{C}^N \dashrightarrow \mathbb{P}(\mathbb{C}^N)$ is the rational map given by projectivization. This description tells us when is $I(r)$ constant, namely when V is a line so its projectivization is a point. The function $I(r)$ measures the area of the image of $V \cap B(0, r)$ in $\mathbb{P}(\mathbb{C}^N)$.

5.2.14. Exercise (Wirtinger inequality).

- (i) Let ω be a skew-symmetric form, and g a positive-definite inner product on a real vector space P . Show that there exist pairwise orthogonal (for g) unit vectors $e_i, f_i \in P$ and real scalars λ_i such that

$$\omega = \sum_i \lambda_i e_i^\vee \wedge f_i^\vee$$

where for a vector $p \in P$, $p^\vee := g(p, -)$ denotes the associated linear form. Show additionally that the absolute values $|\lambda_i|$ are uniquely determined as a set, possibly with multiplicities.

- (ii) Let ω be a Kähler form on a complex vector space V and let g be the associated inner product on V viewed as a real vector space. Show that for any real $2k$ -dimensional subspace $P \subset V$ we have

$$\frac{1}{k!} \omega^k \Big|_P \leq \text{dVol}_g(P)$$

where $\text{dVol}_g(P)$ denotes the volume form induced by g on P , with equality if and only if P is a complex subspace of V . *Hint: Estimate $|\lambda_i|$ from the previous part by considering $\omega(e_i, f_i)$ and the relation to the inner product.*

5.2.15. Aside: dynamical degrees. The quantities defined in Eqn. (5.2.10) are called the *dynamical degrees* of f . They form a log-concave sequence in p , namely:

$$\delta_{p-1}(f) \delta_{p+1}(f) \leq \delta_p(f)^2$$

as follows from the following inequality of Khovanskii–Teissier–Gromov [Gro90, 1.6.C1]. For any Kähler metrics ω_1, ω_2 on X , the sequence

$$\int_X \omega_1^a \wedge \omega_2^{\dim_{\mathbb{C}} X - a} \text{ is log-concave in } a.$$

Note that in the proof of Gromov’s theorem, we only showed that $\lim_{n \rightarrow \infty} \frac{\log \text{Vol}(\Gamma_{f,n})}{n} = \max_p \log \delta_p(f) \leq \log \rho(f)$. The last inequality is, in fact, an equality. This follows from Yomdin’s theorem (see Remark 5.2.4(i)), but can be established also from the next result (I am grateful to Serge Cantat for suggesting to consider $X \times X$).

5.2.16. Lemma (Bounds on (p, q) spectral radius). *Let $\rho_{p,q}(f)$ denote the spectral radius of f^* on the group $H^{p,q}(X)$ of the Hodge decomposition of $H^{p+q}(X; \mathbb{C})$. Then we have the inequality:*

$$\rho_{p,q}(f)^2 \leq \max_{i=0, \dots, p+q} \delta_i(f) \cdot \delta_{p+q-i}(f)$$

Proof. Suppose that $\beta \in H^{p,q}(X)$ is such that $f^* \beta = \lambda \beta$. It is clear that $\rho_{p,q}(f)$ is the largest $|\lambda|$ that occurs as such an eigenvalue.

Consider $X \times X$, with automorphism $f \times f$, and let $\pi_i, i = 1, 2$ be projections to the corresponding factors. The cohomology class

$$\beta \boxtimes \bar{\beta} := \pi_1^* \beta \wedge \pi_2^* \bar{\beta}$$

is in $H^{p+q, p+q}(X \times X)$, is not zero, and has eigenvalue $|\lambda|^2$ under $f \times f$. In §5.2.9 we showed, in particular, that the spectral radius on a (k, k) -group is equal to δ_k , therefore we have that

$$|\lambda|^2 \leq \delta_{p+q}(f \times f)$$

To bound the last expression, note that $\pi_1^* \omega + \pi_2^* \omega$ is a Kähler metric on $X \times X$ when ω is a Kähler metric on X . Plugging this Kähler metric into the definition of $\delta_{p+q}(f \times f)$ and expanding the expression in Eqn. (5.2.10) leads directly to:

$$\delta_k(f \times f) = \max_{i=0 \dots k} \delta_i(f) \cdot \delta_{k-i}(f)$$

which implies the desired claim. \square

5.3. Basic properties of K3 automorphisms

5.3.1. Volumes on K3s. Recall that Ω denotes the (unique up to scale) holomorphic 2-form on a K3 surface X . Then $\Omega \wedge \bar{\Omega}$ defines a volume form on X which is invariant under any automorphism (see Proposition 5.3.2 below).

Suppose that X is algebraic and defined over \mathbb{R} . Since Ω is also in this case an algebraic differential form, it induces a volume form on $X(\mathbb{R})$ which is again invariant under any automorphism.

5.3.2. Proposition (Phase of area form). *Suppose that $f: X \rightarrow X$ is a K3 surface automorphism.*

- (i) *There exists $\delta(f)$ such that $f^* \Omega = \delta(f) \Omega$ and $|\delta(f)| = 1$. Furthermore $\delta(f)$ equals the eigenvalue of f^* on $H^{2,0}$.*
- (ii) *If $f(p) = p$ for some $p \in X$, then $\det(Df_p) = \delta(f)$.*
- (iii) *If X is algebraic then $\delta(f)$ is a root of unity.*

Proof. Because the holomorphic form Ω is unique up to scaling, its pullback must be proportional to it. Because the total volume with respect to $\Omega \wedge \bar{\Omega}$ is preserved, the proportionality constant must have absolute value 1.

For part (ii), note that the determinant of the derivative map at a fixed point can be computed using the action on a non-degenerate volume form. So part (i) implies part (ii).

For part (iii), decompose $H^2(X, \mathbb{Q}) = \text{NS}(X)_{\mathbb{Q}} \oplus \mathcal{T}(X)_{\mathbb{Q}}$ where $\mathcal{T}(X)$ is called the transcendental lattice (see §2.4.2) and this decomposition

is preserved by f^* . The Hodge decomposition carries over to $\mathcal{T}(X)_{\mathbb{C}} = H^{2,0} \oplus \mathcal{T}^{1,1} \oplus H^{0,2}$ and if X is algebraic, then the signature of the metric on $\mathcal{T}^{1,1}$ is strictly negative-definite. Indeed the signature of $H^{1,1}$ is $(1, 19)$ and the only positive direction went into $\text{NS}(X)$ by the algebraicity assumption.

It follows that f^* acts as an isometry on $\mathcal{T}(X)$ when endowed with the positive-definite metric associated to the Hodge decomposition. Since f^* also preserves the integral structure it follows that it has finite order and so its eigenvalues are roots of unity. Since $H^{2,0}$ is one-dimensional, is preserved by f^* , and spanned by Ω , the result follows. \square

5.3.3. Salem numbers. The spectral radius of an automorphism of a complex surface is a special kind of algebraic number. Namely the real algebraic integer $\lambda > 1$ is a *Salem number* if it is a unit (i.e. λ^{-1} is also an algebraic integer) and all Galois conjugates of λ other than λ^{-1} are on the unit circle in \mathbb{C} . Note that λ^{-1} is also a Galois conjugate of λ , since the product of all Galois conjugates of a unit must equal 1.

The minimal polynomial $S(t) \in \mathbb{Z}[t]$ of λ has even degree $2d$ and obeys the symmetry $S(t) = t^{2d}S(\frac{1}{t})$, because its roots are symmetric under $\lambda' \mapsto \frac{1}{\lambda'}$. It follows that we can write $S(t) = t^d R(t + \frac{1}{t})$ for some degree d polynomial $R(t) \in \mathbb{Z}[t]$. Indeed, in the ring $\mathbb{Z}[t, \frac{1}{t}]$ we have an involution $\iota(t) = \frac{1}{t}$, and the fixed point ring $\mathbb{Z}[t, \frac{1}{t}]^{\iota}$ has two bases (as \mathbb{Z} -module): one given by $t^i + \frac{1}{t^i}$ and another given by $(t + \frac{1}{t})^i$ (for $i = 0, 1, \dots$). Expressing $\frac{1}{t^d}S(t)$ in the second basis gives the desired polynomial $R(y)$.

The irreducible polynomial $S(t)$ is called a Salem polynomial, and the associated $R(y)$ is called its associated Salem trace polynomial. Note that $R(y)$ will have one root outside $[-2, 2]$ corresponding to $\lambda + \frac{1}{\lambda}$ and all the other roots will be in the interval $[-2, 2]$, corresponding to expressions of the form $\lambda' + \frac{1}{\lambda'}$ with $|\lambda'| = 1$.

5.3.4. Proposition (Salem numbers and entropy). *If the K3 surface automorphism f has positive entropy, then its spectral radius $\rho(f)$ is a Salem number.*

Proof. It is clear that the eigenvalues of f^* on H^2 are algebraic integers, since it preserves $H^2(X; \mathbb{Z})$. Note also that f^* preserves the indefinite inner product coming from cup product on H^2 .

Next, let λ denote the largest eigenvalue (in absolute value) of f^* on $H^{1,1}$, and v_{λ} the corresponding eigenvector. Then λ^{-1} is also an eigenvalue of f^* on $H^{1,1}$, since f^* preserves v_{λ}^{\perp} and acts as multiplication by λ^{-1} on $H^{1,1}/v_{\lambda}^{\perp}$ since it preserves cup product. Then the matrix

of the cup products of eigenvectors of λ, λ^{-1} must be, up to scaling, equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. On its orthogonal complement in $H^{1,1}$, f^* acts as an isometry for a positive-definite inner product. Similarly, the action on $H^{2,0} \oplus H^{0,2}$ also preserves a positive-definite inner product, so all the other eigenvalues of f^* have absolute value 1. \square

6. Elliptic dynamics on K3s

Outline of section. We now describe two results about automorphisms K3 surfaces that build, or make use of, quasi-periodic (or elliptic) dynamics.

McMullen [McM02] constructed the first (and only, at the moment) examples of positive-entropy K3 surface automorphisms with invariant open sets in which the dynamics has zero entropy. In fact, he showed that there exist examples with domains in which the automorphism is conjugated to an isometry of a polydisc. This construction is outlined in §6.1.

Cantat showed that when the full automorphism group of a K3 surface is sufficiently large, its ergodic invariant measures and closed invariant sets are particularly simple. The proof exploits the dynamics of translations on tori and is outlined in §6.2.

6.1. Siegel domains on K3s

6.1.1. Setup. Throughout, we will be concerned with an automorphism

$$F: \mathbb{I}_{3,19} \rightarrow \mathbb{I}_{3,19}$$

of the unique even unimodular lattice of signature $(3, 19)$. Denote by

$$S(t) := \det(t\mathbf{1} - F)$$

its characteristic polynomial.

6.1.2. Theorem (Torelli for automorphisms). *Suppose that $S(t)$ is a Salem polynomial. Then:*

- (i) *There exists a K3 surface X with an automorphism*

$$f: X \rightarrow X$$

and a marking $\iota: \mathbb{I}_{3,19} \rightarrow H^2(X; \mathbb{Z})$, such that ι conjugates the action of F and f^ .*

- (ii) *The Néron–Severi group of X is trivial, in particular X is not projective.*
- (iii) *There exists an F -invariant positive-definite 2-plane $P_2 \subset \mathbb{I}_{3,19}(\mathbb{R})$ such that, after complexification, F acts with eigenvalues $\delta, \bar{\delta}$. The phase of the area form $\delta(f)$ from [Proposition 5.3.2](#) can be arranged to equal one of these eigenvalues.*

Note that a Salem polynomial is by definition irreducible. The choice of eigenvalue δ or $\bar{\delta}$ above corresponds to the choice of either X or its complex conjugate \bar{X} .

Proof. It follows from the assumptions that F has two eigenvalues λ, λ^{-1} with λ a Salem number, and the corresponding eigenvectors $v_\lambda, v_{\lambda^{-1}}$ are isotropic. It follows that we have an F -invariant decomposition

$$\mathbb{I}_{3,19}(\mathbb{R}) = \mathbb{R}v_\lambda \oplus \mathbb{R}v_{\lambda^{-1}} \oplus P_2 \oplus P_{18}$$

where the inner product is positive-definite on P_2 and negative-definite on P_{18} . After complexification, F will decompose $P_2(\mathbb{C})$ into two eigenspaces $P_2(\mathbb{C})_\delta, P_2(\mathbb{C})_{\bar{\delta}}$ with eigenvalues $\delta, \bar{\delta}$. By the surjectivity of the period map [Theorem 4.3.1](#), there exists a K3 surface X and a marking $\iota: \mathbb{I}_{3,19} \rightarrow H^2(X; \mathbb{Z})$ such that $\iota(P_2(\mathbb{C})_\delta) = H^{2,0}$.

The Néron–Severi group of X is trivial, since its pullback to $\mathbb{I}_{3,19}$ under ι would have to be F -invariant and this contradicts the irreducibility of $S(t)$, the characteristic polynomial of F . To see that the pullback of the Néron–Severi group of X is F -invariant, note that it can be constructed as $\iota^{-1} \text{NS}(X) = P_2^\perp \cap \mathbb{I}_{3,19}$, and both P_2 and the lattice are F -invariant. It follows in particular that the Kähler cone of X is one component of the vectors in $H^{1,1}$ which have positive square.

Note that F preserves the pullback of the Kähler cone since by assumption Salem numbers are greater than 1, hence positive. It follows from the Torelli [Theorem 4.3.3](#) that there exists an automorphism f of X , such that the action of f^* on $H^2(X)$ is conjugated by ι to that of F on $\mathbb{I}_{3,19}$. \square

From now on, $S(t)$ is always assumed to be a Salem polynomial of degree 22.

6.1.3. Sufficient conditions for a Siegel domain. We now describe two essential assumptions on the automorphism F . We then explain in [§6.1.8](#) why the assumptions guarantee that the K3 automorphism provided by [Theorem 6.1.2](#) has a Siegel domain. We then explain in [§6.1.9](#) how to construct an F obeying the assumptions.

The first assumption is that

$$(6.1.4) \quad \operatorname{tr} \left(F \right) = -1 \text{ on } \mathbb{I}_{3,19}.$$

For the second assumption, let $\delta = \delta(f)$ be the phase provided by [Theorem 6.1.2](#). Then the algebraic numbers α, β defined by

$$(6.1.5) \quad \begin{cases} \alpha \cdot \beta = \delta & \text{satisfy } |\alpha| = |\beta| = 1 \\ \alpha + \beta = \frac{1+\delta+\delta^2}{1+\delta} & \text{and are multiplicatively independent.} \end{cases}$$

The assumption that α, β are multiplicatively independent means that if $\alpha^i \beta^j = 1$, for some $i, j \in \mathbb{Z}$, then necessarily $i = j = 0$.

6.1.6. Lefschetz number calculations. Assume that F has been constructed to satisfy the above two assumptions, which translate directly to the same properties of f^* acting on $H^2(X)$. The Lefschetz number of f (see [\[GH78, p. 421\]](#)) satisfies

$$L(f) := \operatorname{tr} \left(f^*, H^0 \oplus H^2 \oplus H^4 \right) = 1 - 1 + 1 = 1$$

and therefore f has exactly one fixed point $p \in X$. Indeed all isolated fixed points of a holomorphic map have positive index, and f has only isolated fixed points since a positive-dimensional fixed-point set would give a non-trivial element of the Néron–Severi group.

The holomorphic Lefschetz number of f (see [\[GH78, p. 424\]](#)) satisfies

$$L(f, \mathcal{O}) := \operatorname{tr} \left(f^*, H^{0,0} \oplus H^{0,2} \right) = 1 + \bar{\delta}$$

but can also be expressed in terms of the derivative at the unique fixed point:

$$L(f, \mathcal{O}) = \frac{1}{\det(1 - Df_p)}$$

Let α, β denote the eigenvalues of Df_p on $T_p^{1,0}X$. Then $\alpha \cdot \beta = \delta$ because $f^*\Omega = \delta \cdot \Omega$, and combining the two expressions for the holomorphic Lefschetz number gives

$$\frac{1}{(1 - \alpha)(1 - \beta)} = 1 + \bar{\delta} \implies \alpha + \beta = \frac{1 + \delta + \delta^2}{1 + \delta}$$

using in the course of calculations that $\bar{\delta} = \frac{1}{\delta}$.

6.1.7. Diophantine condition. Recall that p was the fixed point of f and the eigenvalues of Df_p are α, β . By the assumption in Eqn. (6.1.5) α, β are multiplicatively independent and by construction they are algebraic numbers. In fact, they satisfy the stronger Diophantine condition

$$|\alpha^{k_1} \beta^{k_2} - 1| \geq \frac{1}{C \max(k_1, k_2)^M} \text{ for some } C, M > 0$$

and any k_i not both equal to zero. Indeed, these estimates follow from results of Fel'dman [Fel68, Thm. 1], themselves based on the Gel'fond–Baker method (see [BW07] for an introduction). Note that Fel'dman shows an estimate of the form

$$|k_0 \log \alpha_0 + \cdots + k_n \log \alpha_n| \geq \exp \left(-C + M \log (\max |k_i|) \right)$$

where α_i are multiplicatively independent algebraic numbers, $\log \alpha_i$ are *some* fixed choices of their logarithms, and C, M depend on the previous choices. The k_i are arbitrary integers, not all zero. Note that $2\pi\sqrt{-1} \in \log(1)$ is a possible choice of the logarithm of 1. To recover the desired Diophantine inequality, fix a choice of $\log \alpha, \log \beta$ and note that

$$|\alpha^{k_1} \beta^{k_2} - 1| \geq \frac{1}{10} \inf_{k_0} \left| k_1 \log \alpha + k_2 \log \beta + k_0 2\pi\sqrt{-1} \right| \geq \frac{1}{C' \max(k_1, k_2)^M}$$

which is what we wanted

6.1.8. Existence of Siegel domain. The assumptions of Sternberg's linearization theorem [Ste61, §5] are exactly the kind of Diophantine condition obtained in §6.1.7. As they are satisfied, it follows that there exists an f -invariant open neighborhood U_p of p in X , and a biholomorphism to a polydisc $h: U_p \rightarrow \Delta^2 \subset \mathbb{C}^2$, such that h conjugates the action of f on U_p to that of $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ on Δ^2 .

Note that Sternberg's theorem is an extension to higher dimensions of Siegel's linearization theorem in one complex variable and the proofs follows a similar method.

6.1.9. Constructing a lattice automorphism. It remains to construct an automorphism $F: \mathbb{I}_{3,19} \rightarrow \mathbb{I}_{3,19}$ satisfying the requirements in §6.1.3. Given a Salem polynomial $S(t)$, the multiplication by t action on $\mathbb{Z}[t]/S(t) =: \mathcal{O}_S$ has characteristic polynomial $S(t)$ by construction. Therefore, we must exhibit a degree 22 Salem polynomial $S(t)$ and endow \mathcal{O}_S with an inner product making it an even, unimodular lattice, with further requirements.

Note that the condition in Eqn. (6.1.4) can be checked at the level of the polynomial $S(t)$. Indeed, writing $S(t) = \sum_{i=0}^{22} a_i t^i$ it follows that $\text{tr}(F) = -a_{21} = -a_1$ using the symmetry of the coefficients.

On the other hand, the condition in Eqn. (6.1.5) also depends on the choice of inner product on \mathcal{O}_S . We omit a direct construction and refer to McMullen’s detailed presentation in [McM02], particularly §8-9. To end, let us note that an explicit Salem polynomial for which the construction can be performed is:

$$(6.1.10) \quad S(t) = 1 + t - t^3 - 2t^4 - 3t^5 - 3t^6 - 2t^7 + 2t^9 + 4t^{10} \\ + 5t^{11} \\ + 4t^{12} + 2t^{13} - 2t^{15} - 3t^{16} - 3t^{17} - 2t^{18} - t^{19} + t^{21} + t^{22}$$

6.2. Twists along elliptic fibrations

We present some results of Cantat [Can01b] that originated in earlier work of Wang [Wan95]. Suppose that the automorphism group of a K3 surface contains two independent “twist” automorphisms (see §6.2.2). Then Theorem 6.2.4 describes the topological and measure-theoretic dynamics of the full automorphism group of the K3 surface.

The idea of understanding a group action through its unipotent elements is a classical one in homogeneous dynamics and has been used in the non-homogeneous setting as well, e.g. by Goldman [Gol97].

6.2.1. Setup. Throughout X is a K3 surface and $X \xrightarrow{\pi} \mathbb{P}^1$ denotes an elliptic fibration. Namely all but finitely many of the fibers are smooth connected genus 1 curves. Two elliptic fibrations will be viewed as different if the homology classes of their fibers have non-trivial intersection.

6.2.2. Twist automorphisms. A *twist automorphism* associated to an elliptic fibration $X \xrightarrow{\pi} \mathbb{P}^1$ is a map $T: X \rightarrow X$ that preserves the fibers of π (i.e. commutes with π) and that is of infinite order.

Alternatively, one can define a twist automorphism to be one that induces a parabolic map on $H^{1,1}$, see §5.1.4.

6.2.3. Jacobians of elliptically fibered K3s. We review a useful construction for K3 surfaces. Starting from an elliptic fibration $X \xrightarrow{\pi} \mathbb{P}^1$, there exists by [Huy16, 11.4.5] an associated elliptic fibration $J(X) \xrightarrow{\pi_J} \mathbb{P}^1$ where $J(X)$ is also, remarkably, a K3 surface with the following properties. For any $p \in \mathbb{P}^1$ the fibers $\pi^{-1}(p)$ and $\pi_J^{-1}(p)$ are isomorphic and additionally, there exists a section $\sigma: \mathbb{P}^1 \rightarrow J(X)$. Note that $J(X)$ has a modular interpretation.

6.2.4. Theorem (Cantat). *Suppose that X has two distinct elliptic fibrations, with each fibration admitting a non-trivial twist automorphism. Then:*

- (i) *The Lebesgue measure on X is $\text{Aut}(X)$ -ergodic.*
- (ii) *For any point $x \in X$, its orbit closure $\overline{\text{Aut}(X)x}$ is either finite, or a real 2-dimensional submanifold, or all of X .*
- (iii) *Any ergodic $\text{Aut}(X)$ -invariant probability measure on X is either supported on finitely many points, or Lebesgue on a totally real surface⁴, or Lebesgue on X .*

The proof of the theorem is based on two elementary facts. The first one holds for general translations on compact abelian groups, and the second will show that the translations along fibers of an elliptic fibration are “truly varying”.

6.2.5. Exercise (Dynamics on compact abelian groups). Suppose that \mathbb{T} is a compact abelian group and $f_\alpha: \mathbb{T} \rightarrow \mathbb{T}$ is translation by $\alpha \in \mathbb{T}$. Let $\mathbb{T}_\alpha \subset \mathbb{T}$ be the smallest compact abelian group that contains α . Then

- (i) Any orbit closure $\overline{f_\alpha^{\mathbb{Z}}t}$ is a translate of \mathbb{T}_α .
- (ii) Any ergodic f_α -invariant probability measure on \mathbb{T} is the translation of Haar measure on \mathbb{T}_α .

6.2.6. Local structure of twists. Suppose that $T: X \rightarrow X$ is a twist automorphism along an elliptic fibration $X \xrightarrow{\pi} \mathbb{P}^1$. All but finitely many fibers of π are elliptic curves and hence isomorphic to $\mathbb{R}^2/\mathbb{Z}^2$ in a way that preserves the group structure. Furthermore, on small enough open sets $U \subset \mathbb{P}^1$ not containing the singular points, the fibration can be trivialized as $U \times \mathbb{R}^2/\mathbb{Z}^2 \rightarrow U$.

Since T is holomorphic, it preserves the group structure on the elliptic curve fibers of π . Therefore, in the constructed trivialization, there exists $\alpha: U \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ such that the action of T is as $(p, t) \mapsto (p, t + \alpha(p))$. Note that α is a real-analytic function by construction.

6.2.7. Proposition (Twists are not isotrivial). *With notation as above, the map $\alpha: U \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ is an open map.*

Proof. Consider the construction of α . First pick a local section of the fibration, i.e. a holomorphic map $s: U \rightarrow \pi^{-1}(U)$ such that $\pi \circ s(p) = p$. This determines some trivialization $\pi^{-1}(U) \xrightarrow{\psi} U \times \mathbb{R}^2/\mathbb{Z}^2$ satisfying

⁴Recall that a totally real surface $S \subset X$ is such that at every $s \in S$, we have $T_s X = T_s S \oplus \sqrt{-1}T_s S$ (where we are considering tangent spaces of real manifolds and $\sqrt{-1}$ is the complex structure on X).

the condition $\psi(s(p)) = (p, 0)$ and compatible with the group structure. Next apply the automorphism and use the trivialization to find that

$$(p, \alpha(p)) = \psi\left(T(s(p))\right)$$

and since non-trivial holomorphic maps are open, as are homeomorphisms and fibration maps, it follows that α is open, provided it is not constant.

To check that α is indeed not constant, one can appeal to the argument in [Can01b], last paragraph of Proof of Prop. 2.2. We sketch the argument, with a slight twist. Assume by contradiction that α is constant for some open set $U \subset \mathbb{P}^1$. Then by analyticity, it is constant on any other such open set on which it is defined. Taking the associated Jacobian K3 $J(X)$ (see §6.2.3), we see that the twist automorphism determines another section $\sigma_\alpha: \mathbb{P}^1 \rightarrow J(X)$ besides the zero section of the Jacobian fibration. The class $[\sigma_\alpha(\mathbb{P}^1)] \in H^2(J(X))$ intersects non-trivially the classes of the fibers.

The twist automorphism T of X extends to a twist automorphism T_J of $J(X)$. It follows from a cohomological calculation that

$$[\sigma_\alpha(\mathbb{P}^1)] \cdot \left(T_J^n[\sigma_\alpha(\mathbb{P}^1)]\right) \text{ grows quadratically in } n.$$

Like in [Can01b, p. 207] one checks that the contribution to the intersection at the singular fibers is uniformly bounded. It follows that for n large enough σ_α and $T_J^n \circ \sigma_\alpha$ intersect at one point away from the singular fibers. By local constancy of α it follows that the sections agree. Therefore α is a torsion point for all elliptic curve fibers in the fibration, so T_J (and thus T) is of finite order. This is a contradiction to T being a twist. \square

We can now sketch the arguments for the main result.

Proof of Theorem 6.2.4. We only illustrate the ideas by establishing (i). Subsequent parts are based on similar arguments but require a more in-depth analysis of the possibilities.

Suppose that $A \subset X$ is an $\text{Aut}(X)$ -invariant subset of positive Lebesgue measure. Let T_1, T_2 be twists along distinct elliptic fibrations π_1, π_2 . It suffices to check that for each of the π_i and Lebesgue-a.e. $p \in \mathbb{P}^1$, the set $A \cap \pi_i^{-1}(p)$ has either full, or null Lebesgue measure in $\pi_i^{-1}(p)$.

Assuming this claim, note that by Fubini we can choose a set S_1 of positive Lebesgue measure in \mathbb{P}^1 such that $\pi_1^{-1}(s) \cap A$ full Lebesgue measure in $\pi_1^{-1}(s)$, $\forall s \in S_1$ (by first arranging that it has positive Lebesgue measure). Therefore A contains $\pi_1^{-1}(S_1)$, up to Lebesgue null

sets. All but finitely many fibers of π_2 intersect all elliptic curve fibers of π_1 , therefore Lebesgue a.e. fiber of π_2 intersects A in a set of positive Lebesgue measure. Hence Lebesgue a.e. fiber of π_2 intersects A in a set of full Lebesgue measure, showing that A has full Lebesgue measure.

It remains to establish the initial claim. It is local on the base \mathbb{P}^1 , hence we can assume that the twist dynamics has been trivialized as in §6.2.6. We showed that the map α is open, hence outside countably many points and real-analytic curves on the base, the twist map is ergodic for Lebesgue measure on the fibers. Since A is $\text{Aut}(X)$ -invariant, hence twist-invariant, the claim follows. \square

7. Hyperbolic dynamics on K3s

Outline of section. We specialize now to a single automorphism $f: X \rightarrow X$ which is furthermore of positive entropy, or equivalently it acts as a hyperbolic matrix on $H^{1,1}(X)$, see §5.1.4.

The basic facts about such maps, due to Cantat [Can01a], are described in §7.1. This involves the construction of currents that are expanded/contracted by the dynamics and the associated measure of maximal entropy. If the measure of maximal entropy is in the same class as Lebesgue measure, then Cantat–Dupont [CD20b] showed that the automorphism must necessarily be a Kummer example (see §5.1.2). A different proof of this result, from [FT19] using Ricci-flat metrics, is described in §7.2.

7.1. Currents

Recall that in §5.1.4, hyperbolic automorphisms were defined to be those which admit an eigenvalue λ in cohomology with $|\lambda| > 1$. The basic results concerning such automorphisms are contained in the following theorem. This section provides the necessary background and sketches a partial proof.

7.1.1. Theorem (Cantat [Can01a]). *Suppose that $f: X \rightarrow X$ is a positive entropy automorphism of a K3 surface. Let $v_{\pm} \in H^{1,1}(X)$ be non-zero cohomology classes with $f_*v_{\pm} = \lambda^{\pm 1}v_{\pm}$, where $\lambda > 1$.*

(i) *There exist unique closed currents η_{\pm} , with $[\eta_{\pm}] = v_{\pm}$, satisfying*

$$f_*\eta_{\pm} = \lambda^{\pm 1}\eta_{\pm}$$

and with locally L^1 potentials.

(ii) *There exist unique closed positive currents in the cohomology classes v_{\pm} , and they agree with η_{\pm} from (i).*

- (iii) The currents η_{\pm} have locally Hölder potentials and satisfy $\eta_+^2 = 0 = \eta_-^2$; in cohomology we have $v_+^2 = 0 = v_-^2$.
- (iv) The measure $\mu := \eta_+ \wedge \eta_-$, when normalized to be a probability measure, is the unique measure of maximal entropy. It is mixing, hence ergodic.

7.1.2. Remark.

- (i) The eigenvalue λ of f_* on cohomology is also equal to its spectral radius. Indeed, any element of the orthogonal group $\mathbf{O}_{1,n}(\mathbb{R})$ has at most one eigenvalue λ with $|\lambda| > 1$, and the eigenvector v in this case is necessarily a null vector for the indefinite quadratic form. Then the action of f_* preserves the line spanned by v , and also its orthogonal complement v^{\perp} , and acts as an isometry on v^{\perp}/v , which is equipped with a natural positive-definite quadratic form. Note also that λ^{-1} has to be an eigenvalue as well, by looking for instance at determinants. It follows from [Theorem 5.2.3](#) that the topological entropy of f is $\log \lambda$.
- (ii) It is more delicate to show that the measure $\mu := \eta_+ \wedge \eta_-$ has maximal entropy, i.e. $\log \lambda$, and that it is unique with this property, see [[Can01a](#), Thm. 6.2]. One needs some further tools from Pesin theory, in particular the existence of stable/unstable manifolds, and the Ledrappier–Young relations between entropy and conditional measures. See also the classical work of Bedford–Lyubich–Smillie [[BLS93](#)] where these kinds of arguments were introduced in holomorphic dynamics.
- (iii) The currents provided by [Theorem 7.1.1](#) are in general of rather weak regularity, for instance they are not smooth. In fact, the measures obtained by restricting the currents to 1-dimensional complex curves must have fractional Hausdorff dimension, unless X is a Kummer example as in [§2.2.6](#). For more on this, see [[FT18](#)], which is based on [Theorem 7.2.2](#) below.

7.1.3. Basic facts about positive currents. Recall that currents are defined as continuous linear functionals on the space of all smooth forms on X . For a current η the differential $d\eta$ is defined using integration by parts. Its action on smooth forms is according to $\langle d\eta, \phi \rangle := -\langle \eta, d\phi \rangle$.

We will be interested⁵ in $(1, 1)$ -currents, i.e. functionals on the space of smooth $(1, 1)$ -forms. Such a current is *positive* if for any smooth

⁵For convenience, we work in complex dimension 2. There are multiple notions of positivity in higher dimensions, see [§7.1.9](#).

(1, 0)-form α we have

$$\langle \eta, \sqrt{-1}\alpha \wedge \bar{\alpha} \rangle \geq 0$$

7.1.4. Exercise (Measure coefficients). Suppose that η is a positive (1, 1)-current on a complex surface. Write it in local coordinates as

$$\eta = \sqrt{-1} \sum \eta_{i,\bar{j}} dz^i \bar{d}z^{\bar{j}}$$

where $\eta_{i,\bar{j}}$ are generalized functions.

Show that $\eta_{i,\bar{j}}$ are in fact locally finite measures. *Hint: A linear functional which is positive on positive functions is given by a measure.*

7.1.5. The mass of a current. Suppose that ω is a Kähler metric on X and η is a positive current. Then its *mass* relative to ω is defined to be $\int \omega \wedge \eta$, which is a positive number by the positivity of ω . This is analogous to the mass of a measure.

Recall that the space of positive measures of total mass bounded by a constant is weakly compact. This implies that the space of positive currents of mass relative to ω bounded by a constant is also weakly compact.

7.1.6. Cohomology and currents. One can compute the cohomology of a compact manifold X using its De Rham complex of smooth differential forms. An analogous discussion holds when replacing smooth forms by currents, and the cohomology groups are canonically identified.

Furthermore, in the case when X is a Kähler manifold, the decomposition into (p, q) -components in cohomology is compatible with the same decomposition for smooth forms, or for currents.

7.1.7. Bedford–Taylor theory. While in general it is not possible to define the product of two currents, in complex analysis this is sometimes possible. Recall that a *local potential* of a (1, 1)-current η is a function $\phi \in L^1_{loc}$ defined in some chart such that $\eta = \sqrt{-1}\partial\bar{\partial}\phi$ in the sense of distributions.

Bedford–Taylor theory (see [BT76] for a starting point) defines a product of currents with continuous potentials locally, as follows. If in a chart $\eta_i = \sqrt{-1}\partial\bar{\partial}\phi_i$ then set

$$\eta_1 \wedge \eta_2 = \sqrt{-1}\partial\bar{\partial}(\phi\eta_2)$$

where $\phi\eta_2$ is well-defined since ϕ is continuous and the coefficients of η_2 are measures. Since the symmetry is broken in the definition, one must check that $\eta_1 \wedge \eta_2 = \eta_2 \wedge \eta_1$ and that $\eta_1 \wedge \eta_2 \geq 0$ if $\eta_i \geq 0$ for $i = 1, 2$.

This discussion gives meaning to the expressions $\eta_+^2 = 0 = \eta_-^2$ and $\mu = \eta_+ \wedge \eta_-$ in Theorem 7.1.1, provided we establish continuity of potentials. Note that the measure μ is singular with respect to the

invariant Lebesgue measure $\Omega \wedge \bar{\Omega}$, unless (X, f) is a Kummer example (see §7.2).

Proof of Theorem 7.1.1. We only treat the first three parts of the theorem. The arguments below apply equally well to f^{-1} , so we treat only the case of v_+ and η_+ .

For part (i), note that the operator $\frac{1}{\lambda}f_*$ acts on the space of closed currents in the cohomology class v_+ , which have locally L^1 potentials. Define a distance on this space by

$$(7.1.8) \quad \text{dist}(\eta, \eta') := \int_X |\phi| \, d\text{Vol}$$

where $\eta = \eta' + \sqrt{-1}\partial\bar{\partial}\phi$ normalized as $\int_X \phi \, d\text{Vol} = 0$.

The space is complete for this distance, and $\frac{1}{\lambda}f_*$ acts as a uniform contraction. Therefore, there exists a unique fixed point.

Part (ii) follows analogously, but now considering the action of $\frac{1}{\lambda}f_*$ on the space of closed positive currents. Closed positive currents have locally L^1 potentials by [Hö7, Cor. 3.2.8], so it is a subspace of the space considered in (i). The subspace is compact for the same distance function, and since $\frac{1}{\lambda}f_*$ acts as a uniformly contracting bijection, the space is either a single point or empty (since its diameter must vanish). The space is nonempty, since for example it contains a weak limit of the sequence

$$\eta_n := \frac{c_+}{n} \sum_{i=0}^n \frac{1}{\lambda^i} (f_*)^i \omega$$

where ω is a Kähler metric on X . Here c_+ is a nonzero constant of proportionality depending only on $[\omega] \cdot v_+$. It follows that there is a unique closed positive current η_+ , and it must coincide with the current constructed in (i).

To establish (iii), it suffices to prove the existence of locally Hölder potentials. The fact that $v_+^2 = 0$ follows since f_* preserves the intersection pairing in cohomology, and scales v_+ by λ . Similarly, $\eta_+^2 = 0$ since $\eta_+^2 \geq 0$ as a current, and $[\eta_+]^2 = v_+^2 = 0$.

Finally, consider the action of $\frac{1}{\lambda}f_*$ on the space of currents in v_+ with Hölder potentials of exponent α . Equip it with the distance

$$\text{dist}(\eta, \eta') = \|\phi\|_{C^0} + \|\phi\|_{C^\alpha}$$

with the same assumptions as in Eqn. (7.1.8). Let now L be a Lipschitz constant for f^{-1} , for example take $L := \|Df^{-1}\|_{C^0(X)}$, using the same background metric as for the definition of the Hölder distance. We have

that

$$\begin{aligned}
\text{dist} \left(\frac{1}{\lambda} f_* \eta, \frac{1}{\lambda} f_* \eta' \right) &= \frac{1}{\lambda} \|\phi\|_{C^0} + \frac{1}{\lambda} \sup_{x,y \in X} \frac{|f_* \phi(x) - f_* \phi(y)|}{|x - y|^\alpha} \\
&= \frac{1}{\lambda} \|\phi\|_{C^0} + \frac{1}{\lambda} \sup_{x,y \in X} \frac{|\phi(f^{-1}x) - \phi(f^{-1}y)|}{|x - y|^\alpha} \\
&\leq \frac{1}{\lambda} \|\phi\|_{C^0} + \frac{1}{\lambda} L^\alpha \|\phi\|_{C^\alpha} \\
&\leq \left(\frac{L^\alpha}{\lambda} \right) \text{dist}(\eta, \eta')
\end{aligned}$$

Taking α sufficiently close to 0 that $\frac{L^\alpha}{\lambda} < 1$, it follows that the map acts as a strict contraction and has a unique fixed point. It agrees with the fixed point constructed in the previous parts (i) and (ii). \square

7.1.9. Aside: Further notions of positivity. A detailed treatment of the following ideas is provided by Demailly in [Dem12, III.1]. For simplicity, we consider an n -dimensional complex vector space V – the corresponding notions on a complex manifold are defined by considering tangent spaces at every point. Positive throughout is understood as non-negative, i.e. greater than or equal to zero.

First, observe that any complex vector space V has a canonical orientation when viewed as a real vector space: if dz_1, \dots, dz_n is a basis of the complex dual V^\vee , then the orientation is given by

$$(\sqrt{-1}dz_1 \wedge \overline{dz_1}) \wedge \dots \wedge (\sqrt{-1}dz_n \wedge \overline{dz_n}) = 2^n (dx_1 \wedge dy_1) \wedge \dots \wedge (dx_n \wedge dy_n)$$

Call a positive volume form any (n, n) -form which is a positive real scalar multiple of this fixed volume. Note here that there is another possible choice given by $dx_1 \cdots dx_n \wedge dy_1 \cdots dy_n$ – this ambiguity is ultimately responsible for rather involved signs in calculations.

Define a (p, p) -form to be *strongly positive* if it is a positive linear combination of expressions:

$$(\sqrt{-1}\alpha_1 \wedge \overline{\alpha_1}) \wedge \dots \wedge (\sqrt{-1}\alpha_p \wedge \overline{\alpha_p}) \text{ for } \alpha_i \in V^\vee.$$

Above, α_i are the same as $(1, 0)$ -forms.

Define a (p, p) -form γ to be *positive* if for any strongly positive $(n - p, n - p)$ -form β we have that $\gamma \wedge \beta$ is a positive volume form.

Here are some useful properties:

- Strongly positive forms are positive.
- The convex cones of positive (p, p) -forms and strongly positive $(n - p, n - p)$ -forms are dual to each other. Furthermore, both cones have interior in the corresponding real vector spaces.

- The wedge product of strongly positive forms is strongly positive. This can fail for positive forms.
- The two notions of positivity agree for $p = 0, 1, n - 1, n$. In other cases, there exist positive forms which are not strongly positive.

7.1.10. Example (Positive, but not strongly positive forms). Take a $(p, 0)$ -form β , then $\gamma := (\sqrt{-1})^{p^2} \beta \wedge \bar{\beta}$ is positive. However, it is strongly positive if and only if β is decomposable, i.e. if and only if there exist $(1, 0)$ -forms α_i such that $\beta = \alpha_1 \wedge \cdots \wedge \alpha_p$. On \mathbb{C}^4 , one can then take $\beta = dz_1 \wedge dz_2 + dz_3 \wedge dz_4$, which is not decomposable (since $\beta^2 \neq 0$) and obtain a positive, but not strongly positive form.

7.2. Rigidity of Kummer examples

7.2.1. Setup. On a K3 surface X equipped with a hyperbolic automorphism f , there are two natural f -invariant probability measures: dVol coming from the holomorphic symplectic form, and the measure of maximal entropy μ . It is immediate to check that if (X, f) is a Kummer example, then $\mu = \text{dVol}$; this holds even with the general definition introduced in §5.1.2. It turns out that the converse is also true, as was established by Cantat & Dupont [CD20b].

7.2.2. Theorem (Rigidity of Kummer examples). *If the measure of maximal entropy μ is in the Lebesgue measure class, then (X, f) is a Kummer example.*

The assumption only says that μ is proportional to dVol on a set of positive Lebesgue measure. After some preliminaries, we sketch below a proof from [FT19], using the Ricci-flat metrics on K3 surfaces. For simplicity, we will make the stronger assumption $\mu = \text{dVol}$.

7.2.3. Lyapunov exponents. For a more detailed treatment of the next topic, see Ledrappier's lecture notes [Led84] or [Fil19a]. Suppose that (X, g) is a compact Riemannian manifold and $f: X \rightarrow X$ is a diffeomorphism. For a point $x \in X$, we expect $\|D_x f^n\|$ to grow exponentially in n , being the product (by the chain rule) of n matrices of bounded size. If this is the case, denote by

$$\sigma_1(x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|D_x f^n\|$$

and call it the top Lyapunov exponent of f at x .

Let now m be an ergodic f -invariant probability measure. Then the Oseledets theorem guarantees that $\sigma_1(x)$ is well-defined for m -a.e.

$x \in X$ and equals the same value denoted $\sigma_1(m)$. Furthermore, define

$$(7.2.4) \quad I_n := \int_X \log \|D_x f^n\| dm(x)$$

and observe that this sequence is *subadditive*:

$$\begin{aligned} I_{k+l} &= \int_X \log \|D_x f^{k+l}\| dm(x) \leq \\ &\leq \int_X \left(\log \|D_x f^k\| + \log \|D_{f^k x} f^l\| \right) dm(x) = \\ &= I_k + \int_X \log \|D_x f^l\| d(f^k_* m)(x) = \\ &= I_k + I_l \end{aligned}$$

where we have used the inequality for matrix norms $\|AB\| \leq \|A\| \|B\|$ after taking logarithms. It follows by Fekete's lemma that $\lim_{n \rightarrow \infty} \frac{1}{n} I_n$ exists and equals $\inf_n \frac{1}{n} I_n$, and the Oseledec's theorem guarantees that

$$\sigma_1(m) = \lim_{n \rightarrow +\infty} \frac{1}{n} I_n = \inf_{n \rightarrow +\infty} \frac{1}{n} I_n$$

When X is n -dimensional, one can define analogously n Lyapunov exponents $\sigma_1 \geq \dots \geq \sigma_n$ using the exterior power derivative matrices $\Lambda^k(Df)$ – their growth rate will be $\sigma_1 + \dots + \sigma_k$.

7.2.5. Ledrappier–Young formula. For an arbitrary ergodic measure m , the Ledrappier–Young formula [LY85a] relates its measure-theoretic entropy $h(m)$, its Lyapunov exponents $\sigma_i(m)$, and the Hausdorff dimension of conditional measures of m along appropriate foliations. The general shape of the formula is

$$(7.2.6) \quad h(m) = \sum_{\sigma_i(m) > 0} \sigma_i(m) \cdot \dim_i(m)$$

where the sum is over the positive Lyapunov exponents, now listed without multiplicities. The quantities $\dim_i(m)$ are determined from Hausdorff dimensions as follows. There exists a (measurable) family of invariant unstable manifolds $\mathcal{W}^{\geq \sigma_i}$, expanded by the forward dynamics, and the manifolds are nested, namely $\mathcal{W}^{\geq \sigma_i} \subseteq \mathcal{W}^{\geq \sigma_{i+1}}$ if $\sigma_i > \sigma_{i+1}$. There exist conditional measures supported on the leaves of $\mathcal{W}^{\geq \sigma_i}$ and their Hausdorff dimension is given by $\dim_1 + \dots + \dim_i(m)$ where $\dim_i(m)$ is the quantity from Eqn. (7.2.6). Furthermore, it is proved in [LY85b] that $0 \leq \dim_i(m) \leq \text{mult}(\sigma_i)$ where $\text{mult}(\sigma_i)$ denotes the multiplicity of the Lyapunov exponent σ_i .

7.2.7. Volume-preserving case. For the case of interest to us, namely when X is a K3 surface, there is one non-negative Lyapunov exponent which has multiplicity 2. The multiplicity is 2 because X is a complex manifold, so the derivative cocycle commutes with multiplication by the complex structure, and there is only one non-negative exponent because the derivative cocycle preserves the volume measure, so the sum of all exponents (for any measure) has to vanish. When $m = \text{dVol}$, we have $\dim_i(m) = 2$ and in fact Ledrappier–Young [LY85a] prove that this last property is equivalent to m being in the Lebesgue class. Finally, by Gromov–Yomdin Theorem 5.2.3 we know the entropy satisfies $h(\mu) = \log \lambda$, where λ is the spectral radius of f^* on cohomology. We conclude, under the assumption $\mu = \text{dVol}$, that

$$\sigma = \frac{h}{2} = \frac{\log \lambda}{2}$$

which will be essential to the argument.

Combining this last equality with the characterization of σ as the infimum of the I_n (see §7.2.3 for notation) it follows that

$$(7.2.8) \quad \int_X \log \|D_x f^N\| d\mu(x) =: I_N \geq N \frac{h}{2}$$

7.3. Proof of Kummer rigidity

We can now proceed to the proof of Theorem 7.2.2. We make the stronger assumption that $\mu = \text{dVol}$; removing it requires a lot more work, see [FT19, §5].

Let $[\eta_{\pm}]$ be the cohomology classes expanded/contracted by the automorphism. We make the additional assumption that the cohomology class $[\eta_+] + [\eta_-]$ contains a Kähler metric (see Remark 7.3.2 for how to address the general situation). By applying the automorphism f , it is clear that the cohomology class $e^t[\eta_+] + e^{-t}[\eta_-]$ contains a Kähler metric for all $t \in \mathbb{R}$, so let ω_t denote the Ricci-flat metric in that class. We normalize the cohomology classes such that $[\eta_+]^2 = 0 = [\eta_-]^2$ and $[\eta_+][\eta_-] = 1$, so that $[\omega_t]^2 = 2$.

Recall that $h = \log \lambda$ is the topological entropy of f . Then note that $f_*[\omega_t] = [\omega_{t+h}]$ by definitions, and in fact

$$f_*\omega_t = \omega_{t+h}$$

since the Ricci-flat metrics are unique in their cohomology class.

To a point $x \in X$ we can associate the following quantity, which is the local expansion factor of ω_0 relative to ω_t . Namely, there exists

$\sigma(x, t) \geq 0$ such that in an orthonormal basis at x we have:

$$(7.3.1) \quad \begin{aligned} \omega_0(x) &= |dz_1|^2 + |dz_2|^2 \\ \omega_t(x) &= |e^{\sigma(x,t)} dz_1|^2 + |e^{-\sigma(x,t)} dz_2|^2 \end{aligned}$$

With this notation, and using ω_0 as our background metric, it follows that $\log \|D_x f^N\| = \sigma(x, Nh)$. Furthermore

$$\omega_0 \wedge \omega_{Nh} = \left(e^{2\sigma(x, Nh)} + e^{-2\sigma(x, Nh)} \right) d\text{Vol}$$

and we can compute the integral in cohomology: $\int \omega_0 \wedge \omega_{Nh} = e^{Nh} + e^{-Nh}$. We can now put together the information and apply Jensen's inequality:

$$\begin{aligned} \log \left(e^{Nh} + e^{-Nh} \right) &= \log \left(\int_X \omega_0 \wedge \omega_{Nh} \right) \\ &\geq \int_X \log \left(\frac{\omega_0 \wedge \omega_{Nh}}{d\text{Vol}} \right) d\text{Vol} \\ &= \int_X \log \left(e^{2\sigma(x, N)} + e^{-2\sigma(x, N)} \right) d\text{Vol} \end{aligned}$$

So far we have not used the assumption $\mu = d\text{Vol}$, but now we can do so in the form of the inequality Eqn. (7.2.8) that bounds from below $I_N = \int \sigma(x, Nh) d\text{Vol}$ by $N\frac{h}{2}$. Combined with the fact that $\log(e^x + e^{-x})$ is convex and increasing for $x \geq 0$, and using Jensen again gives:

$$\begin{aligned} \int_X \log \left(e^{2\sigma(x, Nh)} + e^{-2\sigma(x, Nh)} \right) d\text{Vol} &\geq \log \left(e^{2I_N} + e^{-2I_N} \right) \\ &\geq \log \left(e^{Nh} + e^{-Nh} \right) \end{aligned}$$

We conclude that we must have had equality throughout all the inequalities, and furthermore $\sigma(x, Nh) = \frac{Nh}{2}$ independently of x or N .

Returning to the pointwise description of ω_0 and ω_{Nh} from Eqn. (7.3.1), one sees that f is uniformly expanding and moreover preserves two holomorphic foliations given by the most expanded/contracted direction of f . From here one concludes that (X, f) is a Kummer example, using results of Cantat [Can01a, Thm. 7.4]. Alternatively, one can now verify that the Ricci-flat metrics are flat, see [FT19, §3.2], and deduce that we have a Kummer example. \square

7.3.2. Remark. A smooth K3 surface does not admit an everywhere defined holomorphic foliation, or a flat metric. In the proof above, one has to work with singular versions of the K3 surface to carry out the argument in the general case. Namely, there is an orbifold quotient $X \xrightarrow{\nu} Y$, and orbifold-Kähler metrics ω_Y on Y , such that

the cohomology class $[\eta_+] + [\eta_-]$ is represented by $\nu^*[\omega_Y]$. One can furthermore arrange the orbifold-Kähler metrics to be Ricci-flat and carry out the proof on Y instead of X . Similarly, the holomorphic foliations used to recognize the Kummer examples are built first on the orbifold Y .

On the other hand, the argument as presented already implies that for a generic $(2, 2, 2)$ -example as in §5.1.1, the measure of maximal entropy μ cannot equal $d\text{Vol}$. Indeed, for a hyperbolic automorphism of a generic $(2, 2, 2)$ example, the class $[\eta_+] + [\eta_-]$ will be Kähler, even ample. If we had $\mu = d\text{Vol}$, then the above argument would yield everywhere-defined foliations on X , which is impossible.

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