COHOMOLOGY OF THE UNIVERSAL ABELIAN SURFACE WITH APPLICATIONS TO ARITHMETIC STATISTICS

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ABSTRACT. The moduli stack $A_2$ of principally polarized abelian surfaces comes equipped with the universal abelian surface $\pi : X_2 \to A_2$. The fiber of $\pi$ over a point corresponding to an abelian surface $A$ in $A_2$ is $A$ itself. We determine the $\ell$-adic cohomology of $X_2$ as a Galois representation, and hence the singular cohomology of $X_2(C)$. Similarly, we consider the bundles $X_2^n \to A_2$ and $X_2^{\text{Sym}(n)} \to A_2$ for all $n \geq 1$, where the fiber over a point corresponding to an abelian surface $A$ is $A^n$ and $\text{Sym}^n A$ respectively. We describe how to compute the $\ell$-adic cohomology of $X_2^n$ and $X_2^{\text{Sym}(n)}$ and explicitly calculate it in low degrees for all $n$ and in all degrees for $n = 2$. These results yield new information regarding the arithmetic statistics on abelian surfaces, including an exact calculation of the expected value and variance as well as asymptotics for higher moments of the number of $\mathbb{F}_q$-points.

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1. INTRODUCTION

An abelian surface is an abelian variety of dimension 2. Over $\mathbb{C}$, all abelian surfaces are isomorphic to $\mathbb{C}^2/L$ for some lattice $L$ with real rank 4. The fine moduli stack $A_2$ of principally polarized abelian surfaces is a smooth Deligne–Mumford stack defined over $\mathbb{Z}$. It comes equipped with a
universal bundle $X_2 \to A_2$. The stack $X_2$ has the abelian surface $A$ as the fiber over the corresponding point in $A_2$. Using the projection map $X_2 \to A_2$, we can take $n$th fiber powers $X_2^n$ of $X_2$ over $A_2$, which has the $n$th power $A^n$ of an abelian surface over the corresponding point in $A_2$. Since each $A^n$ has an action of $S_n$ permuting the coordinates (which is not a free action), taking the quotient $X_2^n / S_n$ gives a new stack $X_2^{\text{Sym}(n)}$, which has $\text{Sym}^n A$ as a fiber over the point corresponding to $A$ in $A_2$.

Our main theorems are the computations of the $\ell$-adic cohomology of the universal abelian surface and related spaces as Galois representations (up to semi-simplification). Given the $\ell$-adic cohomology of any space $X$ of study in this paper, Artin’s comparison theorem ([Mil08b, Theorem 21.1]) and the transfer isomorphism give the singular cohomology of $X$ as vector spaces for all $\ell$.

### Theorem 1.1.
The cohomology of the universal abelian surface $X_2$ is given by

$$H^k(\mathcal{X}(2); \mathbb{Q}) \otimes \mathbb{Q}_\ell \cong H^k_{\text{et}}(\mathcal{X}; \mathbb{Q}_\ell)$$

as vector spaces for all $k \geq 0$. From now on, all cohomology will denote $\ell$-adic cohomology and we drop the subscripts to write $H^*(X; \mathcal{V})$ in place of both $H^*_{\text{et}}(\mathcal{X}; \mathbb{Q})$ and $H^*_\text{et}(\mathcal{X}_{\mathbb{Q}}; \mathcal{V})$ for any $\ell$-adic local system $\mathcal{V}$ on $\mathcal{X}$ with $\ell$ coprime to $q$. (See Remark 2.4 for details justifying this notation.)

### Theorem 1.2.
For all $n \geq 1$, the cohomology of the universal $n$th fiber product of abelian surfaces is

$$H^k(X_2^n; \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & k = 0 \\ 0 & k = 1, 3, k > 7 \\ 2\mathbb{Q}_\ell(-1) & k = 2 \\ 2\mathbb{Q}_\ell(-2) & k = 4 \\ \mathbb{Q}_\ell(-5) & k = 5 \\ \mathbb{Q}_\ell(-3) & k = 6 \end{cases}$$

up to semi-simplification, where $\mathbb{Q}_\ell = \mathbb{Q}_\ell(0)$ is the trivial Galois representation, $\mathbb{Q}_\ell(1)$ is the $\ell$-adic cyclotomic character, and $\mathbb{Q}_\ell(-1)$ is its dual. For all $n \in \mathbb{N}$, $\mathbb{Q}_\ell(n)$ is the $n$th tensor power of $\mathbb{Q}_\ell(1)$ and $\mathbb{Q}_\ell(-n)$ is the $n$th tensor power of $\mathbb{Q}_\ell(-1)$. For all $n \in \mathbb{Z}$, $\mathbb{Q}_\ell(n) \cong \mathbb{Q}_\ell$ as a $\mathbb{Q}_\ell$-vector space. Denote $\mathbb{Q}_\ell(n)^\otimes m$ by $m\mathbb{Q}_\ell(n)$.

Applying similar techniques gives the cohomology of the $n$th fiber product of $X_2$ in low degrees.

### Theorem 1.2.
For all $n \geq 1$, the cohomology of the universal $n$th fiber product of abelian surfaces is

$$H^k(X_2^n; \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & k = 0 \\ 0 & k = 1, 3 \\ \left(\binom{n+1}{2} + 1\right)\mathbb{Q}_\ell(-1) & k = 2 \\ \left(\frac{2(n+1)(n^2+n+2)}{8} + \binom{n+1}{2}\right)\mathbb{Q}_\ell(-2) & k = 4 \\ \left(\binom{n+1}{2}\mathbb{Q}_\ell(-5) \oplus \binom{n}{2}\mathbb{Q}_\ell(-4)\right) & k = 5 \end{cases}$$

up to semi-simplification.

For $n$ large enough compared to the degree, the cohomology of the universal $n$th symmetric power of abelian surfaces is independent of $n$; the following theorem determines the cohomology in low degrees.
Theorem 1.3. For all \( n \geq k \) for \( k \) even and for all \( n \geq k - 1 \) for \( k \) odd,

\[
H^k(\chi_{2^m}^{\text{Sym}(n)}; \mathbb{Q}_\ell) = \begin{cases} 
\mathbb{Q}_\ell & k = 0 \\
0 & k = 1, 3 \\
2\mathbb{Q}_\ell(-1) & k = 2 \\
5\mathbb{Q}_\ell(-2) & k = 4 \\
\mathbb{Q}_\ell(-5) + \mathbb{Q}_\ell(-4) & k = 5
\end{cases}
\]

up to semi-simplification.

The proofs of these theorems use the Leray spectral sequence of the morphisms \( \pi : \mathcal{X} \to \mathcal{A}_2 \), with \( \mathcal{X} = \mathcal{X}_2, \mathcal{X}_2^n \), and \( \chi_{2^m}^{\text{Sym}(n)} \) respectively. The spectral sequence takes as input the cohomology of local systems of \( \mathcal{A}_2 \), which has been computed by Petersen in [Pet15]. Then it still remains to determine the local systems involved in the latter two cases, converting this problem about cohomology into a series of problems about the representation theory of \( \text{Sp}(4, \mathbb{Z}) \). For \( \chi_{2^m}^{\text{Sym}(n)} \), we give recursive formulas (in \( n \)) for the relevant local systems in Subsection 4.1. For \( \chi_{2^m}^{\text{Sym}(n)} \), we show that the local systems \( R^k\pi_*\mathbb{Q}_\ell \) stabilize for \( n \geq k \). In both cases, we use these facts to prove Theorems 1.2 and 1.3. The cohomology in higher degrees is quite involved to determine for all general \( n \) and involve Galois representations attached to certain (Siegel) modular forms, but the methods of this paper give a finite computation for the relevant local systems for each fixed \( n \). We work out the case \( n = 2 \) completely – see Theorems 4.17 and 5.5 for the cohomology of \( \chi_{2^2}^{\text{Sym}(2)} \) and \( \chi_{2^2}^{\text{Sym}(2)} \) respectively.

Arithmetic Statistics. We fix throughout a finite field \( \mathbb{F}_q \). The Weil conjectures give bounds on the number of \( \mathbb{F}_q \)-points on any projective variety over \( \mathbb{F}_q \). Applied to an abelian surface \( A \) they assert that

\[
\#A(\mathbb{F}_q) = q^2 + a_3q^{3/2} + a_2q + a_1q^{1/2} + 1
\]

where \( a_i \) are some sums of \( n_i \) roots of unity with \( n_i = 4, 6, 4 \) for \( i = 1, 2, 3 \) respectively. A simple corollary is

\[
|\#A(\mathbb{F}_q) - (q^2 + 1)| \leq 4q^{3/2} + 6q + 4q^{1/2},
\]

constraining the possible values that \( \#A(\mathbb{F}_q) \) can take. The exact set of possible values of \( \#A(\mathbb{F}_q) \) is given by Honda–Tate theory, which yields a bijection between isogeny classes of simple abelian varieties (of all dimensions) over \( \mathbb{F}_q \) and Weil \( q \)-polynomials. In particular, for a Weil \( q \)-polynomial \( f \), there is some abelian variety \( V \) such that the characteristic polynomial \( f_V \) of the Frobenius endomorphism of \( V \) is given by \( f_V = f^e \) for some \( e \geq 1 \), for which \( \#V(\mathbb{F}_q) = f_V(1) \). While the restriction of this bijection to simple abelian surfaces is known, it is too long to restate here. See [Rü90] and [Wat69], or [DGS14] Section 2 for an overview.

It is evident that the distribution of \( \#A(\mathbb{F}_q) \) as the abelian surface \( A \) varies over \( \mathcal{A}_2(\mathbb{F}_q) \) is non-trivial, but that studying the counts \( \#\chi_{2^m}^{\text{Sym}(n)}(\mathbb{F}_q) \) for \( n \geq 1 \) will shed light on this distribution. Our main tool to obtain these point counts is the Grothendieck–Lefschetz–Behrend trace formula ([Beh93, Theorem 3.1.2]). A first observation through standard applications of the Weil conjectures and the trace formula is that \( \#\chi_{2^m}^{\text{Sym}(n)}(\mathbb{F}_q) = q^d + O(q^{d-\frac{1}{2}}) \) where \( d = \dim \chi_{2^m}^{\text{Sym}(n)} \), because \( \chi_{2^m}^{\text{Sym}(n)} \) is finitely covered by a smooth, irreducible, quasiprojective variety. However, applying the trace formula to our cohomological theorems immediately gives more precise asymptotics for \( \#\chi_{2^m}^{\text{Sym}(n)}(\mathbb{F}_q) \) as well as new arithmetic statistics about the number of \( \mathbb{F}_q \)-points of abelian surfaces. Below, we consider expected values of random variables on \( \mathcal{A}_2(\mathbb{F}_q) \) by giving \( \mathcal{A}_2(\mathbb{F}_q) \) a natural
probability measure where each isomorphism class of an abelian surface $A$ has probability inversely proportional to the size of its $F_q$-automorphism group; see Lemma 6.8 for more details.

**Corollary 1.4.** The expected number of $F_q$-points on abelian surfaces defined over $F_q$ is

$$
E[\#A(F_q)] = q^2 + q + 1 - \frac{1}{q^3 + q^2}.
$$

For each prime power $q > 0$, there is a simple abelian surface $A_q$ over $F_q$ with $\#A_q(F_q) = q^2 + q + 1$ by the Honda–Tate correspondence for surfaces ([Rüc90, Theorem 1.1]) which corresponds to the case $a_3 = a_1 = 0, a_2 = 1$ of the Weil conjectures. Although $E[\#A(F_q)]$ is not realized by an abelian surface for any fixed $q$, the minimal difference between the expected value and the $F_q$-point count of an arbitrary abelian surface goes to 0 as $q$ increases, i.e.

$$
\min_{[A] \in A_2(F_q)} |\#A(F_q) - E[\#(F_q)]| \to 0 \quad \text{as} \quad q \to \infty.
$$

The trace formula is also used to compute the exact expected value of $\# A^2(F_q)$ and an asymptotic estimate for the expected value of $\# A^n(F_q)$ for $n > 2$. Here, the $F_q$-points of the $n$th power $A^n$ of an abelian surface $A$ are ordered $n$-tuples of (not necessarily distinct) $F_q$-points of $A$. Because $\# A^n(F_q) = (\# A(F^n))^n$ for any abelian surface $A$, the following corollary gives the exact second moment of the number of $F_q$-points on abelian surfaces and asymptotic estimates on the $n$th moment for all $n \geq 3$.

**Corollary 1.5.** The expected value of $\# A^2(F_q)$ is

$$
E[\# A^2(F_q)] = q^4 + 3q^3 + 6q^2 + 3q - \frac{5q^2 + 5q + 3}{q^3 + q^2}
$$

and for all $n \geq 1$,

$$
E[\# A^n(F_q)] = q^{2n} + \left(\frac{n + 1}{2}\right)q^{2n-1} + \left(\frac{n(n + 1)(n^2 + n + 2)}{8}\right)q^{2n-2} + O(q^{2n-3}).
$$

However, note that computing the $n$th moment for large $n$ involves representations attached to (Siegel) modular forms; therefore, the recursive formulas for local systems in the cohomological computations do not completely determine these moments.

The same methods give the exact expected value of $\# \text{Sym}^2 A(F_q)$ and an asymptotic estimate for the expected value of $\# \text{Sym}^n A(F_q)$ for $n > 2$. The $F_q$-points of the symmetric power $\text{Sym}^n A$ of an abelian surface $A$ are the unordered $n$-tuples of (not necessarily distinct) $F_q$-points of $A$ which are defined as an $n$-tuple over $F_q$. This means that the $n$-tuple contains all Galois conjugates of each point of the $n$-tuple.

**Corollary 1.6.** The expected value of $\# \text{Sym}^2 A(F_q)$ is

$$
E[\# \text{Sym}^2 A(F_q)] = q^4 + q^3 + 4q^2 + q - \frac{q^2 + 3q + 1}{q^3 + q^2}
$$

and for all $n \geq 4$,

$$
E[\# \text{Sym}^n A(F_q)] = q^{2n} + q^{2n-1} + 4q^{2n-2} + O(q^{2n-3}).
$$

Because Corollary 1.5 gives the exact second moment of $\# A(F_q)$, we also obtain the variance:

**Corollary 1.7.** The variance of $\# A(F_q)$ is

$$
\text{Var}(\# A(F_q)) = q^3 + 3q^2 + q - \frac{3q^2 + 3q + 1}{q^3 + q^2} - \frac{1}{(q^3 + q^2)^2}.
$$
These statistics are computed in Subsection 6.1 by studying $\# \mathcal{X}(\mathbb{F}_q)$ for various stacks $\mathcal{X}$. All $\mathbb{F}_q$-point counts in this paper are weighted by the inverse of the size of their automorphism groups, as explained in Section 6. For $\mathcal{X}_n^2$, there is a way to give these weighted counts an unweighted interpretation by counting the number of $\mathbb{F}_q$-isomorphism classes of objects parametrized by $\mathbb{F}_q$-points of $\mathcal{X}_n^2$ instead of the $\mathbb{F}_q$-isomorphism classes. These ideas are standard (e.g. see [vdGvdV92 Section 5] or [KS99 Section 10.7] for this in other contexts), but we detail the argument for $n$-tuples of points in abelian surfaces in Subsection 6.3.

**Related work.** The cohomologies and point counts of the moduli space of abelian varieties and the universal abelian variety are often studied via Siegel modular forms. For example, the cohomology of local systems on the moduli space of elliptic curves is known classically (e.g. the Eichler–Shimura isomorphism) and yields connections between modular forms and point counts of elliptic curves over finite fields. (See [vdG13] and [HT18] for a survey on current developments in this area.) The approach involving Siegel modular forms is implicit in this paper, as they appear Petersen’s computations in [Pet15], but we do not directly pursue this direction.

Recent work in arithmetic statistics of abelian varieties also take a different flavor than of this paper. Honda–Tate theory has been used to determine some probabilistic data about the group structure of abelian surfaces ([DG5+14]), upper and lower bounds on the number of $\mathbb{F}_q$-points on abelian varieties ([AHL13]), sizes of isogeny classes of abelian surfaces ([XY20]), and others. Certainly, this is not the only current approach in this direction – for example, [CFHS12] takes a heuristic approach to determining the probability that the number of $\mathbb{F}_q$-points on a genus 2 curve is prime.

On the other hand, cohomological methods have been applied to related spaces to deduce arithmetic-statistical results or heuristics, such as the number of points on curves of genus $g$ ([AEK+15]) and the average number of points on smooth cubic surfaces ([Das19]). For a survey in the case of counting genus $g$ curves and its connection to the cohomology of the relevant moduli spaces, see [vdG15].

**Outline of paper.** In Section 2, we give a description of the spaces of study and the cohomological tools used throughout the paper. In Section 3, we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2 and completely work out the cohomology of $\mathcal{X}_2^2$ as an example. In Section 5, we carry out analogous arguments to prove Theorem 1.3 and work out the cohomology of $\mathcal{X}_2^{\text{sym}(2)}$. In Section 6, we deduce new arithmetic statistics results about abelian surfaces using the previous sections, including Corollaries 1.4, 1.5, 1.6, and 1.7.

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2. $A_2$, Its Cohomology, and Cohomological Tools

In this section, we describe the spaces that we study in this paper and outline the cohomological tools that will be used throughout the paper.

2.1. Spaces of interest. Denote the moduli stack of principally polarized abelian surfaces by $A_2$. There is an explicit, complex analytic construction of this space: let $\mathcal{H}_2$ be the Siegel upper half space of degree 2 with the usual action of $\text{Sp}(4, \mathbb{Z})$,

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (C\tau + D)^{-1}(A\tau + B).
$$

To each $\tau \in \mathcal{H}_2$, we can associate a lattice $L_\tau \subseteq \mathbb{C}^2$ and therefore a complex torus $A_\tau$. It turns out that $A_\tau$ comes with a natural principal polarization $H_\tau$, making $(A_\tau, H_\tau)$ into a principally polarized abelian variety. For any $\tau_1, \tau_2 \in \mathcal{H}_2$, the abelian surfaces $\langle A_{\tau_1}, H_{\tau_1} \rangle$ and $\langle A_{\tau_2}, H_{\tau_2} \rangle$ are isomorphic if and only if $\tau_1$ and $\tau_2$ are in the same $\text{Sp}(4, \mathbb{Z})$-orbit.

The stabilizer of each point is finite. Therefore, the group $\text{Sp}(4, \mathbb{Z})$ on $\mathcal{H}_2$ is not free. For example, $-I_4$ fixes every $\tau \in \mathcal{H}_2$. However, the stabilizer of each point is finite. Therefore, $A_2(C)$ is a stack-theoretic quotient $[\text{Sp}(4, \mathbb{Z})/\mathcal{H}_2]$, with the underlying topological space given by the quotient $\text{Sp}(4, \mathbb{Z})/\mathcal{H}_2$, which we call $(A_2)_{\text{an}}^{C}$.

Let $f_M : A_\tau \to A_\tau$ be the isomorphism given by $M \in \text{Sp}(4, \mathbb{Z})$. After identifying $\mathbb{C}^2$ with the $\mathbb{R}$-span of the basis $\{e_1, e_2, e_3, e_4\}$ of $L_\tau$, one can check that $f_M$ acts by left multiplication of $M$ on $\mathbb{C}^2$, which descends to a map on $\mathbb{C}^2/L_\tau$. This induces an action on the cohomology of each $A_\tau$, which is well-known:

**Fact 2.1.** For all $k \geq 0$ and for all complex abelian surfaces $A$, $H^k(A; \mathbb{Q}) \cong \wedge^k V$ as a representation of $\text{Sp}(4, \mathbb{Z})$, where $V$ is the standard representation.

In addition, the cohomology of $A_2$ is known:

**Theorem 2.2** ([LW85, Corollary 5.2.3], [vdG13, Section 10]).

$$
H^k(A_2; \mathbb{Q}_\ell) = \begin{cases} 
\mathbb{Q}_\ell & k = 0 \\
\mathbb{Q}_\ell(-1) & k = 2 \\
0 & \text{otherwise}.
\end{cases}
$$

Next, we denote the universal abelian surface by $X_2$ and give an explicit, complex analytic construction. Take the action of $\text{Sp}(4, \mathbb{Z}) \ltimes \mathbb{Z}^4$ on $\mathcal{H}_2 \times \mathbb{C}^2$, where $\mathbb{Z}^4$ acts by translation on each $L_\tau \subseteq \mathbb{C}^2$. This gives a stack-theoretic quotient $[\text{Sp}(4, \mathbb{Z}) \ltimes \mathcal{H}_2 \times \mathbb{C}^2]$, denoted $X_2(C)$, and an underlying analytic space $\text{Sp}(4, \mathbb{Z}) \ltimes \mathbb{Z}^4 \backslash (\mathcal{H}_2 \times \mathbb{C}^2)$, denoted $(X_2)^{\text{an}}_C$. Note that the fiber of the natural projection $X_2(C) \to A_2(C)$ over a point corresponding to the surface $A$ is $A$ itself.

There is a natural projection map $X_2 \to A_2$, and hence the product $X_2^n$ of $X_2$ with itself $n$ times fibered over $A_2$. As before, the group $\text{Sp}(4, \mathbb{Z}) \ltimes (\mathbb{Z}^4)^n$ acts on $\mathcal{H}_2 \times (\mathbb{C}^2)^n$ in the obvious way, and so there is a stack-theoretic quotient $X_2^n(C) = [\text{Sp}(4, \mathbb{Z}) \ltimes (\mathbb{Z}^4)^n / \mathcal{H}_2 \times (\mathbb{C}^2)^n]$, with the underlying analytic space $(X_2^n)^{\text{an}}_C$ given by the usual quotient. The fiber of $X_2^n(C) \to A_2(C)$ over a point corresponding to the surface $A$ is the $n$th power $A^n$.

Also consider the stack $A_2^{\text{Sym}(n)}$. Over $C$, each fiber $A^n$ of the projection morphism $X_2^n \to A_2$ has an action of $S_n$ permuting the coordinates, giving another stack-theoretic quotient $X_2^{\text{Sym}(n)} = [X_2^n / S_n]$. The fiber of $X_2^{\text{Sym}(n)} \to A_2$ of the point corresponding to the abelian surface $A$ is $\text{Sym}^n A$, the $n$th symmetric power of $A$. As usual, there is an underlying analytic space $(X_2^{\text{Sym}(n)})^{\text{an}}_C = (X_2^n)^{\text{an}}_C / S_n$. 

For any $N \geq 2$ and abelian surface $A$, let $A[N]$ be the kernel of the multiplication by $N$ map on $A$. Consider the moduli stack $A_2[N]$ of principally polarized abelian surfaces with symplectic level $N$ structure, i.e. pairs $(A, \alpha)$ where $\alpha$ is an isomorphism from $A[N]$ to a fixed symplectic module $((\mathbb{Z}/N\mathbb{Z})^4, \langle \cdot, \cdot \rangle)$. Let $\text{Sp}(4, \mathbb{Z})[N] = \text{ker}(\text{Sp}(4, \mathbb{Z}) \to \text{Sp}(4, \mathbb{Z}/N\mathbb{Z}))$. Then $A_2[N](\mathbb{C})$ and its universal family $A_2[N](\mathbb{C})$ are stack-theoretic quotients $[\text{Sp}(4, \mathbb{Z})[N]\backslash \mathbb{H}_2]$ and $[\text{Sp}(4, \mathbb{Z})[N] \times \mathbb{Z}^4 \backslash \mathbb{H}_2 \times \mathbb{C}^2]$ by the same reasoning as above. In general, we obtain even more than the stack-theoretic quotient in the case $N \geq 3$. Both $A_2[N]$ and $A_2[N]$ are quasiprojective schemes over $\mathbb{Z}[1/N]$. (\cite{FC90} Chapter IV).

On the other hand, consider the moduli stack $A_{2,N}$ of principally polarized abelian surfaces with principal level $N$ structure, i.e. pairs $(A, \beta)$ where $\beta : A[N] \to (\mathbb{Z}/N\mathbb{Z})^4$ is an isomorphism. For $N \geq 3$, $A_{2,N}$ and its universal family $A_{2,N}$ are quasiprojective schemes over $\mathbb{Z}[1/N]$ (\cite{FC90} Chapter I). This shows that $\mathcal{X} = A_2$, $A_2^{\alpha}$, and $A_2^{\text{Sym}(n)}$ are all finite quotients of quasi-projective schemes over $\mathbb{Z}[1/N]$ with $N \geq 3$. (For more details, also see \cite{Os12} Theorem 2.1.11.) Over characteristic zero, quotient stacks with finite automorphism groups at every point are Deligne–Mumford stacks (\cite{Edi00} Corollary 2.2). Over positive characteristic, quotient stacks are a priori Artin stacks with a smooth atlas. Because any base change of an étale morphism is étale, any stack $\mathcal{X}$ considered in this paper obtained from a stack over $\mathbb{Z}[1/N]$ via base-change to $\mathbb{F}_q$ (where $N$ and $q$ are coprime) has an étale atlas; therefore, $\mathcal{X}_{\mathbb{F}_q}$ is a Deligne–Mumford stack.

In fact, $A_2$ and $A_2^{\alpha}$ are complements of normal crossing divisors in smooth, proper stacks over $\mathbb{Z}$ (see \cite{FC90} Chapter VI), making both $A_2$ and $A_2^{\alpha}$ as well as its finite quotient $A_2^{\text{Sym}(n)}$ smooth stacks over any finite field $\mathbb{F}_q$. The points in $\mathcal{X}(\mathbb{F}_q)$ can be identified with the fixed points of $\text{Frob}_q$ in $A_2(\mathbb{F}_q)$. This identification of $\mathcal{X}(\mathbb{F}_q)$ agrees with the definition of $\mathbb{F}_q$-points as morphisms $\text{Spec}(\mathbb{F}_q) \to \mathcal{X}$. Over any field $k$, there are the following moduli interpretations of the $k$-points of these stacks. When we write an abelian surface $A$, we mean $A$ with a principal polarization.

1. $A_2(k)$ is the groupoid of $k$-isomorphism classes of abelian surfaces $A$ defined over $k$.

2. $A_2[N](k)$ is the groupoid of $k$-isomorphism classes of pairs $(A, \alpha)$ where $\alpha : A[N] \to (\mathbb{Z}/N\mathbb{Z})^4$ is a symplectic isomorphism,

3. $A_2^{\alpha}(k)$ is the groupoid of $k$-isomorphism classes of pairs $(A, p)$ with $p \in A^n$, defined over $k$.

4. $A_2^{\text{Sym}(n)}(k)$ is the groupoid of $k$-isomorphism classes of pairs $(A, p)$ with $p \in A^n/S_n$, defined over $k$.

Remark 2.3. We note that for some $\{p_1, \ldots, p_n\} \in (A^n/S_n)(k)$, a lift $(p_1, \ldots, p_n) \in A^n$ may not be a $k$-point of $A^n$. For instance, if $\text{Gal}(\overline{k}/k)$ permutes the points $p_1, \ldots, p_n \in A^n$, then $\{p_1, \ldots, p_n\}$ will be a $k$-point of $\text{Sym}^n A$, but not necessarily of $A^n$.

Although this is possibly not the most efficient framework, we will access all stacks discussed by taking quotient stacks of the respective quasiprojective varieties throughout this paper in an effort to keep the arguments as concrete as possible.

2.2. Local systems on $A_2$. Representations of $\pi_1^{\text{orb}}(A_2) = \text{Sp}(4, \mathbb{Z})$ give rise to local systems on $A_2$. In the rest of this paper, we focus on the irreducible representations of $\text{Sp}(4, \mathbb{Z})$ obtained as restrictions of irreducible representations of $\text{Sp}(4, \mathbb{C})$. We briefly review the construction of such local systems as given in \cite{Pet15}.
By Weyl’s construction (see [FH04 Section 17.3]), they are all given in the following way: for any \( a \geq b \geq 0 \), there is an irreducible representation \( V_{a,b} \) with highest weight \( aL_1 + bL_2 \), using the notation of [FH04 Chapter 17]. In particular, \( V_{a,b} \) is a summand of \( V_{1,0}^{\otimes (a+b)} \) by construction where \( V_{1,0} \) is the 4-dimensional standard representation of \( \text{Sp}(4, \mathbb{Z}) \).

Let \( \pi : \mathcal{A} \rightarrow \mathcal{A}_2 \). By the proper base change theorem ([Mil80 Corollary VI.2.5]), the stalk of \( R^1\pi_*\mathbb{Q}_p \) at \( [A] \in \mathcal{A}_2 \) is isomorphic to \( H^1(\mathcal{A}; \mathbb{Q}_p) \). Define \( V_{1,0} \) to be the local system \( R^1\pi_*\mathbb{Q}_p \). The underlying \( \text{Sp}(4, \mathbb{Z}) \)-representation of each stalk of \( V_{1,0} \) is the standard representation \( V_{1,0} \). Then \( V_{1,0} \) is a local system with Hodge weight 1 equipped with a \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-equivariant symplectic pairing

\[
\bigwedge^2 V_{1,0} \rightarrow \mathbb{Q}_p(-1)
\]

coming from the Weil pairing of each \( [A] \in \mathcal{A}_2 \). Applying Weyl’s construction to the local system \( V_{1,0} \) yields local systems \( V_{a,b} \) for all \( a \geq b \geq 0 \). Each \( V_{a,b} \) is a summand in \( V_{1,0}^{\otimes (a+b)} \), so \( V_{a,b} \) has Hodge weight \( a + b \). The underlying \( \text{Sp}(4, \mathbb{Z}) \)-representation of \( V_{a,b} \) is \( V_{a,b} \) as the notation suggests. For all \( n \in \mathbb{Z} \), let \( V_{a,b}(n) := V_{a,b} \otimes \mathbb{Q}_p(n) \) be the \( n \)th Tate twist of \( V_{a,b} \).

2.3. Cohomological tools. In this subsection, we list the tools we will need in subsequent sections regarding cohomology computations. First, we set some notation used for the remainder of the paper. We will always denote by \( A \) an abelian surface. By \( H^p(\mathcal{X}; H^q(\mathcal{X}; \mathbb{Q}_p)) \) for a stack \( \mathcal{X} \) and some space \( X \) with a natural \( \pi^{\text{orb}}(\mathcal{X}) \)-action, we will always mean the cohomology of the local system associated to the \( \pi^{\text{orb}}(\mathcal{X}) \)-representation \( H^q(\mathcal{X}; \mathbb{Q}_p) \). The prime \( \ell \) will always be taken to be coprime to \( q \) when working with the base change \( \mathcal{X}_{\overline{\mathbb{F}}_\ell} \).

Remark 2.4. Let \( \mathcal{X} = \mathcal{A}_2 \) or \( \mathcal{X}_n^2 \) and let \( V \) be an \( \ell \)-adic local system on \( \mathcal{X} \). Because \( \mathcal{X} \) is a complement of a normal crossing divisor of a smooth, proper stack over \( \mathbb{Z} \) ([FC90 Chapter VI]), \( H^q_{\text{et}}(\mathcal{X}_{\overline{\mathbb{Q}}}; V) \) is unramified at every prime \( p \neq \ell \) ([Pet15 p. 11]), i.e. the action of \( \text{Frob}_p \) is well-defined. There is an isomorphism

\[
H^q_{\text{et}}(\mathcal{X}_{\overline{\mathbb{Q}}}; V) \cong H^q_{\text{et}}(\mathcal{X}_{\overline{\mathbb{F}}_\ell}; V)
\]

such that the action of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subseteq \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the left side factors through the surjection \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \), where \( \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \) acts on the right side. Passing both sides through the Hochschild–Serre spectral sequence shows that the same isomorphism holds for \( \mathcal{X}_n^{\text{Sym}(n)} = [\mathcal{X}_n^2/S_n] \).

The next two statements are well-known for all \( g \geq 1 \) but we specialize to the case \( g = 2 \). Both theorems (for general \( g \geq 1 \)) can be found in [HTT88].

Theorem 2.5 (Poincaré Duality for \( \mathcal{A}_2 \)). For any \( a \geq b \geq 0 \),

\[
H^k_{\text{et}}(\mathcal{A}_2; V_{a,b}) \cong H^{6-k}_{\text{et}}(\mathcal{A}_2; V_{a,b})^* \otimes \mathbb{Q}_\ell (-3 - a - b).
\]

Theorem 2.6 (Deligne’s weight bounds). The mixed Hodge structures on the groups \( H^k(\mathcal{A}_2; V_{a,b}) \) have weights larger than or equal to \( k + a + b \).

In Sections 3 and 4 we compute the étale cohomology \( H^*(\mathcal{X}; \mathbb{Q}_p) \), with \( \mathcal{X} = \mathcal{A}_2, \mathcal{X}_n^2, \) and \( \mathcal{X}_n^{\text{Sym}(n)} \) respectively. In all of these cases, there are morphisms \( \pi : \mathcal{X} \rightarrow \mathcal{A}_2 \) to which we want to apply the Leray spectral sequence to obtain the desired results.

Theorem 2.7 ([Del68]). Let \( f : X \rightarrow Y \) be a smooth projective morphism of complex varieties. Then the Leray spectral sequence for \( f \) degenerates on the \( E_2 \)-page.
For \( N \geq 3 \), the projection \( \pi : \mathcal{X}_2[N]_\mathbb{Q} \rightarrow A_2[N]_\mathbb{Q} \) is a projective morphism of quasi-projective varieties. Combined with a corollary of the proper base change theorem ([Mil08b, Corollary VI.4.3]), this implies the following useful result:

**Corollary 2.8.** For all \( n \geq 1, N \geq 3 \), the Leray spectral sequence for \( \pi : \mathcal{X}_2[N]_\mathbb{Q} \rightarrow A_2[N]_\mathbb{Q} \) degenerates on the \( E_2 \)-page.

Finally, we state the main tool of this paper.

**Proposition 2.9.** Let \( \mathcal{X} = \mathcal{X}_2^n \) (resp. \( \mathcal{X}_2^{\text{Sym}(n)} \)) and \( \pi : \mathcal{X} \rightarrow A_2 \). There is a spectral sequence

\[
E_2^{p,q} = H^p(A_2; H^q(Z_n; \mathbb{Q}_\ell)) \implies H^{p+q}(\mathcal{X}; \mathbb{Q}_\ell)
\]

with \( Z_n = A^n \) (resp. \( Z_n = \text{Sym}^n A \)), which degenerates on the \( E_2 \)-page.

**Proof.** Let \( N \geq 3 \) and let \( \mathcal{X}_2[N]^n \) be the \( n \)th fiber power of \( \mathcal{X}_2[N] \) over \( A_2[N] \) with respect to the projection map \( \mathcal{X}_2[N] \rightarrow A_2[N] \). Then \( A_2[N] \) and \( \mathcal{X}_2[N]^n \) are quasi-projective varieties. By the standard Leray spectral sequence for étale cohomology ([Mil085 Theorem 12.7]) with \( \pi_N : \mathcal{X}_2[N]^n \rightarrow A_2[N]_\mathbb{Q} \),

\[
H^p(A_2[N]_\mathbb{Q}; R^q(\pi_N)_* \mathbb{Q}_\ell) \implies H^{p+q}(\mathcal{X}_2[N]^n_\mathbb{Q}; \mathbb{Q}_\ell)
\]

and this spectral sequence degenerates on the \( E_2 \)-page by Corollary 2.8. Applying a corollary of the proper base change theorem ([Mil080 Corollary VI.2.5]) with the torsion (constant) sheaf \( \mathbb{Z}/\ell^n \mathbb{Z} \), taking inverse limits, and tensoring with \( \mathbb{Q}_\ell \),

\[
\bigoplus_{p+q=k} H^p(A_2[N]_\mathbb{Q}; H^q(A^n; \mathbb{Q}_\ell)) \cong H^k(\mathcal{X}_2[N]^n_\mathbb{Q}; \mathbb{Q}_\ell)
\]

as Galois representations up to semi-simplification. Then by the transfer isomorphism for the \( \text{Sp}(4, \mathbb{Z}/N\mathbb{Z}) \)-quotients \( \mathcal{X}_2[N]^n_\mathbb{Q} \rightarrow (\mathcal{X}_2^n)_\mathbb{Q} \) and \( A_2[N]_\mathbb{Q} \rightarrow (A_2)_\mathbb{Q} \) with coefficients in a field of characteristic zero,

\[
\bigoplus_{p+q=k} H^p(A_2[N]_\mathbb{Q}; H^q(A^n; \mathbb{Q}_\ell))^{\text{Sp}(4, \mathbb{Z}/N\mathbb{Z})} \cong H^k(\mathcal{X}_2[N]^n_\mathbb{Q}; \mathbb{Q}_\ell)^{\text{Sp}(4, \mathbb{Z}/N\mathbb{Z})} \cong H^k((\mathcal{X}_2^n)_\mathbb{Q}; \mathbb{Q}_\ell)
\]

and

\[
H^p(A_2[N]_\mathbb{Q}; H^q(A^n; \mathbb{Q}_\ell))^{\text{Sp}(4, \mathbb{Z}/N\mathbb{Z})} \cong H^p((A_2)_\mathbb{Q}; H^q(A^n; \mathbb{Q}_\ell)),
\]

where on the left, \( H^q(A^n) \) is the local system corresponding to the respective \( \text{Sp}(4, \mathbb{Z})[N] \)-representation, while on the right, \( H^q(A^n) \) is the local system corresponding to the respective \( \text{Sp}(4, \mathbb{Z}) \)-representation. Therefore, taking \( \text{Sp}(4, \mathbb{Z}/N\mathbb{Z}) \)-invariants in the spectral sequence for \( \pi_N \), which one can do by naturality of that sequence, gives the following \( E_2 \)-page of a spectral sequence

\[
E_2^{p,q} = H^p((A_2)_\mathbb{Q}; H^q(A^n; \mathbb{Q}_\ell)) \implies H^{p+q}((\mathcal{X}_2^n)_\mathbb{Q}; \mathbb{Q}_\ell)
\]

degenerating on the \( E_2 \)-page. By Remark 2.4 the spectral sequence for \( (\mathcal{X}_2^n)_\mathbb{Q} \rightarrow (A_2)_\mathbb{Q} \) must also degenerate on the \( E_2 \)-page; we now write \( \mathcal{X} \) in place of both \( \mathcal{X}_2[N]_\mathbb{Q} \) and \( (\mathcal{X}_2^n)_\mathbb{Q} \) for all stacks \( \mathcal{X} \) throughout this proof. Lastly, again by the transfer isomorphism,

\[
H^k(\mathcal{X}_2^{\text{Sym}(n)}; \mathbb{Q}_\ell) \cong H^k(\mathcal{X}_2^n; \mathbb{Q}_\ell)^{S_n}.
\]

Because \( S_n \) acts trivially on \( A_2 \),

\[
H^p(A_2; H^q(A^n))^{S_n} \cong H^p(A_2; H^q(A^n)^{S_n}) \cong H^p(A_2; H^q(\text{Sym}^n A)),
\]

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where $H^q(Sym^n A)$ is again an $Sp(4, \mathbb{Z})$-representation. Therefore, again by naturality, taking $S_n$-invariants in the spectral sequence for $\mathcal{X}_2^n \to A_2$ gives

$$E_2^{p,q} = H^p(A_2; H^q(Sym^n A)) \Rightarrow H^{p+q}(\mathcal{X}_2^{\text{Sym}(n)}; \mathbb{Q}_\ell)$$

which degenerates on the $E_2$-page. \hfill $\Box$

Remark 2.10. This spectral sequence has appeared elsewhere in the literature (e.g. see [GHT18 Sections 4, 6]).

3. Cohomology of the Universal Abelian Surface

In this section, we study the cohomology of $\mathcal{X}_2$ using $\pi : \mathcal{X}_2 \to A_2$. We first need to compute the following local systems.

Lemma 3.1. The local systems $H^k(A; \mathbb{Q}_\ell)$ on $A_2$ are

$$H^k(A; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & k = 0 \\ V_{1,0} & k = 1 \\ Q_{\ell}(-1) \oplus V_{1,1} & k = 2 \\ V_{1,0}(-1) & k = 3 \\ Q_{\ell}(-2) & k = 4 \\ 0 & k > 4. \end{cases}$$

Proof. By Fact 2.1, $H^q(A; \mathbb{Q}_\ell) \cong \bigwedge^q V_{1,0}$, the local system given by the corresponding representation of $Sp(4, \mathbb{Z})$. Then we can decompose this local system as we would the corresponding representation into irreducible ones. This precise decomposition (as $Sp(4, \mathbb{Z})$-representations) is given in [FH04 Chapter 16].

It remains to determine the Galois action on each local system. For appropriate constants $m_{a,b}(H^k(A))$, the decomposition

$$\psi : H^k(A; \mathbb{Q}_\ell) \cong \bigoplus_{a,b \geq 0} m_{a,b}(H^k(A))V_{a,b}$$

gives an isomorphism on the stalks as $Sp(4, \mathbb{Z})$-representations but does not respect the Galois action. Any Frobp $\in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by an element of $GSp(4, \mathbb{Q}_\ell)$ on $H^1(A; \mathbb{Q}_\ell)$ because the symplectic form $\bigwedge^2 V_{1,0} \to \mathbb{Q}_\ell(1)$ of Subsection 2.2 is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant. With this in mind, we can make the isomorphism $\psi$ respect the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by taking appropriate Tate twists of the summands $V_{a,b}$ on the right side. \hfill $\Box$

Our main tool is [Pet15 Theorem 2.1], restated below for convenience. Before we do so, we need to establish some notation, which agrees with that of [Pet15].

Let $s_k$ be the dimension of the space of cusp forms of $SL(2, \mathbb{Z})$ of weight $k$. For $j \geq 0$ and $k \geq 3$, let $s_{j,k}$ be the dimension of the space of vector-valued Siegel cusp forms for $Sp(4, \mathbb{Z})$ transforming according to the representation $\text{Sym}^j \otimes \text{det}^k$. Let $\rho_f$ be the 2-dimensional $\ell$-adic Galois representation of weight $k - 1$ of the normalized cusp eigenform $f$ for $SL(2, \mathbb{Z})$, as given by [Del69], and let $S_k = \bigoplus_f \rho_f$ be the direct sum of such representations for $k$. Let $\tau_f$ be the 4-dimensional $\ell$-adic Galois representation of the vector-valued Siegel cusp eigenform $f$ type $\text{Sym}^j \otimes \text{det}^k$ as given by [Wei05], and let $S_{j,k} = \bigoplus_f \tau_f$. Let $s'_k$ be the number of normalized cusp eigenforms of weight $k$ for $SL(2, \mathbb{Z})$ for which $L(f, 1/2)$ vanishes. Let $S_{j,k} = \text{gr}^W_{j+2k-3}$. In particular, $S_{j,k} = S_{j,k}$ if $j \neq 0$ or $k = 1$. 


(mod 2). Otherwise, $S_{j,k}$ is a subrepresentation of $S_{j,k}$ that can be determined in a prescribed way — see [Pet15] for more details.

Because every abelian surface has an involution which acts by multiplication by $(-1)^k$ on each stalk of $V_{1,0}^{\otimes k}$, and each $V_{a,b}$ is a summand of $V_{1,0}^{\otimes (a+b)}$, the cohomology $H^p(A_2; V_{a,b})$ vanishes if $a + b$ is odd.

**Theorem 3.2** (Petersen, [Pet15] Theorem 2.1). Suppose $(a, b) \neq (0, 0)$, and that $a + b$ is even. Then

1. In degrees $k \neq 2, 3, 4$,
   $$H^k_c(A_2; V_{a,b}) = 0.$$ (1)

2. In degree 4,
   $$H^4_c(A_2; V_{a,b}) = \begin{cases} s_{a+b+4}Q(−b − 2) & a = b \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$ (2)

3. In degree 3, up to semi-simplification,
   $$H^3_c(A_2; V_{a,b}) = S_{a−b−3} \oplus s_{a+b+4}S_{a−b−2}−b−1 \oplus s_{a+3}$$
   $$\oplus \begin{cases} s'_{a+b+4}Q(−b−1) & a = b \text{ even} \\ s_{a+b+4}Q(−b−1) & \text{otherwise} \end{cases}$$
   $$\oplus \begin{cases} Q & a = b \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$
   $$\oplus \begin{cases} Q(−1) & b = 0 \\ 0 & \text{otherwise} \end{cases}.$$ (3)

4. In degree 2, up to semi-simplification,
   $$H^2_c(A_2; V_{a,b}) = S_{b+2} \oplus s_{a−b+2}Q$$
   $$\oplus \begin{cases} s'_{a+b+4}Q(−b−1) & a = b \text{ even} \\ 0 & \text{otherwise} \end{cases}$$
   $$\oplus \begin{cases} Q & a > b > 0 \text{ and } a, b \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$ (4)

**Remark 3.3.** Although we do not need the full power of Theorem 3.2 to compute $H^*(A_2; \mathbb{Q}_l)$, the existence of such a result is important for the calculations in Sections 4 and 5.

We now give examples of computations using Theorem 3.2. The next corollary gives all applications of this theorem that we explicitly use in the rest of the paper. All results are up to semi-simplification.

**Corollary 3.4.**

$$H^k(A_2; V_{1,1}) = \begin{cases} Q(−5) & k = 3 \\ 0 & \text{otherwise}, \end{cases}$$

$$H^k(A_2; V_{2,0}) = \begin{cases} Q(−4) & k = 3 \\ 0 & \text{otherwise}, \end{cases}$$

$$H^k(A_2; V_{2,2}) = 0 \quad \text{for all } k.$$
Proof. The smallest weight possible for nonzero cusp forms of \( \text{SL}(2, \mathbb{Z}) \) is 12. (For example, see [Ser73, Theorem 7.4].) Thus \( s_k' = s_k = 0 \) and \( S_k = 0 \) for all \( k < 12 \).

By Theorem 3.2,

\[
\begin{align*}
H^1_c(A_2; V_{1,1}) &= 0, \\
H^2_c(A_2; V_{1,1}) &= \mathcal{S}_{0,4} \oplus s_6 S_2(-2) \oplus S_4 \oplus s_6 Q_\ell(-2) \oplus Q_\ell = \mathcal{S}_{0,4} \oplus Q_\ell, \\
H^3_c(A_2; V_{1,1}) &= S_3 \oplus s_2 Q_\ell = 0, \\
H^4_c(A_2; V_{2,0}) &= 0, \\
H^5_c(A_2; V_{2,0}) &= \mathcal{S}_{2,3} \oplus S_6 S_4(-1) \oplus S_5 \oplus s_6 Q_\ell(-1) \oplus Q_\ell(-1) = \mathcal{S}_{2,3} \oplus Q_\ell(-1), \\
H^6_c(A_2; V_{2,0}) &= S_2 \oplus s_4 Q_\ell = 0, \\
H^7_c(A_2; V_{2,2}) &= s_6 Q_\ell(-4) = 0, \\
H^8_c(A_2; V_{2,2}) &= \mathcal{S}_{0,5} \oplus s_8 S_2(-3) \oplus S_5 \oplus s_8' Q_\ell(-3) = \mathcal{S}_{0,5}, \\
H^9_c(A_2; V_{2,2}) &= S_4 \oplus s_2 Q_\ell \oplus s_8 Q_\ell(-3) = 0.
\end{align*}
\]

By computations in [Wak12, p. 249],

\[
\mathcal{S}_{0,4} \subseteq S_{0,4} = 0, \\
\mathcal{S}_{0,5} \subseteq S_{0,5} = 0.
\]

By [Ibu12] Lemma 2.1, \( S_{j,k} \) is 0 for all \( 0 \leq k \leq 4 \) and \( j \leq 14 \), and so \( \mathcal{S}_{2,3} \subseteq S_{2,3} = 0. \)

Finally, by Poincaré Duality (Theorem 2.5),

\[
\begin{align*}
H^{6-k}(A_2; V_{1,1})^* &\cong H^k_c(A_2; V_{1,1}) \otimes Q_\ell(5) = \begin{cases} Q_\ell(5) & k = 3 \\ 0 & \text{otherwise} \end{cases}, \\
H^{6-k}(A_2; V_{2,0})^* &\cong H^k_c(A_2; V_{2,0}) \otimes Q_\ell(5) = \begin{cases} Q_\ell(4) & k = 3 \\ 0 & \text{otherwise} \end{cases} \quad \Box
\end{align*}
\]

With the above preliminaries in hand, we can now prove our first main result, Theorem 1.1. We restate the theorem here for convenience.

**Theorem 1.1** The cohomology of the universal abelian surface \( \mathcal{X}_2 \) is given by

\[
H^k(\mathcal{X}_2; Q_\ell) = \begin{cases} Q_\ell & k = 0 \\ 0 & k = 1, 3, k > 7 \\ 2Q_\ell(-1) & k = 2 \\ 2Q_\ell(-2) & k = 4 \\ Q_\ell(-5) & k = 5 \\ Q_\ell(-3) & k = 6 \end{cases}
\]

up to semi-simplification.

**Proof.** By Proposition 2.9 there is a spectral sequence

\[
E_2^{p,q} = H^p(A_2; H^q(A; Q_\ell)) \implies H^{p+q}(\mathcal{X}_2; Q_\ell)
\]
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4. COHOMOLOGY OF FIBER POWERS OF THE UNIVERSAL ABELIAN SURFACE

In this section, we compute $H^*(X^n_2; \mathbb{Q}_\ell)$. In particular, we give a procedure for computing this for general $n$, and then give the specific results that follow for the case $n = 2$. For brevity, we omit the coefficients when it is clear that we mean the constant ones and write $H^*(X)$ to mean $H^*(X; \mathbb{Q}_\ell)$.

4.1. Computations for general $n$. Let $\pi : X_2 \to A_2$ and let $\pi^n : X^n_2 \to A_2$. We first need to consider the following local systems.

**Lemma 4.1.** The local systems $H^k(A^n)$ on $A_2$ are

$$H^k(A^n) \cong \bigoplus_{\sum_{i=1}^n k_i = k} \left( \bigotimes_i H^k_i(A) \right)$$

where we say $H^k(A^{n-1}) = 0$ if $k < 0$. For appropriate constants $m_{a,b}(H^k(A^n))$,

$$H^k(A^n) \cong \bigoplus_{a \geq b \geq 0} m_{a,b}(H^k(A^n))V_{a,b} \left( \frac{a + b - k}{2} \right).$$

**Proof.** The Künneth isomorphism applied to the local system $H^k(A^n)$ says

$$H^k(A^n) \cong \bigoplus_{k_1 + k_2 = k} H^{k_1}(A^{n-1}) \otimes H^{k_2}(A)$$

$$\cong \bigoplus_{k_2 = 0}^4 H^{k_2}(A) \otimes H^{k-k_2}(A^{n-1}),$$
Lemma 4.3. For $a \geq b \geq 0$,

$$V_{1,0} \otimes V_{a,b} = V_{a-1,b} \oplus V_{a+1,b} \oplus V_{a,b-1} \oplus V_{a,b+1}.$$  

Proof. By Proposition 4.2

$$V_{1,0} \otimes V_{a,b} = \bigoplus_{\nu} N_{(1,0)(a,b)} \nu V_{\nu}$$

with $N_{(1,0)(a,b)} \nu = \sum_{\xi,\sigma,\tau} c_{\xi,\sigma}^{(1,0)} c_{\sigma,\tau}^{(a,b)} c_{\tau,\nu}^{(a,b)}$. Suppose $\zeta, \sigma, \tau$, and $\nu$ are partitions such that the corresponding summand above is nonzero. Then since $\zeta, \sigma \subseteq (1,0)$ and $\zeta, \tau \subseteq (a,b)$, we must have $\zeta = (\xi_1, \xi_2)$, $\sigma = (\sigma_1, \sigma_2)$, and $\tau = (\tau_1, \tau_2)$.
First, consider \( e^{(1,0)}_{c,\sigma} \). Then, \( \zeta, \sigma \leq (1,0) \), with \( \zeta_1 + \zeta_2 + \sigma_1 + \sigma_2 = 1 \). This forces \( \zeta \) and \( \sigma \) to satisfy (1) \( \zeta = (1,0) \) and \( \sigma = (0,0) \) or (2) \( \zeta = (0,0) \) and \( \sigma = (1,0) \). In each case, there is only one skew semi-standard Young tableau of shape \((1,0)/\zeta\) with content \( \sigma \): (1) the empty tableau, and (2) the unique tableau with one box. So \( e^{(1,0)}_{c,\sigma} = 1 \).

Next, we consider \( e^{(a,b)}_{c,\tau} = e^{(a,b)}_{c,\zeta} \). A tableau with shape \((a,b)/\tau\) with content (1) \( \zeta = (1,0) \) or (2) \( \zeta = (0,0) \) must satisfy (1) \((a + b) - (\tau_1 + \tau_2) = 1\) or (2) \((a + b) - (\tau_1 + \tau_2) = 0\), respectively.

(1) If \( \zeta = (1,0) \), then \( \tau = (a - 1, b) \) or \( \tau = (a, b - 1) \). In both cases, a tableau with shape \((a,b)/\tau\) with content \( \zeta \) must be the unique tableau with one box.

(2) If \( \zeta = (0,0) \), then \( \tau = (a, b) \). A tableau with shape \((a,b)/\tau\) with content \( \zeta \) must be the empty tableau.

All three possibilities here give \( e^{(a,b)}_{c,\zeta} = 1 \). Lastly, we consider \( e^{(a,b)}_{c,\tau} = e^{(a,b)}_{c,\sigma} \).

(1) If \( \zeta = (1,0) \) and \( \sigma = (0,0) \), then \( \nu = \tau = (a - 1, b) \) or \( (a,b - 1) \).

(2) If \( \zeta = (0,0) \), \( \sigma = (1,0) \), and \( \tau = (a, b) \), then \( \nu = (a + 1, b) \) or \( (a,b + 1) \).

In all of the above, a tableau with shape \( \nu/\tau \) with content \( \sigma \) must be the unique tableau with one box. Combining all of this casework,

\[
N_{(1,0)(a,b)\nu} = \begin{cases} 
1 & \nu = (a - 1, b), (a, b - 1), (a + 1, b), (a, b + 1) \\
0 & \text{otherwise}. 
\end{cases}
\]

**Lemma 4.4.** If \((a,b) \neq (0,0), (1,1)\), then

\[
V_{1,1} \otimes V_{a,b} = V_{a-1,b-1} \oplus V_{a-1,b+1} \oplus V_{a,b} \oplus V_{a+1,b-1} \oplus V_{a+1,b+1}.
\]

For \((a,b) = (0,0)\) or \((1,1)\),

\[
V_{1,1} \otimes V_{0,0} = V_{1,1}, \quad V_{1,1} \otimes V_{1,1} = V_{0,0} \oplus V_{2,0} \oplus V_{2,2}.
\]

**Proof.** The cases \((a,b) = (0,0), (1,1)\) can be checked computationally. Let \((a,b) \neq (0,0), (1,1)\).

Suppose \( \zeta, \sigma, \tau, \) and \( \nu \) are partitions such that \( e^{(1,1)}_{c,\tau} e^{(a,b)}_{c,\zeta} e^{(a,b)}_{c,\sigma} \neq 0 \). Then since \( \zeta, \sigma \leq (1,1), \zeta, \tau \leq (a,b) \), we must have \( \zeta = (\zeta_1, \zeta_2), \sigma = (\sigma_1, \sigma_2), \) and \( \tau = (\tau_1, \tau_2) \).

First, consider \( e^{(1,1)}_{c,\sigma} \). We have \( \zeta, \sigma \leq (1,1) \) with \( \zeta_1 + \zeta_2 + \sigma_1 + \sigma_2 = 2 \). This forces the three possibilities: (1) \( \zeta = (1,1) \) and \( \sigma = (0,0) \), (2) \( \zeta = (1,0) \) and \( \sigma = (1,0) \), or (3) \( \zeta = (0,0) \) and \( \sigma = (1,1) \).

Next, consider \( e^{(a,b)}_{c,\tau} = e^{(a,b)}_{c,\zeta} \), and the above three possibilities.

(1) If \( \zeta = (1,1) \), then a tableau with shape \((a,b)/\zeta\) has \( a - 1 \) cells in the first row and \( b - 1 \) cells in the second row. In order the tableau to satisfy the lattice word condition and have the labels be increasing within each row, the entire first row must be labelled 1. Therefore, the content \((\tau_1, \tau_2)\) must satisfy \( \tau_1 \geq a - 1 \) with \( \tau_1 + \tau_2 = a + b - 2 \). Because \( \tau \leq (a,b) \), this forces two possibilities: \( \tau = (a - 1, b - 1) \) or \( \tau = (a, b - 2) \).

Suppose \( \tau = (a, b - 2) \). We count tableau with shape \((a,b)/\tau\) and content \( \zeta = (1,1) \). The tableau of shape \((a,b)/(a,b - 2)\) has one row with two cells, and the lattice word condition imposes that all cells of the first row must be labeled 1. Therefore, there are no such tableau with content \( \zeta = (1,1) \). For \( \tau = (a - 1, b - 1) \), see Figure 3 for the unique tableau with shape \((a,b)/(a-1, b-1)\) and content \( \zeta = (1,1) \).

(2) If \( \zeta = (1,0) \), then a tableau with shape \((a,b)/\tau\) and content \( \zeta = (1,0) \) must satisfy \( \tau = (a - 1, b) \) or \( (a, b - 1) \). In both cases, such a tableau is the unique tableaux with one cell.
For any Definition 4.5.

(3) If $\zeta = (0, 0)$, then a tableau with shape $(a, b)/\tau$ and content $\zeta = (0, 0)$ must satisfy $\tau = (a, b)$. In this case, such a tableau must be the empty one.

Lastly, consider $c_{\sigma, \tau}^{(n, b)} = c_{\tau, \sigma}^{(n, b)}$.

(1) If $\zeta = (1, 1)$, $\sigma = (0, 0)$, and $\tau = (a - 1, b - 1)$, then a tableau with shape $\nu/\tau$ and content $\sigma = (0, 0)$ must satisfy $\nu = (a - 1, b - 1)$. The only such tableau is the empty one.

(2) If $\zeta = (1, 0)$, $\sigma = (1, 0)$, and $\tau = (a - 1, b)$ or $(a - 1, b - 1)$, then a tableau with shape $\nu/\tau$ and content $\sigma = (1, 0)$ must satisfy $\nu_1 + \nu_2 = a + b$ with $\tau \subseteq \nu = (\nu_1, \nu_2)$. If $\tau = (a - 1, b)$, then $\nu = (a, b)$. In this case, such a tableau must be the unique one with one cell.

(3) If $\zeta = (0, 0)$, $\sigma = (1, 1)$, and $\tau = (a, b)$, then by similar reasons as case (1) of $c_{\zeta, \tau}^{(n, b)}$, $\nu = (a + 1, b + 1)$. In this case, such a tableau of shape $\nu/\tau$ and content $\sigma$ must be that of Figure 3.

Combining all of the casework above implies

$$N_{(1, 1)(a, b)\nu} = \begin{cases} 1 & \nu = (a - 1, b - 1), (a - 1, b + 1), (a, b), (a + 1, b - 1), (a + 1, b + 1) \\ 0 & \text{otherwise} \end{cases}$$

with which we apply Proposition 4.2 and conclude.

We are now able to give a recursive formula (in $n$) for the multiplicity of a given $V_{a, b}$ in $H^k(A^n; \mathbb{Q}_l)$ as $\text{Sp}(4, \mathbb{Z})$-representations. We do so in pieces after establishing some notation.

**Definition 4.5.** For any $(a, b) \in \mathbb{N}^2$, $S \subseteq \mathbb{N}^2$, and any $\text{Sp}(4, \mathbb{Z})$-representation $V$ obtained as a restriction of an $\text{Sp}(4, \mathbb{C})$-representation, let

$$m_{a, b}(V) = \begin{cases} \langle V_{a, b}, V \rangle & a \geq b \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 4.6.** Lemma 3.1 determines $m_{a, b}(H^q(A))$ for all $a \geq b \geq 0$ and all $q \geq 0$.

**Definition 4.7.** Denote the indicator function by $1_S(a, b)$, with

$$1_S(a, b) = \begin{cases} 1 & (a, b) \in S \\ 0 & (a, b) \notin S. \end{cases}$$

The following two lemmas establish formulas that are necessary to give a recursive formula for $m_{a, b}(H^k(A^n))$ in $n$. The proofs are completely straight-forward but are included for completeness.

**Lemma 4.8.** For any $a \geq b \geq 0$ and $k \geq 0$,

$$\langle V_{a, b}, V_{1, 0} \otimes V \rangle = m_{a + 1, b}(V) + m_{a, b + 1}(V) + m_{a - 1, b}(V) + m_{a, b - 1}(V).$$
Proof. Let \( I(a, b) = \{(a + 1, b), (a, b + 1), (a - 1, b), (a, b - 1)\} \). By Lemma 4.3

\[
\langle V_{a,b}, V_{1,0} \otimes V \rangle = \left\langle V_{a,b}, V_{1,0} \otimes \left( \bigoplus_{(A,B) \in I(a,b)} m_{A,B}(V) V_{A,B} \right) \right\rangle \\
= \sum_{(A,B) \in I(a,b)} m_{A,B}(V) \langle V_{a,b}, V_{1,0} \otimes V_{A,B} \rangle \\
= \sum_{(A,B) \in I(a,b)} m_{A,B}(V) (V_{a,b} + V_{A,B} + V_{A+1,B} + V_{A,B+1}) \\
= m_{a+1,b}(V) + m_{a,b+1}(V) + m_{a-1,b}(V) + m_{a,b-1}(V).
\]

Lemma 4.9. For any \( a \geq b \geq 0 \),

\[
\langle V_{a,b}, (Q \oplus V_{1,1}) \otimes V \rangle \\
= (2 - I((1,1),(0,0))) (a,b) m_{a,b}(V) + m_{a-1,b-1}(V) + m_{a-1,b+1}(V) + m_{a+1,b-1}(V) + m_{a+1,b+1}(V).
\]

Proof. For all \( a \geq b \geq 0 \), \( \langle V_{a,b}, Q \oplus V_{1,1} \rangle = m_{a,b}(V) \). Now let \( J(a,b) = \{(a-1,b-1), (a-1,b+1), (a+1,b-1), (a+1,b+1)\} \). By Lemma 4.4

\[
\langle V_{a,b}, V_{1,1} \otimes V \rangle = \left\langle V_{a,b}, V_{1,1} \otimes \left( \bigoplus_{(A,B) \in J(a,b)} m_{A,B}(V) V_{A,B} \right) \right\rangle \\
= \sum_{(A,B) \in J(a,b)} m_{A,B}(V) \langle V_{a,b}, V_{1,1} \otimes V_{A,B} \rangle.
\]

For all \( (A,B) \neq (1,1), (0,0), \) Lemma 4.4 says

\[
m_{A,B}(V) \langle V_{a,b}, V_{1,1} \otimes V_{A,B} \rangle = m_{A,B}(V) \left\langle V_{a,b}, \bigoplus_{(C,D) \in J(A,B)} V_{C,D} \right\rangle \\
= m_{A,B}(V).
\]

If \( (0,0) \in J(a,b) \), then \( (a,b) = (0,0) \) or \( (1,1) \). If \( (1,1) \in J(a,b) \), then \( (a,b) = (0,0), (1,1), (2,0), \) or \( (2,2) \). Therefore, the above gives that outside of these cases,

\[
\langle V_{a,b}, V_{1,1} \otimes V \rangle = \sum_{(A,B) \in J(a,b)} m_{A,B}(V) \\
= m_{a-1,b-1}(V) + m_{a-1,b+1}(V) + m_{a,b}(V) + m_{a+1,b-1}(V) + m_{a+1,b+1}(V)
\]

which gives the lemma, since \( (a,b) \neq (1,1), (0,0) \) and \( I((1,1),(0,0)) (a,b) = 0 \).

If \( (a,b) = (2,0) \) or \( (2,2) \), then \( (1,1) \in J(a,b) \) but \( (0,0) \notin J(a,b) \). In this case, Lemma 4.4 gives

\[
m_{1,1}(V) \langle V_{a,b}, V_{1,1} \otimes V_{1,1} \rangle = m_{1,1}(V) \langle V_{a,b}, V_{0,0} \oplus V_{2,0} \oplus V_{2,2} \rangle \\
= m_{1,1}(V).
\]

Therefore, by the same calculation as the above, the lemma holds.
If \((a, b) = (0, 0)\) or \((1,1)\), then both \((0, 0), \,(1, 1) \in J(a, b)\). In this case,
\[
m_{0,0}(V)(V_{a,b}, V_{1,1} \otimes V_{0,0}) = m_{0,0}(V)(V_{a,b}, V_{1,1}) = (1 - 1_{(0,0)}(a, b))m_{0,0}(V),
\]
\[
m_{1,1}(V)(V_{a,b}, V_{1,1} \otimes V_{0,0}) = m_{1,1}(V)(V_{a,b}, V_{0,0} \oplus V_{2,0} \oplus V_{2,2}) = (1 - 1_{(1,1)}(a, b))m_{1,1}(V).
\]
In both cases,
\[
m_{a,b}(V)(V_{a,b}, V_{1,1} \otimes V_{a,b}) = 0 = (1 - 1_{(0,0),(1,1)}(a, b))m_{a,b}(V)
\]
and so the lemma holds here as well. \(\square\)

Combining all of the lemmas of this subsection shows that we have determined a recursive formula for all parts of the first identity of the following proposition.

**Proposition 4.10.** Let \(a \geq b \geq 0, n \geq 1, \text{ and } k \geq 0\). Viewing \(H^*(A^n)\) as \(\text{Sp}(4, \mathbb{Z})\)-representations,
\[
m_{a,b}(H^k(A^n)) = m_{a,b}(H^k(A^{n-1})) + \langle V_{a,b}, V_{1,0} \otimes H^{k-1}(A^{n-1}) \rangle + \langle V_{a,b}, (Q_\ell \otimes V_{1,1}) \otimes H^{k-2}(A^{n-1}) \rangle \\
+ \langle V_{a,b}, V_{1,0} \otimes H^{k-3}(A^{n-1}) \rangle + m_{a,b}(H^{k-4}(A^{n-1}))
\]
and
\[
m_{0,0}(H^k(A^n)) = m_{0,0}(H^k(A^{n-1})) + m_{1,0}(H^{k-1}(A^{n-1})) + m_{0,0}(H^{k-2}(A^{n-1})) + m_{1,1}(H^{k-2}(A^{n-1})) \\
+ m_{1,0}(H^{k-3}(A^{n-1})) + m_{0,0}(H^{k-4}(A^{n-1})).
\]

**Proof.** Apply the lemmas above to compute, for \(j = 1, 3\),
\[
\langle V_{0,0}, V_{1,0} \otimes H^{k-j}(A^{n-1}) \rangle = m_{1,0}(H^{k-j}(A^{n-1})),
\]
\[
\langle V_{0,0}, (Q_\ell \otimes V_{1,1}) \otimes H^{k-2}(A^{n-1}) \rangle = m_{0,0}(H^{k-2}(A^{n-1})) + m_{1,1}(H^{k-2}(A^{n-1})).
\]
The second identity of the proposition is obtained by plugging these into the first identity with \(a = b = 0\). \(\square\)

We can also more explicitly describe the representations \(V_{a,b}\) that occur in \(H^k(A^n)\). These descriptions are necessary to prove Theorem 1.2.

**Proposition 4.11.** If \(m_{a,b}(H^k(A^n)) \neq 0\), then \(a + b \equiv k \pmod{2}\) and \(a + b \leq k\). For all \(n \geq k\), all such \(a \geq b \geq 0\) and \(k \geq 0\) give \(m_{a,b}(H^k(A^n)) \neq 0\).

**Proof.** To prove that if \(m_{a,b}(H^k(A^n)) \neq 0\) then \(a + b \equiv k \pmod{2}\) and \(a + b \leq k\), we proceed by induction on \(n\). For \(n = 1\), the claim is true by Lemma 3.1. Now assume the claim for \(n - 1\). Suppose \(a + b \equiv k \pmod{2}\) or \(a + b > k\). Then using Lemmas 4.8 and 4.9 and letting \(V(k) = H^k(A^{n-1})\) and \(j = 1\) or \(3\),
\[
\langle V_{a,b}, V_{1,0} \otimes V(k-j) \rangle = m_{a+1,b}(V(k-j)) + m_{a,b+1}(V(k-j)) \\
+ m_{a-1,b}(V(k-j)) + m_{a,b-1}(V(k-j)) = 0,
\]
\[
\langle V_{a,b}, (Q_\ell \otimes V_{1,1}) \otimes V(k-2) \rangle = (2 - 1_{(1,1),(0,0)}(a, b))m_{a,b}(V(k-2)) + m_{a-1,b-1}(V(k-2)) \\
+ m_{a-1,b+1}(V(k-2)) + m_{a+1,b-1}(V(k-2)) + m_{a+1,b+1}(V(k-2)) = 0.
\]
Both of the above are zero by the inductive hypothesis. Then by Proposition 4.10,

\[ m_{a,b}(H^k(A^n)) = m_{a,b}(V(k)) + (V_{a,b}, V_{1,0} \otimes V(k-1)) + (V_{a,b}, (Q_2 \oplus V_{1,1}) \otimes V(k-2)) + (V_{a,b}, V_{1,0} \otimes V(k-3)) + m_{a,b}(V(k-4)) = m_{a,b}(V(k)) + m_{a,b}(V(k-4)) = 0. \]

For fixed \( k \geq 0 \), we next show that \( m_{a,b}(H^k(A^n)) \neq 0 \) for \( n \geq k \), by induction on \( n \). For \( n = 1 \), the claim is again true by Lemma 3.1. Assume the claim holds for \( M \) and \( k \) s.t. \( a + b \equiv k \pmod{2} \), and \( a + b \leq k \). If \( b \geq 1 \), then by Proposition 4.10, Lemma 4.12 and the inductive hypothesis,

\[ m_{a,b}(H^k(A^n)) \geq (V_{a,b}, V_{1,0} \otimes H^{k-1}(A^{n-1})) \geq m_{a,b-1}(H^{k-1}(A^{n-1})) \geq 0. \]

If \( a \neq 0 \) and \( b = 0 \), then

\[ m_{a,b}(H^k(A^n)) \geq (V_{a,b}, V_{1,0} \otimes H^{k-1}(A^{n-1})) \geq m_{a-1,b}(H^{k-1}(A^{n-1})) \geq 0. \]

If \( (a, b) = (0, 0) \) and \( k \geq 2 \), then

\[ m_{0,0}(H^k(A^n)) \geq (V_{0,0}, (Q_2 \oplus V_{1,1}) \otimes H^{k-2}(A^{n-1})) \geq (2 - 1_{\{1,1\},(0,0)}) (0, 0) m_{0,0}(H^{k-2}(A^{n-1})) \geq 0. \]

If \( k = 0 \), then \( H^0(A^n) = Q_2 \) for all \( n \) and so \( m_{0,0}(H^0(A^n)) = 1 \).

**Lemma 4.12.** Let \( a \geq b \geq 0 \) and \( M > 0 \).

1. \( m_{a,b}(H^0(A^M)) = 1_{\{0,0\}}(a, b) \).
2. \( m_{1,0}(H^1(A^M)) = M \).
3. \( m_{0,0}(H^2(A^M)) = 2^M \).
4. \( m_{1,1}(H^2(A^M)) = 2^M \).
5. \( m_{2,0}(H^2(A^M)) = \binom{M}{2} \).
6. \( m_{1,0}(H^3(A^M)) = \binom{M}{3} + 2(M+1) + M^2 \).
7. \( m_{0,0}(H^4(A^M)) = \frac{M(M+1)(M^2+M+2)}{8} \).

**Proof.** All proofs are by induction on \( M \). We can check manually that all claims hold for \( M = 1 \). Assume that they hold for \( M - 1 \).

1. For all \( M \), \( H^0(A^M) = Q_2 \), the constant local system.
2. By Proposition 4.10 and Lemma 4.8, \( m_{1,0}(H^1(A^M)) = m_{1,0}(H^1(A^{M-1})) + (V_{1,0}, V_{1,0} \otimes H^0(A^{M-1})) = (M - 1) + (m_{2,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) + m_{0,0}(H^0(A^{M-1}))) = M \).
3. By Proposition 4.10, \( m_{0,0}(H^2(A^M)) = m_{0,0}(H^2(A^{M-1})) + m_{1,0}(H^1(A^{M-1})) + m_{0,0}(H^1(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) = \binom{M}{2} + (M - 1) + 1 = \binom{M + 1}{2} \).
4. By Proposition 4.10 and Lemma 4.8, \( m_{1,1}(H^2(A^M)) = m_{1,1}(H^2(A^{M-1})) + (V_{1,1}, V_{1,0} \otimes (H^1(A^{M-1}))) + (V_{1,1}, (Q_2 \oplus V_{1,1}) \otimes (H^0(A^{M-1}))) = \binom{M}{2} + (m_{2,1}(H^1(A^{M-1})) + m_{1,0}(H^1(A^{M-1}))) + m_{1,1}(H^0(A^{M-1})) + m_{0,0}(H^0(A^{M-1})) + m_{2,0}(H^0(A^{M-1})) + m_{2,2}(H^0(A^{M-1})) = \binom{M}{2} + (m_{2,1}(H^1(A^{M-1})) + (M - 1) + 1 = \binom{M + 1}{2} \).
where the last equality follows by Proposition \ref{prop:4.11}, which gives that $m_{2,1}(H^1(A^{M-1})) = 0$.

(5) By Proposition \ref{prop:4.10} Lemma \ref{lem:4.8} Lemma \ref{lem:4.9} \cite{1} and \cite{2},

\[
m_{2,0}(H^2(A^M)) = m_{2,0}(H^2(A^{M-1})) + (V_{2,0}, V_{1,0} \otimes (H^1(A^{M-1}))) + (V_{2,0}, (Q_{\ell} \oplus V_{1,1}) \otimes (H^0(A^{M-1})))
\]

\[
= \left(\binom{M-1}{2}\right) + (m_{3,0}(H^1(A^{M-1})) + m_{2,1}(H^1(A^{M-1})) + m_{1,0}(H^1(A^{M-1})))
\]

\[
+ (2m_{2,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) + m_{3,1}(H^0(A^{M-1})))
\]

\[
= \left(\binom{M-1}{2}\right) + (M - 1) = \left(\binom{M}{2}\right).
\]

(6) By Proposition \ref{prop:4.10} Lemma \ref{lem:4.8} Lemma \ref{lem:4.9} and \cite{1} - \cite{5},

\[
m_{1,0}(H^3(A^M)) = m_{1,0}(H^3(A^{M-1})) + (V_{1,0}, V_{1,0} \otimes H^2(A^{M-1}))
\]

\[
+ (V_{1,0}, (Q_{\ell} \oplus V_{1,1}) \otimes H^1(A^{M-1})) + (V_{1,0}, V_{1,0} \otimes H^0(A^{M-1})))
\]

\[
= m_{1,0}(H^3(A^{M-1})) + (m_{2,0}(H^2(A^{M-1})) + m_{1,1}(H^2(A^{M-1}) + m_{0,0}(H^2(A^{M-1})))
\]

\[
+ (2m_{2,0}(H^1(A^{M-1})) + m_{2,1}(H^1(A^{M-1})))
\]

\[
+ (m_{2,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) + m_{0,0}(H^0(A^{M-1})))
\]

\[
= \left(\binom{M-1}{3}\right) + 2\left(\binom{M}{3}\right) + (M - 1)^2 + \left(\binom{M-1}{2}\right) + \left(\binom{M}{2}\right) + (M - 1)
\]

\[
= \left(\binom{M+1}{3}\right) + M^2
\]

(7) By Proposition \ref{prop:4.10} \cite{1} - \cite{4}, and \cite{5},

\[
m_{0,0}(H^4(A^M)) = m_{0,0}(H^4(A^{M-1})) + m_{1,0}(H^3(A^{M-1})) + m_{0,0}(H^2(A^{M-1})) + m_{1,1}(H^2(A^{M-1}))
\]

\[
+ m_{1,0}(H^1(A^{M-1})) + m_{0,0}(H^0(A^{M-1})))
\]

\[
= \frac{(M - 1)(M)(M^2 - M + 2)}{8} + \left(\binom{M-1}{3}\right) + 2\left(\binom{M}{3}\right) + (M - 1)^2
\]

\[
+ \left(\binom{M}{2}\right) + \left(\binom{M}{2}\right) + (M - 1) + 1
\]

\[
= \frac{M(M + 1)(M^2 + M + 2)}{8}.
\]

\[
\square
\]

We are now ready to compute $H^k(A^n_2; Q_\ell)$ for all $n \geq 1$ and $0 \leq k \leq 5$.

**Theorem 1.2** For all $n \geq 1$,

\[
H^k(A^n_2; Q_\ell) = \begin{cases}
Q_\ell & k = 0 \\
0 & k = 1, 3 \\
\binom{n+1}{2} Q_\ell(-1) & k = 2 \\
\binom{n+1}{2} Q_\ell(-2) & k = 4 \\
\binom{n+1}{2} Q_\ell(-5) & k = 5
\end{cases}
\]
up to semi-simplification.

Proof. Denote the \((p, q)\)-entry on the \(E_2\)-sheet of the Leray spectral sequence of \(\pi^n : X_2^n \to A_2\) by \(E_2^{p,q}(n)\). We compute many entries on the \(E_2\)-sheet and list the nonzero results in Figure 4, from which the theorem follows directly. The special case of \(n = 1\) is given in Figure 4. All computations here are up to semi-simplification.

1. \(E_2^{p,0}(n) = H^p(A_2; Q_\ell)\) for all \(p\).

Proof. By definition and Lemmas 4.1 and 4.12

\[ E_2^{p,0}(n) = H^p(A_2; H^0(A^n)) = \bigoplus_{a+b \geq 0} m_{a,b} H^0(A^n) H^p(A_2; V_{a,b} \left( \frac{a+b}{2} \right)) = H^p(A_2; Q_\ell). \]

2. \(E_2^{p,q}(n) = 0\) for all \(p \geq 0, q \equiv 1 \pmod{2}\).

Proof. Suppose \(m_{a,b} \neq 0\). By Proposition 4.11, \(a+b \equiv 1 \pmod{2}\). Therefore

\[ E_2^{p,q}(n) = H^p(A_2; H^q(A^n)) = \bigoplus_{a+b \geq 0 \atop a+b \equiv 1 \pmod{2}} m_{a,b} H^q(A^n) H^p(A_2; V_{a,b} \left( \frac{a+b-q}{2} \right)) = 0 \]

where the last equality follows since \(H^p(A_2; V_{a,b}) = 0\). (See the remark before Theorem 3.2) \(\square\)

3. \(E_2^{2,2}(n) = (n+1) H^p(A_2; Q_\ell)(-1)\) and \(E_2^{p,4}(n) = \frac{n(n+1)(a^2+n+2)}{8} H^p(A_2; Q_\ell)(-2)\) for \(p = 0, 1, 2\).

Proof. Let \(q = 2, 4\). By definition and Proposition 4.11

\[ E_2^{p,q}(n) = H^p(A_2; H^q(A^n)) = \bigoplus_{a+b \leq q \atop a+b \equiv 0 \pmod{2}} m_{a,b} H^q(A^n) H^p(A_2; V_{a,b} \left( \frac{a+b-q}{2} \right)). \]

By Theorem 3.2, \(H^p(A_2; V_{a,b}) = 0\) for \((a, b) \neq 0\) and \(p = 0, 1\), so

\[ E_2^{p,q}(n) = m_{0,0} H^q(A^n) H^p(A_2; Q_\ell) \left( -\frac{q}{2} \right) \]

for \(p = 0, 1\). The claim then follows by Lemma 4.12. For \(p = 2\), Theorem 3.2 gives that \(H^2(A_2; V_{a,b}) \neq 0\) only if \(a = b\) even, and so

\[ E_2^{2,q}(n) = m_{0,0} H^q(A^n) H^2(A_2; Q_\ell) \left( -\frac{q}{2} \right) \oplus \begin{cases} 0 & q = 2 \\ m_{2,2} H^4(A^n) H^2(A_2; V_{2,2}) & q = 4 \end{cases} \]

In the case \(q = 2\), the claim then follows from Lemma 4.12. In the case \(q = 4\), the claim follows from Corollary 3.4 since \(H^2(A_2; V_{2,2}) = 0\). \(\square\)

4. \(E_2^{3,2}(n) = (n+1) Q_\ell(-5) \oplus (\binom{n}{2}) Q_\ell(-4)\) and \(E_2^{4,2}(n) = 0\).
Figure 4. Some low degree terms of the $E_2$-page of the Leray spectral sequence for $\pi^n : X^n_\ell \rightarrow A_2$ determined in Theorem 4.2. Note that $E_2^{p,q} = 0$ for all $q \geq 0$ and $E_2^{p,q} = 0$ for all $p \geq 4$ and $q = 0, 1, 2, 3$.


$$E_2^{p,2}(n) = H^p(A_2; H^2(A^n)) = \bigoplus_{a \geq b \geq 0} m_{a,b}(H^2(A^n))H^p(A_2; V_{a,b}) (a + b - 2) \begin{pmatrix} 2 \end{pmatrix}$$

$$= m_{0,0}(H^2(A^n))H^0(A_2; V_{0,0})(-1) + m_{1,1}(H^2(A^n))H^0(A_2; V_{1,1}) \oplus m_{2,0}(H^2(A^n))H^0(A_2; V_{2,0})$$

$$= \begin{pmatrix} n + 1 \end{pmatrix} H^0(A_2; V_{1,1}) \oplus \begin{pmatrix} n \end{pmatrix} H^0(A_2; V_{2,0}).$$

Then the claim follows from Corollary 3.4.

Remark 4.13. Theorem 4.2 is consistent with the stabilization result [GHT18, Theorem 6.1], which says that the rational cohomology of $X^n_\ell$ stabilizes in degrees $k < g = 2$.

4.2. Explicit computations for $n = 2$. Once one has computed $m_{a,b}(H^k(A^n))$ for fixed $n$ and for all $k \geq 0$, $a \geq b \geq 0$, one can in theory apply Theorem 3.2 and determine $H^*(X^n_\ell; Q_\ell)$. In this subsection, we detail the results of this process for $n = 2$.

Lemma 4.14. The local systems $H^k(A_2; Q_\ell)$ on $A_2$ are

$$H^k(A_2; Q_\ell) \cong \begin{cases} V_{0,0} & k = 0 \\ 2V_{1,0} & k = 1 \\ 3V_{0,0}(-1) \oplus 3V_{1,1} \oplus V_{2,0} & k = 2 \\ 6V_{1,0}(-1) \oplus 2V_{2,1} & k = 3 \\ 6V_{0,0}(-2) \oplus 4V_{1,1}(-1) \oplus 3V_{2,0}(-1) \oplus V_{2,2} & k = 4 \\ 6V_{1,0}(-2) \oplus 2V_{2,1}(-1) & k = 5 \\ 3V_{0,0}(-3) \oplus 3V_{1,1}(-2) \oplus V_{2,0}(-2) & k = 6 \\ 2V_{1,0}(-3) & k = 7 \\ V_{0,0}(-4) & k = 8 \\ 0 & k > 8. \end{cases}$$

Proof. This is a direct computation using Lemma 4.1, Lemma 4.8, and Lemma 4.9.\qed
Using Lemma 4.14, we can compute all entries of the $E_2$-page of the Leray spectral sequence for $\pi^2 : A_2^2 \to A_2$. As always, the following results are up to semi-simplification.

**Lemma 4.15.** For $q = 1, 3, 5, 7,$ and all $p$,

$$H^p(A_2; H^q(A^n)) = 0.$$

For $q = 0, 8$,

$$H^p(A_2; H^q(A^2)) \cong \begin{cases} \mathbb{Q}_\ell \left(-\frac{q}{2}\right) & p = 0 \\ \mathbb{Q}_\ell \left(-\frac{q}{2} - 1\right) & p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

For $q = 2, 6$,

$$H^p(A_2; H^q(A^2)) \cong \begin{cases} 3\mathbb{Q}_\ell \left(-\frac{q}{2}\right) & p = 0 \\ 3\mathbb{Q}_\ell \left(-\frac{q}{2} - 1\right) & p = 2 \\ 3\mathbb{Q}_\ell \left(-\frac{q^2 - 2}{2} - 5\right) \oplus \mathbb{Q}_\ell \left(-\frac{q^2 - 2}{2} - 4\right) & p = 3 \\ 0 & \text{otherwise.} \end{cases}$$

For $q = 4$,

$$H^p(A_2; H^q(A^2)) \cong \begin{cases} 6\mathbb{Q}_\ell(-2) & p = 0 \\ 6\mathbb{Q}_\ell(-3) & p = 2 \\ 3\mathbb{Q}_\ell(-5) \oplus 4\mathbb{Q}_\ell(-6) & p = 3 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** By definition, $E_2^{p,q} = H^p(A_2; H^q(A^2)) = \bigoplus_{a+b\geq 0} m_{a,b}(H^q(A^n))H^p(A_2; V_{a,b})$. Suppose $q$ is odd. By Proposition 4.11 if $m_{a,b}(H^q(A^2)) \neq 0$, then $a + b \equiv 1 \pmod{2}$. For such $(a,b)$, the remarks before Theorem 3.2 imply that $H^p(A_2; V_{a,b}) = 0$ for all $p$.

For $0 \leq q \leq 8$ even, combine Lemma 4.1, Lemma 4.14, Theorem 2.2 and Corollary 3.4.

**Remark 4.16.** The computations in Lemma 4.15 are consistent with Theorem 1.2.

By the usual argument, we obtain the following theorem using these preliminaries.

**Theorem 4.17.** The cohomology of the second fiber power $A_2^2$ of the universal abelian surface is given by

$$H^k(A_2^2; \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & k = 0 \\ 0 & k = 1, 3, k > 10 \\ 4\mathbb{Q}_\ell(-1) & k = 2 \\ 9\mathbb{Q}_\ell(-2) & k = 4 \\ 3\mathbb{Q}_\ell(-5) \oplus \mathbb{Q}_\ell(-4) & k = 5 \\ 9\mathbb{Q}_\ell(-3) & k = 6 \\ 3\mathbb{Q}_\ell(-5) \oplus 4\mathbb{Q}_\ell(-6) & k = 7 \\ 4\mathbb{Q}_\ell(-4) & k = 8 \\ 3\mathbb{Q}_\ell(-7) \oplus \mathbb{Q}_\ell(-6) & k = 9 \\ \mathbb{Q}_\ell(-5) & k = 10 \end{cases}$$

up to semi-simplification.
Proof. By Proposition 2.9, there is a spectral sequence
\[ E_2^{p,q} = H^p(A_2; H^q(A^2)) \implies H^{p+q}(A_2^2; \mathbb{Q}_\ell) \]
which degenerates on the \(E_2\)-page. Combining the lemmas in this section gives the entries of the \(E_2\)-page as recorded in Figure 5. \(\square\)

5. Cohomology of \(\mathcal{X}_2^{\text{Sym}(n)}\)

In this section, we compute \(H^*(\mathcal{X}_2^{\text{Sym}(n)}; \mathbb{Q}_\ell)\). As opposed to Section 4, we show that the cohomology in fixed degree stabilizes as \(n\) increases and explicitly give the computations for small degree. Afterwards, we give a complete description of the cohomology for the case \(n = 2\). For brevity, we will often drop the constant coefficients if the context is clear, writing \(H_k(A^n)\) instead of \(H_k(A^n; \mathbb{Q}_\ell)\).

5.1. Computations for general \(n\). Let \(\pi^n : \mathcal{X}_2^{\text{Sym}(n)} \to A_2\) and \(\pi : \mathcal{X}_2^2 \to A_2\). Let \(A = \mathbb{C}^2/L\) be an abelian surface. For all \(n \geq 1\), the symmetric group \(S_n\) acts on \(A^n\) by permuting the coordinates, which induces an action of \(S_n\) on \(H^*(A^n)\). The \(S_n\)-action on \(A^n\) is not free but this is not a problem in the context of stacks. On the other hand,

\[ H^k(A^n; \mathbb{Q}_\ell) \cong \bigoplus_{(k_1, \ldots, k_n)} H^{k_i}(A; \mathbb{Q}_\ell)^{S_n} \]

by the Künneth formula.

Lemma 5.1. The local systems \(H^k(\text{Sym}^n A; \mathbb{Q}_\ell)\) on \(A_2\) are

\[ H^k(\text{Sym}^n A; \mathbb{Q}_\ell) \cong H^k(A^n; \mathbb{Q}_\ell)^{S_n} \cong \bigoplus_{(k_0, \ldots, k_4) \sum_{i=0}^4 k_i = k; k_i \geq 0} \bigotimes_{i=0}^4 \text{Sym}^{k_i} H^i(A; \mathbb{Q}_\ell). \]
Proof. The first isomorphism in the lemma statement follows from the transfer isomorphism for Sym\(^n A \cong A^n / S_n\). For the second isomorphism, observe that the induced action of \(S_n\) on \(H^k(A^n; \mathbb{Q}_\ell)\) is given by
\[
\sigma \cdot (c_1 \otimes \cdots \otimes c_n) = c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(n)} \in \bigotimes_{i=1}^n H^{k_{\sigma(i)}}(A; \mathbb{Q}_\ell) \subseteq H^k(A^n; \mathbb{Q}_\ell)
\]
for all \(c_1 \otimes \cdots \otimes c_n \in \bigotimes_{i=1}^n H^{k_i}(A; \mathbb{Q}_\ell)\). There is a projection \(p: H^k(A^n; \mathbb{Q}_\ell) \to H^k(A^n; \mathbb{Q}_\ell)^{S_n}\) given by averaging. For each \((k_1, \ldots, k_n)\) with \(\sum_{i=1}^n k_i = k\) and \(k_1 \geq \cdots \geq k_n\), consider the subspace
\[
W_{k_1, \ldots, k_n} := \bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^n H^{\sigma(k_i)}(A; \mathbb{Q}_\ell) \subseteq H^k(A^n; \mathbb{Q}_\ell).
\]
For the summand corresponding to \(1 \in S_n\) above can be written as \(\bigotimes_{i=0}^4 H^i(A; \mathbb{Q}_\ell)^{\otimes m_i}\), with \(m_i = \#\{k_j : k_j = i\}\). The image \(p(W_{k_1, \ldots, k_n})\) is spanned by the image \(p(c)\) of simple tensors \(c \in \bigotimes_{i=0}^4 H^i(A; \mathbb{Q}_\ell)^{\otimes m_i}\) and so \(p\) restricted to \(\bigotimes_{i=0}^4 H^i(A; \mathbb{Q}_\ell)^{\otimes m_i}\) is surjective onto \(p(W_{k_1, \ldots, k_n})\) with kernel \(\langle c - \sigma \cdot c : \sigma \in \prod_{i=0}^4 S_{m_i} \rangle\). This implies that
\[
p(W_{k_1, \ldots, k_n}) \cong \bigotimes_{i=0}^4 \left(H^i(A; \mathbb{Q}_\ell)^{\otimes m_i} / \langle c - \sigma \cdot c : \sigma \in S_{m_i} \rangle\right) \cong \bigotimes_{i=0}^4 \text{Sym}^{m_i} H^i(A; \mathbb{Q}_\ell). \quad \square
\]

Lemma 5.2. For fixed \(k\), \(H^k(\mathcal{X}_2^{\text{Sym}(n)}; \mathbb{Q}_\ell)\) stabilizes for \(n \geq k\).

Proof. Fix \(k \in \mathbb{N}\). For each \(n \in \mathbb{N}\), consider the set
\[
S(n) := \left\{(k_0, \ldots, k_4) \in \mathbb{N}^5 : \sum_{i=0}^4 ik_i = k, \sum_{i=0}^4 k_i = n\right\}.
\]
If \(n \geq k\), then there is a bijection \(S(k) \to S(n)\) given by sending each \((k_0, \ldots, k_4)\) to \((k_0 + (n - k), k_1, \ldots, k_4)\). Using this bijection and the fact that \(\text{Sym}^n H^0(A; \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell\) for any \(m \geq 0\), compute for all \(n \geq k\) that
\[
H^k(A^k; \mathbb{Q}_\ell)_{S_k} \cong \bigoplus_{(k_0, \ldots, k_4) \in S(k)} \text{Sym}^{k_i} H^i(A; \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \otimes \left(\bigotimes_{i=1}^4 \text{Sym}^{k_i} H^i(A; \mathbb{Q}_\ell)\right)
\]
\[
\cong \left(\bigoplus_{(k_0, \ldots, k_4) \in S(n)} \mathbb{Q}_\ell \otimes \left(\bigotimes_{i=1}^4 \text{Sym}^{k_i} H^i(A; \mathbb{Q}_\ell)\right)\right) \cong H^k(A^n; \mathbb{Q}_\ell)^{S_n}.
\]
So for fixed \(k\), \(H^k(\text{Sym}^n A; \mathbb{Q}_\ell)\) stabilizes for all \(n \geq k\).

By Proposition 2.9
\[
H^k(\mathcal{X}_2^{\text{Sym}(n)}; \mathbb{Q}_\ell) = \bigoplus_{p+q=k} H^p(A_2; H^q(\text{Sym}^n A; \mathbb{Q}_\ell)) = \bigoplus_{p+q=k} H^p(A_2; H^q(\text{Sym}^k A; \mathbb{Q}_\ell))
\]
where the last equality follows because \(q \leq k \leq n\). \(\square\)

For small \(k\), this simplifies the computations for \(H^k(\text{Sym}^n A; \mathbb{Q}_\ell)\).
Proposition 5.3. For all $n \geq 1$

$$H^0(\text{Sym}^n A; \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell,$$
$$H^1(\text{Sym}^n A; \mathbb{Q}_\ell) \cong V_{1,0}.$$

For all $n \geq 2$,

$$H^2(\text{Sym}^n A; \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1) \oplus V_{1,1} \oplus V_{2,0}.$$

For all $n \geq 3$,

$$H^3(\text{Sym}^n A; \mathbb{Q}_\ell) \cong 3V_{1,0}(-1) \oplus V_{2,1} \oplus V_{3,0}.$$

For all $n \geq 4$,

$$H^4(\text{Sym}^n A; \mathbb{Q}_\ell) \cong 4\mathbb{Q}_\ell(-2) \oplus 3V_{1,1}(-1) \oplus 3V_{2,0}(-1) \oplus V_{2,2} \oplus V_{3,1} \oplus V_{4,0}.$$

Proof. The necessary facts from the representation theory of $\text{Sp}(4, \mathbb{Z})$ are Lemma 4.3 and the fact that for any $a \geq 0$,

$$\text{Sym}^a V_{1,0} \cong V_{a,0}, \quad \text{Sym}^a V_{1,1} \cong \bigoplus_{k=0}^{\lfloor a/2 \rfloor} V_{a-2k,a-2k}.$$

These two isomorphisms follow from [FH04, Exercises 16.10, 16.11]. Applying these facts to the direct sum given by Lemma 5.1 gives the decomposition into irreducible $\text{Sp}(4, \mathbb{Z})$-representations as claimed. Finally, add appropriate Tate twists as in the proof of Lemma 3.1. $\square$

Proposition 5.3 provides the inputs to the computation of the cohomology of $\chi_{2^{\text{Sym}(n)}}$ in the usual way.

Theorem 1.3 For all $n \geq k$ for $k$ even and for all $n \geq k - 1$ for $k$ odd,

$$H^k(\chi_{2^{\text{Sym}(n)}}; \mathbb{Q}_\ell) = \begin{cases} 
\mathbb{Q}_\ell & k = 0 \\
0 & k = 1, 3 \\
2\mathbb{Q}_\ell(-1) & k = 2 \\
5\mathbb{Q}_\ell(-2) & k = 4 \\
\mathbb{Q}_\ell(-5) \oplus \mathbb{Q}_\ell(-4) & k = 5
\end{cases}$$

up to semi-simplification.

Proof. Denote the $(p,q)$-entry on the $E_2$-sheet of the Leray spectral sequence (given by Proposition 2.9) of $\pi^n : \chi_{2^{\text{Sym}(n)}} \rightarrow \mathcal{A}_2$ by $E_2^{p,q}(n) = H^p(\mathcal{A}_2; H^q(\text{Sym}^n A))$. This spectral sequence degenerates on the $E_2$-page. Applying Proposition 5.3 and Corollary 3.4 yields $E_2^{p,q}(n)$ for $n \geq q$ and $q = 0, 2, 4$, which we record in Figure 6. Observe also that $E_2^{p,q}(n) = 0$ for all $n \geq 0$ if $q$ is odd or if $p > 4$ by Theorem 3.2. The theorem now follows directly. $\square$

5.2. Explicit computations for $n = 2$. We compute $H^*(\chi_{2^{\text{Sym}(2)}}; \mathbb{Q}_\ell)$ completely. We first need the following.
Lemma 5.4. The local systems $H^k(Sym^2 A; \mathbb{Q}_\ell)$ on $A_2$ are

$$H^k(Sym^2 A; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & k = 0 \\ \mathbb{V}_{1,0} & k = 1 \\ \mathbb{Q}_\ell(-1) \oplus \mathbb{V}_{1,1} \oplus \mathbb{V}_{2,0} & k = 2 \\ 3\mathbb{V}_{1,0}(-1) \oplus \mathbb{V}_{2,1} & k = 3 \\ 4\mathbb{Q}_\ell(-2) \oplus 2\mathbb{V}_{1,1}(-1) \oplus \mathbb{V}_{2,0}(-1) \oplus \mathbb{V}_{2,2} & k = 4 \\ 3\mathbb{V}_{1,0}(-2) \oplus \mathbb{V}_{2,1}(-1) & k = 5 \\ \mathbb{Q}_\ell(-3) \oplus \mathbb{V}_{1,1}(-2) \oplus \mathbb{V}_{2,0}(-2) & k = 6 \\ \mathbb{V}_{1,0}(-3) & k = 7 \\ \mathbb{Q}_\ell(-4) & k = 8 \\ 0 & k > 8 \end{cases}$$

Proof. This is a direct computation using Lemma 5.1. 

As usual, we want to compute $H^p(A_2; H^q(Sym^n A))$ for all $p, q \geq 0$. With Lemma 5.4, this process is completely analogous to that of Sections 3 and 4. Therefore, we list the results below and omit the explanations.
The cohomology of $\mathcal{X}_{2}^{{\text{Sym}}(2)}$ is given by

\[
H^k(\mathcal{X}_{2}^{{\text{Sym}}(2)}; \mathcal{Q}_\ell) = \begin{cases} 
\mathcal{Q}_\ell & k = 0 \\
0 & k = 1, 3, k > 10 \\
2\mathcal{Q}_\ell(-1) & k = 2 \\
5\mathcal{Q}_\ell(-2) & k = 4 \\
\mathcal{Q}_\ell(-5) \oplus \mathcal{Q}_\ell(-4) & k = 5 \\
5\mathcal{Q}_\ell(-3) & k = 6 \\
2\mathcal{Q}_\ell(-6) \oplus \mathcal{Q}_\ell(-5) & k = 7 \\
2\mathcal{Q}_\ell(-4) & k = 8 \\
\mathcal{Q}_\ell(-7) \oplus \mathcal{Q}_\ell(-6) & k = 9 \\
\mathcal{Q}_\ell(-5) & k = 10 
\end{cases}
\]

up to semi-simplification.

**Proof.** By Proposition 2.9, there is a spectral sequence with $E_2^{p,q} = H^n(\mathcal{A}_2; H^q(\text{Sym}^2 A))$ which degenerates on the $E_2$-page and converges to $H^*(\mathcal{X}_{2}^{{\text{Sym}}(2)}; \mathcal{Q}_\ell)$. Using Lemma 5.4 and Corollary 3.4, we can compute $E_2^{p,q}$ for all $p, q \geq 0$. The results are recorded in Figure 7, from which the theorem follows directly. \qed

**Remark 5.6.** Note for all $n \geq 2$, the $E_2^{p,q}$ term in the spectral sequence for $\mathcal{X}_{2}^{{\text{Sym}}(n)} \to \mathcal{A}_2$ for $q = 0, 1, 2, 3$ remain stable and are given in the corresponding entries in Figure 7.

6. **ARITHMETIC STATISTICS**

In this section, we apply the cohomological results of the previous sections to obtain arithmetic statistics results about abelian surfaces over finite fields. We also discuss the possible applications giving arithmetic statistics about torsion on abelian surfaces using level structures. Finally, we
give a different interpretation of the (suitably weighted) point counts of $\mathbf{F}_q$-isomorphism classes obtained in this section as unweighted counts of $\overline{\mathbf{F}}_q$-isomorphism classes.

Given the étale cohomology of a variety over a finite field $\mathbf{F}_q$, one can use the Grothendieck–Lefschetz trace formula to immediately deduce the number of $\mathbf{F}_q$-points on the variety. Even though the spaces $\mathcal{X}$ studied in this paper are not varieties but rather algebraic stacks, there is fortunately an applicable generalization, the Grothendieck–Lefschetz–Behrend trace formula, which gives the groupoid cardinality of their $\mathbf{F}_q$-points.

**Definition 6.1.** Let $X$ be a groupoid. The groupoid cardinality $\# X$ is defined as

$$\# X = \sum_{x \in X} \frac{1}{\# \text{Aut}(x)}.$$

The following definition is necessary in order to state the Grothendieck–Lefschetz–Behrend trace formula.

**Definition 6.2.** Let $\mathcal{X}$ be a smooth Deligne–Mumford stack of finite type over $\mathbf{F}_q$. Let $\mathcal{X}_{\mathbf{F}_q, \text{sm}}$ be the smooth site associated to $\mathcal{X}_{\mathbf{F}_q}$. The arithmetic Frobenius acting on $H^i(\mathcal{X}_{\mathbf{F}_q, \text{sm}}, \mathbf{Q}_\ell)$ is denoted by $\Phi_q : H^i(\mathcal{X}_{\mathbf{F}_q, \text{sm}}, \mathbf{Q}_\ell) \to H^i(\mathcal{X}_{\mathbf{F}_q, \text{sm}}, \mathbf{Q}_\ell)$. The action of $\Phi_q$ on $\mathbf{Q}_\ell(1)$ is multiplication by $q$. For any $n \in \mathbb{Z}$, the action of $\Phi_q$ on $\mathbf{Q}_\ell(n)$ is multiplication by $q^n$.

**Remark 6.3.** For Deligne–Mumford stacks, the étale and smooth cohomology of constructible sheaves coincide (see [LMB00, Chapter 12]), so we may use them interchangeably. All stacks in this paper are Deligne–Mumford stacks over $\mathbf{F}_q$. We will omit the distinction and just write $H^* (\mathcal{X}; \mathbf{Q}_\ell)$ for étale (or smooth) cohomology of $\mathcal{X}_{\mathbf{F}_q}$.

The main tool of this section is the following trace formula.

**Theorem 6.4** (Grothendieck–Lefschetz–Behrend trace formula, [Beh93, Theorem 3.1.2]). Let $\mathcal{X}$ be a smooth Deligne–Mumford stack of finite type and constant dimension over the finite field $\mathbf{F}_q$. Then

$$q^{\dim \mathcal{X}} \sum_{k \geq 0} (-1)^k (\text{tr} \Phi_q | H^k (\mathcal{X}_{\mathbf{F}_q, \text{sm}}, \mathbf{Q}_\ell)) = \sum_{\xi \in \mathcal{X}(\mathbf{F}_q)} \frac{1}{\# \text{Aut}(\xi)} = \# \mathcal{X}(\mathbf{F}_q).$$

### 6.1. Applying the Grothendieck–Lefschetz–Behrend trace formula

In this subsection, we apply the trace formula (Theorem 6.4) to deduce corollaries of the cohomological results of the previous sections. Recall that the interpretation of the $\mathbf{F}_q$-points of each stack in question is given in Section 2.1. This first count is well-known, but we list it below for completeness.

**Theorem 6.5.**

$$\# \mathcal{X}_{\mathbf{F}_q} = q^3 + q^2.$$

**Proof.** Using Theorem 2.2 apply the trace formula (Theorem 6.4), noting that $\dim \mathcal{X}_2 = 3$ and that the values $\text{tr} (\Phi_q | H^k (\mathcal{X}_2; \mathbf{Q}_\ell))$ are known.

The point counts in the rest of this section are new to the best of our knowledge. Although all cohomology computations in the previous sections are only up to semi-simplification, the trace of a linear operator does not change under semi-simplification. Therefore, we apply the trace formula (Theorem 6.4) to the semi-simplification without making this distinction.
Theorem 6.6.
\[
\# \mathcal{X}_2(F_q) = q^5 + 2q^4 + 2q^3 + q^2 - 1,
\]
\[
\# \mathcal{X}_2^2(F_q) = q^7 + 4q^6 + 9q^5 + 9q^4 + 3q^3 - 5q^2 - 5q - 3,
\]
\[
\# \mathcal{X}_2^{\text{Sym}(2)} = q^7 + 2q^6 + 5q^5 + 5q^4 + q^3 - q^2 - 3q - 1.
\]

**Proof.** In all cases, the counts follow from the trace formula (Theorem 6.4). Applying Theorem 1.1 and the fact that \(\dim X_2 = 5\),
\[
\# \mathcal{X}_2(F_q) = q^5(1 + 2q^{-1} + 2q^{-2} - q^{-5} + q^{-3}) = q^5 + 2q^4 + 2q^3 + q^2 - 1.
\]
Applying Theorem 4.17 and the fact that \(\dim X_2^2 = 7\),
\[
\# \mathcal{X}_2^2(F_q) = q^7(1 + 4q^{-1} + 9q^{-2} - (3q^{-5} + q^{-4}) + 9q^{-3})
\]
\[
+ q^7(-3q^{-5} + 4q^{-6} + 4q^{-4} - (3q^{-7} + q^{-6}) + q^{-5})
\]
\[
= q^7 + 4q^6 + 9q^5 + 9q^4 + 3q^3 - 5q^2 - 5q - 3.
\]
Applying Theorem 5.5 and the fact that \(\dim X_2^{\text{Sym}(2)} = 7\),
\[
\# \mathcal{X}_2^{\text{Sym}(2)}(F_q) = q^7(1 + 2q^{-1} + 5q^{-2} - (q^{-4} + q^{-5}) + 5q^{-3})
\]
\[
+ q^7(-q^{-5} + 2q^{-6} + 2q^{-4} - (q^{-7} + q^{-6}) + q^{-5})
\]
\[
= q^7 + 2q^6 + 5q^5 + 5q^4 + q^3 - q^2 - 3q - 1.
\]

We can also piece together the partial information we have about \(H^k(\mathcal{X}_2^n; Q_\ell)\) and \(H^k(\mathcal{X}_2^{\text{Sym}(n)}; Q_\ell)\) to give an approximation of \(\# \mathcal{X}_2^n(F_q)\) and \(\# \mathcal{X}_2^{\text{Sym}(n)}(F_q)\), for fixed \(n\) and asymptotic in \(q\).

**Theorem 6.7.** For all \(n \geq 1\),
\[
\# \mathcal{X}_2^n(F_q) = q^3 + 2n + \left(\frac{n+1}{2}\right) + 1 \quad \frac{n(n+1)(n^2+n+2)}{8} + \left(\frac{n+1}{2}\right) q^{1+2n} + O(q^{2n})
\]
and for all \(n \geq 4\),
\[
\# \mathcal{X}_2^{\text{Sym}(n)}(F_q) = \# \mathcal{X}_2^{\text{Sym}(n)} = q^3 + 2n + 5q^{1+2n} + O(q^{2n}).
\]

**Proof.** By Theorem 2.6, for all \(p \geq 0\) and \(a \geq b \geq 0\),
\[
|\text{tr}(\Phi_q[H^p(A_2; \mathcal{V}_{a,b})])| \leq \dim H^p(A_2; \mathcal{V}_{a,b})q^{-\frac{p+a+b}{2}}
\]
and so for any \(N \geq 0\) such that \(N - p \equiv a + b \pmod{2}\),
\[
|\text{tr}(\Phi_q[H^p(A_2; \mathcal{V}_{a,b})(a + b - (N - p)/2])| \leq \dim H^p(A_2; \mathcal{V}_{a,b})q^{-N/2}.
\]
For any \(N \geq 0\) and \(\mathcal{X} = \mathcal{X}_2^n\) or \(\mathcal{X}^{\text{Sym}(n)}\), this estimate, the trace formula (Theorem 6.4), the properties of the Leray spectral sequence for \(\mathcal{X} \to A_2\) (Proposition 2.9), and Proposition 4.11 imply
\[
\# \mathcal{X}(F_q) = q^{3+2n} \sum_{0 \leq k \leq N} (-1)^k \text{tr}(\Phi_q[H^k(\mathcal{X}; Q_\ell)]) + O\left(q^{3+2n - \frac{N+1}{2}}\right).
\]
For \(\mathcal{X} = \mathcal{X}_2^n\), applying Theorem 1.2 with \(N = 5\) gives
\[
\# \mathcal{X}_2^n(F_q) = q^{3+2n} + \left(\frac{n+1}{2}\right) q^{2+2n} + \frac{n(n+1)(n^2+n+2)}{8} + \left(\frac{n+1}{2}\right) q^{1+2n} + O(q^{2n})
\]
and for $\mathcal{X} = X_2^{\text{Sym}(n)}$, the same computation using Theorem 1.3 with $N = 5$ gives

$$\# X_2^{\text{Sym}(n)} = q^{3+2n} + 2q^{2+2n} + 5q^{1+2n} + O(q^{2n}).$$

These point counts imply the arithmetic statistics results outlined in Section 1, which we discuss for the remainder of this subsection.

**Lemma 6.8.** Define a probability measure $P$ on $A_2(F_q)$ by

$$P([A_0]) = \frac{1}{\# A_2(F_q) \# \text{Aut}_{F_q}(A_0)}$$

for each $F_q$-isomorphism class $[A_0] \in A_2(F_q)$. The expected value of $F_q$-points on abelian surfaces with respect to this probability measure is

$$E[\# A(F_q)] = \left( \frac{\sum (A,p) \in \mathcal{X}(F_q) - \# \text{Aut}_{F_q}(A,p)}{\sum A \in F_q - \# \text{Aut}_{F_q}(A)} \right) = \frac{\# A_2(F_q)}{\# A_2(F_q)}.$$

More generally, for $Z$ the fiber in $X$ over a point in $A_2$,

$$E[\# Z(F_q)] = \left( \frac{\sum (Z_0,p) \in \mathcal{X}(F_q) - \# \text{Aut}_{F_q}(Z,p)}{\sum A \in F_q - \# \text{Aut}_{F_q}(A)} \right) = \frac{\# X(F_q)}{\# A_2(F_q)}.$$

**Proof.** Consider a representative abelian surface $A_0$ in a fixed $F_q$-isomorphism class $[A_0]$. There is an action of $\text{Aut}_{F_q}(A_0)$ on its fiber $Z_0$ in $X$. For any $p_0 \in Z_0(F_q)$, its $F_q$-isomorphism class is precisely its orbit under the action of $\text{Aut}_{F_q}(A_0)$. So the contribution of $A_0$ (and its corresponding fiber $Z_0$) to $E[\# Z(F_q)]$ is

$$\frac{1}{\# A_2(F_q)} \sum_{[p] \in Z_0(F_q)/\sim} \frac{1}{\# \text{Aut}_{F_q}(Z,p)} \sum_{p \in Z_0(F_q)/\sim} \# \text{Orb}(p) = \frac{\# Z_0(F_q)}{\# A_2(F_q) \# \text{Aut}_{F_q}(A_0)}$$

where the sums are over $F_q$-isomorphism classes of points $p \in Z_0(F_q)$ and the first equality follows from the orbit-stabilizer theorem.

Lemma 6.8 and the results of this section immediately imply the statistics given in Section 1; we restate them here for convenience.

**Corollary 1.4** The expected number of $F_q$-points on abelian surfaces defined over $F_q$ is

$$E[\# A(F_q)] = q^2 + q + 1 - \frac{1}{q^3 + q^2}.$$

Because $# A^n(F_q) = # A(F_q)^n$ for all abelian surfaces $A$, the following corollary gives asymptotics for all moments of $# A(F_q)$ as well as the exact second moment.

**Corollary 1.5** The expected value of $# A^2(F_q)$ is

$$E[\# A^2(F_q)] = q^4 + 3q^3 + 6q^2 + 3q - \frac{5q^2 + 5q + 3}{q^3 + q^2}$$

and for all $n \geq 1$,

$$E[\# A^n(F_q)] = q^{2n} + \left( \frac{n+1}{2} \right) q^{2n-1} + \left( \frac{n(n+1)(n^2+n+2)}{8} \right) q^{2n-2} + O(q^{2n-3}).$$
Recall that \( \text{Sym}^n A(F_q) \) for an abelian surface \( A \) and any \( n \geq 1 \) is the set of \( n \)-tuples defined over \( F_q \) as tuples, i.e. the \( n \) points are permuted by \( \text{Frob}_q \).

**Corollary 1.6** The expected value of \( \# \text{Sym}^2 A(F_q) \) is

\[
E[\# \text{Sym}^2 A(F_q)] = q^4 + q^3 + 4q^2 + q - \frac{q^2 + 3q + 1}{q^4 + q^2}
\]

and for all \( n \geq 4 \),

\[
E[\# \text{Sym}^n A(F_q)] = q^{2n} + q^{2n-1} + 4q^{2n-2} + O(q^{2n-3}).
\]

The fact that we have determined the exact value for the second moment means we can calculate the variance of \( \# A(F_q) \) using Corollary 1.4.

**Corollary 1.7** The variance of \( \# A(F_q) \) is

\[
\text{Var}(\# A(F_q)) = E[\# A^2(F_q)] - (E[\# A(F_q)])^2 = q^3 + 3q^2 + q - 1 - \frac{3q^2 + 3q + 1}{q^3 + q^2} - \frac{1}{(q^3 + q^2)^2}.
\]

**6.2. Level structures.** Let \( N \geq 2 \) and let \( \pi_N : X_2[N] \to A_2[N] \) be the projection map. By construction, \( \pi_N^{\text{orb}}(A_2[N]) \cong \text{Sp}(4, \mathbb{Z})[N] \). For \( N \geq 3 \), \( \text{Sp}(4, \mathbb{Z})[N] \) is torsion-free, from which it follows that all points in \( A_2[N] \) have trivial stabilizers and \( \pi_N^{\text{orb}}(A_2[N]) \cong \pi_1(A_2[N]) \).

First, all representations of \( \text{Sp}(4, \mathbb{Z})[N] \) obtained by restriction of irreducible representations of \( \text{Sp}(4, \mathbb{C}) \) are still irreducible. In particular, the decomposition given in Lemma 3.1 also holds for the local system \( H^k(A; \mathbb{Q}_l) \) on \( A_2[N] \), obtained by restricting \( \text{Sp}(4, \mathbb{Z}) \)-representations to \( \text{Sp}(4, \mathbb{Z})[N] \).

However, the cohomology of local systems on \( A_2[N] \) is not yet known in general. In the case \( N = 2 \), many parts of the Euler characteristics of local systems \( V_{a,b} \) on \( A_2[2] \) are known and there are conjectures for the rest; this is done in [BFvdG08]. Here, we consider the compactly supported Euler characteristics of such local systems, defined

\[
\epsilon_c(A_2[2]; V_{a,b}) := \sum_{k \geq 0} (-1)^k [H^k_c(A_2[2]; V_{a,b})]
\]

taken in the Grothendieck group of an appropriate category, e.g. the category of mixed Hodge structures or of Galois representations.

**Conjecture 6.9** (Bergström–Faber–van der Geer, [BFvdG08, Section 10]). The compactly supported Euler characteristic of \( V_{1,1} \) over \( A_2[2] \) is given by \( 5Q_l(-3) - 10Q_l(-2) \). The compactly supported Euler characteristic of \( V_{0,0} \) over \( A_2[2] \) is given by \( Q_l(-3) + Q_l(-2) - 14Q_l(-1) + 16Q_l \).

The cohomology of \( A_2[2] \) is also computed in [LW85, Theorem 5.2.1]. Assuming these two calculations and using the Leray spectral sequence of \( A_2[2] \) to \( A_2[2] \),

\[
\epsilon_c(A_2[2]; Q_l) = \sum_{k \geq 0} (-1)^k H^k(A_2[2]; Q_l) = \sum_{p,q \geq 0} (-1)^{p+q} H^p_f(A_2[2]; H^q(A; Q_l)) = Q_l(-5) + 2Q_l(-4) - 7Q_l(-3) - 7Q_l(-2) + 2Q_l(-1) + 16Q_l.
\]

These computations plus a version of the trace formula (Theorem 6.4) for compactly supported cohomology imply that

\[
\# A_2[2](F_q) = q^3 + q^2 - 14q + 16,
\]
\[
\# A_2[2](F_q) = q^5 + 2q^4 - 7q^3 - 7q^2 + 2q + 16.
\]
In particular, note that an $F_q$-point on $A_2[N]$ corresponds to an abelian surface $A$ defined over $F_q$ with an ordered basis of its $N$-torsion defined over $F_q$. Therefore, careful analysis of the counts \#$A_2[2](F_q)$ and \#$X_2[2](F_q)$ will yield the average number of abelian surfaces over $F_q$ with 2-torsion defined over $F_q$ and the average number of $F_q$-points on such abelian surfaces.

6.3. **An unweighted interpretation.** The interpretation of \#$X(F_q)$ as a weighted count of $F_q$-isomorphism classes of objects parametrized by $X$ (for $X$ any stack considered in this paper) as in the previous subsections is a natural one. However, there is a way to interpret this count as an unweighted one of $F_q$-isomorphism classes of such objects, for $X = X_n^\ast$ for any $n \geq 1$ and $X = A_2$. We outline the ideas here, although they are standard. (For example, see [vdGvdV92, Section 5] or [KS99] Section 10.7 for this in other contexts.)

**Theorem 6.10** ([Ser02 Proposition III.1.5]). Let $K/k$ be a Galois extension and let $V$ be a quasiprojective variety defined over $k$. Let $E(K/k, V)$ be the $k$-isomorphism classes of varieties $W$ that become isomorphic to $V$ over $K$. Then there is a canonical bijection

$$\theta : E(K/k, V) \to H^1(\text{Gal}(K/k); \text{Aut}_K(V)).$$

In particular, let $\alpha : W \to V$ be a $K$-isomorphism. Define the map

$$\theta_0 : W \mapsto (\gamma \mapsto \alpha_\gamma := \alpha^{-1} \circ \alpha \in Z^1(\text{Gal}(K/k); \text{Aut}_K(V)))$$

where $\alpha_\gamma := \gamma^{-1} \circ \alpha \circ \gamma$. The map $\alpha \mapsto (\gamma \mapsto a_\gamma)$ descends to $\theta$.

**Remark 6.11.** In our setting, we take $K = F_q$ and $L = F_q$, and $V$ to be either a principally polarized abelian surface $(A, \lambda)$, or a tuple $(A, \lambda, p)$ with $p \in A^n(F_q)$. Let $E(F_q/F_q, (A, \lambda, p))$ be the $F_q$-isomorphism classes of tuples of principally polarized abelian varieties $(A', \lambda')$ and points $p \in (A')^n(F_q)$. The only issue to check is that the map $\theta$ is still well-defined – see [Mil08a, p. 157].

Fix some $n \in \mathbb{Z}_{\geq 0}$, a principally polarized abelian surface $(A_0, \lambda_0)$ defined over $F_q$, and a point $p_0 \in A_0^n(F_q)$. (If $n = 0$, we set $p_0 = \emptyset$.)

**Lemma 6.12.** Let $\text{Aut}_{F_q}(A_0, \lambda_0, p_0)$ act on itself by twisted conjugacy,

$$\varphi \cdot \psi = \varphi^{\text{Frob}_q} \psi \varphi^{-1}.$$ 

Let $\varphi_1 \sim \varphi_2$ in $\text{Aut}_{F_q}(A_0, \lambda_0, p_0)$ if they are in the same orbit. There is a bijection

$$\tau : Z^1(\text{Gal}(F_q/F_q), \text{Aut}_{F_q}(A_0, \lambda_0, p)) \to \text{Aut}_{F_q}(A_0, \lambda_0, p)$$

given by $\tau : (\gamma \mapsto a_\gamma) \mapsto a_{\text{Frob}_q}$ which descends to a bijection

$$H^1(\text{Gal}(F_q/F_q), \text{Aut}_{F_q}(A_0, \lambda_0, p)) \to \text{Aut}_{F_q}(A_0, \lambda_0, p)/\sim.$$ 

**Proof.** Because $\text{Gal}(F_q/F_q)$ is topologically generated by $\text{Frob}_q$, a cocycle $(\gamma \mapsto a_\gamma)$ is determined by the image of $\text{Frob}_q$, given by an arbitrary $a_{\text{Frob}_q} \in \text{Aut}_{F_q}(\bar{A})$. This gives that $\tau$ is a bijection.

Two cohomologous cocycles $\xi_1$ and $\xi_2$ are given by twists by some $\varphi \in \text{Aut}_{F_q}(A_0, \lambda_0, p)$, i.e. $\varphi^{\text{Frob}_q} \xi_1(\text{Frob}_q) = \xi_2(\text{Frob}_q) \varphi$, or after rearranging, $\xi_2(\text{Frob}_q) = \varphi^{\text{Frob}_q} \xi_1(\text{Frob}_q) \varphi^{-1}$. \qed

**Proposition 6.13.** For all $n \geq 0$,

$$\#X_2^n(F_q) = \#\{F_q\text{-isomorphism classes of } (A, \lambda, p), \text{ with } p \in A^n(F_q), \text{ defined over } F_q\},$$

where $X_2^n := A_2$. 

Theorem 6.10 and Lemma 6.12

References


Proof. First, fix some \((A, \lambda, p) \in E(\mathbb{F}_q/\mathbb{F}_q, (A_0, \lambda_0, p_0))\) and let \(\alpha : A \to A_0\) be an \(\mathbb{F}_q\)-isomorphism. By definition, \(\tau_0(A, \lambda, p) = \alpha \text{Frob}_q \alpha^{-1}\). Take \(\varphi \in \text{Stab}(\alpha \text{Frob}_q \alpha^{-1})\) to see that

\[
\alpha_{\text{Frob}_q}^{-1} \varphi = \varphi \alpha_{\text{Frob}_q} \alpha^{-1} \varphi^{-1} = (\text{Frob}_q)^{-1}(\varphi \alpha)(\text{Frob}_q)(\varphi^{-1})^{-1}
\]

which is equivalent to the condition \((\alpha^{-1} \varphi \alpha) \text{Frob}_q = \text{Frob}_q(\alpha^{-1} \varphi \alpha)\), i.e. \(\alpha^{-1} \varphi \alpha \in \text{Aut}_{\mathbb{F}_q}(A, \lambda, p)\). Therefore, the stabilizer of \(\tau_0(A, \lambda, p)\) under the twisted conjugacy action of \(\text{Aut}_{\mathbb{F}_q}(A_0, \lambda_0, p_0)\) is the subgroup \(\alpha \text{Aut}_{\mathbb{F}_q}(A, \lambda, p) \alpha^{-1}\) and \# \text{Stab}(\tau_0(A, \lambda, p)) = \# \text{Aut}_{\mathbb{F}_q}(A, \lambda, p).

Consider the following computation:

\[
\sum_{(A, \lambda, p)} \frac{1}{\# \text{Aut}_{\mathbb{F}_q}(A, \lambda, p)} = \sum_{\varphi \in \text{Aut}_{\mathbb{F}_q}(A_0, \lambda_0, p_0)/\sim} \frac{1}{\# \text{Stab}(\varphi)} = \sum_{\varphi \in \text{Aut}_{\mathbb{F}_q}(A_0, \lambda_0, p_0)/\sim} \frac{\# \text{Orb}(\varphi)}{\# \text{Aut}_{\mathbb{F}_q}(A_0, \lambda_0, p_0)} = \frac{1}{\# \text{Aut}_{\mathbb{F}_q}(A_0, \lambda_0, p_0)} \sum_{\varphi \in \text{Aut}_{\mathbb{F}_q}(A_0, \lambda_0, p_0)/\sim} \# \text{Orb}(\varphi) = 1.
\]

The sum in the first line above is over \(E(\mathbb{F}_q/\mathbb{F}_q, (A_0, \lambda_0, p_0))\). Then

\[
\# A^n_2(\mathbb{F}_q) = \sum_{(A, \lambda, p) \in A^n_2(\mathbb{F}_q)} \frac{1}{\# \text{Aut}_{\mathbb{F}_q}(A, \lambda, p)}
\]

\[
= \sum_{(A_0, \lambda_0, p_0) \in A^n_2(\mathbb{F}_q)} \left( \sum_{(A, \lambda, p) \in E(\mathbb{F}_q/\mathbb{F}_q, (A_0, \lambda_0, p_0))} \frac{1}{\# \text{Aut}_{\mathbb{F}_q}(A, \lambda, p)} \right)
\]

\[
= \sum_{(A_0, \lambda_0, p_0) \in A^n_2(\mathbb{F}_q)} 1 = \# \{\text{\(\mathbb{F}_q\)-isomorphism classes of \((A, \lambda, p)\), with \(p \in A^n(\mathbb{F}_q)\), defined over \(\mathbb{F}_q\)}\}.
\]

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