COHOMOLOGY OF THE UNIVERSAL ABELIAN SURFACE WITH APPLICATIONS TO ARITHMETIC STATISTICS

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ABSTRACT. The moduli stack $\mathcal{A}_2$ of principally polarized abelian surfaces comes equipped with the universal abelian surface $\pi : \mathcal{X}_2 \to \mathcal{A}_2$. The fiber of π over a point corresponding to an abelian surface A in $\mathcal{A}_2$ is A itself. We determine the ℓ-adic cohomology of $\mathcal{X}_2$ as a Galois representation. Similarly, we consider the bundles $\mathcal{X}_2^n \to \mathcal{A}_2$ and $\mathcal{X}_2^{\mathrm{Sym}(n)} \to \mathcal{A}_2$ for all $n \geq 1$, where the fiber over a point corresponding to an abelian surface A is A^n and $\mathrm{Sym}^n A$ respectively. We describe how to compute the ℓ-adic cohomology of $\mathcal{X}_2^n$ and $\mathcal{X}_2^{\mathrm{Sym}(n)}$ and explicitly calculate it in low degrees for all n and in all degrees for $n = 2$. These results yield new information regarding the arithmetic statistics on abelian surfaces, including an exact calculation of the expected value and variance as well as asymptotics for higher moments of the number of $\mathbf{F}_q$-points.

CONTENTS

1. INTRODUCTION

An *abelian surface* is an abelian variety of dimension 2. Over $\mathbf{C}$, all abelian surfaces are isomorphic to $\mathbf{C}^2/L$ for some lattice L with real rank 4. The fine moduli stack $\mathcal{A}_2$ of principally polarized abelian surfaces is a smooth Deligne–Mumford stack defined over $\mathbf{Z}$. It comes equipped with a universal bundle $\mathcal{X}_2 \to \mathcal{A}_2$. The fiber over the point corresponding to an abelian surface A in $\mathcal{A}_2$ is A itself. Using the projection map $\mathcal{X}_2 \to \mathcal{A}_2$, we can take nth fiber powers $\mathcal{X}_2^n$ of $\mathcal{X}_2$ over $\mathcal{A}_2$, which has the nth power A^n of an abelian surface over the corresponding point in $\mathcal{A}_2$. Since each A^n has an action of S_n permuting the coordinates (which is not a free action), taking the quotient $\mathcal{X}_2^n/S_n$ gives a new stack $\mathcal{X}_2^{\mathrm{Sym}(n)}$, which has $\mathrm{Sym}^n A$ as a fiber over the point corresponding to A in $\mathcal{A}_2$.

Our main theorems are the computations of the ℓ-adic cohomology of the universal abelian surface and related spaces as Galois representations (up to semi-simplification). From now on, all cohomology will denote ℓ-adic cohomology and we drop the subscripts to write $H^*(\mathcal{X};\mathbf{V})$ in place of both $H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{Q}}};\mathbf{V})$ and $H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{F}}_q};\mathbf{V})$ for any ℓ-adic local system $\mathbf{V}$ on $\mathcal{X}$ with ℓ coprime to q. (See Remark 2.3 for details justifying this notation.)

Theorem 1.1. The cohomology of the universal abelian surface $\mathcal{X}_2$ is given by

$$H^k(\mathcal{X}_2;\mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k=0 \\ 0 & k=1,3,k\geq 7 \\ 2\mathbf{Q}_\ell(-1) & k=2 \\ 2\mathbf{Q}_\ell(-2) & k=4 \\ \mathbf{Q}_\ell(-5) & k=5 \\ \mathbf{Q}_\ell(-3) & k=6 \end{cases}$$

up to semi-simplification, where $\mathbf{Q}_\ell = \mathbf{Q}_\ell(0)$ is the trivial Galois representation, $\mathbf{Q}_\ell(1)$ is the ℓ-adic cyclotomic character, and $\mathbf{Q}_\ell(-1)$ is its dual. For all $n \in \mathbf{N}$, $\mathbf{Q}_\ell(n)$ is the nth tensor power of $\mathbf{Q}_\ell(1)$ and $\mathbf{Q}_\ell(-n)$ is the nth tensor power of $\mathbf{Q}_\ell(-1)$. For all $n \in \mathbf{Z}$, $\mathbf{Q}_\ell(n) \cong \mathbf{Q}_\ell$ as a $\mathbf{Q}_\ell$-vector space. Denote $\mathbf{Q}_\ell(n)^{\oplus m}$ by $m\mathbf{Q}_\ell(n)$.

Applying similar techniques gives the cohomology of the nth fiber product of $\mathcal{X}_2$ in low degrees. The following theorem applies these techniques to explicitly compute $H^k(\mathcal{X}_2^n;\mathbf{Q}_\ell)$ for $0 \leq k \leq 5$ for all $n \geq 1$. Here and in the rest of the paper, we use the convention that $\binom{n}{m} = 0$ if $n < m$.

Theorem 1.2. For all $n \geq 1$, the cohomology of the universal nth fiber product of abelian surfaces is

$$H^k(\mathcal{X}_2^n;\mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k=0 \\ 0 & k=1,3 \\ \left(\binom{n+1}{2}+1\right)\mathbf{Q}_\ell(-1) & k=2 \\ \left(\frac{n(n+1)(n^2+n+2)}{8}+\binom{n+1}{2}\right)\mathbf{Q}_\ell(-2) & k=4 \\ \binom{n+1}{2}\mathbf{Q}_\ell(-5)\oplus\binom{n}{2}\mathbf{Q}_\ell(-4) & k=5 \end{cases}$$

up to semi-simplification.

The cohomology of the universal nth symmetric power of abelian surfaces *stabilizes* as n increases, by which we mean that the cohomology $H^k(\mathcal{X}^{\mathrm{Sym}(n)};\mathbf{Q}_\ell)$ is independent of n for n large enough compared to the degree k. As with the cohomology of the nth fiber product of $\mathcal{X}_2$, we explicitly compute $H^k(\mathcal{X}_2^{\mathrm{Sym}(n)};\mathbf{Q}_\ell)$ for $0 \leq k \leq 5$ and all n large enough compared to k.

Theorem 1.3. For all $n \geq k$ for k even and for all $n \geq k-1$ for k odd,

$$H^k(\mathcal{X}_2^{\mathrm{Sym}(n)}; \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ 0 & k = 1, 3 \\ 3\mathbf{Q}_\ell(-1) & k = 2 \\ 9\mathbf{Q}_\ell(-2) & k = 4 \\ 2\mathbf{Q}_\ell(-5) & k = 5 \end{cases}$$

up to semi-simplification.

The proofs of these theorems use the Leray spectral sequence of the morphisms $\pi : \mathcal{X} \to \mathcal{A}_2$, with $\mathcal{X} = \mathcal{X}_2$, $\mathcal{X}_2^n$, and $\mathcal{X}_2^{\mathrm{Sym}(n)}$ respectively. The spectral sequence takes as input the cohomology of local systems of $\mathcal{A}_2$, which has been computed by Petersen in [Pet15]. Then it still remains to determine the local systems involved in the latter two cases, converting this problem about cohomology into a series of problems about the representation theory of $\mathrm{Sp}(4)$. For $\mathcal{X}_2^n$, we give recursive formulas (in n) for the relevant local systems in Subsection 4.1. For $\mathcal{X}_2^{\mathrm{Sym}(n)}$, we show that the local systems $R^k\pi_*\mathbf{Q}_\ell$ stabilize for $n \geq k$. In both cases, we use these facts to prove Theorems 1.2 and 1.3.

The cohomology in higher degrees is quite involved to determine for general $n \geq 0$ and $k \geq 0$ and involve Galois representations attached to certain (Siegel) modular forms. However, the methods of this paper give a finite computation for the relevant local systems for each fixed n. This means that given enough information about the inputs to Petersen's theorem ([Pet15, Theorem 2.1]), it is possible to compute the cohomology groups for larger values k and n using the results of this paper. We work out the case $n = 2$ completely – see Theorems 4.16 and 5.5 for the cohomology of $\mathcal{X}_2^2$ and $\mathcal{X}_2^{\mathrm{Sym}(2)}$ respectively.

Arithmetic Statistics. We fix throughout a finite field $\mathbf{F}_q$. The Weil conjectures give bounds on the number of $\mathbf{F}_q$-points on any projective variety over $\mathbf{F}_q$. Applied to an abelian surface A they assert that

$$\#A(\mathbf{F}_q) = q^2 + a_3 q^{3/2} + a_2 q + a_1 q^{1/2} + 1$$

where a_i are some sums of n_i roots of unity with $n_i = 4, 6, 4$ for $i = 1, 2, 3$ respectively. A simple corollary is

$$|\#A(\mathbf{F}_q) - (q^2+1)| \leq 4q^{3/2} + 6q + 4q^{1/2},$$

constraining the possible values that $\#A(\mathbf{F}_q)$ can take. The exact set of possible values of $\#A(\mathbf{F}_q)$ is given by *Honda–Tate theory*, which yields a bijection between isogeny classes of simple abelian varieties (of all dimensions) over $\mathbf{F}_q$ and Weil q-polynomials. In particular, for a Weil q-polynomial f, there is some abelian variety V such that the characteristic polynomial f_V of the Frobenius endomorphism of V is given by $f_V = f^e$ for some $e \geq 1$, for which $\#V(\mathbf{F}_q) = f_V(1)$. While the restriction of this bijection to simple abelian surfaces is known, we omit it for brevity and refer the reader to [Rüc90], [Wat69] and [DGS+14, Section 2] for an overview.

Studying the counts $\#\mathcal{X}_2^n(\mathbf{F}_q)$ and $\#\mathcal{X}_2^{\mathrm{Sym}(n)}(\mathbf{F}_q)$ for $n \geq 1$ will give more information about the distribution of $\#A(\mathbf{F}_q)$ as the abelian surface A varies over $\mathcal{A}_2(\mathbf{F}_q)$. Our main tool to obtain these point counts is the Grothendieck–Lefschetz–Behrend trace formula ([Beh93, Theorem 3.1.2]). A first observation through standard applications of the Weil conjectures and the trace formula is that $\#\mathcal{X}_2^n(\mathbf{F}_q) = q^d + O(q^{d-\frac{1}{2}})$ where $d = \dim \mathcal{X}_2^n$, because $\mathcal{X}_2^n$ is finitely covered by a smooth, irreducible, quasiprojective variety. However, applying the trace formula to our cohomological theorems immediately gives more precise asymptotics for $\#\mathcal{X}_2^n(\mathbf{F}_q)$ as well as new arithmetic statistics about the number of $\mathbf{F}_q$-points of abelian surfaces. Below, we consider expected values of random variables on $\mathcal{A}_2(\mathbf{F}_q)$ by giving $\mathcal{A}_2(\mathbf{F}_q)$ a natural probability measure where each isomorphism class of an abelian surface A has probability inversely proportional to the size of its $\mathbf{F}_q$-automorphism group; see Lemma 6.8 for more details.

Corollary 1.4. The expected number of $\mathbf{F}_q$-points on abelian surfaces defined over $\mathbf{F}_q$ is

$$\mathbf{E}[\#A(\mathbf{F}_q)] = q^2 + q + 1 - \frac{1}{q^3 + q^2}.$$

For each prime power $q > 0$, there is a simple abelian surface A_q over $\mathbf{F}_q$ with $\#A_q(\mathbf{F}_q) = q^2+q+1$ by the Honda–Tate correspondence for surfaces ([Rüc90, Theorem 1.1]) which corresponds to the case $a_3 = a_1 = 0$, $a_2 = 1$ of the Weil conjectures. Although $\mathbf{E}[\#A(\mathbf{F}_q)]$ is not realized by an abelian surface for any fixed q, the minimal difference between the expected value and the $\mathbf{F}_q$-point count of an arbitrary abelian surface goes to 0 as q increases, i.e.

$$\min_{[A]\in\mathcal{A}_2(\mathbf{F}_q)} |\#A(\mathbf{F}_q) - \mathbf{E}[\#A(\mathbf{F}_q)]| \to 0 \quad \text{as } q \to \infty.$$

For any abelian surface A and $n > 1$, $A^n(\mathbf{F}_q)$ denotes the set of $\mathbf{F}_q$-points of the nth power A^n of A, which are ordered n-tuples of (not necessarily distinct) $\mathbf{F}_q$-points of A. The trace formula is also used to compute the exact expected value of $\#A^2(\mathbf{F}_q)$ and an asymptotic estimate for the expected value of $\#A^n(\mathbf{F}_q)$ for $n > 2$. Because $\#A^n(\mathbf{F}_q) = (\#A(\mathbf{F}_q))^n$ for any abelian surface A, the following corollary gives the exact second moment of the number of $\mathbf{F}_q$-points on abelian surfaces and asymptotic estimates on the nth moment for all $n \geq 3$.

Corollary 1.5. The expected value of $\#A^2(\mathbf{F}_q)$ is

$$\mathbf{E}[\#A^2(\mathbf{F}_q)] = q^4 + 3q^3 + 6q^2 + 3q - \frac{5q^2 + 5q + 3}{q^3 + q^2}$$

and for all $n \geq 1$,

$$\mathbf{E}[\#A^n(\mathbf{F}_q)] = q^{2n} + \binom{n+1}{2} q^{2n-1} + \left(\frac{n(n+1)(n^2+n+2)}{8}\right) q^{2n-2} + O(q^{2n-3}).$$

Note that computing the nth moment for large n involves representations attached to (Siegel) modular forms; therefore, the recursive formulas for local systems in the cohomological computations do not completely determine these moments. However for fixed $n \geq 0$, the trace formula does give the exact nth moment of $\#A(\mathbf{F}_q)$ if $H^k(\mathcal{X}^n; \mathbf{Q}_\ell)$ for all $k \geq 0$ are known.

For any abelian surface A and $n > 1$, $\operatorname{Sym}^n A(\mathbf{F}_q)$ denotes the set of $\mathbf{F}_q$-points of the symmetric power $\operatorname{Sym}^n A$ of an abelian surface A, which are the unordered n-tuples of (not necessarily distinct) $\overline{\mathbf{F}}_q$-points of A defined as an n-tuple over $\mathbf{F}_q$. This means that the n-tuple contains all Galois conjugates of each point of the n-tuple. The same methods give the exact expected value of $\#\operatorname{Sym}^2 A(\mathbf{F}_q)$ and an asymptotic estimate for the expected value of $\#\operatorname{Sym}^n A(\mathbf{F}_q)$ for $n > 2$.

Corollary 1.6. The expected value of $\#\operatorname{Sym}^n A(\mathbf{F}_q)$ for $n = 2$ is

$$\mathbf{E}[\#\operatorname{Sym}^2 A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^{\mathrm{Sym}(2)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^4 + 2q^3 + 4q^2 + 2q + 1 - \frac{3q^2 + 2q + 2}{q^3 + q^2}.$$

For $n = 3$,

$$\mathbf{E}[\#\operatorname{Sym}^3 A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^{\mathrm{Sym}(3)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^6 + 2q^5 + O(q^4)$$

and for all $n \geq 4$,

$$\mathbf{E}[\#\operatorname{Sym}^n A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^{\operatorname{Sym}(n)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^{2n} + 2q^{2n-1} + 7q^{2n-2} + O(q^{2n-3}).$$

Because Corollary 1.5 gives the exact second moment of $\#A(\mathbf{F}_q)$, we also obtain the variance:

Corollary 1.7. The variance of $\#A(\mathbf{F}_q)$ is

$$\operatorname{Var}(\#A(\mathbf{F}_q)) = q^3 + 3q^2 + q - 1 - \frac{3q^2 + 3q + 1}{q^3 + q^2} - \frac{1}{(q^3 + q^2)^2}.$$

These statistics are computed in Subsection 6.1 by studying $\#\mathcal{X}(\mathbf{F}_q)$ for various stacks $\mathcal{X}$. All $\mathbf{F}_q$-point counts in this paper are weighted by the inverse of the size of their automorphism groups, as explained in Section 6.

Related work. The studies of the cohomologies of the moduli space of abelian varieties and the universal abelian variety, point counts over finite fields of these varieties, and Siegel modular forms are tightly intertwined. For example, the cohomology of local systems on the moduli space of elliptic curves is known classically (e.g. the Eichler–Shimura isomorphism) and yields connections between modular forms and point counts of elliptic curves over finite fields. (See [vdG13] and [HT18] for a survey on current developments in this area.) In the case of abelian surfaces, Faber and van der Geer first pursued this approach in [FvdG04a, FvdG04b], giving conjectural formulas for the class of the ℓ-adic cohomology of local systems of $\mathcal{A}_2$ in the Grothendieck group of ℓ-adic Galois representations based on computer-generated point counts over finite fields. These conjectures were proven by Weissauer ([Wei09]) in the case of local systems with regular highest weight, and by Petersen ([Pet15]) in the most general case. The connections between these ideas are implicit in our paper; we fully take advantage of the work mentioned above, in both the computations of the cohomology of the relevant spaces and the resulting arithmetic statistics.

Recent work in arithmetic statistics of abelian varieties also take a different flavor than of this paper. Honda–Tate theory has been used to determine some probabilistic data about the group structure of abelian surfaces ([DGS+14]), upper and lower bounds on the number of $\mathbf{F}_q$-points on abelian varieties ([AHL13]), sizes of isogeny classes of abelian surfaces ([XY20]), and others. Certainly, this is not the only current approach in this direction – for example, [CFHS12] takes a heuristic approach to determining the probability that the number of $\mathbf{F}_q$-points on a genus 2 curve is prime.

On the other hand, cohomological methods have been applied to related spaces to deduce arithmetic statistical results or heuristics, such as the number of points on curves of genus g ([AEK+15]) and the average number of points on smooth cubic surfaces ([Das19]). For a survey in the case of counting genus g curves and its connection to the cohomology of the relevant moduli spaces, see [vdG15].

Outline of paper. In Section 2, we give a description of the spaces of study and the cohomological tools used throughout the paper. In Section 3, we prove Theorem 1.1 and in Section 4, we prove Theorem 1.2 and completely work out the cohomology of $\mathcal{X}_2^2$ as an example. In Section 5, we carry out analogous arguments to prove Theorem 1.3 and work out the cohomology of $\mathcal{X}_2^{\operatorname{Sym}(2)}$. In Section 6, we deduce new arithmetic statistics results about abelian surfaces using the previous sections, including Corollaries 1.4, 1.5, 1.6, and 1.7.

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2. $\mathcal{A}_2$, its Cohomology, and Cohomological Tools

In this section, we describe the spaces that we study in this paper and outline the cohomological tools that will be used throughout the paper.

2.1. **Spaces of interest.** Denote the moduli stack of principally polarized abelian surfaces by $\mathcal{A}_2$. For concreteness, we discuss an explicit construction of the set of complex points $\mathcal{A}_2(\mathbf{C})$ of $\mathcal{A}_2$: let $\mathcal{H}_2$ be the Siegel upper half space of degree 2 with the usual action of $\mathrm{Sp}(4,\mathbf{Z})$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (C\tau + D)^{-1}(A\tau + B).$$

To each $\tau \in \mathcal{H}_2$, we can associate a lattice $L_\tau \subseteq \mathbf{C}^2$ and therefore a complex torus A_τ. It turns out that A_τ comes with a natural principal polarization H_τ, making (A_τ, H_τ) into a principally polarized abelian variety. For any τ_1, $\tau_2 \in \mathcal{H}_2$, the abelian surfaces (A_{τ_1}, H_{τ_1}) and (A_{τ_2}, H_{τ_2}) are isomorphic if and only if τ_1 and τ_2 are in the same $\mathrm{Sp}(4,\mathbf{Z})$-orbit.

The action of $\mathrm{Sp}(4,\mathbf{Z})$ on $\mathcal{H}_2$ is not free. For example, $-I_4$ fixes every $\tau \in \mathcal{H}_2$. However, the stabilizer of each point is finite. Therefore, $\mathcal{A}_2(\mathbf{C})$ is the set of points of the orbifold $\mathrm{Sp}(4,\mathbf{Z})\backslash\mathcal{H}_2$, with the underlying analytic space of $\mathrm{Sp}(4,\mathbf{Z})\backslash\mathcal{H}_2$ denoted $(\mathcal{A}_2)^{\mathrm{an}}_{\mathbf{C}}$.

The cohomology of $\mathcal{A}_2$ is known:

Theorem 2.1 ([LW85, Corollary 5.2.3], [vdG13, Section 10])**.**

$$H^k(\mathcal{A}_2;\mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ \mathbf{Q}_\ell(-1) & k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Next, we denote the universal abelian surface by $\mathcal{X}_2$ and give an explicit construction for the complex points $\mathcal{X}_2(\mathbf{C})$ of $\mathcal{X}_2$. Take the action of $\mathrm{Sp}(4,\mathbf{Z}) \ltimes \mathbf{Z}^4$ on $\mathcal{H}_2 \times \mathbf{C}^2$, where $\mathbf{Z}^4$ acts by translation on each $L_\tau \subseteq \mathbf{C}^2$. Then $\mathcal{X}_2(\mathbf{C})$ is the set of points of the orbifold $\mathrm{Sp}(4,\mathbf{Z}) \ltimes \mathbf{Z}^4\backslash\mathcal{H}_2 \times \mathbf{C}^2$ with the underlying analytic space of $(\mathrm{Sp}(4,\mathbf{Z}) \ltimes \mathbf{Z}^4)\backslash(\mathcal{H}_2 \times \mathbf{C}^2)$ denoted $(\mathcal{X}_2)^{\mathrm{an}}_{\mathbf{C}}$. Note that the fiber of the natural projection $\mathcal{X}_2(\mathbf{C}) \to \mathcal{A}_2(\mathbf{C})$ over a point corresponding to the surface A is the set $A(\mathbf{C})$ of $\mathbf{C}$-points of A itself.

The stack $\mathcal{X}_2^n$ is defined to be the fiber product of $\mathcal{X}_2 \to \mathcal{A}_2$ with itself n times and comes with a natural map $\mathcal{X}_2^n \to \mathcal{A}_2$. As before, the group $\mathrm{Sp}(4,\mathbf{Z}) \ltimes (\mathbf{Z}^4)^n$ acts on $\mathcal{H}_2 \times (\mathbf{C}^2)^n$ in the obvious way, and so the set $\mathcal{X}_2^n(\mathbf{C})$ of the $\mathbf{C}$-points of $\mathcal{X}_2^n$ is the set of points of the orbifold $\mathrm{Sp}(4,\mathbf{Z})\ltimes(\mathbf{Z}^4)^n\backslash\mathcal{H}_2\times(\mathbf{C}^2)^n$ with the underlying analytic space $(\mathcal{X}_2^n)^{\mathrm{an}}_{\mathbf{C}}$ given by the usual quotient.

The fiber of $\mathcal{X}_2^n(\mathbf{C}) \to \mathcal{A}_2(\mathbf{C})$ over a point corresponding to the surface A is the set $A^n(\mathbf{C})$ of $\mathbf{C}$-points of the nth power A^n.

Also consider the stack $\mathcal{X}_2^{\mathrm{Sym}(n)}$. Each fiber A^n of the projection morphism $\mathcal{X}_2^n \to \mathcal{A}_2$ has an action of S_n permuting the coordinates, giving a stack-theoretic quotient $\mathcal{X}_2^{\mathrm{Sym}(n)} = [\mathcal{X}_2^n/S_n]$. The fiber of $\mathcal{X}_2^{\mathrm{Sym}(n)} \to \mathcal{A}_2$ of the point corresponding to the abelian surface A is $\mathrm{Sym}^n A$, the stack quotient $[A^n/S_n]$. As usual, there is an underlying analytic space $(\mathcal{X}_2^{\mathrm{Sym}(n)})^{\mathrm{an}}_{\mathbf{C}} = (\mathcal{X}_2^n)^{\mathrm{an}}_{\mathbf{C}}/S_n$.

For any $N \geq 2$ and abelian surface A, let $A[N]$ be the kernel of the multiplication by N map on A. Consider the moduli stack $\mathcal{A}_2[N]$ of principally polarized abelian surfaces with symplectic level N structure, i.e. pairs (A, α) where α is an isomorphism from $A[N]$ to a fixed symplectic module $((\mathbf{Z}/N\mathbf{Z})^4, \langle \cdot, \cdot \rangle)$. Let $\mathrm{Sp}(4, \mathbf{Z})[N] = \ker(\mathrm{Sp}(4, \mathbf{Z}) \to \mathrm{Sp}(4, \mathbf{Z}/N\mathbf{Z}))$. If $N \geq 3$, both $\mathcal{A}_2[N]$ and $\mathcal{X}_2[N]$ are quasiprojective schemes over $\mathbf{Z}[\frac{1}{N}, \zeta_N]$ ([FC90, Chapter IV]).

On the other hand, consider the moduli stack $\mathcal{A}_{2,N}$ of principally polarized abelian surfaces with principal level N structure, i.e. pairs (A, β) where $\beta : A[N] \to (\mathbf{Z}/N\mathbf{Z})^4$ is an isomorphism. For $N \geq 3$, $\mathcal{A}_{2,N}$ and its universal family $\mathcal{X}_{2,N}$ are quasiprojective schemes over $\mathbf{Z}[\frac{1}{N}]$ ([FC90, Chapter I]). This shows that over $\mathbf{Z}[\frac{1}{N}]$ with $N \geq 3$, the stacks $\mathcal{X} = \mathcal{A}_2$, $\mathcal{X}_2^n$, and $\mathcal{X}_2^{\mathrm{Sym}(n)}$ are all finite quotients of quasiprojective schemes (cf. [Ols12, Theorem 2.1.11]). Over characteristic zero, quotient stacks with finite automorphism groups at every point are Deligne–Mumford stacks ([Edi00, Corollary 2.2]). Over positive characteristic, quotient stacks are a priori Artin stacks with a smooth atlas. Because any base change of an étale morphism is étale, any stack $\mathcal{X}$ considered in this paper obtained from a stack over $\mathbf{Z}[\frac{1}{N}]$ via base-change to $\mathbf{F}_q$ (where N and q are coprime) has an étale atlas; therefore, $\mathcal{X}_{\mathbf{F}_q}$ is a Deligne–Mumford stack.

In fact, $\mathcal{A}_2$ and $\mathcal{X}_2^n$ are complements of normal crossing divisors in smooth, proper stacks over $\mathbf{Z}$ (see [FC90, Chapter VI]), making both $\mathcal{A}_2$ and $\mathcal{X}_2^n$ as well as its finite quotient $\mathcal{X}_2^{\mathrm{Sym}(n)}$ smooth stacks over any finite field $\mathbf{F}_q$. Over any field k, there are the following moduli interpretations of the k-points of these stacks. When we write an abelian surface A, we mean A with a principal polarization.

(1) $[\mathcal{A}_2(k)]$ is the set of k-isomorphism classes of abelian surfaces A defined over k.

(2) $[\mathcal{A}_2[N](k)]$ is the set of k-isomorphism classes of pairs (A, α) where $\alpha : A[N] \to (\mathbf{Z}/N\mathbf{Z})^4$ is a symplectic isomorphism,

(3) $[\mathcal{X}_2^n(k)]$ is the set of k-isomorphism classes of pairs (A, p) with $p \in A^n$, defined over k.

(4) $[\mathcal{X}_2^{\mathrm{Sym}(n)}(k)]$ is the set of k-isomorphism classes of pairs (A, p) with $p \in A^n/S_n$, defined over k.

Remark 2.2. We note that for some $\{p_1, \ldots, p_n\} \in (A^n/S_n)(k)$, a lift $(p_1, \ldots, p_n) \in A^n$ may not be a k-point of A^n. For instance, if $\mathrm{Gal}(\overline{k}/k)$ permutes the points $p_1, \ldots, p_n \in A^n$, then $\{p_1, \ldots, p_n\}$ will be a k-point of $\mathrm{Sym}^n A$, but not necessarily of A^n.

Although this is possibly not the most efficient framework, we will access all stacks discussed by taking quotient stacks of the respective quasiprojective schemes throughout this paper in an effort to keep the arguments as concrete as possible.

2.2. **Local systems on $\mathcal{A}_2$.** Representations of $\mathrm{GSp}(4, \mathbf{Q}_\ell)$ give rise to ℓ-adic local systems on $(\mathcal{A}_2)_{\mathbf{Z}[1/\ell]}$ ([FC90, p. 238]). The local systems are considered instead as representations of $\mathrm{Sp}(4, \mathbf{Q}_\ell)$ in [Pet15];

we also study the underlying $\mathrm{Sp}(4,\mathbf{Q}_\ell)$-representations of local systems at various points in this paper. In this section we review the construction of local systems on $(\mathcal{A}_2)_{\mathbf{Z}[1/\ell]}$; the reader may also consult [BFvdG14, Section 4] or [vdG11, Section 4].

By Weyl's construction (see [FH04, Section 17.3]), all irreducible representations of $\mathrm{Sp}(4,\mathbf{Q}_\ell)$ are given in the following way: for any $a \geq b \geq 0$, there is an irreducible representation $W_{a,b}$ with highest weight $aL_1 + bL_2$, using the notation of [FH04, Chapter 17]. In particular, $W_{a,b}$ is a summand of $W_{1,0}^{\otimes(a+b)}$ and is the irreducible representation of highest weight in $\mathrm{Sym}^{a-b}(W_{1,0}) \otimes \mathrm{Sym}^b(\bigwedge^2 W_{1,0})$ by construction where $W_{1,0}$ is the 4-dimensional standard representation of $\mathrm{Sp}(4,\mathbf{Q}_\ell)$. As explained in [FvdG04a, Section 1], we can lift $W_{a,b}$ to a representation of $\mathrm{GSp}(4,\mathbf{Q}_\ell)$ of dominant weight $aL_1 + bL_2 - (a+b)\eta$ where η is the *multiplier representation*, which we denote by $V_{a,b}$. The multiplier representation η is defined as $\eta : \mathrm{GSp}(4,\mathbf{Q}_\ell) \to \mathbf{Q}_\ell^\times$ where for any $M \in \mathrm{GSp}(4,\mathbf{Q}_\ell)$ written as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A, B, C, D \in \mathrm{GL}(2)$, $\eta(M)$ satisfies

$$AD^T - BC^T = \eta(M) I_2.$$

In particular, this makes $V_{1,0}$ the contragredient representation of the standard representation V of $\mathrm{GSp}(4,\mathbf{Q}_\ell)$, i.e. $V_{1,0} \cong V \otimes \eta^{-1}$.

Let $\pi : \mathcal{X}_2 \to \mathcal{A}_2$. By the proper base change theorem ([Mil80, Corollary VI.2.5]), the stalk of $R^1\pi_*\mathbf{Q}_\ell$ at $[A] \in \mathcal{A}_2$ is isomorphic to $H^1(A;\mathbf{Q}_\ell)$. Define $\mathbf{V}_{1,0}$ to be the local system $R^1\pi_*\mathbf{Q}_\ell$. The underlying $\mathrm{GSp}(4,\mathbf{Q}_\ell)$-representation of each stalk of $\mathbf{V}_{1,0}$ is $V_{1,0}$ and $\mathbf{V}_{1,0}$ is a local system equipped with a symplectic pairing

$$\mathbf{V}_{1,0} \wedge \mathbf{V}_{1,0} \to \mathbf{Q}_\ell(-1).$$

Applying Weyl's construction to the local system $\mathbf{V}_{1,0}$ yields local systems $\mathbf{V}_{a,b}$ for all $a \geq b \geq 0$. Each $\mathbf{V}_{a,b}$ is a summand in $\mathbf{V}_{1,0}^{\otimes(a+b)}$, so $\mathbf{V}_{a,b}$ has Hodge weight $a+b$. The underlying $\mathrm{Sp}(4,\mathbf{Q}_\ell)$-representation of $\mathbf{V}_{a,b}$ is $W_{a,b}$ and the underlying $\mathrm{GSp}(4,\mathbf{Q}_\ell)$-representation of $\mathbf{V}_{a,b}$ is $V_{a,b}$.

For all $n \in \mathbf{Z}$, let $\mathbf{V}_{a,b}(n) := \mathbf{V}_{a,b} \otimes \mathbf{Q}_\ell(n)$ be the nth Tate twist of $\mathbf{V}_{a,b}$. Tate twists also correspond to tensoring the local systems with the multiplier representation η, i.e. $\mathbf{V}_{a,b}(n) = \mathbf{V}_{a,b} \otimes \eta^n$.

2.3. **Cohomological tools.** In this subsection, we list the tools we will need in subsequent sections regarding cohomology computations. First, we set some notation used for the remainder of the paper. We will always denote by A an abelian surface. By $H^*(\mathcal{A}_2; H^q(X;\mathbf{Q}_\ell))$ for some morphism $f : \mathcal{Y} \to \mathcal{A}_2$ with a fiber X over some point in $\mathcal{A}_2$ where $R^q f_*\mathbf{Q}_\ell$ is locally constant, we will always mean the cohomology of $R^q f_*\mathbf{Q}_\ell$ (see the proof of Proposition 2.8). The prime ℓ will always be taken to be coprime to q when working with the base change $\mathcal{X}_{\overline{\mathbf{F}}_q}$.

Remark 2.3. Let $\mathcal{X} = \mathcal{A}_2$ or $\mathcal{X}_2^n$ and let $\mathbf{V}$ be an ℓ-adic local system on $\mathcal{X}$. Since $\mathcal{X}$ is a complement of a normal crossing divisor of a smooth, proper stack over $\mathbf{Z}$ ([FC90, Chapter VI]), $H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{Q}}};\mathbf{V})$ is unramified at every prime $p \neq \ell$ ([Pet15, p. 11]), i.e. the action of Frob_p is well-defined. There is an isomorphism

$$H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{Q}}};\mathbf{V}) \cong H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{F}}_p};\mathbf{V})$$

such that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the left side factors through the surjection $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, where $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ acts on the right side.

To obtain the analogous isomorphism for $\mathcal{X}_2^{\mathrm{Sym}(n)}$, we recall that $\mathcal{X}_2^{\mathrm{Sym}(n)}$ is a quotient stack of a scheme $\mathcal{X}_{2,N}^n$ of some finite group G. We now study the Hochschild–Serre spectral sequence for the quotient $\mathcal{X}_{2,N}^n \to \mathcal{X}_2^{\mathrm{Sym}(n)}$, where $\mathcal{X}_{2,N}^n$ is a quasiprojective scheme which is a complement of a normal crossing divisor of a smooth, proper scheme over $\mathbf{Z}$ ([FC90, Chapter VI]). These spectral sequences are given by

$$E_2^{p,q} = H^p(G; H^q((\mathcal{X}_{2,N}^n)_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)) \implies H^{p+q}((\mathcal{X}_2^{\mathrm{Sym}(n)})_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell),$$
$$E_2^{p,q} = H^p(G; H^q((\mathcal{X}_{2,N}^n)_{\overline{\mathbf{F}}_p}; \mathbf{Q}_\ell)) \implies H^{p+q}((\mathcal{X}_2^{\mathrm{Sym}(n)})_{\overline{\mathbf{F}}_p}; \mathbf{Q}_\ell).$$

Therefore up to semi-simplification, there is the analogous isomorphism

$$H^*_{\text{ét}}((\mathcal{X}_2^{\mathrm{Sym}(n)})_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell) \cong H^*_{\text{ét}}(\mathcal{X}_{2,N}^n)_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)^G \cong H^*_{\text{ét}}(\mathcal{X}_{2,N}^n)_{\overline{\mathbf{F}}_p}; \mathbf{Q}_\ell)^G \cong H^*_{\text{ét}}((\mathcal{X}_2^{\mathrm{Sym}(n)})_{\overline{\mathbf{F}}_p}; \mathbf{Q}_\ell).$$

As stated in the Introduction, we write $H^*(\mathcal{X}; \mathbf{V})$ to denote both $H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{Q}}}; \mathbf{V})$ and $H^*_{\text{ét}}(\mathcal{X}_{\overline{\mathbf{F}}_q}; \mathbf{V})$ for any ℓ-adic local system $\mathbf{V}$ with ℓ coprime to q. As such, all ℓ-adic local systems $\mathbf{V}$ in this paper are local systems on $(\mathcal{A}_2)_{\overline{\mathbf{Q}}}$ or $(\mathcal{A}_2)_{\overline{\mathbf{F}}_q}$.

The next two statements are well-known for all $g \geq 1$ but we specialize to the case $g = 2$. Both theorems (for general $g \geq 1$) can be found in [HT18].

Theorem 2.4 (Poincaré Duality for $\mathcal{A}_2$)**.** For any $a \geq b \geq 0$,

$$H_c^k(\mathcal{A}_2; \mathbf{V}_{a,b}) \cong H^{6-k}(\mathcal{A}_2; \mathbf{V}_{a,b})^* \otimes \mathbf{Q}_\ell\,(-3-a-b)\,.$$

(Recall that $\dim \mathcal{A}_2 = 3$.)

Theorem 2.5 (Deligne's weight bounds)**.** The mixed Hodge structures on the groups $H^k(\mathcal{A}_2; \mathbf{V}_{a,b})$ have weights larger than or equal to $k + a + b$.

In Sections 3, 4, and 5, we compute the étale cohomology $H^*(\mathcal{X}; \mathbf{Q}_\ell)$, with $\mathcal{X} = \mathcal{X}_2$, $\mathcal{X}_2^n$, and $\mathcal{X}_2^{\mathrm{Sym}(n)}$ respectively. In all of these cases, there are morphisms $\pi : \mathcal{X} \to \mathcal{A}_2$ to which we want to apply the Leray spectral sequence to obtain the desired results.

Theorem 2.6 ([Del68])**.** Let $f : X \to Y$ be a smooth projective morphism of complex varieties. Then the Leray spectral sequence for f degenerates on the E_2-page.

For $N \geq 3$, the projection $\pi : \mathcal{X}_2[N]^n_{\mathbf{C}} \to \mathcal{A}_2[N]_{\mathbf{C}}$ is a projective morphism of quasi-projective varieties. Combined with a corollary of the proper base change theorem ([Mil80, Corollary VI.4.3]), this implies the following useful result:

Corollary 2.7. For all $n \geq 1$, $N \geq 3$, the Leray spectral sequence for $\pi : \mathcal{X}_2[N]^n_{\overline{\mathbf{Q}}} \to \mathcal{A}_2[N]_{\overline{\mathbf{Q}}}$ degenerates on the E_2-page.

Finally, we state the main tool of this paper. The following proposition gives a Leray spectral sequence for each morphism of stacks $\pi : \mathcal{X} \to \mathcal{A}_2$ with $\mathcal{X} = \mathcal{X}_2^n$ or $\mathcal{X}_2^{\mathrm{Sym}(n)}$ using the Leray spectral sequence for schemes in étale cohomology.

Proposition 2.8. Let $\mathcal{X} = \mathcal{X}_2^n$ (resp. $\mathcal{X}_2^{\mathrm{Sym}(n)}$) and $\pi : \mathcal{X} \to \mathcal{A}_2$. There is a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{A}_2; H^q(Z_n; \mathbf{Q}_\ell)) \implies H^{p+q}(\mathcal{X}; \mathbf{Q}_\ell)$$

with $Z_n = A^n$ (resp. $Z_n = \mathrm{Sym}^n A$), which degenerates on the E_2-page.

Proof. Let $N \geq 3$ and let $\mathcal{X}_2[N]^n$ be the nth fiber power of $\mathcal{X}_2[N]$ over $\mathcal{A}_2[N]$ with respect to the projection map $\mathcal{X}_2[N] \to \mathcal{A}_2[N]$. Then $\mathcal{A}_2[N]$ and $\mathcal{X}_2[N]^n$ are quasi-projective schemes. By the standard Leray spectral sequence for étale cohomology ([Mil08b, Theorem 12.7]) with $\pi_N : \mathcal{X}_2[N]^n_{\overline{\mathbf{Q}}} \to \mathcal{A}_2[N]_{\overline{\mathbf{Q}}}$,

$$H^p(\mathcal{A}_2[N]_{\overline{\mathbf{Q}}}; R^q(\pi_N)_*\mathbf{Q}_\ell) \implies H^{p+q}(\mathcal{X}_2[N]^n_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)$$

and this spectral sequence degenerates on the E_2-page by Corollary 2.7. Applying a corollary of the proper base change theorem ([Mil80, Corollary VI.2.5]) with the torsion (constant) sheaf $\mathbf{Z}/\ell^m\mathbf{Z}$, taking inverse limits, and tensoring with $\mathbf{Q}_\ell$,

$$\bigoplus_{p+q=k} H^p(\mathcal{A}_2[N]_{\overline{\mathbf{Q}}}; H^q(A^n; \mathbf{Q}_\ell)) \cong H^k(\mathcal{X}_2[N]^n_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)$$

as Galois representations up to semi-simplification. Then by the Hochschild–Serre spectral sequence for the $\mathrm{Sp}(4, \mathbf{Z}/N\mathbf{Z})$-quotient $\mathcal{X}_2[N]^n_{\overline{\mathbf{Q}}} \to (\mathcal{X}_2^n)_{\overline{\mathbf{Q}}}$, where $\mathrm{Sp}(4, \mathbf{Z}/N\mathbf{Z})$ acts diagonally on $\mathcal{X}_2^n[N]$, and the $\mathrm{Sp}(4, \mathbf{Z}/N\mathbf{Z})$-quotient $\mathcal{A}_2[N]_{\overline{\mathbf{Q}}} \to (\mathcal{A}_2)_{\overline{\mathbf{Q}}}$,

$$\bigoplus_{p+q=k} H^p(\mathcal{A}_2[N]_{\overline{\mathbf{Q}}}; H^q(A^n; \mathbf{Q}_\ell))^{\mathrm{Sp}(4,\mathbf{Z}/N\mathbf{Z})} \cong H^k(\mathcal{X}_2[N]^n_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)^{\mathrm{Sp}(4,\mathbf{Z}/N\mathbf{Z})} \cong H^k((\mathcal{X}_2^n)_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)$$

and

$$H^p(\mathcal{A}_2[N]_{\overline{\mathbf{Q}}}; H^q(A^n; \mathbf{Q}_\ell))^{\mathrm{Sp}(4,\mathbf{Z}/N\mathbf{Z})} \cong H^p((\mathcal{A}_2)_{\overline{\mathbf{Q}}}; H^q(A^n; \mathbf{Q}_\ell)),$$

where on the left, $H^q(A^n)$ is the local system corresponding to the respective $\mathrm{Sp}(4, \mathbf{Z})[N]$-representation, while on the right, $H^q(A^n)$ is the local system corresponding to the respective $\mathrm{Sp}(4, \mathbf{Z})$-representation. Therefore, taking $\mathrm{Sp}(4, \mathbf{Z}/N\mathbf{Z})$-invariants in the spectral sequence for π_N, which one can do by naturality of that sequence, gives the following E_2-page of a spectral sequence

$$E_2^{p,q} = H^p((\mathcal{A}_2)_{\overline{\mathbf{Q}}}; H^q(A^n; \mathbf{Q}_\ell)) \implies H^{p+q}((\mathcal{X}_2^n)_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell)$$

degenerating on the E_2-page. By Remark 2.3, the spectral sequence for $(\mathcal{X}_2^n)_{\overline{\mathbf{F}}_q} \to (\mathcal{A}_2)_{\overline{\mathbf{F}}_q}$ must also degenerate on the E_2-page. Therefore, we have now proven the Proposition for $\mathcal{X} = (\mathcal{X}_2^n)_{\overline{\mathbf{Q}}}$ and $(\mathcal{X}_2^n)_{\overline{\mathbf{F}}_q}$. Lastly, again by the Hochschild–Serre spectral sequence,

$$H^k(\mathcal{X}_2^{\mathrm{Sym}(n)}; \mathbf{Q}_\ell) \cong H^k(\mathcal{X}_2^n; \mathbf{Q}_\ell)^{S_n}.$$

Because S_n acts trivially on $\mathcal{A}_2$,

$$H^p((\mathcal{A}_2)_{\overline{\mathbf{Q}}}; H^q(A^n))^{S_n} \cong H^p((\mathcal{A}_2)_{\overline{\mathbf{Q}}}; H^q(A^n)^{S_n}) \cong H^p((\mathcal{A}_2)_{\overline{\mathbf{Q}}}; H^q(\mathrm{Sym}^n A))$$

and

$$H^p((\mathcal{A}_2)_{\overline{\mathbf{F}}_q}; H^q(A^n))^{S_n} \cong H^p((\mathcal{A}_2)_{\overline{\mathbf{F}}_q}; H^q(A^n)^{S_n}) \cong H^p((\mathcal{A}_2)_{\overline{\mathbf{F}}_q}; H^q(\mathrm{Sym}^n A))$$

where $H^q(\mathrm{Sym}^n A)$ is again an $\mathrm{Sp}(4, \mathbf{Z})$-representation. Again by naturality, taking S_n-invariants in the spectral sequence for $(\mathcal{X}_2^n)_{\overline{\mathbf{Q}}} \to (\mathcal{A}_2)_{\overline{\mathbf{Q}}}$ and $(\mathcal{X}_2^n)_{\overline{\mathbf{F}}_q} \to (\mathcal{A}_2)_{\overline{\mathbf{F}}_q}$ gives

$$E_2^{p,q} = H^p((\mathcal{A}_2)_{\overline{\mathbf{Q}}}; H^q(\mathrm{Sym}^n A)) \implies H^{p+q}((\mathcal{X}_2^{\mathrm{Sym}(n)})_{\overline{\mathbf{Q}}}; \mathbf{Q}_\ell),$$
$$E_2^{p,q} = H^p((\mathcal{A}_2)_{\overline{\mathbf{F}}_q}; H^q(\mathrm{Sym}^n A)) \implies H^{p+q}((\mathcal{X}_2^{\mathrm{Sym}(n)})_{\overline{\mathbf{F}}_q}; \mathbf{Q}_\ell)$$

which both degenerate on the E_2-page. □

3. Cohomology of the Universal Abelian Surface

In this section, we study the cohomology of $\mathcal{X}_2$ using $\pi : \mathcal{X}_2 \to \mathcal{A}_2$. We first need to compute the following local systems.

Lemma 3.1. There are isomorphisms of local systems

$$H^k(A; \mathbf{Q}_\ell) \cong \begin{cases} \mathbf{Q}_\ell & k = 0 \\ \mathbf{V}_{1,0} & k = 1 \\ \mathbf{Q}_\ell(-1) \oplus \mathbf{V}_{1,1} & k = 2 \\ \mathbf{V}_{1,0}(-1) & k = 3 \\ \mathbf{Q}_\ell(-2) & k = 4 \\ 0 & k > 4. \end{cases}$$

Proof. By [FC90, p. 238], smooth ℓ-adic sheaves on $\mathcal{A}_2$ correspond to continuous representations of the arithmetic fundamental group $\pi_1(\mathcal{A}_2)$ of $\mathcal{A}_2$ after choosing a base point; we can view $\mathrm{GSp}(4)$ as the arithmetic fundamental group of $\mathcal{A}_2$ after a choice of base point (cf. [vdG11, p. 6]). For any abelian surface A, there is an isomorphism $\bigwedge^k H^1(A; \mathbf{Q}_\ell) \to H^k(A; \mathbf{Q}_\ell)$ for all $k \geq 0$ given by the cup-product pairing by [Mil08a, Theorem 12.1]. Therefore, there is an isomorphism of local systems between $H^k(A; \mathbf{Q}_\ell)$ and $\bigwedge^k \mathbf{V}_{1,0}$, the local system corresponding to the $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation $\bigwedge^k V_{1,0}$. We deccompose the local system $\bigwedge^k \mathbf{V}_{1,0}$ into a direct sum of local systems of the form $\mathbf{V}_{a,b}(n)$ with $a \geq b \geq 0$ and $n \in \mathbf{Z}$ corresponding to irreducible $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representations.

We first consider the decomposition of the $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation $\bigwedge^k V_{1,0}$ into irreducible $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representations, where $V_{a,b}$ denotes the irreducible $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation corresponding to the partition $a \geq b \geq 0$ as explained in Section 2.2. Recall also that $W_{a,b}$ denotes the irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation corresponding to the partition $a \geq b \geq 0$. The decomposition of $\bigwedge^k W_{1,0}$ as $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation is

$$\bigwedge^k W_{1,0} \cong \begin{cases} W_{0,0} & k = 0, 4, \\ W_{1,0} & k = 1, 3, \\ W_{0,0} \oplus W_{1,1} & k = 2 \end{cases}$$

as given in [FH04, Chapter 16].

The $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation decomposition above determines the corresponding $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation $\bigwedge^k V_{1,0}$ up to tensoring by the multiplier representation η (see Section 2.2). In particular, this means that if $W_{a,b} \subseteq \bigwedge^k W_{1,0}$ for some $a \geq b \geq 0$ then there exists some $q \in \mathbf{Z}$ such that $V_{a,b} \otimes \eta^q \subseteq \bigwedge^k V_{1,0}$. To determine the q for each summand $V_{a,b} \otimes \eta^q \subseteq \bigwedge^k V_{1,0}$, it will suffice to consider the action of all scalar matrices $mI_4 \in \mathrm{GSp}(4, \mathbf{Q}_\ell)$ on $\bigwedge^k V_{1,0}$.

Recall that $V_{1,0}$ is the contragredient of the standard representation as a $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation V, i.e. $V_{1,0} \cong V \otimes \eta^{-1}$. Apply the definition of the multiplier representation from Section 2.2 to see that $\eta(mI_4) = m^2$. For any $v \in V_{a,b} \subseteq V_{1,0}^{\otimes(a+b)} = V^{\otimes(a+b)} \otimes \eta^{-(a+b)}$,

$$mI_4 \cdot v = (m^{(a+b)} v) \cdot m^{-2(a+b)} = m^{-(a+b)} v$$

and so $mI_4 \cdot w = m^{-(a+b)+2q} w$ for any $w \in V_{a,b} \otimes \eta^q \subseteq V_{1,0}^{\otimes(a+b)} \otimes \eta^q$. On the other hand, for any $w \in V_{a,b} \otimes \eta^q \subseteq \bigwedge^k V_{1,0} \subseteq V_{1,0}^{\otimes k}$,

$$mI_4 \cdot w = (m^k w) \cdot m^{-2k} = m^{-k} w.$$

This shows that $m^{-(a+b)+2q} w = m^{-k} w$, which implies that $-k = -(a+b) + 2q$ and so $q = \frac{(a+b)-k}{2}$. Therefore as $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representations,

$$\bigwedge^k V_{1,0} \cong \begin{cases} V_{0,0} & k = 0, \\ V_{1,0} & k = 1, \\ V_{0,0} \otimes \eta^{-1} \oplus V_{1,1} & k = 2, \\ V_{1,0} \otimes \eta^{-1} & k = 3, \\ V_{0,0} \otimes \eta^{-2} & k = 4, \\ 0 & k > 4. \end{cases}$$

Tensoring by $\eta(q)$ corresponds to tensoring with $\mathbf{Q}_\ell(q)$ for all $n \in \mathbf{Z}$ (see Section 2.2). Rewriting the above decomposition of $\bigwedge^k V_{1,0}$ in terms of local systems proves this lemma. □

Our main tool is [Pet15, Theorem 2.1], restated below for convenience. Before we do so, we need to establish some notation, which agrees with that of [Pet15].

Let s_k be the dimension of the space of cusp forms of $\mathrm{SL}(2, \mathbf{Z})$ of weight k. For $j \geq 0$ and $k \geq 3$, let $s_{j,k}$ be the dimension of the space of vector-valued Siegel cusp forms for $\mathrm{Sp}(4, \mathbf{Z})$ transforming according to the representation $\mathrm{Sym}^j \otimes \det^k$. Let ρ_f be the 2-dimensional ℓ-adic Galois representation of weight $k-1$ of the normalized cusp eigenform f for $\mathrm{SL}(2, \mathbf{Z})$, as given by [Del69], and let $S_k = \bigoplus_f \rho_f$ be the direct sum of such representations for k. Let τ_f be the 4-dimensional ℓ-adic Galois representation of the vector-valued Siegel cusp eigenform f of type $\mathrm{Sym}^j \otimes \det^k$ as given by [Wei05], and let $S_{j,k} = \bigoplus_f \tau_f$. Let s'_k be the number of normalized cusp eigenforms of weight k for $\mathrm{SL}(2, \mathbf{Z})$ for which $L(f, \frac{1}{2})$ vanishes.

Finally, we need to describe the Galois representations $\overline{S}_{j,k}$. These representations satisfy the condition that $\overline{S}_{j,k} = S_{j,k}$ if $j \neq 0$ or $k \equiv 1 \pmod 2$. Otherwise, $\overline{S}_{j,k}$ is a subrepresentation of $S_{j,k}$ that can be determined in a prescribed way. Because the only property of $\overline{S}_{j,k}$ we will use in this paper is that it is a subrepresentation of $S_{j,k}$, we refer the reader to [Pet15, p. 3] for the specific definition.

Because every abelian surface has an involution which acts by multiplication by $(-1)^k$ on each stalk of $\mathbf{V}_{1,0}^{\otimes k}$, and each $\mathbf{V}_{a,b}$ is a summand of $\mathbf{V}_{1,0}^{\otimes(a+b)}$, the cohomology $H^p(\mathcal{A}_2; \mathbf{V}_{a,b})$ vanishes if $a+b$ is odd. (See [FvdG04a, §1] or [Pet15, p. 3].)

Theorem 3.2 (Petersen, [Pet15, Theorem 2.1])**.** Suppose $(a, b) \neq (0, 0)$, and that $a+b$ is even. Then

(1) In degrees $k \neq 2, 3, 4$,

$$H_c^k(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0.$$

(2) In degree 4,

$$H_c^4(\mathcal{A}_2; \mathbf{V}_{a,b}) = \begin{cases} s_{a+b+4}\mathbf{Q}_\ell(-b-2) & a = b \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

(3) In degree 3, up to semi-simplification,

$$\begin{aligned} H^3_c(\mathcal{A}_2;\mathbf{V}_{a,b}) =& \overline{S}_{a-b,b+3} \oplus s_{a+b+4}S_{a-b+2}(-b-1) \oplus S_{a+3} \\ &\oplus \begin{cases} s'_{a+b+4}\mathbf{Q}_\ell(-b-1) & a=b \text{ even} \\ s_{a+b+4}\mathbf{Q}_\ell(-b-1) & \text{otherwise} \end{cases} \\ &\oplus \begin{cases} \mathbf{Q}_\ell & a=b \text{ odd} \\ 0 & \text{otherwise} \end{cases} \\ &\oplus \begin{cases} \mathbf{Q}_\ell(-1) & b=0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(4) In degree 2, up to semi-simplification,

$$\begin{aligned} H^2_c(\mathcal{A}_2;\mathbf{V}_{a,b}) =& S_{b+2} \oplus s_{a-b+2}\mathbf{Q}_\ell \\ &\oplus \begin{cases} s'_{a+b+4}\mathbf{Q}_\ell(-b-1) & a=b \text{ even} \\ 0 & \text{otherwise} \end{cases} \\ &\oplus \begin{cases} \mathbf{Q}_\ell & a>b>0 \text{ and } a,b \text{ even} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 3.3. Although we do not need the full power of Theorem 3.2 to compute $H^*(\mathcal{X}_2;\mathbf{Q}_\ell)$, the existence of such a result is important for the calculations in Sections 4 and 5.

We now give examples of computations using Theorem 3.2. The next corollary gives all applications of this theorem that we explicitly use in the rest of the paper. All results are up to semi-simplification.

Corollary 3.4. For all integers $k \geq 0$,

(1) $$H^1(\mathcal{A}_2;\mathbf{V}_{a,b}) = 0 \qquad \text{for all } a \geq b \geq 0,$$

(2) $$H^k(\mathcal{A}_2;\mathbf{V}_{1,1}) = \begin{cases} \mathbf{Q}_\ell(-5) & k=3 \\ 0 & \text{otherwise,} \end{cases}$$

(3) $$H^k(\mathcal{A}_2;\mathbf{V}_{2,0}) = \begin{cases} \mathbf{Q}_\ell(-4) & k=3 \\ 0 & \text{otherwise,} \end{cases}$$

(4) $$H^k(\mathcal{A}_2;\mathbf{V}_{2,2}) = 0 \qquad \text{for all } k \geq 0.$$

Proof. The smallest weight possible for nonzero cusp forms of $\mathrm{SL}(2,\mathbf{Z})$ is 12. (For example, see [Ser73, Theorem 7.4].) Thus $s'_k = s_k = 0$ and $S_k = 0$ for all $k < 12$.

By Theorem 3.2,

$$\begin{aligned}
&H^5_c(\mathcal{A}_2;\mathbf{V}_{a,b}) = 0 \qquad \text{for all } a \geq b \geq 0,\\
&H^4_c(\mathcal{A}_2;\mathbf{V}_{1,1}) = 0,\\
&H^3_c(\mathcal{A}_2;\mathbf{V}_{1,1}) = \overline{S}_{0,4} \oplus s_6 S_2(-2) \oplus S_4 \oplus s_6\mathbf{Q}_\ell(-2) \oplus \mathbf{Q}_\ell = \overline{S}_{0,4} \oplus \mathbf{Q}_\ell,\\
&H^2_c(\mathcal{A}_2;\mathbf{V}_{1,1}) = S_3 \oplus s_2\mathbf{Q}_\ell = 0,\\
&H^4_c(\mathcal{A}_2;\mathbf{V}_{2,0}) = 0,\\
&H^3_c(\mathcal{A}_2;\mathbf{V}_{2,0}) = \overline{S}_{2,3} \oplus s_6 S_4(-1) \oplus S_5 \oplus s_6\mathbf{Q}_\ell(-1) \oplus \mathbf{Q}_\ell(-1) = \overline{S}_{2,3} \oplus \mathbf{Q}_\ell(-1),\\
&H^2_c(\mathcal{A}_2;\mathbf{V}_{2,0}) = S_2 \oplus s_4\mathbf{Q}_\ell = 0,\\
&H^4_c(\mathcal{A}_2;\mathbf{V}_{2,2}) = s_8\mathbf{Q}_\ell(-4) = 0,\\
&H^3_c(\mathcal{A}_2;\mathbf{V}_{2,2}) = \overline{S}_{0,5} \oplus s_8 S_2(-3) \oplus S_5 \oplus s'_8\mathbf{Q}_\ell(-3) = \overline{S}_{0,5},\\
&H^2_c(\mathcal{A}_2;\mathbf{V}_{2,2}) = S_4 \oplus s_2\mathbf{Q}_\ell \oplus s'_8\mathbf{Q}_\ell(-3) = 0.
\end{aligned}$$

By computations in [Wak12, p. 249],

$$\begin{aligned}
\overline{S}_{0,4} &\subseteq S_{0,4} = 0,\\
\overline{S}_{0,5} &\subseteq S_{0,5} = 0.
\end{aligned}$$

By [Ibu12, Lemma 2.1], $S_{j,k} = 0$ for all $0 \leq k \leq 4$ and $j \leq 14$, and so

$$\overline{S}_{2,3} \subseteq S_{2,3} = 0.$$

Finally, by Poincaré Duality (Theorem 2.4),

$$\begin{aligned}
&H^1(\mathcal{A}_2;\mathbf{V}_{a,b})^* \cong H^5_c(\mathcal{A}_2;\mathbf{V}_{a,b}) \otimes \mathbf{Q}_\ell(3+a+b) = 0 \qquad \text{for all } a \geq b \geq 0,\\
&H^k(\mathcal{A}_2;\mathbf{V}_{1,1})^* \cong H^{6-k}_c(\mathcal{A}_2;\mathbf{V}_{1,1}) \otimes \mathbf{Q}_\ell(5) = \begin{cases} \mathbf{Q}_\ell(5) & k=3\\ 0 & \text{otherwise,}\end{cases}\\
&H^k(\mathcal{A}_2;\mathbf{V}_{2,0})^* \cong H^{6-k}_c(\mathcal{A}_2;\mathbf{V}_{2,0}) \otimes \mathbf{Q}_\ell(5) = \begin{cases} \mathbf{Q}_\ell(4) & k=3\\ 0 & \text{otherwise,}\end{cases}\\
&H^k(\mathcal{A}_2;\mathbf{V}_{2,2})^* \cong H^{6-k}_c(\mathcal{A}_2;\mathbf{V}_{2,2}) \otimes \mathbf{Q}_\ell(7) = 0 \qquad \text{for all } k \geq 0.
\end{aligned}$$

□

With the above preliminaries in hand, we can now prove our first main result, Theorem 1.1. We restate the theorem here for convenience.

Theorem 1.1. The cohomology of the universal abelian surface $\mathcal{X}_2$ is given by

$$H^k(\mathcal{X}_2;\mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k=0\\ 0 & k=1,3,k\geq 7\\ 2\mathbf{Q}_\ell(-1) & k=2\\ 2\mathbf{Q}_\ell(-2) & k=4\\ \mathbf{Q}_\ell(-5) & k=5\\ \mathbf{Q}_\ell(-3) & k=6\end{cases}$$

up to semi-simplification.

$q \backslash p$	0	1	2	3
4	$\mathbf{Q}_\ell(-2)$		$\mathbf{Q}_\ell(-3)$	
3				
2	$\mathbf{Q}_\ell(-1)$		$\mathbf{Q}_\ell(-2)$	$\mathbf{Q}_\ell(-5)$
1				
0	$\mathbf{Q}_\ell$		$\mathbf{Q}_\ell(-1)$	

FIGURE 1. The nonzero terms on the E_2-page of the Leray spectral sequence for $\pi : \mathcal{X}_2 \to \mathcal{A}_2$.

Proof. By Proposition 2.8, there is a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{A}_2; H^q(A; \mathbf{Q}_\ell)) \implies H^{p+q}(\mathcal{X}_2; \mathbf{Q}_\ell)$$

which degenerates on the E_2-page. Combining Lemma 3.1, Theorem 2.1, and Corollary 3.4 shows that the E_2-page is as in Figure 1. The theorem now follows directly, since up to semi-simplification,

$$H^k(\mathcal{X}_2; \mathbf{Q}_\ell) = \bigoplus_{p+q=k} E_2^{p,q}. \qquad \square$$

4. COHOMOLOGY OF FIBER POWERS OF THE UNIVERSAL ABELIAN SURFACE

In this section, we compute $H^*(\mathcal{X}_2^n; \mathbf{Q}_\ell)$. In particular, we give a procedure for computing this for general n, and then give the specific results that follow for the case $n = 2$. For brevity, we omit the coefficients when it is clear that we mean the constant ones and write $H^*(X)$ to mean $H^*(X; \mathbf{Q}_\ell)$.

4.1. **Computations for general** n**.** Let $\pi : \mathcal{X}_2 \to \mathcal{A}_2$ and let $\pi^n : \mathcal{X}_2^n \to \mathcal{A}_2$. We first need to consider the following local systems.

Lemma 4.1. There are isomorphisms of local systems

$$\begin{aligned} H^k(A^n) &\cong \bigoplus_{\sum_{i=1}^n k_i = k} \left(\bigotimes_i H^{k_i}(A) \right) \\ &\cong H^k(A^{n-1}) \oplus \left(\mathbf{V}_{1,0} \otimes H^{k-1}(A^{n-1})\right) \oplus \left((\mathbf{Q}_\ell(-1) \oplus \mathbf{V}_{1,1}) \otimes H^{k-2}(A^{n-1})\right) \\ &\quad \oplus \left(\mathbf{V}_{1,0}(-1) \otimes H^{k-3}(A^{n-1})\right) \oplus H^{k-4}(A^{n-1})(-2) \end{aligned}$$

where we say $H^k(A^{n-1}) = 0$ if $k < 0$. For appropriate constants $m_{a,b}(H^k(A^n))$,

$$H^k(A^n) \cong \bigoplus_{a \geq b \geq 0} m_{a,b}(H^k(A^n)) \mathbf{V}_{a,b} \left(\frac{a+b-k}{2} \right). \tag{5}$$

Proof. The Künneth isomorphism applied to the local system $H^k(A^n)$ says

$$H^k(A^n) \cong \bigoplus_{k_1+k_2=k} H^{k_1}(A^{n-1}) \otimes H^{k_2}(A)$$
$$\cong \bigoplus_{k_2=0}^{4} H^{k_2}(A) \otimes H^{k-k_2}(A^{n-1}),$$

from which the first half of the lemma follows from Lemma 3.1.

For the third isomorphism, the same proof as that of Lemma 3.1 applies as follows. As explained in Section 2.2, $W_{a,b}$ denotes the irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation corresponding to the partition $a \geq b \geq 0$ and $V_{a,b}$ denotes an irreducible $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation whose underlying $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation is $W_{a,b}$. For each summand in the first isomorphism, suppose as $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations

$$W_{a,b} \subseteq \bigotimes_i H^{k_i}(A) \cong \bigotimes_i \left(\bigwedge^{k_i} H^1(A) \right)$$

which implies that for some $q \in \mathbf{Z}$, $V_{a,b} \otimes \eta^q \subseteq \bigotimes_i \left(\bigwedge^{k_i} V_{1,0} \right) \subseteq V_{1,0}^{\otimes k}$ as $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representations. Exactly the same computation as in the proof of Lemma 3.1 by applying scalar matrices $mI_4 \in \mathrm{GSp}(4, \mathbf{Q}_\ell)$ to both $V_{1,0}^{\otimes k}$ and $V_{a,b} \otimes \eta^q$ shows that $q = \frac{a+b-k}{2}$. Finally, note that tensoring with η corresponds to Tate twists, which proves the last isomorphism. $\square$

In order to apply Theorem 3.2 like in Section 3, we need to decompose each $H^k(A^n; \mathbf{Q}_\ell)$ into irreducible representations of $\mathrm{Sp}(4, \mathbf{Q}_\ell)$. Consider the restriction of the irreducible $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation $V_{a,b}$ (corresponding to the partition $a \geq b \geq 0$) to $\mathrm{Sp}(4, \mathbf{Q}_\ell)$; as in Section 2.2, we denote this irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation by $W_{a,b}$. We account for the $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representation structures at the end by using Lemma 4.1(5). We set $W_{a,b} := 0$ if $a < b$ or $b < 0$. Lemma 4.1 gives rise to a recursive computation (in n) for this decomposition to which we will apply the following two lemmas:

Lemma 4.2. For $a \geq b \geq 0$,

$$W_{1,0} \otimes W_{a,b} = W_{a-1,b} \oplus W_{a+1,b} \oplus W_{a,b-1} \oplus W_{a,b+1}.$$

Lemma 4.3. If $a > b \geq 0$, then

$$W_{1,1} \otimes W_{a,b} = W_{a-1,b-1} \oplus W_{a-1,b+1} \oplus W_{a,b} \oplus W_{a+1,b-1} \oplus W_{a+1,b+1}.$$

If $a \geq 0$, then

$$W_{1,1} \otimes W_{a,a} = W_{a-1,a-1} \oplus W_{a+1,a-1} \oplus W_{a+1,a+1}.$$

The proofs of Lemmas 4.2 and 4.3 apply combinatorial theorems to decompose tensor products of irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations into a direct sum of irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations. We summarize the necessary combinatorial results and prove Lemmas 4.2 and 4.3 in Appendix A; the rest of this paper is independent of the content of Appendix A.

We are now able to give a recursive formula (in n) for the multiplicity of a given $W_{a,b}$ in $H^k(A^n; \mathbf{Q}_\ell)$ as $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations. We do so in pieces after establishing some notation.

Definition 4.4. For any $(a, b) \in \mathbf{N}^2$ and any $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation V, let

$$m_{a,b}(V) = \langle W_{a,b}, V \rangle.$$

Recall that if $a < b$ or $b < 0$, then $W_{a,b} = 0$ so $m_{a,b}(V) = 0$ for all V.

Remark 4.5. Lemma 3.1 determines $m_{a,b}(H^q(A))$ for all $a \geq b \geq 0$ and all $q \geq 0$.

Definition 4.6. For any set S, denote the indicator function by $\mathbf{1}_S(a,b)$ with

$$\mathbf{1}_S(a,b) = \begin{cases} 1 & (a,b) \in S \\ 0 & (a,b) \notin S. \end{cases}$$

The following two lemmas establish formulas that are necessary to give a recursive formula for $m_{a,b}(H^k(A^n))$ in n. The proofs are completely straight-forward but are included for completeness.

Lemma 4.7. For any $a \geq b \geq 0$, $k \geq 0$, and $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation V,

$$\langle W_{a,b}, W_{1,0} \otimes V \rangle = m_{a+1,b}(V) + m_{a,b+1}(V) + m_{a-1,b}(V) + m_{a,b-1}(V).$$

Proof. By Section 2.2, V is a direct sum of irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations, which are all of the form $W_{A,B}$ for some $A \geq B \geq 0$. Let $I(a,b) = \{(a+1,b), (a,b+1), (a-1,b), (a,b-1)\}$. By Lemma 4.2,

$$\begin{aligned} \langle W_{a,b}, W_{1,0} \otimes V \rangle &= \left\langle W_{a,b}, W_{1,0} \otimes \left(\bigoplus_{(A,B) \in I(a,b)} m_{A,B}(V) W_{A,B} \right) \right\rangle \\ &= \sum_{(A,B) \in I(a,b)} m_{A,B}(V) \langle W_{a,b}, W_{1,0} \otimes W_{A,B} \rangle \\ &= \sum_{(A,B) \in I(a,b)} m_{A,B}(V) \langle W_{a,b}, W_{A+1,B} \oplus W_{A,B+1} \oplus W_{A-1,B} \oplus W_{A,B-1} \rangle \\ &= m_{a+1,b}(V) + m_{a,b+1}(V) + m_{a-1,b}(V) + m_{a,b-1}(V). \end{aligned}$$

□

Lemma 4.8. For any $a \geq b \geq 0$ and $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation V,

$$\begin{aligned} &\langle W_{a,b}, (W_{0,0} \oplus W_{1,1}) \otimes V \rangle \\ &\quad = (2 - \mathbf{1}_{\{a=b\}}(a,b)) m_{a,b}(V) + m_{a-1,b-1}(V) + m_{a-1,b+1}(V) + m_{a+1,b-1}(V) + m_{a+1,b+1}(V). \end{aligned}$$

Proof. By Section 2.2, V is a direct sum of irreducible $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations, which are all of the form $W_{A,B}$ for some $A \geq B \geq 0$. For all $a \geq b \geq 0$, $\langle W_{a,b}, W_{0,0} \otimes V \rangle = m_{a,b}(V)$. Now let $J(a,b) = \{(a-1,b-1), (a-1,b+1), (a,b), (a+1,b-1), (a+1,b+1)\}$. By Lemma 4.3,

$$\begin{aligned} \langle W_{a,b}, W_{1,1} \otimes V \rangle &= \left\langle W_{a,b}, W_{1,1} \otimes \left(\bigoplus_{(A,B) \in J(a,b)} m_{A,B}(V) W_{A,B} \right) \right\rangle \\ &= \sum_{(A,B) \in J(a,b)} m_{A,B}(V) \langle W_{a,b}, W_{1,1} \otimes W_{A,B} \rangle. \end{aligned}$$

For all $(A,B) \in J(a,b)$ with $A > B$, Lemma 4.3 says

$$m_{A,B}(V) \langle W_{a,b}, W_{1,1} \otimes W_{A,B} \rangle = m_{A,B}(V) \left\langle W_{a,b}, \bigoplus_{(C,D) \in J(A,B)} W_{C,D} \right\rangle = m_{A,B}(V).$$

If $A > B$ for all $(A, B) \in J(a, b)$ then $a \neq b$ and

$$\begin{aligned}\langle W_{a,b}, (\mathbf{Q}_\ell \oplus W_{1,1}) \otimes V\rangle &= \langle W_{a,b}, \mathbf{Q}_\ell \otimes V\rangle + \langle W_{a,b}, W_{1,1} \otimes V\rangle \\ &= m_{a,b}(V) + \sum_{(A,B)\in J(a,b)} m_{A,B}(V) \\ &= 2m_{a,b}(V) + m_{a-1,b-1}(V) + m_{a-1,b+1}(V) + m_{a+1,b-1}(V) + m_{a+1,b+1}(V),\end{aligned}$$

which proves the lemma in this case.

If $(A, A) \in J(a, b)$ with $A \geq 0$, then one of the following cases occur:

(1) $A + 1 = a = b$ and

$$\begin{aligned}m_{A,A}(V)\langle W_{a,b}, W_{1,1} \otimes W_{A,A}\rangle &= m_{A,A}(V)\langle W_{A+1,A+1}, W_{A-1,A-1} \oplus W_{A+1,A-1} \oplus W_{A+1,A+1}\rangle \\ &= m_{a-1,a-1}(V).\end{aligned}$$

(2) $A + 1 = a = b + 2$ and

$$\begin{aligned}m_{A,A}(V)\langle W_{a,b}, W_{1,1} \otimes W_{A,A}\rangle &= m_{A,A}(V)\langle W_{A+1,A-1}, W_{A-1,A-1} \oplus W_{A+1,A-1} \oplus W_{A+1,A+1}\rangle \\ &= m_{a-1,b+1}(V).\end{aligned}$$

(3) $A = a = b$ and

$$\begin{aligned}m_{A,A}(V)\langle W_{a,b}, W_{1,1} \otimes W_{A,A}\rangle &= m_{A,A}(V)\langle W_{A,A}, W_{A-1,A-1} \oplus W_{A+1,A-1} \oplus W_{A+1,A+1}\rangle \\ &= 0.\end{aligned}$$

(4) $A - 1 = a = b - 2$, but this implies that $a < b$.

(5) $A - 1 = a = b$ and

$$\begin{aligned}m_{A,A}(V)\langle W_{a,b}, W_{1,1} \otimes W_{A,A}\rangle &= m_{A,A}(V)\langle W_{A-1,A-1}, W_{A-1,A-1} \oplus W_{A+1,A-1} \oplus W_{A+1,A+1}\rangle \\ &= m_{a+1,a+1}(V).\end{aligned}$$

Rearranging, these cases reduce to one of the following:

(1) $a = b$ and $(a-1, a-1), (a, a), (a+1, a+1) \in J(a, a)$, so

$$\begin{aligned}\langle W_{a,a}, (W_{0,0} \oplus W_{1,1}) \otimes V\rangle &= \langle W_{a,a}, W_{0,0} \otimes V\rangle + \langle W_{a,a}, W_{1,1} \otimes V\rangle \\ &= m_{a,a}(V) + \sum_{(A,B)\in J(a,a)} m_{A,B}(V)\langle W_{a,a}, W_{1,1} \otimes W_{A,B}\rangle \\ &= m_{a,a}(V) + \left(\sum_{(A,A)\in J(a,a)} m_{A,A}(V)\langle W_{a,a}, W_{1,1} \otimes W_{A,A}\rangle\right) \\ &\quad + m_{a+1,a-1}(V)\langle W_{a,a}, W_{1,1} \otimes W_{a+1,a-1}\rangle \\ &= m_{a,a}(V) + m_{a-1,a-1}(V) + m_{a+1,a+1}(V) + m_{a+1,a-1}(V)\end{aligned}$$

which proves the lemma in this case.

(2) $a = b+2$ and $(a-1, a-1) \in J(a,b)$, so

$$\begin{aligned}\langle W_{a,b}, (W_{0,0} \oplus W_{1,1}) \otimes V\rangle &= \langle W_{a,b}, W_{0,0} \otimes V\rangle + \langle W_{a,b}, W_{1,1} \otimes V\rangle \\ &= m_{a,b}(V) + \sum_{(A,B)\in J(a,b)} m_{A,B}(V)\langle W_{a,b}, W_{1,1} \otimes W_{A,B}\rangle \\ &= m_{a,b}(V) + m_{a-1,b+1}(V)\langle W_{a,b}, W_{1,1} \otimes W_{a-1,a-1}\rangle + \sum_{\substack{(A,B)\in J(a,b)\\(A,B)\neq(a-1,a-1)}} m_{A,B}(V) \\ &= m_{a,b}(V) + \sum_{(A,B)\in J(a,b)} m_{A,B}(V)\end{aligned}$$

which proves the lemma in this case. □

Combining all of the lemmas of this subsection shows that we have determined a recursive formula for all parts of the first identity of the following proposition.

Proposition 4.9. Let $a \geq b \geq 0$, $n \geq 1$, and $k \geq 0$. Viewing $H^*(A^N)$ as $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations,

(6)
$$\begin{aligned}m_{a,b}(H^k(A^n)) = m_{a,b}(H^k(A^{n-1})) + \langle W_{a,b}, W_{1,0} \otimes H^{k-1}(A^{n-1})\rangle + \langle W_{a,b}, (W_{0,0} \oplus W_{1,1}) \otimes H^{k-2}(A^{n-1})\rangle \\ +\langle W_{a,b}, W_{1,0} \otimes H^{k-3}(A^{n-1})\rangle + m_{a,b}(H^{k-4}(A^{n-1}))\end{aligned}$$

and

$$\begin{aligned}m_{0,0}(H^k(A^n)) = m_{0,0}(H^k(A^{n-1})) + m_{1,0}(H^{k-1}(A^{n-1})) + m_{0,0}(H^{k-2}(A^{n-1})) \\ +m_{1,1}(H^{k-2}(A^{n-1})) + m_{1,0}(H^{k-3}(A^{n-1})) + m_{0,0}(H^{k-4}(A^{n-1})).\end{aligned} \tag{7}$$

Proof. Apply $m_{a,b}$ to the isomorphism given by Lemma 4.1 to obtain Equation (6).

For $j = 1, 3$, apply Lemma 4.7 to compute

$$\langle W_{0,0}, W_{1,0} \otimes H^{k-j}(A^{n-1})\rangle = m_{1,0}(H^{k-j}(A^{n-1}))$$

and apply Lemma 4.8 to compute

$$\langle W_{0,0}, (W_{0,0} \oplus W_{1,1}) \otimes H^{k-2}(A^{n-1})\rangle = m_{0,0}(H^{k-2}(A^{n-1})) + m_{1,1}(H^{k-2}(A^{n-1})).$$

Plug in $a = b = 0$ to Equation (6) with these calculations to obtain Equation (7). □

We can also more explicitly describe the representations $W_{a,b}$ that occur in $H^k(A^n)$. These descriptions are necessary to prove Theorem 1.2.

Proposition 4.10. If $m_{a,b}(H^k(A^n)) \neq 0$, then $a + b \equiv k \pmod 2$ and $a + b \leq k$. For all $n \geq k$, all such $a \geq b \geq 0$ and $k \geq 0$ give $m_{a,b}(H^k(A^n)) \neq 0$.

Proof. To prove that if $m_{a,b}(H^k(A^n)) \neq 0$ then $a + b \equiv k \pmod 2$, we proceed by induction on n. For $n = 1$, the claim is true by Lemma 3.1. Now assume the claim for $n - 1$. Suppose $a + b \equiv k + 1 \pmod 2$ or $a + b > k$ and let $V(k) = H^k(A^{n-1})$. Then using Lemma 4.7 with $j = 1$ or 3,

$$\begin{aligned}\langle W_{a,b}, W_{1,0} \otimes V(k-j))\rangle &= m_{a+1,b}(V(k-j)) + m_{a,b+1}(V(k-j)) \\ &\quad + m_{a-1,b}(V(k-j)) + m_{a,b-1}(V(k-j)).\end{aligned}$$

Here, each summand is of the form $m_{\alpha,\beta}(V(k-j))$ for some $(\alpha, \beta) \in I(a,b)$ using the notation of the proof of Lemma 4.7. All such tuples satisfy $\alpha + \beta \equiv a + b + 1 \pmod 2$, and so $\alpha + \beta \equiv k$

$\pmod 2$. Since $k-j \equiv k+1 \pmod 2$, the inductive hypothesis shows that each $m_{\alpha,\beta}(V(k-j)) = 0$. Therefore, $\langle W_{a,b}, W_{1,0} \otimes V(k-j))\rangle = 0$.

By Lemma 4.8 ,

$$\langle W_{a,b}, (W_{0,0} \oplus W_{1,1}) \otimes V(k-2)\rangle = (2 - \mathbf{1}_{\{a=b\}}(a,b))m_{a,b}(V(k-2)) + m_{a-1,b-1}(V(k-2)) + m_{a-1,b+1}(V(k-2)) + m_{a+1,b-1}(V(k-2)) + m_{a+1,b+1}(V(k-2)).$$

Each summand is a multiple of $m_{\alpha,\beta}(V(k-2))$ for some $(\alpha,\beta) \in J(a,b)$ using the notation of the proof of Lemma 4.8. All such tuples satisfy $\alpha+\beta \equiv a+b \pmod 2$, and so $\alpha+\beta \equiv k+1 \pmod 2$. Since $\alpha+\beta \not\equiv k-2 \pmod 2$, the inductive hypothesis shows that each $m_{\alpha,\beta}(V(k-2)) = 0$. Therefore, $\langle W_{a,b}, (W_{0,0} \oplus W_{1,1}) \otimes V(k-2)\rangle = 0$.

By Proposition 4.9(6),

$$\begin{aligned} m_{a,b}(H^k(A^n)) &= m_{a,b}(V(k)) + \langle W_{a,b}, W_{1,0} \otimes V(k-1))\rangle + \langle W_{a,b}, (W_{0,0} \oplus W_{1,1}) \otimes V(k-2)\rangle \\ &\quad + \langle W_{a,b}, W_{1,0} \otimes V(k-3)\rangle + m_{a,b}(V(k-4)) \\ &= m_{a,b}(V(k)) + m_{a,b}(V(k-4)) = 0. \end{aligned}$$

For fixed $k \geq 0$ and $a \geq b \geq 0$ with $a+b \equiv k \pmod 2$ and $a+b \leq k$, we next show that $m_{a,b}(H^k(A^n)) \neq 0$ for $n \geq k$ by induction on n. For $n=1$, the claim is again true by Lemma 3.1. Assume the claim holds for $n-1$ and take any $k \leq n$. If $b \geq 1$, then by Proposition 4.9(6), Lemma 4.7, and the inductive hypothesis,

$$m_{a,b}(H^k(A^n)) \geq \langle W_{a,b}, W_{1,0} \otimes H^{k-1}(A^{n-1})\rangle \geq m_{a,b-1}(H^{k-1}(A^{n-1})) > 0.$$

If $a \neq 0$ and $b = 0$, then

$$m_{a,b}(H^k(A^n)) \geq \langle W_{a,b}, W_{1,0} \otimes H^{k-1}(A^{n-1})\rangle \geq m_{a-1,b}(H^{k-1}(A^{n-1})) > 0.$$

If $(a,b) = (0,0)$ and $k \geq 2$, then

$$m_{0,0}(H^k(A^n)) \geq \langle W_{0,0}, (W_{0,0} \oplus W_{1,1}) \otimes H^{k-2}(A^{n-1})\rangle \geq (2 - \mathbf{1}_{\{(1,1),(0,0)\}}(0,0))m_{0,0}(H^{k-2}(A^{n-1})) > 0.$$

If $k=0$, then $H^0(A^n) = W_{0,0}$ for all n and so $m_{0,0}(H^0(A^n)) = 1$. □

Throughout the rest of this section, let $\binom{n}{m} = 0$ if $n < m$. The following lemma contains the recursive calculations of $m_{a,b}(H^k(A^n))$ for select values of a, b, and k which will be used in the proof of Theorem 1.2, the main theorem of this section.

Lemma 4.11. Let $a \geq b \geq 0$ and $M > 0$.

(a) $m_{a,b}(H^0(A^M)) = \mathbf{1}_{\{(0,0)\}}(a,b)$.
(b) $m_{1,0}(H^1(A^M)) = M$.
(c) $m_{0,0}(H^2(A^M)) = \binom{M+1}{2}$.
(d) $m_{1,1}(H^2(A^M)) = \binom{M+1}{2}$.
(e) $m_{2,0}(H^2(A^M)) = \binom{M}{2}$.
(f) $m_{1,0}(H^3(A^M)) = \binom{M}{3} + 2\binom{M+1}{3} + M^2$.
(g) $m_{0,0}(H^4(A^M)) = \frac{M(M+1)(M^2+M+2)}{8}$.

Proof. All proofs are by induction on M. We can check manually that all claims hold for $M=1$. Assume that they hold for $M-1$.

(1) For all M, $H^0(A^M) = W_{0,0}$, the trivial $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation.
(2) By Proposition 4.9(6), Lemma 4.7 and (a),

$$\begin{aligned} m_{1,0}(H^1(A^M)) &= m_{1,0}(H^1(A^{M-1})) + \langle W_{1,0}, W_{1,0} \otimes H^0(A^{M-1})\rangle \\ &= (M-1) + \big(m_{2,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) + m_{0,0}(H^0(A^{M-1}))\big) = M. \end{aligned}$$

(3) By Proposition 4.9(7), (a), and (b),

$$\begin{aligned} m_{0,0}(H^2(A^M)) &= m_{0,0}(H^2(A^{M-1})) + m_{1,0}(H^1(A^{M-1})) + m_{0,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) \\ &= \binom{M}{2} + (M-1) + 1 = \binom{M+1}{2}. \end{aligned}$$

(4) By Proposition 4.9(6), Lemma 4.7, Lemma 4.8, (a), and (b),

$$\begin{aligned} m_{1,1}(H^2(A^M)) &= m_{1,1}(H^2(A^{M-1})) + \langle W_{1,1}, W_{1,0} \otimes (H^1(A^{M-1}))\rangle + \langle W_{1,1}, (\mathbf{Q}_\ell \oplus W_{1,1}) \otimes (H^0(A^{M-1}))\rangle \\ &= \binom{M}{2} + \big(m_{2,1}(H^1(A^{M-1})) + m_{1,0}(H^1(A^{M-1}))\big) \\ &\quad + \big(m_{1,1}(H^0(A^{M-1})) + m_{0,0}(H^0(A^{M-1})) + m_{2,0}(H^0(A^{M-1})) + m_{2,2}(H^0(A^{M-1}))\big) \\ &= \binom{M}{2} + \big(m_{2,1}(H^1(A^{M-1})) + (M-1)\big) + (1) = \binom{M+1}{2} \end{aligned}$$

where the last equality follows by Proposition 4.10, which gives that $m_{2,1}(H^1(A^{M-1})) = 0$.

(5) By Proposition 4.9(6), Lemma 4.7, Lemma 4.8, (a), and (b),

$$\begin{aligned} m_{2,0}(H^2(A^M)) &= m_{2,0}(H^2(A^{M-1})) + \langle W_{2,0}, W_{1,0} \otimes (H^1(A^{M-1}))\rangle + \langle W_{2,0}, (\mathbf{Q}_\ell \oplus W_{1,1}) \otimes (H^0(A^{M-1}))\rangle \\ &= \binom{M-1}{2} + \big(m_{3,0}(H^1(A^{M-1})) + m_{2,1}(H^1(A^{M-1})) + m_{1,0}(H^1(A^{M-1}))\big) \\ &\quad + \big(2m_{2,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) + m_{3,1}(H^0(A^{M-1}))\big) \\ &= \binom{M-1}{2} + (M-1) = \binom{M}{2}. \end{aligned}$$

where the last equality follows by Proposition 4.10, which gives that $m_{3,0}(H^1(A^{M-1})) = m_{2,1}(H^1(A^{M-1})) = 0$.

(6) By Proposition 4.9(6), Lemma 4.7, Lemma 4.8, and (a) - (e),

$$\begin{aligned} m_{1,0}(H^3(A^M)) &= m_{1,0}(H^3(A^{M-1})) + \langle W_{1,0}, W_{1,0} \otimes H^2(A^{M-1})\rangle \\ &\quad + \langle W_{1,0}, (\mathbf{Q}_\ell \oplus W_{1,1}) \otimes H^1(A^{M-1})\rangle + \langle W_{1,0}, W_{1,0} \otimes H^0(A^{M-1})\rangle \\ &= m_{1,0}(H^3(A^{M-1})) + \big(m_{2,0}(H^2(A^{M-1})) + m_{1,1}(H^2(A^{M-1})) + m_{0,0}(H^2(A^{M-1}))\big) \\ &\quad + \big(2m_{1,0}(H^1(A^{M-1})) + m_{2,1}(H^1(A^{M-1}))\big) \\ &\quad + \big(m_{2,0}(H^0(A^{M-1})) + m_{1,1}(H^0(A^{M-1})) + m_{0,0}(H^0(A^{M-1}))\big) \\ &= \left(\binom{M-1}{3} + 2\binom{M}{3} + (M-1)^2\right) + \left(\binom{M-1}{2} + \binom{M}{2} + \binom{M}{2}\right) \\ &\quad + \big(2(M-1) + m_{2,1}(H^1(A^{M-1}))\big) + 1 \\ &= \binom{M}{3} + 2\binom{M+1}{3} + M^2 \end{aligned}$$

where again, the last equality uses that $m_{2,1}(H^1(A^{M-1})) = 0$ by Proposition 4.10.

(7) By Proposition 4.9(7), (a) - (d), and (f),

$$\begin{aligned}
m_{0,0}(H^4(A^M)) &= m_{0,0}(H^4(A^{M-1})) + m_{1,0}(H^3(A^{M-1})) + m_{0,0}(H^2(A^{M-1})) + m_{1,1}(H^2(A^{M-1})) \\
&\quad + m_{1,0}(H^1(A^{M-1})) + m_{0,0}(H^0(A^{M-1})) \\
&= \frac{(M-1)(M)(M^2-M+2)}{8} + \left(\binom{M-1}{3} + 2\binom{M}{3} + (M-1)^2\right) \\
&\quad + \binom{M}{2} + \binom{M}{2} + (M-1) + 1 \\
&= \frac{M(M+1)(M^2+M+2)}{8}.
\end{aligned}$$

□

We are now ready to compute $H^k(\mathcal{X}_2^n; \mathbf{Q}_\ell)$ for all $n \geq 1$ and $0 \leq k \leq 5$.

Theorem 1.2. For all $n \geq 1$,

$$H^k(\mathcal{X}_2^n; \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ 0 & k = 1, 3 \\ \left(\binom{n+1}{2} + 1\right) \mathbf{Q}_\ell(-1) & k = 2 \\ \left(\frac{n(n+1)(n^2+n+2)}{8} + \binom{n+1}{2}\right) \mathbf{Q}_\ell(-2) & k = 4 \\ \binom{n+1}{2} \mathbf{Q}_\ell(-5) \oplus \binom{n}{2} \mathbf{Q}_\ell(-4) & k = 5 \end{cases}$$

up to semi-simplification.

Proof. Recall that the (p, q)-entry of the E_2-sheet of the Leray spectral sequence of $\pi^n : \mathcal{X}_2^n \to \mathcal{A}_2$ is $H^p(\mathcal{A}_2; H^q(A^n))$ by Proposition 2.8; denote this entry by $E_2^{p,q}(n)$. We compute many entries on the E_2-sheet and list the nonzero results in Figure 2, from which the theorem follows directly. The special case of $n = 1$ is given in Figure 1. All computations here are up to semi-simplification.

(1) $E_2^{p,0}(n) = H^p(\mathcal{A}_2; \mathbf{Q}_\ell)$ for all p.

Proof. By definition and Lemmas 4.1 and 4.11,

$$E_2^{p,0}(n) = H^p(\mathcal{A}_2; H^0(A^n)) = \bigoplus_{a \geq b \geq 0} m_{a,b}(H^0(A^n)) H^p\left(\mathcal{A}_2; \mathbf{V}_{a,b}\left(\frac{a+b}{2}\right)\right) = H^p(\mathcal{A}_2; \mathbf{Q}_\ell).$$

□

(2) $E_2^{p,q}(n) = 0$ for all $p \geq 0$, $q \equiv 1 \pmod 2$.

Proof. Suppose $m_{a,b}(H^q(A^n)) \neq 0$. By Proposition 4.10, $a + b \equiv 1 \pmod 2$. Therefore

$$E_2^{p,q}(n) = H^p(\mathcal{A}_2; H^q(A^n)) = \bigoplus_{\substack{a \geq b \geq 0 \\ a+b \equiv 1 \pmod 2}} m_{a,b}(H^q(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{a,b})\left(\frac{a+b-q}{2}\right) = 0$$

where the last equality follows since $H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0$. (See the remark before Theorem 3.2.)

□

(3) $E_2^{1,q}(n) = 0$ for all $q \geq 0$.

Proof. By definition and Corollary 3.4(1),

$$E_2^{1,q}(n) = H^1(\mathcal{A}_2; H^q(A^n)) = \bigoplus_{a \geq b \geq 0} m_{a,b}(H^q(A^n)) H^1(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0.$$

□

(4) $E_2^{p,2}(n) = \binom{n+1}{2} H^p(\mathcal{A}_2; \mathbf{Q}_\ell)(-1)$ and $E_2^{p,4}(n) = \frac{n(n+1)(n^2+n+2)}{8} H^p(\mathcal{A}_2; \mathbf{Q}_\ell)(-2)$ for $p = 0, 1, 2$.

Proof. Let $q = 2, 4$. By definition and Proposition 4.10,

$$E_2^{p,q}(n) = H^p(\mathcal{A}_2; H^q(A^n)) = \bigoplus_{\substack{a+b\le q \\ a+b\equiv 0 \pmod 2}} m_{a,b}(H^q(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) \left(\frac{a+b-q}{2}\right).$$

By Theorem 3.2, $H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0$ for $(a, b) \neq 0$ and $p = 0, 1$, so

$$E_2^{p,q}(n) = m_{0,0}(H^q(A^n)) H^p(\mathcal{A}_2; \mathbf{Q}_\ell) \left(-\frac{q}{2}\right)$$

for $p = 0, 1$. The claim then follows by Lemma 4.11. For $p = 2$, Theorem 3.2 gives that $H^2(\mathcal{A}_2; \mathbf{V}_{a,b}) \neq 0$ only if $a = b$ even, and so

$$E_2^{2,q}(n) = m_{0,0}(H^q(A^n)) H^2(\mathcal{A}_2; \mathbf{Q}_\ell) \left(-\frac{q}{2}\right) \oplus \begin{cases} 0 & q = 2 \\ m_{2,2}(H^4(A^n)) H^2(\mathcal{A}_2; \mathbf{V}_{2,2}) & q = 4. \end{cases}$$

In the case $q = 2$, the claim then follows from Lemma 4.11. In the case $q = 4$, the claim follows from Corollary 3.4(4) since $H^2(\mathcal{A}_2; \mathbf{V}_{2,2}) = 0$. □

(5) $E_2^{3,2}(n) = \binom{n+1}{2} \mathbf{Q}_\ell(-5) \oplus \binom{n}{2} \mathbf{Q}_\ell(-4)$ and $E_2^{4,2}(n) = 0$.

Proof. Let $p = 3, 4$. By definition, Proposition 4.10, Theorem 2.1, and Lemma 4.11,

$$\begin{aligned} E_2^{p,2}(n) = H^p(\mathcal{A}_2; H^2(A^n)) &= \bigoplus_{a\ge b\ge 0} m_{a,b}(H^2(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) \left(\frac{a+b-2}{2}\right) \\ &= m_{0,0}(H^2(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{0,0})(-1) \oplus m_{1,1}(H^2(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{1,1}) \oplus m_{2,0}(H^2(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{2,0}) \\ &= \binom{n+1}{2} H^p(\mathcal{A}_2; \mathbf{V}_{1,1}) \oplus \binom{n}{2} H^p(\mathcal{A}_2; \mathbf{V}_{2,0}). \end{aligned}$$

Then the claim follows from Corollary 3.4(2) and (3). □

(6) $E_2^{p,q}(n) = 0$ for all $p \geq 4$ and $q = 0, 1, 2, 3$.

Proof. By definition and Proposition 4.10,

$$E_2^{p,q}(n) = H^p(\mathcal{A}_2; H^q(A^n)) = \bigoplus_{\substack{a+b\le q \\ a+b\equiv q \pmod 2}} m_{a,b}(H^q(A^n)) H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) \left(\frac{a+b-q}{2}\right).$$

For $q = 0, 1, 2, 3$, the only tuples (a, b) with $a \geq b \geq 0$ and $a + b \leq q$ satisfy $(a, b) \in \{(0, 0), (1, 0), (1, 1), (2, 0), (2, 1)\}$. By the remark before Theorem 3.2, $H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0$ for all $p \geq 0$ if $a + b \equiv 1 \pmod 2$. By Corollary 3.4(2) and (3), $H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0$ for all $p \geq 4$ and $(a, b) = (1, 1), (2, 0)$. By Theorem 2.1, $H^p(\mathcal{A}_2; \mathbf{V}_{0,0}) = 0$ for all $p \geq 3$. Therefore, all summands of the direct sum above are zero. □

□

Remark 4.12. Theorem 1.2 is consistent with the stabilization result [GHT18, Theorem 6.1], which says that the rational cohomology of $\mathcal{X}_g^n$ stabilizes in degrees $k < g = 2$.

q				
4	$\frac{n(n+1)(n^2+n+2)}{8}\mathbf{Q}_\ell(-2)$	0	$\frac{n(n+1)(n^2+n+2)}{8}\mathbf{Q}_\ell(-3)$	?
3	0	0	0	0
2	$\binom{n+1}{2}\mathbf{Q}_\ell(-1)$	0	$\binom{n+1}{2}\mathbf{Q}_\ell(-2)$	$\binom{n+1}{2}\mathbf{Q}_\ell(-5) \oplus \binom{n}{2}\mathbf{Q}_\ell(-4)$
1	0	0	0	0
0	$\mathbf{Q}_\ell$	0	$\mathbf{Q}_\ell(-1)$	0
	0	1	2	3 p

FIGURE 2. Some low degree terms of the E_2-page of the Leray spectral sequence for $\pi^n : \mathcal{X}_2^n \to \mathcal{A}_2$ determined in Theorem 1.2.

4.2. **Explicit computations for** $n = 2$**.** Once one has computed $m_{a,b}(H^k(A^n))$ for fixed n and for all $k \geq 0$, $a \geq b \geq 0$, one can in theory apply Proposition 2.8 and Theorem 3.2 to determine $H^*(\mathcal{X}_2^n; \mathbf{Q}_\ell)$. In this subsection, we detail the results of this process for $n = 2$.

Lemma 4.13. The local systems $H^k(A^2; \mathbf{Q}_\ell)$ on $\mathcal{A}_2$ are

$$H^k(A^2;\mathbf{Q}_\ell) \cong \begin{cases} \mathbf{V}_{0,0} & k=0 \\ 2\mathbf{V}_{1,0} & k=1 \\ 3\mathbf{V}_{0,0}(-1) \oplus 3\mathbf{V}_{1,1} \oplus \mathbf{V}_{2,0} & k=2 \\ 6\mathbf{V}_{1,0}(-1) \oplus 2\mathbf{V}_{2,1} & k=3 \\ 6\mathbf{V}_{0,0}(-2) \oplus 4\mathbf{V}_{1,1}(-1) \oplus 3\mathbf{V}_{2,0}(-1) \oplus \mathbf{V}_{2,2} & k=4 \\ 6\mathbf{V}_{1,0}(-2) \oplus 2\mathbf{V}_{2,1}(-1) & k=5 \\ 3\mathbf{V}_{0,0}(-3) \oplus 3\mathbf{V}_{1,1}(-2) \oplus \mathbf{V}_{2,0}(-2) & k=6 \\ 2\mathbf{V}_{1,0}(-3) & k=7 \\ \mathbf{V}_{0,0}(-4) & k=8 \\ 0 & k>8. \end{cases}$$

Proof. This is a direct computation using Lemma 4.1, Lemma 4.7, Lemma 4.8, and Lemma 3.1. □

Using Lemma 4.13, we can compute all entries of the E_2-page of the Leray spectral sequence for $\pi^2 : \mathcal{X}_2^2 \to \mathcal{A}_2$. As always, the following results are up to semi-simplification.

Lemma 4.14. For $q = 1, 3, 5, 7$, and all p,

$$H^p(\mathcal{A}_2; H^q(A^2)) = 0.$$

For $q = 0, 8$,

$$H^p(\mathcal{A}_2; H^q(A^2)) \cong \begin{cases} \mathbf{Q}_\ell\left(-\frac{q}{2}\right) & p=0 \\ \mathbf{Q}_\ell\left(-\frac{q}{2}-1\right) & p=2 \\ 0 & \text{otherwise.} \end{cases}$$

For $q = 2, 6$,

$$H^p(\mathcal{A}_2; H^q(A^2)) \cong \begin{cases} 3\mathbf{Q}_\ell\left(-\frac{q}{2}\right) & p=0 \\ 3\mathbf{Q}_\ell\left(-\frac{q}{2}-1\right) & p=2 \\ 3\mathbf{Q}_\ell\left(-\frac{q-2}{2}-5\right) \oplus \mathbf{Q}_\ell\left(-\frac{q-2}{2}-4\right) & p=3 \\ 0 & \text{otherwise.} \end{cases}$$

For $q = 4$,

$$H^p(\mathcal{A}_2; H^4(A^2)) \cong \begin{cases} 6\mathbf{Q}_\ell(-2) & p = 0 \\ 6\mathbf{Q}_\ell(-3) & p = 2 \\ 3\mathbf{Q}_\ell(-5) \oplus 4\mathbf{Q}_\ell(-6) & p = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, $E_2^{p,q} = H^p(\mathcal{A}_2; H^q(A^2)) = \bigoplus_{a \geq b \geq 0} m_{a,b}(H^q(A^2)) H^p(\mathcal{A}_2; \mathbf{V}_{a,b})$. Suppose q is odd. By Proposition 4.10, if $m_{a,b}(H^q(A^2)) \neq 0$, then $a + b \equiv 1 \pmod 2$. For such (a, b), the remarks before Theorem 3.2 imply that $H^p(\mathcal{A}_2; \mathbf{V}_{a,b}) = 0$ for all p.

For $0 \leq q \leq 8$ even, combine Lemma 4.1, Lemma 4.13, Theorem 2.1, and Corollary 3.4. □

Remark 4.15. The computations in Lemma 4.14 are consistent with Theorem 1.2.

By the usual argument, we obtain the following theorem using these preliminaries.

Theorem 4.16. The cohomology of the second fiber power $\mathcal{X}_2^2$ of the universal abelian surface is given by

$$H^k(\mathcal{X}_2^2; \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ 0 & k = 1, 3, k > 10 \\ 4\mathbf{Q}_\ell(-1) & k = 2 \\ 9\mathbf{Q}_\ell(-2) & k = 4 \\ 3\mathbf{Q}_\ell(-5) \oplus \mathbf{Q}_\ell(-4) & k = 5 \\ 9\mathbf{Q}_\ell(-3) & k = 6 \\ 3\mathbf{Q}_\ell(-5) \oplus 4\mathbf{Q}_\ell(-6) & k = 7 \\ 4\mathbf{Q}_\ell(-4) & k = 8 \\ 3\mathbf{Q}_\ell(-7) \oplus \mathbf{Q}_\ell(-6) & k = 9 \\ \mathbf{Q}_\ell(-5) & k = 10 \end{cases}$$

up to semi-simplification.

Proof. By Proposition 2.8, there is a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{A}_2; H^q(A^2)) \implies H^{p+q}(\mathcal{X}_2^2; \mathbf{Q}_\ell)$$

which degenerates on the E_2-page. Combining the lemmas in this section gives the entries of the E_2-page as recorded in Figure 3. □

5. Cohomology of $\mathcal{X}_2^{\mathrm{Sym}(n)}$

In this section, we compute $H^*(\mathcal{X}_2^{\mathrm{Sym}(n)}; \mathbf{Q}_\ell)$. As opposed to Section 4, we show that the cohomology in fixed degree stabilizes as n increases and explicitly give the computations for small degree. Afterwards, we give a complete description of the cohomology for the case $n = 2$. For brevity, we will often drop the constant coefficients if the context is clear, writing $H^k(A^n)$ instead of $H^k(A^n; \mathbf{Q}_\ell)$.

q \ p	0	1	2	3
8	$\mathbf{Q}_\ell(-4)$		$\mathbf{Q}_\ell(-5)$	
7				
6	$3\mathbf{Q}_\ell(-3)$		$3\mathbf{Q}_\ell(-4)$	$3\mathbf{Q}_\ell(-7) \oplus \mathbf{Q}_\ell(-6)$
5				
4	$6\mathbf{Q}_\ell(-2)$		$6\mathbf{Q}_\ell(-3)$	$3\mathbf{Q}_\ell(-5) \oplus 4\mathbf{Q}_\ell(-6)$
3				
2	$3\mathbf{Q}_\ell(-1)$		$3\mathbf{Q}_\ell(-2)$	$3\mathbf{Q}_\ell(-5) \oplus \mathbf{Q}_\ell(-4)$
1				
0	$\mathbf{Q}_\ell$		$\mathbf{Q}_\ell(-1)$	

FIGURE 3. The nonzero terms on the E_2-page of the Leray spectral sequence for $\pi^2 : \mathcal{X}_2^2 \to \mathcal{A}_2$.

5.1. **Computations for general** n**.** Let $\pi^n : \mathcal{X}_2^{\mathrm{Sym}(n)} \to \mathcal{A}_2$ and $\pi : \mathcal{X}_2 \to \mathcal{A}_2$. Let A be an abelian surface. For all $n \geq 1$, the symmetric group S_n acts on A^n by permuting the coordinates, which induces an action of S_n on $H^*(A^n)$. The S_n-action on A^n is not free but this is not a problem in the context of stacks. On the other hand,

$$H^k(A^n; \mathbf{Q}_\ell) \cong \bigoplus_{\substack{(k_1,\dots,k_n) \\ \sum k_i = k}} \bigotimes_{i=1}^n H^{k_i}(A; \mathbf{Q}_\ell)$$

by the Künneth formula.

Lemma 5.1. There are isomorphisms of local systems

$$H^k(\mathrm{Sym}^n A; \mathbf{Q}_\ell) \cong H^k(A^n; \mathbf{Q}_\ell)^{S_n} \cong \bigoplus_{\substack{(k_0,\dots,k_4) \\ \sum_{i=0}^4 ik_i = k \\ \sum_{i=0}^4 k_i = n}} \bigwedge^{k_1} H^1(A) \otimes \mathrm{Sym}^{k_2} H^2(A) \otimes \bigwedge^{k_3} H^3(A) \otimes \mathbf{Q}_\ell(-2k_4).$$

Proof. For any abelian surface A, there is an isomorphism $H^k(\mathrm{Sym}^n A; \mathbf{Q}_\ell) \to H^k(A^n; \mathbf{Q}_\ell)^{S_n}$ by the Hochschild–Serre spectral sequence for $A^n \to A^n/S_n \cong \mathrm{Sym}^n A$. For any $\sigma \in S_n$, let $Q_\sigma(x_1, \dots, x_n)$ be the sum of all products $x_i x_j$ with $i < j$ which occur in reversed order in the sequence $\sigma^{-1}(1), \dots, \sigma^{-1}(n)$. The induced action of S_n on $H^k(A^n; \mathbf{Q}_\ell)$ is then given by

$$\sigma \cdot (c_1 \otimes \cdots \otimes c_n) = (-1)^\varepsilon \left(c_{\sigma^{-1}(1)} \otimes \cdots \otimes c_{\sigma^{-1}(n)}\right) \in \bigotimes_{i=1}^n H^{m_{\sigma^{-1}(i)}}(A; \mathbf{Q}_\ell) \subseteq H^k(A^n; \mathbf{Q}_\ell)$$

for all $c_1 \otimes \cdots \otimes c_n \in \bigotimes_{i=1}^n H^{m_i}(A; \mathbf{Q}_\ell)$ with $\varepsilon = Q_\sigma(m_1, \dots, m_n)$ as described in [Mac62]. For example if $\sigma = (\ell, \ell+1)$ is a transposition, then $Q_\sigma(x_1, \dots, x_n) = x_\ell x_{\ell+1}$ which encodes the fact that $H^*(A^n; \mathbf{Q}_\ell)$ is a graded commutative ring with respect to the cup product and that for any $c_1, \dots, c_n \in H^*(A; \mathbf{Q}_\ell)$ with $c_i \in H^{k_i}(A; \mathbf{Q}_\ell)$ for some $k_i \geq 0$ for all $1 \leq i \leq n$, the cup product $c_1 \smile \cdots \smile c_n$ corresponds to the simple tensor $c_1 \otimes \cdots \otimes c_n$ under the Künneth isomorphism. There is a relationship between Q_{σ_1}, Q_{σ_2}, and $Q_{\sigma_1\sigma_2}$ which encodes the fact that S_n acts on $H^k(A^n;\mathbf{Q}_\ell)$; we refer the reader to [Mac62, (1.1)-(1.3)] for more properties of the polynomials Q_σ since we do not use any of them explicitly in this proof.

There is a projection $p : H^k(A^n;\mathbf{Q}_\ell) \to H^k(A^n;\mathbf{Q}_\ell)^{S_n}$ given by averaging. For each fixed $(m_1,\dots,m_n)$ with $\sum_{i=1}^n m_i = k$ and $m_1 \geq \cdots \geq m_n$, consider the S_n-subrepresentation

$$W_{m_1,\dots,m_n} := \bigoplus_{\sigma\in S_n}\bigotimes_{i=1}^{n} H^{m_{\sigma(i)}}(A;\mathbf{Q}_\ell) \subseteq H^k(A^n;\mathbf{Q}_\ell).$$

The summand corresponding to $1 \in S_n$ above can be written as $\bigotimes_{i=0}^4 H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}$ with $k_i = \#\{m_j : m_j = i\}$. For any simple tensor $c \in \bigotimes_{i=1}^n H^{m_{\sigma(i)}}(A;\mathbf{Q}_\ell)$ in $W_{m_1,\dots,m_n}$, it is straightforward to check that $\sigma\cdot c \in \bigotimes_{i=0}^4 H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}$. Because $W_{m_1,\dots,m_n}$ is spanned over $\mathbf{Q}_\ell$ by such simple tensors c and $p(c) = p(\sigma\cdot c)$ for all $\sigma\in S_n$, the image $p(W_{m_1,\dots,m_n})$ is spanned by the images $p(c)$ of simple tensors $c \in \bigotimes_{i=0}^4 H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}$. Therefore, p restricted to $\bigotimes_{i=0}^4 H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}$ is surjective onto $p(W_{m_1,\dots,m_n})$ with kernel $\langle c - \sigma\cdot c : \sigma \in \prod_{i=0}^4 S_{k_i} \leq S_n\rangle$.

For a transposition $\sigma = (\ell_1\ell_2) \in S_{k_j} \leq \prod_{i=0}^4 S_{k_i}$ with $\ell_1 > \ell_2$, observe that

$$Q_\sigma(m_1,\dots,m_n) = m_{\ell_2}m_{\ell_1} + \sum_{\ell=\ell_2+1}^{\ell_1-1}(m_\ell m_{\ell_1} + m_{\ell_2}m_\ell) = j^2 + \sum_{\ell=\ell_2+1}^{\ell_1-1} 2j^2 \equiv j \pmod 2.$$

This implies that for $i \equiv 0 \pmod 2$,

$$H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}/\langle c-\sigma\cdot c : \sigma\in S_{k_i}\rangle \cong \operatorname{Sym}^{k_i} H^i(A;\mathbf{Q}_\ell)$$

and for $i \equiv 1 \pmod 2$,

$$H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}/\langle c-\sigma\cdot c : \sigma\in S_{k_i}\rangle \cong \bigwedge^{k_i} H^i(A;\mathbf{Q}_\ell).$$

Combining all of the above,

$$\begin{aligned} p(W_{m_1,\dots,m_n}) &\cong \bigotimes_{i=0}^{4}\left(H^i(A;\mathbf{Q}_\ell)^{\otimes k_i}/\langle c-\sigma\cdot c : \sigma\in S_{k_i}\rangle\right) \\ &\cong \bigwedge^{k_1} H^1(A;\mathbf{Q}_\ell)\otimes \operatorname{Sym}^{k_2} H^2(A;\mathbf{Q}_\ell)\otimes \bigwedge^{k_3} H^3(A;\mathbf{Q}_\ell)\otimes\mathbf{Q}_\ell. \end{aligned}$$

Therefore, we have proven the desired isomorphisms on the level of $\mathrm{Sp}(4,\mathbf{Q}_\ell)$-representations. To determine the structure as $\mathrm{GSp}(4,\mathbf{Q}_\ell)$-representations and as local systems, we add in appropriate Tate twists as in the proofs of Lemma 3.1 and 4.1. □

Lemma 5.2. For fixed k, $H^k(\mathcal{X}_2^{\mathrm{Sym}(n)};\mathbf{Q}_\ell)$ stabilizes for $n \geq k$ up to semi-simplification.

Proof. Fix $k\in\mathbf{N}$. For each $n\in\mathbf{N}$, consider the set

$$S(n) := \left\{(k_0,\dots,k_4)\in\mathbf{N}^5 : \sum_{i=0}^4 ik_i = k, \sum_{i=0}^4 k_i = n\right\}.$$

If $n \geq k$, then there is a bijection $S(k)\to S(n)$ given by sending each $(k_0,\dots,k_4)\mapsto(k_0+(n-k),k_1,\dots,k_4)$. Using this bijection and the fact that $\operatorname{Sym}^m H^0(A;\mathbf{Q}_\ell)\cong\mathbf{Q}_\ell$ for any $m\geq 0$, compute for all $n \geq k$ that

$$H^k(A^k;\mathbf{Q}_\ell)^{S_k} \cong \bigoplus_{(k_0,\ldots,k_4)\in S(k)} \bigwedge^{k_1} H^1(A;\mathbf{Q}_\ell)\otimes \mathrm{Sym}^{k_2} H^2(A;\mathbf{Q}_\ell)\otimes \bigwedge^{k_3} H^3(A;\mathbf{Q}_\ell)\otimes \mathbf{Q}_\ell(-2k_4)$$

$$\cong \bigoplus_{(k_0,\ldots,k_4)\in S(n)} \bigwedge^{k_1} H^1(A;\mathbf{Q}_\ell)\otimes \mathrm{Sym}^{k_2} H^2(A;\mathbf{Q}_\ell)\otimes \bigwedge^{k_3} H^3(A;\mathbf{Q}_\ell)\otimes \mathbf{Q}_\ell(-2k_4) \cong H^k(A^n;\mathbf{Q}_\ell)^{S_n}.$$

By Lemma 5.1 and the above computation, there is an isomorphism of local systems

$$H^k(\mathrm{Sym}^k A;\mathbf{Q}_\ell) \cong H^k(\mathrm{Sym}^n A;\mathbf{Q}_\ell)$$

for all $n \geq k$.

Up to semi-simplification,

$$H^k(\mathcal{X}_2^{\mathrm{Sym}(n)};\mathbf{Q}_\ell) = \bigoplus_{p+q=k} H^p(\mathcal{A}_2; H^q(\mathrm{Sym}^n A;\mathbf{Q}_\ell)) = \bigoplus_{p+q=k} H^p(\mathcal{A}_2; H^q(\mathrm{Sym}^k A;\mathbf{Q}_\ell))$$

where the first equality follows by Proposition 2.8 and the second equality follows by the first part of the proof which shows that as local systems,

$$H^q(\mathrm{Sym}^q A;\mathbf{Q}_\ell) \cong H^q(\mathrm{Sym}^k A;\mathbf{Q}_\ell) \cong H^q(\mathrm{Sym}^n A;\mathbf{Q}_\ell)$$

for all k, n such that $q \leq k \leq n$. □

For small k, this simplifies the computations for the local systems $H^k(\mathrm{Sym}^n A;\mathbf{Q}_\ell)$.

Proposition 5.3. For all $n \geq 1$,

$$H^0(\mathrm{Sym}^n A;\mathbf{Q}_\ell) \cong \mathbf{Q}_\ell,$$
$$H^1(\mathrm{Sym}^n A;\mathbf{Q}_\ell) \cong \mathbf{V}_{1,0}.$$

For all $n \geq 2$,

$$H^2(\mathrm{Sym}^n A;\mathbf{Q}_\ell) \cong 2\mathbf{Q}_\ell(-1) \oplus 2\mathbf{V}_{1,1}.$$

For all $n \geq 3$,

$$H^3(\mathrm{Sym}^n A;\mathbf{Q}_\ell) \cong 4\mathbf{V}_{1,0}(-1) \oplus \mathbf{V}_{2,1}.$$

For all $n \geq 4$,

$$H^4(\mathrm{Sym}^n A;\mathbf{Q}_\ell) \cong 7\mathbf{Q}_\ell(-2) \oplus 4\mathbf{V}_{1,1}(-1) \oplus 2\mathbf{V}_{2,0}(-1) \oplus 2\mathbf{V}_{2,2}.$$

For all $n \geq 5$,

$$H^5(\mathrm{Sym}^n A;\mathbf{Q}_\ell) \cong 10\mathbf{V}_{1,0}(-2) \oplus 5\mathbf{V}_{2,1}(-1) \oplus \mathbf{V}_{3,2}.$$

Proof. The necessary facts from the representation theory of $\mathrm{Sp}(4,\mathbf{Q}_\ell)$ are Lemma 4.2, Lemma 4.3, and [FH04, Exercise 16.11] which says that $\mathrm{Sym}^a W_{1,1} \cong \bigoplus_{k=0}^{\lfloor \frac{a}{2} \rfloor} W_{a-2k,a-2k}$ for any $a \geq 0$. Applying these facts to the direct sum given by Lemma 5.1 gives the decomposition into irreducible $\mathrm{Sp}(4,\mathbf{Q}_\ell)$-representations as claimed. Finally, add appropriate Tate twists as in the proof of Lemma 3.1.

As an example, we work out the computation for $H^4(\mathrm{Sym}^n A;\mathbf{Q}_\ell)$ for $n \geq 4$ explicitly. For $k=4$, the tuples $(k_0,\ldots,k_4) \in \mathbf{N}^5$ satisfying $\sum_{i=0}^4 ik_i = k = 4$ and $\sum_{i=0}^4 k_i = n$ are

$$(n-1,0,0,0,1),\ (n-2,1,0,1,0),\ (n-2,0,2,0,0),\ (n-3,2,1,0,0),\ (n-4,4,0,0,0).$$

Lemma 5.1 gives

$$H^4(\mathrm{Sym}^n A; \mathbf{Q}_\ell) \cong H^4(A) \oplus \big(H^1(A) \otimes H^3(A)\big) \oplus \mathrm{Sym}^2 H^2(A) \oplus \left(\bigwedge^2 H^1(A) \otimes H^2(A)\right) \oplus \bigwedge^4 H^1(A).$$

Decompose each summand (as $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations) into a direct sum of irreducible representations:

(1) By Lemma 3.1,

$$H^4(A; \mathbf{Q}_\ell) \cong \mathbf{Q}_\ell.$$

(2) By Lemmas 3.1 and 4.2 for the first and second isomorphisms respectively,

$$H^1(A) \otimes H^3(A) \cong W_{1,0} \otimes W_{1,0} \cong \mathbf{Q}_\ell \oplus W_{2,0} \oplus W_{1,1}.$$

(3) By Lemma 3.1 and [FH04, (B.2)] for the first and second isomorphisms respectively,

$$\mathrm{Sym}^2 H^2(A) \cong \mathrm{Sym}^2(\mathbf{Q}_\ell \oplus W_{1,1}) \cong \bigoplus_{a=0}^{2} \mathrm{Sym}^a \mathbf{Q}_\ell \otimes \mathrm{Sym}^{2-a} W_{1,1} = \mathrm{Sym}^2 W_{1,1} \oplus W_{1,1} \oplus \mathbf{Q}_\ell.$$

By [FH04, Exercise 16.11],

$$\mathrm{Sym}^2 W_{1,1} \oplus W_{1,1} \oplus \mathbf{Q}_\ell \cong (W_{2,2} \oplus W_{0,0}) \oplus W_{1,1} \oplus \mathbf{Q}_\ell \cong W_{2,2} \oplus W_{1,1} \oplus 2\mathbf{Q}_\ell.$$

(4) By Lemma 3.1 and distributivity of tensor products over direct sums,

$$\bigwedge^2 H^1(A) \otimes H^2(A) \cong (\mathbf{Q}_\ell \oplus W_{1,1}) \otimes (\mathbf{Q}_\ell \oplus W_{1,1}) \cong \mathbf{Q}_\ell \oplus 2W_{1,1} \oplus (W_{1,1} \otimes W_{1,1}).$$

By Lemma 4.3,

$$\bigwedge^2 H^1(A) \otimes H^2(A) \cong \mathbf{Q}_\ell \oplus 2W_{1,1} \oplus (\mathbf{Q}_\ell \oplus W_{2,0} \oplus W_{2,2}) = 2\mathbf{Q}_\ell \oplus 2W_{1,1} \oplus W_{2,0} \oplus W_{2,2}.$$

(5) By Lemma 3.1,

$$\bigwedge^4 H^1(A) \cong \mathbf{Q}_\ell.$$

Collect all the terms above to see that as $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations,

$$H^4(\mathrm{Sym}^n A; \mathbf{Q}_\ell) \cong 7\mathbf{Q}_\ell \oplus 4W_{1,1} \oplus 2W_{2,0} \oplus 2W_{2,2}$$

as claimed. Now add in Tate twists to the local systems corresponding to the appropriate $\mathrm{GSp}(4, \mathbf{Q}_\ell)$-representations as in the proof of Lemma 3.1. □

Proposition 5.3 provides the inputs to the computation of the cohomology of $\mathcal{X}_2^{\mathrm{Sym}(n)}$ in the same way as in Sections 3 and 4.

Theorem 1.3. For all $n \geq k$ for k even and for all $n \geq k-1$ for k odd,

$$H^k(\mathcal{X}_2^{\mathrm{Sym}(n)}; \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ 0 & k = 1, 3 \\ 3\mathbf{Q}_\ell(-1) & k = 2 \\ 9\mathbf{Q}_\ell(-2) & k = 4 \\ 2\mathbf{Q}_\ell(-5) & k = 5 \end{cases}$$

up to semi-simplification.

q				
4	$7\mathbf{Q}_\ell(-2)$	0	$7\mathbf{Q}_\ell(-3)$	$4\mathbf{Q}_\ell(-6) \oplus 2\mathbf{Q}_\ell(-5)$
3	0	0	0	0
2	$2\mathbf{Q}_\ell(-1)$	0	$2\mathbf{Q}_\ell(-2)$	$2\mathbf{Q}_\ell(-5)$
1	0	0	0	0
0	$\mathbf{Q}_\ell$	0	$\mathbf{Q}_\ell(-1)$	0
	0	1	2	3 p

FIGURE 4. Some nonzero terms $E_2^{p,q}(n)$ of the Leray spectral sequence for $\pi^n : \mathcal{X}_2^{\mathrm{Sym}(n)} \to \mathcal{A}_2$, for $n \geq q$ for q even and for all $n \geq 0$ for q odd. Note that $E_2^{1,q} = 0$ for all $q \geq 0$ and $E_2^{p,q} = 0$ for all $p \geq 4$ and $q = 0, 1, 2, 3$.

Proof. Denote the (p, q)-entry on the E_2-sheet of the Leray spectral sequence (given by Proposition 2.8) of $\pi^n : \mathcal{X}_2^{\mathrm{Sym}(n)} \to \mathcal{A}_2$ by $E_2^{p,q}(n) = H^p(\mathcal{A}_2; H^q(\mathrm{Sym}^n A))$. This spectral sequence degenerates on the E_2-page. Applying Proposition 5.3 and Corollary 3.4 yields $E_2^{p,q}(n)$ for $n \geq q$ and $q = 0$, 2, 4, which we record in Figure 4. Observe also that $E_2^{p,q}(n) = 0$ for all $n \geq 0$ if q is odd or if $p > 4$ by Theorem 3.2. The theorem now follows directly. □

5.2. **Explicit computations for** $n = 2$. We compute $H^*(\mathcal{X}_2^{\mathrm{Sym}(2)}; \mathbf{Q}_\ell)$ completely. We first need the following.

Lemma 5.4. There are isomorphisms of local systems

$$H^k(\mathrm{Sym}^2 A; \mathbf{Q}_\ell) \cong \begin{cases} \mathbf{Q}_\ell & k = 0 \\ \mathbf{V}_{1,0} & k = 1 \\ 2\mathbf{Q}_\ell(-1) \oplus 2\mathbf{V}_{1,1} & k = 2 \\ 3\mathbf{V}_{1,0}(-1) \oplus \mathbf{V}_{2,1} & k = 3 \\ 4\mathbf{Q}_\ell(-2) \oplus 2\mathbf{V}_{1,1}(-1) \oplus \mathbf{V}_{2,0}(-1) \oplus \mathbf{V}_{2,2} & k = 4 \\ 3\mathbf{V}_{1,0}(-2) \oplus \mathbf{V}_{2,1}(-1) & k = 5 \\ 2\mathbf{Q}_\ell(-3) \oplus 2\mathbf{V}_{1,1}(-2) & k = 6 \\ \mathbf{V}_{1,0}(-3) & k = 7 \\ \mathbf{Q}_\ell(-4) & k = 8 \\ 0 & k > 8. \end{cases}$$

Proof. This is a direct computation using Lemma 5.1. □

As usual, we want to compute $H^p(\mathcal{A}_2; H^q(\mathrm{Sym}^n A))$ for all p, $q \geq 0$. With Lemma 5.4, this process is completely analogous to that of Sections 3 and 4. Therefore, we list the results below and omit the explanations.

q \ p	0	1	2	3
8	$\mathbf{Q}_\ell(-4)$		$\mathbf{Q}_\ell(-5)$	
7				
6	$2\mathbf{Q}_\ell(-3)$		$2\mathbf{Q}_\ell(-4)$	$2\mathbf{Q}_\ell(-7)$
5				
4	$4\mathbf{Q}_\ell(-2)$		$4\mathbf{Q}_\ell(-3)$	$2\mathbf{Q}_\ell(-6) \oplus \mathbf{Q}_\ell(-5)$
3				
2	$2\mathbf{Q}_\ell(-1)$		$2\mathbf{Q}_\ell(-2)$	$2\mathbf{Q}_\ell(-5)$
1				
0	$\mathbf{Q}_\ell$		$\mathbf{Q}_\ell(-1)$	

FIGURE 5. The nonzero terms on the E_2-page of the Leray spectral sequence for $\pi^2 : \mathcal{X}_2^{\mathrm{Sym}(2)} \to \mathcal{A}_2$.

Theorem 5.5. The cohomology of $\mathcal{X}_2^{\mathrm{Sym}(2)}$ is given by

$$H^k(\mathcal{X}_2^{\mathrm{Sym}(2)}; \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ 0 & k = 1, 3, k > 10 \\ 3\mathbf{Q}_\ell(-1) & k = 2 \\ 6\mathbf{Q}_\ell(-2) & k = 4 \\ 2\mathbf{Q}_\ell(-5) & k = 5 \\ 6\mathbf{Q}_\ell(-3) & k = 6 \\ 2\mathbf{Q}_\ell(-6) \oplus \mathbf{Q}_\ell(-5) & k = 7 \\ 3\mathbf{Q}_\ell(-4) & k = 8 \\ 2\mathbf{Q}_\ell(-7) & k = 9 \\ \mathbf{Q}_\ell(-5) & k = 10 \end{cases}$$

up to semi-simplification.

Proof. By Proposition 2.8, there is a spectral sequence with $E_2^{p,q} = H^p(\mathcal{A}_2; H^q(\mathrm{Sym}^2 A))$ which degenerates on the E_2-page and converges to $H^*(\mathcal{X}_2^{\mathrm{Sym}(2)}; \mathbf{Q}_\ell)$. Using Lemma 5.4 and Corollary 3.4, we can compute $E_2^{p,q}$ for all $p, q \geq 0$. The results are recorded in Figure 5, from which the theorem follows directly. □

Remark 5.6. Note for all $n \geq 2$, the $E_2^{p,q}$ term in the spectral sequence for $\mathcal{X}_2^{\mathrm{Sym}(n)} \to \mathcal{A}_2$ for $q = 0, 1, 2, 3$ remain stable and are given in the corresponding entries in Figure 5.

6. ARITHMETIC STATISTICS

In this section, we apply the cohomological results of the previous sections to obtain arithmetic statistics results about abelian surfaces over finite fields. In Section 6.2 we point out that the techniques of this paper can be applied to give arithmetic statistics about abelian surfaces with an ordered basis of its N-torsion given the cohomology of local systems of $\mathcal{A}_2[N]$, and apply the conjectural formulas ([BFvdG08]) in the case $N = 2$ as an example.

Given the étale cohomology of a variety over a finite field $\mathbf{F}_q$, one can use the Grothendieck–Lefschetz trace formula to immediately deduce the number of $\mathbf{F}_q$-points on the variety. Even though the spaces $\mathcal{X}$ studied in this paper are not varieties but rather algebraic stacks, there is fortunately an applicable generalization, the Grothendieck–Lefschetz–Behrend trace formula, which gives the *groupoid cardinality* of their $\mathbf{F}_q$-points.

Definition 6.1. Let X be a groupoid. The *groupoid cardinality* $\#X$ is defined as

$$\#X = \sum_{x\in X} \frac{1}{\#\operatorname{Aut}(x)}.$$

The following definition is necessary in order to state the Grothendieck–Lefschetz–Behrend trace formula.

Definition 6.2. Let $\mathcal{X}$ be a smooth Deligne–Mumford stack of finite type over $\mathbf{F}_q$. Let $\mathcal{X}_{\overline{\mathbf{F}}_q,\mathrm{sm}}$ be the smooth site associated to $\mathcal{X}_{\overline{\mathbf{F}}_q}$. The arithmetic Frobenius acting on $H^i(\mathcal{X}_{\overline{\mathbf{F}}_q,\mathrm{sm}}, \mathbf{Q}_\ell)$ is denoted by $\Phi_q : H^i(\mathcal{X}_{\overline{\mathbf{F}}_q,\mathrm{sm}}, \mathbf{Q}_\ell) \to H^i(\mathcal{X}_{\overline{\mathbf{F}}_q,\mathrm{sm}}, \mathbf{Q}_\ell)$. The action of Φ_q on $\mathbf{Q}_\ell(1)$ is multiplication by q. For any $n \in \mathbf{Z}$, the action of Φ_q on $\mathbf{Q}_\ell(n)$ is multiplication by q^n.

Remark 6.3. For Deligne–Mumford stacks, the étale and smooth cohomology of abelian sheaves coincide: the étale and smooth cohomology of an abelian sheaf on schemes coincide by [Sta21, Lemma 03YY] and so the same holds for Deligne–Mumford stacks by étale descent since such stacks admit étale covers by schemes. All stacks in this section are Deligne–Mumford stacks over finite fields $\mathbf{F}_q$ (or their algebraic closures $\overline{\mathbf{F}}_q$) of any characteristic (see Section 2.1). We will omit the distinction and just write $H^*(\mathcal{X}; \mathbf{Q}_\ell)$ for étale (or smooth) cohomology of $\mathcal{X}_{\overline{\mathbf{F}}_q}$.

The main tool of this section is the following trace formula.

Theorem 6.4 (Grothendieck–Lefschetz–Behrend trace formula, [Beh93, Theorem 3.1.2])**.** Let $\mathcal{X}$ be a smooth Deligne–Mumford stack of finite type and constant dimension over the finite field $\mathbf{F}_q$. Then

$$q^{\dim \mathcal{X}} \sum_{k\geq 0} (-1)^k (\operatorname{tr} \Phi_q \mid H^k(\mathcal{X}_{\overline{\mathbf{F}}_q,\mathrm{sm}}, \mathbf{Q}_\ell)) = \sum_{\xi\in[\mathcal{X}(\mathbf{F}_q)]} \frac{1}{\#\operatorname{Aut}(\xi)} =: \#\mathcal{X}(\mathbf{F}_q).$$

6.1. **Applying the Grothendieck–Lefschetz–Behrend trace formula.** In this subsection, we apply the trace formula (Theorem 6.4) to deduce corollaries of the cohomological results of the previous sections. Although all cohomology computations in the previous sections are only up to semi-simplification, the trace of a linear operator does not change under semi-simplification. Therefore, we apply the trace formula (Theorem 6.4) to the semi-simplification without making this distinction.

Recall that the interpretation of the $\mathbf{F}_q$-points of each stack in question is given in Section 2.1. This first count is well-known, but we list it below for completeness.

Theorem 6.5.

$$\#\mathcal{A}_2(\mathbf{F}_q) = q^3 + q^2.$$

Proof. Using Theorem 2.1, apply the trace formula (Theorem 6.4), noting that $\dim \mathcal{A}_2 = 3$ and that the values $\operatorname{tr}(\Phi_q \mid H^k(\mathcal{A}_2; \mathbf{Q}_\ell))$ are known. □

The point counts in the rest of this section are new to the best of our knowledge.

Theorem 6.6.

$$\begin{aligned}\#\mathcal{X}_2(\mathbf{F}_q) &= q^5+2q^4+2q^3+q^2-1,\\ \#\mathcal{X}_2^2(\mathbf{F}_q) &= q^7+4q^6+9q^5+9q^4+3q^3-5q^2-5q-3,\\ \#\mathcal{X}_2^{\mathrm{Sym}(2)}(\mathbf{F}_q) &= q^7+3q^6+6q^5+6q^4+3q^3-2q^2-2q-2.\end{aligned}$$

Proof. In all cases, the counts follow from the trace formula (Theorem 6.4). Applying Theorem 1.1 and the fact that $\dim\mathcal{X}_2=5$,

$$\#\mathcal{X}_2(\mathbf{F}_q)=q^5(1+2q^{-1}+2q^{-2}-q^{-5}+q^{-3})=q^5+2q^4+2q^3+q^2-1.$$

Applying Theorem 4.16 and the fact that $\dim\mathcal{X}_2^2=7$,

$$\begin{aligned}\#\mathcal{X}_2^2(\mathbf{F}_q) &= q^7\left(1+4q^{-1}+9q^{-2}-(3q^{-5}+q^{-4})+9q^{-3}\right)\\ &\quad+q^7\left(-(3q^{-5}+4q^{-6})+4q^{-4}-(3q^{-7}+q^{-6})+q^{-5}\right)\\ &= q^7+4q^6+9q^5+9q^4+3q^3-5q^2-5q-3.\end{aligned}$$

Applying Theorem 5.5 and the fact that $\dim\mathcal{X}_2^{\mathrm{Sym}(2)}=7$,

$$\begin{aligned}\#\mathcal{X}_2^{\mathrm{Sym}(2)}(\mathbf{F}_q) &= q^7\left(1+3q^{-1}+6q^{-2}-2q^{-5}+6q^{-3}\right)\\ &\quad+q^7\left(-(q^{-5}+2q^{-6})+3q^{-4}-2q^{-7}+q^{-5}\right)\\ &= q^7+3q^6+6q^5+6q^4+3q^3-2q^2-2q-2\end{aligned}$$

□

We can also piece together the partial information we have about $H^k(\mathcal{X}_2^n;\mathbf{Q}_\ell)$ and $H^k(\mathcal{X}_2^{\mathrm{Sym}(n)};\mathbf{Q}_\ell)$ to give an approximation of $\#\mathcal{X}_2^n(\mathbf{F}_q)$ and $\#\mathcal{X}_2^{\mathrm{Sym}(n)}(\mathbf{F}_q)$, for fixed n and asymptotic in q.

Theorem 6.7. For all $n\geq 1$,

$$\#\mathcal{X}_2^n(\mathbf{F}_q)=q^{3+2n}+\left(\binom{n+1}{2}+1\right)q^{2+2n}+\left(\frac{n(n+1)(n^2+n+2)}{8}+\binom{n+1}{2}\right)q^{1+2n}+O(q^{2n}).$$

For $n=3$,

$$\#\mathcal{X}_2^{\mathrm{Sym}(3)}(\mathbf{F}_q)=q^9+3q^8+O(q^7)$$

and for all $n\geq 4$,

$$\#\mathcal{X}_2^{\mathrm{Sym}(n)}(\mathbf{F}_q)=q^{3+2n}+3q^{2+2n}+9q^{1+2n}+O(q^{2n}).$$

Proof. By Theorem 2.5, for all $p\geq 0$ and $a\geq b\geq 0$,

$$|\mathrm{tr}(\Phi_q|H^p(\mathcal{A}_2;\mathbf{V}_{a,b}))|\leq \dim H^p(\mathcal{A}_2;\mathbf{V}_{a,b})q^{-\frac{p+a+b}{2}}$$

and so for any $N\geq 0$ such that $N-p\equiv a+b \pmod 2$,

$$\left|\mathrm{tr}\left(\Phi_q\middle|H^p(\mathcal{A}_2;\mathbf{V}_{a,b})\left(\frac{a+b-(N-p)}{2}\right)\right)\right|\leq \dim H^p(\mathcal{A}_2;\mathbf{V}_{a,b})q^{-N/2}.$$

For any $N\geq 0$ and $\mathcal{X}=\mathcal{X}_2^n$ or $\mathcal{X}_2^{\mathrm{Sym}(n)}$, this estimate, the trace formula (Theorem 6.4), the properties of the Leray spectral sequence for $\mathcal{X}\to\mathcal{A}_2$ (Proposition 2.8), and Proposition 4.10 imply

$$\#\mathcal{X}(\mathbf{F}_q)=q^{3+2n}\left(\sum_{0\leq k\leq N}(-1)^k\,\mathrm{tr}\left(\Phi_q|H^k(\mathcal{X};\mathbf{Q}_\ell)\right)\right)+O\left(q^{3+2n-\frac{N+1}{2}}\right).$$

For $\mathcal{X} = \mathcal{X}_2^n$, applying Theorem 1.2 with $N = 5$ gives

$$\#\mathcal{X}_2^n(\mathbf{F}_q) = q^{3+2n} + \left(\binom{n+1}{2} + 1\right) q^{2+2n} + \left(\frac{n(n+1)(n^2+n+2)}{8} + \binom{n+1}{2}\right) q^{1+2n} + O(q^{2n})$$

and for $\mathcal{X} = \mathcal{X}_2^{\mathrm{Sym}(n)}$, the same computation using Theorem 1.3 with $n = N = 3$ gives

$$\#\mathcal{X}_2^{\mathrm{Sym}(3)}(\mathbf{F}_q) = q^9 + 3q^8 + O(q^7)$$

and with $n \geq 4$, $N = 5$ gives

$$\#\mathcal{X}_2^{\mathrm{Sym}(n)}(\mathbf{F}_q) = q^{3+2n} + 3q^{2+2n} + 9q^{1+2n} + O(q^{2n}).$$

Finally, we note that it is possible to compute the exact value of $\#\mathcal{X}_2^{\mathrm{Sym}(3)}$ analogously to the calculation of $\#\mathcal{X}_2^{\mathrm{Sym}(2)}$ in Theorem 6.6 but we omit it for brevity. □

These point counts imply the arithmetic statistics results outlined in Section 1 which we discuss for the remainder of this subsection.

Lemma 6.8. Define a probability measure $\mathbf{P}$ on $[\mathcal{A}_2(\mathbf{F}_q)]$ by

$$\mathbf{P}([A_0]) = \frac{1}{\#\mathcal{A}_2(\mathbf{F}_q)\# \operatorname{Aut}_{\mathbf{F}_q}(A_0)}$$

for each $\mathbf{F}_q$-isomorphism class $[A_0] \in [\mathcal{A}_2(\mathbf{F}_q)]$. For fixed $n \geq 1$, the expected value of the number of $\mathbf{F}_q$-points on nth powers of abelian surfaces with respect to this probability measure is

$$\mathbf{E}[\#A^n(\mathbf{F}_q)] = \frac{\sum_{(A_0,p)\in[\mathcal{X}_2^n(\mathbf{F}_q)]} \frac{1}{\# \operatorname{Aut}_{\mathbf{F}_q}(A_0,p)}}{\sum_{A_0\in[\mathcal{A}_2(\mathbf{F}_q)]} \frac{1}{\# \operatorname{Aut}_{\mathbf{F}_q}(A_0)}} = \frac{\#\mathcal{X}_2^n(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)}.$$

Similarly, the expected value of the groupoid cardinality of $\operatorname{Sym}^n A(\mathbf{F}_q)$ is

$$\mathbf{E}[\# \operatorname{Sym}^n A(\mathbf{F}_q)] = \frac{\sum_{(\operatorname{Sym}^n A_0,p)\in[\mathcal{X}_2^{\mathrm{Sym}(n)}(\mathbf{F}_q)]} \frac{1}{\# \operatorname{Aut}_{\mathbf{F}_q}(\operatorname{Sym}^n A_0,p)}}{\sum_{A_0\in[\mathcal{A}_2(\mathbf{F}_q)]} \frac{1}{\# \operatorname{Aut}_{\mathbf{F}_q}(A_0)}} = \frac{\#\mathcal{X}_2^{\mathrm{Sym}(n)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)}.$$

Proof. Consider a representative abelian surface A_0 in a fixed $\mathbf{F}_q$-isomorphism class $[A_0]$ and let $Z_0 = A_0^n$ or $\operatorname{Sym}^n A_0$. There is an action of $\operatorname{Aut}_{\mathbf{F}_q}(A_0)$ on $Z_0(\mathbf{F}_q)$. For any $p_0 \in Z_0(\mathbf{F}_q)$, its $\mathbf{F}_q$-isomorphism class is precisely its orbit under the action of $\operatorname{Aut}_{\mathbf{F}_q}(A_0)$. Let $\operatorname{Stab}_G(p_0)$ denote the stabilizer of p_0 in the group G. Then the automorphism group of the pair (A_0^n, p_0) is $\operatorname{Stab}_{\operatorname{Aut}_{\mathbf{F}_q}(A_0)}(p_0)$ and the automorphism group of $(\operatorname{Sym}^n A_0, p_0)$ is $\operatorname{Stab}_{\operatorname{Aut}_{\mathbf{F}_q}(A_0)}(p_0) \times \operatorname{Stab}_{S_n}(p_0)$; this is a direct product because the action of S_n and $\operatorname{Aut}_{\mathbf{F}_q}(A_0)$ on A_0^n commute.

Let $N(p_0) = \# \operatorname{Stab}_{S_n}(p_0)$ if $Z_0 = \operatorname{Sym}^n A_0$ and $N(p_0) = 1$ if $Z_0 = A_0^n$. Let $\operatorname{Orb}(p_0)$ denote the orbit of p_0 in $Z_0(\mathbf{F}_q)$ under the action of $\operatorname{Aut}_{\mathbf{F}_q}(A_0)$. The contribution of A_0 (and its corresponding fiber Z_0) to the expected value is

$$\frac{1}{\#\mathcal{A}_2(\mathbf{F}_q)} \sum_{p_0\in[Z_0(\mathbf{F}_q)]} \frac{1}{\# \operatorname{Aut}_{\mathbf{F}_q}(Z_0,p_0)} = \frac{1}{\#\mathcal{A}_2(\mathbf{F}_q)} \sum_{p_0\in[Z_0(\mathbf{F}_q)]} \frac{\# \operatorname{Orb}(p_0)}{N(p_0)\# \operatorname{Aut}_{\mathbf{F}_q}(A_0)} = \frac{\#Z_0(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)\# \operatorname{Aut}_{\mathbf{F}_q}(A_0)}$$

where the first equality follows from the orbit-stabilizer theorem and the second follows from the fact that the groupoid cardinality of $\operatorname{Sym}^n A_0(\mathbf{F}_q)$ is $\sum_{p_0\in \operatorname{Sym}^n A_0(\mathbf{F}_q)} \frac{1}{N(p_0)}$. □

Lemma 6.8 and the results of this section immediately imply the statistics given in Section 1; we restate them here for convenience.

Corollary 1.4. The expected number of $\mathbf{F}_q$-points on abelian surfaces defined over $\mathbf{F}_q$ is

$$\mathbf{E}[\#A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^2 + q + 1 - \frac{1}{q^3+q^2}.$$

Because $\#A^n(\mathbf{F}_q) = \#A(\mathbf{F}_q)^n$ for all abelian surfaces A, the following corollary gives asymptotics for all moments of $\#A(\mathbf{F}_q)$ as well as the exact second moment.

Corollary 1.5. The expected value of $\#A^2(\mathbf{F}_q)$ is

$$\mathbf{E}[\#A^2(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^2(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^4 + 3q^3 + 6q^2 + 3q - \frac{5q^2+5q+3}{q^3+q^2}$$

and for all $n \geq 1$,

$$\mathbf{E}[\#A^n(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^n(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^{2n} + \binom{n+1}{2} q^{2n-1} + \left(\frac{n(n+1)(n^2+n+2)}{8}\right) q^{2n-2} + O(q^{2n-3}).$$

Recall that $\operatorname{Sym}^n A(\mathbf{F}_q)$ for an abelian surface A and any $n \geq 1$ is the set of n-tuples defined over $\mathbf{F}_q$ as tuples, i.e. the n points are permuted by Frob_q.

Corollary 1.6. The expected value of $\#\operatorname{Sym}^n A(\mathbf{F}_q)$ for $n = 2$ is

$$\mathbf{E}[\#\operatorname{Sym}^2 A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^{\operatorname{Sym}(2)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^4 + 2q^3 + 4q^2 + 2q + 1 - \frac{3q^2+2q+2}{q^3+q^2}.$$

For $n = 3$,

$$\mathbf{E}[\#\operatorname{Sym}^3 A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^{\operatorname{Sym}(3)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^6 + 2q^5 + O(q^4)$$

and for all $n \geq 4$,

$$\mathbf{E}[\#\operatorname{Sym}^n A(\mathbf{F}_q)] = \frac{\#\mathcal{X}_2^{\operatorname{Sym}(n)}(\mathbf{F}_q)}{\#\mathcal{A}_2(\mathbf{F}_q)} = q^{2n} + 2q^{2n-1} + 7q^{2n-2} + O(q^{2n-3}).$$

The fact that we have determined the exact value for the second moment means we can calculate the variance of $\#A(\mathbf{F}_q)$ using Corollary 1.4.

Corollary 1.7. The variance of $\#A(\mathbf{F}_q)$ is

$$\operatorname{Var}(\#A(\mathbf{F}_q)) = \mathbf{E}[\#A^2(\mathbf{F}_q)] - (\mathbf{E}[\#A(\mathbf{F}_q)])^2 = q^3 + 3q^2 + q - 1 - \frac{3q^2+3q+1}{q^3+q^2} - \frac{1}{(q^3+q^2)^2}.$$

6.2. **Level structures.** Let $N \geq 2$ and let $\pi_N : \mathcal{X}_2[N] \to \mathcal{A}_2[N]$ be the projection map. For each local system $\mathbf{V}_{a,b}$ on $\mathcal{A}_2$, we can define local systems on $\mathcal{A}_2[N]$ (also denoted $\mathbf{V}_{a,b}$) via pullback by the map $pr : \mathcal{A}_2[N] \to \mathcal{A}_2$. By [Sta21, Lemma 075H], there is an isomorphism of sheaves

$$pr^*(R^q\pi_*\mathbf{Q}_\ell) \cong R^q(\pi_N)_*(pr^*\mathbf{Q}_\ell) \cong R^q(\pi_N)_*\mathbf{Q}_\ell.$$

Therefore, the decomposition given in Lemma 3.1 also holds for the local systems $H^k(A;\mathbf{Q}_\ell)$ on $\mathcal{A}_2[N]$, interpreting all local systems $\mathbf{V}_{a,b}$ as their pullbacks to $\mathcal{A}_2[N]$.

However, the cohomology of local systems on $\mathcal{A}_2[N]$ is not yet known in general. In the case $N = 2$, many parts of the Euler characteristics of local systems $\mathbf{V}_{a,b}$ on $\mathcal{A}_2[2]$ are known and there are conjectures for the rest; this is done in [BFvdG08]. Here, we consider the compactly supported Euler characteristics of such local systems, defined

$$e_c(\mathcal{A}_2[2];\mathbf{V}_{a,b}) := \sum_{k\geq 0}(-1)^k[H_c^k(\mathcal{A}_2[2];\mathbf{V}_{a,b})]$$

taken in the Grothendieck group of an appropriate category, e.g. the category of mixed Hodge structures or of Galois representations.

Conjecture 6.9 (Bergström–Faber–van der Geer, [BFvdG08, Section 10])**.** The compactly supported Euler characteristic of $\mathbf{V}_{1,1}$ over $\mathcal{A}_2[2]$ is given by $5\mathbf{Q}_\ell(-3) - 10\mathbf{Q}_\ell(-2)$. The compactly supported Euler characteristic of $\mathbf{V}_{0,0}$ over $\mathcal{A}_2[2]$ is given by $\mathbf{Q}_\ell(-3) + \mathbf{Q}_\ell(-2) - 14\mathbf{Q}_\ell(-1) + 16\mathbf{Q}_\ell$.

The cohomology of $\mathcal{A}_2[2]$ is also computed in [LW85, Theorem 5.2.1]. Assuming these two calculations and using the Leray spectral sequence of $\mathcal{X}_2[2] \to \mathcal{A}_2[2]$,

$$\begin{aligned} e_c(\mathcal{X}_2[2]; \mathbf{Q}_\ell) = \sum_{k \geq 0} (-1)^k H^k_c(\mathcal{X}_2[2]; \mathbf{Q}_\ell) &= \sum_{p,q \geq 0} (-1)^{p+q} H^p_c(\mathcal{A}_2[2]; H^q(A; \mathbf{Q}_\ell)) \\ &= e_c(\mathcal{A}_2[2]; \mathbf{V}_{0,0}) + e_c(\mathcal{A}_2[2]; \mathbf{V}_{0,0}(-1)) + e_c(\mathcal{A}_2[2]; \mathbf{V}_{0,0}(-2)) + e_c(\mathcal{A}_2[2]; \mathbf{V}_{1,1}) \\ &= \mathbf{Q}_\ell(-5) + 2\mathbf{Q}_\ell(-4) - 7\mathbf{Q}_\ell(-3) - 7\mathbf{Q}_\ell(-2) + 2\mathbf{Q}_\ell(-1) + 16\mathbf{Q}_\ell. \end{aligned}$$

These computations plus a version of the trace formula (Theorem 6.4) for compactly supported cohomology imply that

$$\begin{aligned} \#\mathcal{A}_2[2](\mathbf{F}_q) &= q^3 + q^2 - 14q + 16, \\ \#\mathcal{X}_2[2](\mathbf{F}_q) &= q^5 + 2q^4 - 7q^3 - 7q^2 + 2q + 16. \end{aligned}$$

In particular, note that an $\mathbf{F}_q$-point on $\mathcal{A}_2[N]$ corresponds to an abelian surface A defined over $\mathbf{F}_q$ with an ordered basis of its N-torsion defined over $\mathbf{F}_q$. Therefore, careful analysis of the counts $\#\mathcal{A}_2[2](\mathbf{F}_q)$ and $\#\mathcal{X}_2[2](\mathbf{F}_q)$ will yield the average number of abelian surfaces over $\mathbf{F}_q$ with 2-torsion defined over $\mathbf{F}_q$ and the average number of $\mathbf{F}_q$-points on such abelian surfaces.

Appendix A. Tensor Products of Irreducible $\mathrm{Sp}(4)$-Representations

In this appendix we summarize the combinatorial results decomposing tensor products of irreducible $\mathrm{Sp}(4)$-representations into irreducible ones and prove Lemmas 4.2 and 4.3. As usual, we let $W_{a,b}$ denote the irreducible $\mathrm{Sp}(4)$-representation corresponding to the partition $a \geq b \geq 0$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, so that $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. A *Young diagram of shape* λ is an arrangement of left-justified rows of boxes, such that row i has λ_i-many boxes. A *skew shape* λ/μ, where λ and μ are both partitions with $\mu \subseteq \lambda$ is the arrangement of rows of boxes given by the Young diagram of shape λ, with the Young diagram of shape μ erased. A *skew tableau of shape* λ/μ *with content* $\beta = (\beta_1, \ldots, \beta_k)$ is a labeling of a skew shape λ/μ where β_i-many of the boxes are labeled with the number i. Such a tableau is called *semi-standard* if the labels are nondecreasing along rows and increasing along columns. Given a skew semi-standard tableau of shape λ/μ with content β, we may demand that the concatenation of the reversed rows is a *lattice word*: list the entries of the tableau from right to left, starting from the top row and working down; for any t smaller than the length of this list, the first t elements must contain as many entries i as it contains entries $i + 1$. For example, consider the tableau on the left in Figure 6; the concatenation of the reversed rows is "1121." For $t = 3$, the list of the first 3 entries is "112," and there are more entries labeled "1" than there are "2" in "112" (and of course, more entries labeled "2" than there are "3," and so on). This is true for all $t \leq 4$ for this concatenation of the reversed rows for this tableau.

Skew semi-standard tableaux whose concatenation of the reversed rows is a lattice word are called *Littlewood–Richardson tableaux*. For another description of these tableaux, see [FH04, p. 456]. As an example, we give all Littlewood–Richardson tableaux of shape $(4, 2, 1)/(2, 1)$ and content

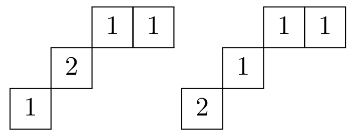

FIGURE 6. All Littlewood–Richardson tableaux of shape $(4,2,1)/(2,1)$ and content $(3,1)$, showing that $c^{(4,2,1)}_{(2,1)(3,1)} = 2$.

$(3,1)$ in Figure 6. Finally, the *Littlewood–Richardson coefficient* $c^{\gamma}_{\alpha\beta}$ is the number of Littlewood–Richardson tableaux of shape γ/α and content β. One important property about Littlewood–Richardson coefficients is that $c^{\gamma}_{\alpha\beta} = c^{\gamma}_{\beta\alpha}$ for all partitions α, β, γ. One can see this by applying the Littlewood–Richardson rule ([FH04, (15.23), (A.8)], [KT87, Theorem 1.4.4]) which says that $c^{\gamma}_{\alpha\beta}$ is the multiplicity of $\mathbf{S}_{\gamma}(V)$ in $\mathbf{S}_{\alpha}(V) \otimes \mathbf{S}_{\beta}(V)$ where $\mathbf{S}_{\gamma}(V)$, $\mathbf{S}_{\alpha}(V)$, and $\mathbf{S}_{\beta}(V)$ are the irreducible representations of $\mathrm{GL}(n)$ corresponding to the partitions α, β, γ using the notation of [FH04, Section 15.3].

Let $\mathscr{P}$ be the set of all partitions. There is a *universal character ring* Λ (a $\mathbf{Z}$-algebra defined in [KT87, Section 1.4]) with a $\mathbf{Z}$-basis $\{\chi_{\mathrm{Sp}}(\lambda)\}_{\lambda \in \mathscr{P}}$ ([KT87, Definition 2.1.1, Proposition 2.1.2]). The structure constants of Λ with respect to the $\mathbf{Z}$-basis $\{\chi_{\mathrm{Sp}}(\lambda)\}_{\lambda \in \mathscr{P}}$ are given by *Newell–Littlewood numbers*:

Theorem A.1 ([Koi89, Theorem 3.1])**.** For any $\mu, \nu \in \mathscr{P}$,

$$\chi_{\mathrm{Sp}}(\mu)\chi_{\mathrm{Sp}}(\nu) = \sum_{\lambda \in \mathscr{P}} N_{\mu\nu\lambda}\chi_{\mathrm{Sp}}(\lambda)$$

with $N_{\mu\nu\lambda} = \sum_{\zeta,\sigma,\tau} c^{\mu}_{\zeta\sigma} c^{\nu}_{\zeta\tau} c^{\lambda}_{\sigma\tau}$. Here, $c^{\gamma}_{\alpha\beta}$ is a Littlewood–Richardson coefficient, i.e. the number of Littlewood–Richardson tableaux of shape γ/α and content β. The constants $N_{\mu\nu\lambda}$ are known as *Newell–Littlewood numbers*.

There is an algebra homomorphism $\pi_{\mathrm{Sp}(2n)} : \Lambda \to R(\mathrm{Sp}(2n))$ where $R(\mathrm{Sp}(2n))$ is the character ring of $\mathrm{Sp}(2n, \mathbf{Q}_\ell)$ called the *specialization homomorphism* ([KT87, Section 2.2]). If $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leq n$ where $\ell(\lambda)$ is the length of the partition λ, then $\pi_{\mathrm{Sp}(2n)}(\chi_{\mathrm{Sp}}(\lambda)) = \chi_{\mathrm{Sp}(2n)}(\lambda)$, the character of the irreducible $\mathrm{Sp}(2n)$-representation corresponding to λ ([KT87, Proposition 2.2.1]). The images $\pi_{\mathrm{Sp}(2n)}(\chi_{\mathrm{Sp}}(\lambda)) \in R(\mathrm{Sp}(2n))$ for $\lambda \in \mathscr{P}$ such that $\ell(\lambda) > n$ are computed in [KT87, Section 2.4] and outlined below:

If $\ell(\lambda) > n$, let λ'_i denote the number of boxes in the ith column of the Young diagram of λ for all $i \geq 1$ and let ℓ be the total number of columns. If there exists $i \geq 1$ for which $\lambda'_i - (i-1) = n+1$, then $\pi_{\mathrm{Sp}(2n)}(\chi_{\mathrm{Sp}}(\lambda)) = 0$.

Now assume $\lambda'_i - (i-1) \neq n+1$ for all $1 \leq i \leq \ell$. For $1 \leq i \leq \ell$, define

$$k_i = \begin{cases} \lambda'_i & \lambda'_i - (i-1) \leq n, \\ 2n + 2i - \lambda'_i & \lambda'_i - (i-1) > n+1 \end{cases}$$

and define $t_i = k_i - (i-1)$. Under these assumptions, $n \geq t_i$ for all $1 \leq i \leq \ell$. If $t_i = t_j$ for some $1 \leq i < j \leq \ell$, then $\pi_{\mathrm{Sp}(2n)}(\chi_{\mathrm{Sp}}(\lambda)) = 0$. Otherwise, reorder the numbers t_i in decreasing order with

$$n \geq t_{i_1} > t_{i_2} > \cdots > t_{i_\ell}.$$

Define $\mu'_k = t_{i_k} + (k-1)$ for $1 \le k \le \ell$. If $\mu'_\ell < 0$ then $\pi_{\mathrm{Sp}(2n)}(\chi_{\mathrm{Sp}}(\lambda)) = 0$. Otherwise, let μ be the Young diagram for which μ'_k is the number of boxes in the kth column. Then $\ell(\mu) \le n$ and

$$\pi_{\mathrm{Sp}(2n)}(\chi_{\mathrm{Sp}}(\lambda)) = (-1)^{\mathrm{sgn}(\sigma)+s}\chi_{\mathrm{Sp}(2n)}(\mu)$$

where σ is the permutation with $\sigma(j) = i_j$ for all $1 \le j \le \ell$ and s is the number of indices $1 \le i \le \ell$ such that $\lambda'_i - (i-1) > n$.

With the arithmetic of Λ given in Theorem A.1 and the specialization homomorphism $\pi_{\mathrm{Sp}(2n)}$, we are ready to prove Lemmas 4.2 and 4.3. Similarly as in Section 4, we set $\chi_{\mathrm{Sp}}(\lambda_1, \ldots, \lambda_n) := 0$ and $\chi_{\mathrm{Sp}(2n)}(\lambda_1, \ldots, \lambda_n) := 0$ if $(\lambda_1, \ldots, \lambda_n)$ is not a partition, or more specifically if $\lambda_i > \lambda_{i-1}$ or $\lambda_i < 0$ for some i.

Proof of Lemma 4.2. By Theorem A.1,

$$\chi_{\mathrm{Sp}}(1)\chi_{\mathrm{Sp}}(a,b) = \sum_{\lambda \in \mathscr{P}} N_{(1)(a,b)\lambda}\chi_{\mathrm{Sp}}(\lambda)$$

with $N_{(1)(a,b)\lambda} = \sum_{\zeta,\sigma,\tau} c^{(1)}_{\zeta\sigma} c^{(a,b)}_{\zeta\tau} c^{\lambda}_{\sigma\tau}$. According to [KT87, p. 509 (1)], the above sum is given by

$$\chi_{\mathrm{Sp}}(1)\chi_{\mathrm{Sp}}(a,b) = \chi_{\mathrm{Sp}}(a-1,b) + \chi_{\mathrm{Sp}}(a+1,b) + \chi_{\mathrm{Sp}}(a,b-1) + \chi_{\mathrm{Sp}}(a,b+1) + \chi_{\mathrm{Sp}}(a,b,1).$$

Applying the specialization homomorphism with [KT87, Proposition 2.2.1] gives

$$\begin{aligned} \chi_{\mathrm{Sp}(4)}(1,0)\chi_{\mathrm{Sp}(4)}(a,b) = {} & \chi_{\mathrm{Sp}(4)}(a-1,b) + \chi_{\mathrm{Sp}(4)}(a+1,b) + \chi_{\mathrm{Sp}(4)}(a,b-1) \\ & + \chi_{\mathrm{Sp}(4)}(a,b+1) + \pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a,b,1)). \end{aligned} \tag{$*$}$$

Now we determine $\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a,b,1))$. Compute that

$$\lambda'_i = \begin{cases} 3 & i = 1, \\ 2 & 2 \le i \le b, \\ 1 & b < i \le a. \end{cases}$$

With $i = 1$ and $2n = 4$,

$$\lambda'_i - (i-1) = 3 = n+1$$

which implies that $\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a,b,1)) = 0$.

The character of the $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representation $W_{1,0} \otimes W_{a,b}$ is the product $\chi_{\mathrm{Sp}(4)}(1)\chi_{\mathrm{Sp}(4)}(a,b)$. Therefore, rewriting $(*)$ on the level of $\mathrm{Sp}(4, \mathbf{Q}_\ell)$-representations proves this lemma. □

Proof of Lemma 4.3. Let $a \ge b \ge 0$. Suppose ζ, σ, τ, and λ are partitions such that $c^{(1,1)}_{\zeta\sigma} c^{(a,b)}_{\zeta\tau} c^{\lambda}_{\sigma\tau} \neq 0$; we first list all possibilities for such partitions ζ, σ, τ, and λ. Throughout this proof, we define $c^{\gamma}_{\alpha\beta} := 0$ and $N_{\alpha\beta\gamma} := 0$ if any of α, β, or γ are tuples of nonnegative integers but are not partitions.

Since $\zeta, \sigma \subseteq (1,1)$, $\zeta, \tau \subseteq (a,b)$, we must have $\zeta = (\zeta_1, \zeta_2)$, $\sigma = (\sigma_1, \sigma_2)$, and $\tau = (\tau_1, \tau_2)$.

Consider $c^{(1,1)}_{\zeta\sigma}$. We have $\zeta, \sigma \subseteq (1,1)$ with $\zeta_1 + \zeta_2 + \sigma_1 + \sigma_2 = 2$. This forces the three possibilities: (1) $\zeta = (1,1)$ and $\sigma = (0,0)$, (2) $\zeta = (1,0)$ and $\sigma = (1,0)$, or (3) $\zeta = (0,0)$ and $\sigma = (1,1)$.

Next, consider $c^{(a,b)}_{\zeta\tau} = c^{(a,b)}_{\tau\zeta}$, and the above three possibilities.

(1) If $\zeta = (1,1)$, then a tableau with shape $(a,b)/\zeta$ has $a-1$ boxes in the first row and $b-1$ boxes in the second row. In order for the tableau to satisfy the lattice word condition and have the labels be increasing within each row, the entire first row must be labeled 1. Therefore, the content (τ_1, τ_2) must satisfy $\tau_1 \ge a-1$ with $\tau_1 + \tau_2 = a + b - 2$. Because $\tau \subseteq (a,b)$, this forces two possibilities: $\tau = (a-1, b-1)$ or $\tau = (a, b-2)$.

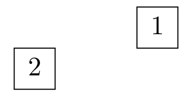

FIGURE 7. The unique Littlewood–Richardson tableaux of shape $(a,b)/(a-1,b-1)$ or $(a+1,b+1)/(a,b)$ and content $(1,1)$.

Suppose $\tau = (a, b-2)$. We count tableau with shape $(a,b)/\tau$ and content $\zeta = (1,1)$. The tableau of shape $(a,b)/(a,b-2)$ has one row with two boxes, and the lattice word condition imposes that all boxes of the first row must be labeled 1. Therefore, there are no such tableau with content $\zeta = (1,1)$. For $\tau = (a-1,b-1)$, see Figure 7 for the unique tableau with shape $(a,b)/(a-1,b-1)$ and content $\zeta = (1,1)$.

(2) If $\zeta = (1,0)$, then a tableau with shape $(a,b)/\tau$ and content $\zeta = (1,0)$ must satisfy $\tau = (a-1,b)$ or $(a,b-1)$. In both cases, such a tableau is the unique tableau with one box.
(3) If $\zeta = (0,0)$, then a tableau with shape $(a,b)/\tau$ and content $\zeta = (0,0)$ must satisfy $\tau = (a,b)$. In this case, such a tableau must be the empty one.

Lastly, consider $c^{\lambda}_{\sigma\tau} = c^{\lambda}_{\tau\sigma}$.

(1) If $\zeta = (1,1)$, $\sigma = (0,0)$, and $\tau = (a-1,b-1)$, then a tableau with shape λ/τ and content $\sigma = (0,0)$ must satisfy $\lambda = (a-1,b-1)$. The only such tableau is the empty one.
(2) If $\zeta = (1,0)$, $\sigma = (1,0)$, and $\tau = (a-1,b)$ or $(a,b-1)$, then a tableau with shape λ/τ and content $\sigma = (1,0)$ must satisfy $\sum_{i\geq 1} \lambda_i = a+b$ with $\tau \subseteq \lambda = (\lambda_1, \lambda_2, \dots)$. If $\tau = (a-1,b)$, then $\lambda = (a,b)$, $(a-1,b+1)$, or $(a-1,b,1)$. If $\tau = (a,b-1)$, then $\lambda = (a+1,b-1)$, (a,b), or $(a,b-1,1)$. In all cases, such a tableau must be the unique one with one box.
(3) If $\zeta = (0,0)$, $\sigma = (1,1)$, and $\tau = (a,b)$, then a tableau with shape λ/τ and content $\sigma = (1,1)$ must satisfy $\sum_{i\geq 1} \lambda_i = a+b+2$ with $\tau = (a,b) \subseteq \lambda = (\lambda_1, \lambda_2, \dots)$. Then λ is one of

$$(a+2,b),\ (a+1,b+1),\ (a,b+2),\ (a+1,b,1),\ (a,b+1,1),\ (a,b,2),\ (a,b,1,1).$$

   (a) Suppose $\lambda = (a+2,b)$ or $(a,b+2)$. The tableau of shape $\lambda/(a,b)$ has one row with two boxes, and the lattice word condition imposes that all boxes of the first row must be labeled 1. Therefore, there are no such tableau with content $\sigma = (1,1)$.
   (b) Suppose $\lambda = (a+1,b+1)$. See Figure 7 for the unique tableau of shape $(a+1,b+1)/(a,b)$ and content $\sigma = (1,1)$.
   (c) Suppose $\lambda = (a+1,b,1)$ or $(a,b+1,1)$. The tableau of shape $\lambda/(a,b)$ has two boxes, each on a distinct row. Therefore, there is a unique tableau of this shape with content $\sigma = (1,1)$.
   (d) Suppose $\lambda = (a,b,2)$. The tableau of shape $(a,b,2)/(a,b)$ has one row with two boxes, and the lattice word condition imposes that all boxes of the first row must be labeled 1. Therefore, there are no such tableau with content $\sigma = (1,1)$.
   (e) Suppose $\lambda = (a,b,1,1)$. The tableau of shape $\lambda/(a,b)$ has two boxes, in two rows and one column. Therefore, there is a unique tableau of shape $\lambda/(a,b)$ and content $\sigma = (1,1)$.

In Figure 8, we record the calculations of the Littlewood–Richardson coefficients $c^{(1,1)}_{\zeta\sigma}$, $c^{(a,b)}_{\zeta\tau}$, and $c^{\lambda}_{\sigma\tau}$ for partitions λ, ζ, σ, and τ found above such that $c^{(1,1)}_{\zeta\sigma} c^{(a,b)}_{\zeta\tau} c^{\lambda}_{\sigma\tau} \neq 0$. Next, we compute

λ	ζ	σ	τ	$c^{(1,1)}_{\zeta\sigma}$	$c^{(a,b)}_{\zeta\tau}$	$c^{\lambda}_{\sigma\tau}$	$\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(\lambda))$
$(a-1,b-1)$	$(1,1)$	$(0,0)$	$(a-1,b-1)$	1	1	1	$\chi_{\mathrm{Sp}(4)}(\lambda)$
$(a-1,b+1)$	$(1,0)$	$(1,0)$	$(a-1,b)$	1	1	1	$\chi_{\mathrm{Sp}(4)}(\lambda)$
(a,b)	$(1,0)$	$(1,0)$	$(a-1,b)$	1	1	1	$\chi_{\mathrm{Sp}(4)}(\lambda)$
	$(1,0)$	$(1,0)$	$(a,b-1)$	1	1	1	
$(a+1,b-1)$	$(1,0)$	$(1,0)$	$(a,b-1)$	1	1	1	$\chi_{\mathrm{Sp}(4)}(\lambda)$
$(a+1,b+1)$	$(0,0)$	$(1,1)$	(a,b)	1	1	1	$\chi_{\mathrm{Sp}(4)}(\lambda)$
$(a,b,1,1)$	$(0,0)$	$(1,1)$	(a,b)	1	1	1	$-\chi_{\mathrm{Sp}(4)}(a,b)$
$(a-1,b,1)$	$(1,0)$	$(1,0)$	$(a-1,b)$	1	1	1	0
$(a,b-1,1)$	$(1,0)$	$(1,0)$	$(a,b-1)$	1	1	1	0
$(a+1,b,1)$	$(0,0)$	$(1,1)$	(a,b)	1	1	1	0
$(a,b+1,1)$	$(0,0)$	$(1,1)$	(a,b)	1	1	1	0

FIGURE 8. All partitions λ, ζ, σ, and τ such that $c^{(1,1)}_{\zeta\sigma}c^{(a,b)}_{\zeta\tau}c^{\lambda}_{\sigma\tau} \neq 0$ for fixed (a,b) with $a \geq b \geq 0$. The values given for $c^{\gamma}_{\alpha\beta}$ assume that α, β, and γ are all partitions. Otherwise, replace the nonzero value given for $c^{\gamma}_{\alpha\beta}$ with 0.

$N_{(1,1)(a,b)\lambda}\chi_{\mathrm{Sp}}(\lambda)$ for some of the partitions λ listed in Figure 8. For any subset $S \subseteq \mathscr{P}$, the indicator function of S is denoted by $\mathbf{1}_S$ (cf. Definition 4.6). Recall that if λ is not a partition, then $\chi_{\mathrm{Sp}}(\lambda) = 0$ by definition.

(1) If $\lambda = (a-1,b-1)$, then $\zeta = (1,1)$, $\sigma = (0,0)$, and $\tau = (a-1,b-1)$, so

$$N_{(1,1)(a,b)(a-1,b-1)} = c^{(1,1)}_{(1,1)(0,0)}c^{(a,b)}_{(1,1)(a-1,b-1)}c^{(a-1,b-1)}_{(0,0)(a-1,b-1)} = \mathbf{1}_{b\geq 1}(a,b)$$

and

$$N_{(1,1)(a,b)(a-1,b-1)}\chi_{\mathrm{Sp}}(a-1,b-1) = \mathbf{1}_{b\geq 1}(a,b)\chi_{\mathrm{Sp}}(a-1,b-1) = \chi_{\mathrm{Sp}}(a-1,b-1).$$

(2) If $\lambda = (a-1,b+1)$, then $\zeta = (1,0)$, $\sigma = (1,0)$, and $\tau = (a-1,b)$, so

$$N_{(1,1)(a,b)(a-1,b+1)} = c^{(1,1)}_{(1,0)(1,0)}c^{(a,b)}_{(1,0)(a-1,b)}c^{(a-1,b+1)}_{(1,0)(a-1,b)} = \mathbf{1}_{a\geq b+2}(a,b)$$

and

$$N_{(1,1)(a,b)(a-1,b+1)}\chi_{\mathrm{Sp}}(a-1,b+1) = \mathbf{1}_{a\geq b+2}(a,b)\chi_{\mathrm{Sp}}(a-1,b+1) = \chi_{\mathrm{Sp}}(a-1,b+1).$$

(3) If $\lambda = (a,b)$, then
   (a) $\zeta = (1,0)$, $\sigma = (1,0)$, and $\tau = (a-1,b)$ or
   (b) $\zeta = (1,0)$, $\sigma = (1,0)$, and $\tau = (a,b-1)$, so

$$N_{(1,1)(a,b)(a,b)} = c^{(1,1)}_{(1,0)(1,0)}c^{(a,b)}_{(1,0)(a-1,b)}c^{(a,b)}_{(1,0)(a-1,b)} + c^{(1,1)}_{(1,0)(1,0)}c^{(a,b)}_{(1,0)(a,b-1)}c^{(a,b)}_{(1,0)(a,b-1)} = \mathbf{1}_{a\geq b+1}(a,b) + \mathbf{1}_{b\geq 1}(a,b)$$

and

$$N_{(1,1)(a,b)(a,b)}\chi_{\mathrm{Sp}}(a,b) = (\mathbf{1}_{a\geq b+1}(a,b) + \mathbf{1}_{b\geq 1}(a,b))\chi_{\mathrm{Sp}}(a,b).$$

(4) If $\lambda = (a+1,b-1)$, then $\zeta = (1,0)$, $\sigma = (1,0)$, and $\tau = (a,b-1)$, so

$$N_{(1,1)(a,b)(a+1,b-1)} = c^{(1,1)}_{(1,0)(1,0)}c^{(a,b)}_{(1,0)(a,b-1)}c^{(a+1,b-1)}_{(1,0)(a,b-1)} = \mathbf{1}_{b\geq 1}(a,b)$$

and

$$N_{(1,1)(a,b)(a+1,b-1)}\chi_{\mathrm{Sp}}(a+1,b-1) = \mathbf{1}_{b\geq 1}(a,b)\chi_{\mathrm{Sp}}(a+1,b-1) = \chi_{\mathrm{Sp}}(a+1,b-1).$$

(5) If $\lambda = (a+1, b+1)$, then $\zeta = (0,0)$, $\sigma = (1,1)$, and $\tau = (a,b)$, so

$$N_{(1,1)(a,b)(a+1,b+1)} = c^{(1,1)}_{(0,0)(1,1)} c^{(a,b)}_{(0,0)(a,b)} c^{(a+1,b+1)}_{(1,1)(a,b)} = 1$$

and

$$N_{(1,1)(a,b)(a+1,b+1)} \chi_{\mathrm{Sp}}(a+1, b+1) = \chi_{\mathrm{Sp}}(a+1, b+1).$$

(6) If $\lambda = (a, b, 1, 1)$, then $\zeta = (0,0)$, $\sigma = (1,1)$, and $\tau = (a,b)$, so

$$N_{(1,1)(a,b)(a,b,1,1)} = c^{(1,1)}_{(0,0)(1,1)} c^{(a,b)}_{(0,0)(a,b)} c^{(a,b,1,1)}_{(1,1)(a,b)} = \mathbf{1}_{b \geq 1}(a,b)$$

and

$$N_{(1,1)(a,b)(a,b,1,1)} \chi_{\mathrm{Sp}}(a, b, 1, 1) = \mathbf{1}_{b \geq 1}(a,b) \chi_{\mathrm{Sp}}(a, b, 1, 1).$$

Aside from the six partitions λ considered above, the remaining partitions λ such that $N_{(1,1)(a,b)\lambda} \neq 0$ are $\lambda = (a+\varepsilon_1, b+\varepsilon_2, 1)$ for $(\varepsilon_1, \varepsilon_2) = (\pm 1, 0), (0, \pm 1)$. For such $\lambda = (a+\varepsilon_1, b+\varepsilon_2, 1)$, compute that

$$\lambda'_i = \begin{cases} 3 & i = 1, \\ 2 & 2 \leq i \leq b + \varepsilon_2, \\ 1 & b + \varepsilon_2 < i \leq a + \varepsilon_1. \end{cases}$$

With $i = 1$,

$$\lambda'_i - (i-1) = 3 = n+1$$

which implies that $\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a+\varepsilon_1, b+\varepsilon_2, 1)) = 0$.

Next, we determine $\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a, b, 1, 1))$. Compute that

$$\lambda'_i = \begin{cases} 4 & i = 1, \\ 2 & 2 \leq i \leq b \\ 1 & b < i \leq a. \end{cases}$$

Then $\lambda'_i - (i-1) \neq n+1 = 3$ for all i. Compute that

$$k_i = \begin{cases} 2 & 1 \leq i \leq b \\ 1 & b < i \leq a. \end{cases}$$

The values $t_i = k_i - (i-1)$ are all distinct and decreasing so compute that

$$\mu'_i = t_i + (i-1) = k_i = \begin{cases} 2 & 1 \leq i \leq b \\ 1 & b < i \leq a. \end{cases}$$

The Young diagram defined by the values μ'_i is the diagram of the partition (a, b). Since the t_i are strictly decreasing, the permutation σ reordering these terms is trivial. The only index i such that $\lambda'_i - (i-1) > n = 2$ is $i = 1$, and so the number s of such indices is 1. Finally, [KT87, Proposition 2.4.1] implies that

$$\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a, b, 1, 1)) = (-1)^{\mathrm{sgn}(\sigma)+s} \chi_{\mathrm{Sp}(4)}(a, b) = -\chi_{\mathrm{Sp}(4)}(a, b).$$

We record the results of these calculations of $\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(\lambda))$ in Figure 8. Recall that $\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(\lambda)) = \chi_{\mathrm{Sp}(4)}(\lambda)$ if $\ell(\lambda) \leq 2$ by [KT87, Proposition 2.2.1].

Applying Theorem A.1 and applying the specialization homomorphism $\pi_{\mathrm{Sp}(4)}$ with [KT87, Proposition 2.2.1] and the above computations,

$$\begin{aligned}
\chi_{\mathrm{Sp}(4)}(1,1)\chi_{\mathrm{Sp}(4)}(a,b) &= \sum_{\lambda\in\mathscr{P}} N_{(1,1)(a,b)\lambda}\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(\lambda)) \\
&= (\mathbf{1}_{a\geq b+1}(a,b)+\mathbf{1}_{b\geq 1}(a,b))\chi_{\mathrm{Sp}(4)}(a,b)+\chi_{\mathrm{Sp}(4)}(a-1,b-1)+\chi_{\mathrm{Sp}(4)}(a-1,b+1) \\
&\quad + \chi_{\mathrm{Sp}(4)}(a+1,b-1)+\chi_{\mathrm{Sp}(4)}(a+1,b+1) \\
&\quad + \left(\sum_{(\varepsilon_1,\varepsilon_2)=(\pm1,0),(0,\pm1)} N_{(1,1)(a,b)(a+\varepsilon_1,b+\varepsilon_2,1)}\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a+\varepsilon_1,b+\varepsilon_2,1))\right) \\
&\quad + \mathbf{1}_{b\geq 1}(a,b)\pi_{\mathrm{Sp}(4)}(\chi_{\mathrm{Sp}}(a,b,1,1)) \\
&= (\mathbf{1}_{a\geq b+1}(a,b)+\mathbf{1}_{b\geq 1}(a,b))\chi_{\mathrm{Sp}(4)}(a,b)+\chi_{\mathrm{Sp}(4)}(a-1,b-1)+\chi_{\mathrm{Sp}(4)}(a-1,b+1) \\
&\quad + \chi_{\mathrm{Sp}(4)}(a+1,b-1)+\chi_{\mathrm{Sp}(4)}(a+1,b+1)-\mathbf{1}_{b\geq 1}(a,b)\chi_{\mathrm{Sp}(4)}(a,b) \\
&= \mathbf{1}_{a\geq b+1}(a,b)\chi_{\mathrm{Sp}(4)}(a,b)+\chi_{\mathrm{Sp}(4)}(a-1,b-1)+\chi_{\mathrm{Sp}(4)}(a-1,b+1) \\
&\quad + \chi_{\mathrm{Sp}(4)}(a+1,b-1)+\chi_{\mathrm{Sp}(4)}(a+1,b+1).
\end{aligned}$$

The character of the $\mathrm{Sp}(4,\mathbf{Q}_\ell)$-representation $W_{1,1}\otimes W_{a,b}$ is the product $\chi_{\mathrm{Sp}(4)}(1,1)\chi_{\mathrm{Sp}(4)}(a,b)$. Therefore, rewriting the above equality on the level of $\mathrm{Sp}(4,\mathbf{Q}_\ell)$-representations proves this lemma. □

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