1. Introduction

This topic proposal is a collection of case studies computing the cohomology of three moduli spaces: configuration spaces of complex projective varieties, the parameter space of nonsingular quadrics in $\mathbb{CP}^2$, and the universal abelian surface. We devote the next three sections (Sections 2, 3, 4) respectively to each of these examples, and attempt to give a flavor of the arguments involved.

In particular, the study of moduli spaces of certain algebraic varieties often leads to applications to arithmetic statistics, yielding average information about the number of $\mathbb{F}_q$-points on such varieties. We highlight these consequences for abelian surfaces in Section 4.

2. Configuration Spaces

For any topological space $X$, there is an associated space parametrizing (ordered or unordered) $n$-tuples of distinct points of $X$. We focus on the ordered case.

Definition 2.1. The ordered configuration space of $n$-points on $X$ is defined as

$$\text{PConf}_n(X) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}.$$  

The integral cohomology of $\text{PConf}_n(C)$ is given by a well-known result of Arnol'd.

Theorem 2.2 ([Arn69]). For $i, j \in [n]$ with $i \neq j$, let

$$\omega_{i,j} = \frac{1}{2\pi i} \left( \frac{dz_i - dz_j}{z_i - z_j} \right)$$

be 1-forms corresponding to "loops around \{z_i = z_j\}". Then the integral cohomology ring of $\text{PConf}_n(C)$ is given by

$$H^*(\text{PConf}_n(C); \mathbb{Z}) \cong \bigwedge^* \mathbb{Z}[\omega_{i,j} : i \neq j] / \langle \omega_{k,\ell} \wedge \omega_{\ell,m} + \omega_{\ell,m} \wedge \omega_{m,k} + \omega_{m,k} \wedge \omega_{k,\ell} : k, \ell, m \text{ distinct} \rangle.$$  

For any complex projective variety $X$ of real dimension $m$, the calculation of $H^*(\text{PConf}_n(X); \mathbb{Z})$ reduces to a generalization of the above case to $\mathbb{R}^m$ ([Coh88 Theorem 6.1]).

Theorem 2.3 ([Tot96 Theorems 1, 2]). Let $p_a : X^n \rightarrow X$ be the projection onto the $a$th coordinate and let $p_{ab} : X^n \rightarrow X^2$ be the projection onto the $a$th and $b$th coordinates. The $E_2$-page of the Leray spectral sequence for $i : \text{PConf}_n(X) \rightarrow X^n$ is given by a bigraded algebra $H^*(X^n; \mathbb{Z})[G_{ab} : 1 \leq a, b \leq n]/R$, where $R$ consists of the relations

1. $G_{ab} = (-1)^m G_{ba}$ and $G_{ab}^2 = 0$,
2. $G_{ab} G_{ac} + G_{bc} G_{ba} + G_{ca} G_{cb} = 0$ for $a, b, c$ distinct, and
3. $p_a^* (x) G_{ab} = p_b^* (x) G_{ab}$ for $a \neq b$ and $x \in H^*(X; \mathbb{Z})$,

where $H^i(X^n)$ has degree $(i, 0)$ and each $G_{ab}$ has degree $(0, m - 1)$. The first nontrivial differential $d_m$ is given by $dG_{a,b} = p_{a,b}^* \Delta$, where $\Delta$ is the class of the diagonal in $H^m(X^2; \mathbb{Z})$.

Proof Sketch. For any partition $I$ of $[n]$ with $n - s$ parts, define

$$X_I^{-s} := \{(x_k) \in X^n : x_i = x_j \text{ if } i, j \text{ belong to the same subset of } I\} \cong X^{n-s}.$$
If $J$ is a refinement of $I$ with $n - t$ parts, then $X^{n-s}_I \subseteq X^{n-s}_J$. First compute the stalk of $R^t\mathbb{Z}$ at $x \in X^{n-s}_I$ for an arbitrary partition $I$ of $[n]$. Applying the exponential map shows that for $U$ a nice neighborhood of $x \in X^n$,

$$(R^ti_*\mathbb{Z})_x \cong H^q(U \cap \text{PConf}_n(X); \mathbb{Z}) \cong H^q(\text{PConf}_{k+1}(R^m) \times \cdots \times \text{PConf}_{n-s}(R^m); \mathbb{Z}).$$

Cohen’s generalization of Theorem 2.2 gives that $H^*(\text{PConf}(R^m); \mathbb{Z})$ is a graded commutative algebra over $\mathbb{Z}$ generated by $G_{ab}$, $1 \leq a, b \leq k$, satisfying the relations (1) and (2) above. By the Künneth formula, $R^ti_*\mathbb{Z} = 0$ unless $q = r(m - 1)$ for some $r \in \mathbb{Z}_{\geq 0}$.

For fixed partition $I$ with $n - s$ parts and all $0 \leq r \leq s$, there is an interpretation of the cohomology of $\text{PConf}_{n-s}(R^m)$ in degree $r(m - 1)$ as “coming from top degree classes.”

**Lemma 2.4** ([Tol96 Lemma 3]). Fix $0 \leq r \leq s$. Let $J$ be a partition of $[n]$ that refines $I$ which has $n - r$ parts. There is a natural map of spaces $f_j : \prod_{k=1}^{n-s} \text{PConf}_{k}(R^m) \rightarrow \prod_{k=1}^{n-r} \text{PConf}_{k}(R^m)$, where the top degree of cohomology of the target space is $r(m - 1)$. These maps together induce an isomorphism

$$(f^*_j)_{j \in S_{n-r-1}(t)} : \bigoplus_{j \in S_{n-r-1}(t)} H^{r(m-1)}(\bigotimes_{k=1}^{n-r} \text{PConf}_{k}(R^m); \mathbb{Z}) \cong H^{r(m-1)}(\bigotimes_{k=1}^{n-s} \text{PConf}_{k}(R^m); \mathbb{Z})$$

where $S_{n-r-1}(t)$ is the set of all partitions $J$ of $[n]$ which have $n - r$ parts and refine $I$.

Applying Lemma 2.4 stalk-wise to the sheaf $R^{r(m-1)}i_*\mathbb{Z}$ gives a decomposition into a direct sum of sheaves with stalks at $x \in X^{n-s}_I$ as given by the lemma. Each summand corresponding to a partition $J$ is constant on $X^{n-r}_J$ and 0 outside of $X^{n-r}_J$. Therefore by summing over partitions $J$ of $[n]$ with $n - r$ parts,

$$E_2^{p, r(m-1)} \cong H^p(X^n; R^{r(m-1)}i_*\mathbb{Z}) \cong \bigoplus_{|J|=n-r} H^p(X^{n-r}_J; \mathbb{Z}) \otimes H^{r(m-1)}(\bigotimes_{k=1}^{n-r} \text{PConf}_{k}(R^m); \mathbb{Z})$$

$$\cong \bigoplus_{|J|=n-r} H^p(X^{n-r}_J; \mathbb{Z}) \bigotimes_{k=1}^r \prod_{k=1}^r G_{a_k b_k} : a_k, b_k \text{ in the } k\text{th set in } J, a_k < b_k \bigg\}.$$

There is an isomorphism $\Phi$ from $H^p(X^n; \mathbb{Z}) \{\prod_{k=1}^r G_{a_k b_k} \}_{/ (p'_a(x)G_{ab} - p'_b(x)G_{ab} : x \in H^p(X))}$ to the above given by the following. For any $z \prod_{k=1}^r G_{a_k b_k}$ with $z \in H^p(X^n; \mathbb{Z})$, there is a unique partition $J$ of $[n]$ with $n - r$ parts such that $a_k \sim b_k$ in $J$ for all $k$. Let $i_j : X^{n-r}_J \rightarrow X^n$ be the inclusion map and

$$\Phi : z \left( \prod_k G_{a_k b_k} \right) \mapsto i_j^*(z) \left( \prod_k G_{a_k b_k} \right)$$

and extend linearly.

The $E_2$-page is generated as an algebra by $E_2^{p,0}$ and $E_2^{0,m-1}$. Therefore, it is enough to determine $d_m : E_2^{0,m-1} \rightarrow E_2^{m,0}$. Consider the commutative diagram:

$$\begin{array}{ccc}
\text{PConf}_n(X) & \longrightarrow & X^n \\
\downarrow\pi_j & & \downarrow\pi_j \\
X^2 - \Delta & \longrightarrow & X^2
\end{array}$$

The class $G_{ij} \in H^0(X^{n-1}; \mathbb{Z})G_{ij}$ is the pullback of $1 \in E_2^{0,m-1}$ of the spectral sequence of the inclusion on the bottom for which the differential $d_m$ is multiplication by the class of the diagonal $\Delta$. Therefore, $d_m : G_{ij} \mapsto p_{ij}^*(\Delta)$. \hfill \Box

Further, there are no more nontrivial differentials because $X$ is a complex projective variety.
Theorem 2.5 ([Lot96 Theorem 3]). If $X$ is a complex projective variety, then the Leray spectral sequence for $\text{PConf}_n(X) \to X^n$ degenerates on the $E_{m+1}$-page. The cohomology of $\text{PConf}_n(X)$ only depends on $H^*(X)$.

Proof Sketch. If $X$ is a complex projective variety, then the Leray spectral sequence has a weight filtration, which the differentials respect. The $E_2^{p,r}(m-1)$-term is pure of weight $p + rm$. This forces the sequence to degenerate on the $E_{m+1}$-page.

3. Space of Nonsingular Quadrics in $\mathbb{CP}^2$.

A nonsingular algebraic hypersurface of degree $d$ in $\mathbb{CP}^n$ is the vanishing locus of a homogeneous, degree $d$ polynomial in $n + 1$ variables which has nonzero discriminant. The parameter space of such nonsingular hypersurfaces is the projectivization of $\Pi_{d,n} \setminus \Sigma_{d,n}$, where $\Pi_{d,n} = C^{D(d,n)}$ is identified with all homogeneous polynomials of degree $d$ in $n + 1$ variables over $\mathbb{C}$ and $\Sigma_{d,n}$ is the discriminant locus. Call this parameter space $N_{d,n} \subseteq \mathbb{CP}^{D(d,n)-1}$.

The goal of this section is to compute the cohomology of $N_{d,n}$ for the special case of $d = n = 2$, closely following [Vas99].

Theorem 3.1 ([Vas99 Proposition 1]). The integral cohomology of $N_{2,2}$ is given by

$$H^k(N_{2,2}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 5 \\ 0 & k = 1, 2, 4, k > 5 \\ \mathbb{Z}/2\mathbb{Z} & k = 3. \end{cases}$$

Using the Künneth isomorphism and the fact that $N_{d,n} \times C^*$ and $\Pi_{d,n} \setminus \Sigma_{d,n}$ are diffeomorphic, it suffices to compute $H^*(\Pi_{d,n} \setminus \Sigma_{d,n}; \mathbb{Z})$. Letting $\Sigma_{d,n}$ be a one-point compactification of $\Sigma_{d,n}$,

$$\tilde{H}^k(\Pi_{d,n} \setminus \Sigma_{d,n}; \mathbb{Z}) \cong \tilde{H}^k(S^{2D(d,n)} \setminus \Sigma_{d,n}; \mathbb{Z}) \cong \tilde{H}_{2D(d,n)-k-1}(\Sigma_{d,n}; \mathbb{Z}) \text{ by Alexander duality} \cong \tilde{H}_{2D(d,n)-k-1}(\Sigma_{d,n}; \mathbb{Z}).$$

The last isomorphism follows because the Borel–Moore homology $\tilde{H}_{2D(d,n)-k-1}(\Sigma_{d,n}; \mathbb{Q})$ is the reduced homology of $\Sigma_{d,n}$. In the case of $n = d = 2$, the spectral sequence that computes $\tilde{H}_*(\Sigma_{d,n}; \mathbb{Q})$ is simpler to describe than the general case, so we specialize to this case from now on. Define $\Pi := \Pi_{2,2}$ to be the set of all homogeneous quadratic forms in 3 variables and $\Sigma := \Sigma_{2,2}$ its discriminant locus.

Theorem 3.2 ([Vas99 Proposition 1]). The integral cohomology of $\Pi \setminus \Sigma$ is given by

$$H^k(\Pi \setminus \Sigma; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 1, 5, 6 \\ 0 & k = 2, k > 6 \\ \mathbb{Z}/2\mathbb{Z} & k = 3, 4. \end{cases}$$

Proof Sketch. The main idea of Vassiliev’s method is to construct a space $\sigma$ admitting a filtration defined by the possible “types of singularities” of quadrics in $\mathbb{CP}^2$ with a map $\sigma \to \Sigma$ inducing isomorphisms on $H_*$. For any $f \in \Sigma$, denote the singular set of $f$ by $\text{Sing}(f) \subseteq \mathbb{CP}^2$. There are three possible types of singular sets of $f$:

1. $\text{Sing}(f)$ is a point, i.e. $f$ defines two lines intersecting at one point,
2. $\text{Sing}(f)$ is a line, i.e. $f$ defines a double line,
3. $\text{Sing}(f)$ is all of $\mathbb{CP}^2$, i.e. $f \equiv 0$.

The possible singular sets of $f \in S_i$ are parametrized by $A_1 := \mathbb{CP}^2$, $A_2 := \mathbb{CP}^{2\nu}$, and $A_3 := \{\bullet\}$ for the three types respectively. Consider the join $\mathbb{CP}^2 \ast \mathbb{CP}^{2\nu} \ast \{\bullet\}$ and simplices of $\mathbb{CP}^2 \ast \mathbb{CP}^{2\nu} \ast \{\bullet\}$ with vertices $\{v_i\}$ (possibly consisting of less than 3 points) such that $v_i \in A_i$ for each vertex $v_i$. A simplex with vertices $\{v_i\}$ is called coherent if $v_i \subseteq v_j$ for all $i < j$, where we identify the vertices $v_i$ with the corresponding subsets of $\mathbb{CP}^2$. Define $\Lambda$ to be the union of all coherent simplices with the subset topology inherited from $\mathbb{CP}^2 \ast \mathbb{CP}^{2\nu} \ast \{\bullet\}$.
For all sets of the form $K = \text{Sing}(g) \subseteq \mathbb{C}P^2$ for some $g \in \Sigma$, define

$L(K) := \{ f \in \Sigma : K \subseteq \text{Sing}(f) \} \subseteq \Sigma,$

$\Lambda(K) := \text{union of coherent simplices with vertices } \{v_i\} \text{ with } v_i \subseteq K \subseteq \Lambda,$

and finally

$\sigma := \bigcup_{f \in \Sigma} L(\text{Sing}(f)) \times \Lambda(\text{Sing}(f)) \subseteq \Sigma \times \Lambda.$

There is a natural filtration on $\sigma$ which determines the desired spectral sequence. Let

$F_i = \bigcup_{K \in A_j \text{ with } j \leq i} L(K) \times \Lambda(K)$

and so $F_1 \subseteq F_2 \subseteq F_3 = \sigma$. The spectral sequence associated to this filtration of $\sigma$ has $E^1$-page

$E^1_{p,q} = \tilde{H}_{p+q}(F_p \setminus F_{p-1}; \mathbb{Z}) \implies \tilde{H}_*(\sigma; \mathbb{Z}).$

For each $p \in [3]$, the homology of $F_p \setminus F_{p-1}$ can be computed by fitting it in a fiber bundle over a simpler space. To do so, define

$\Phi_i = \bigcup_{K \in A_{j} \text{ with } j \leq i} \Lambda(K)$

which form a filtration of $\Lambda$. There is a pair of fiber bundles (see Figure 1) which combined with the Thom isomorphism give the appropriate $E^1$-page as in Figure 2.

As for the differentials, there are three potentially nontrivial ones:

$d^1_{2,7} : \tilde{H}_9(F_7 \setminus F_1) \rightarrow \tilde{H}_8(F_1), \quad d^1_{2,5} : \tilde{H}_7(F_2 \setminus F_1) \rightarrow \tilde{H}_6(F_1), \quad d^1_{3,5} : \tilde{H}_6(F_3 \setminus F_2) \rightarrow \tilde{H}_7(F_2 \setminus F_1).$

Each differential corresponds to the connecting homomorphism of the long exact sequences of the pairs $(F_i \setminus F_{i-1}, F_{i-1} \setminus F_{i-2})$, which gives a nice geometric description of the images of the generators.
under each map. By analyzing the vector bundles $F_1 \backslash F_{i-1} \to \Phi_i \to \Phi_{i-1}$ carefully, we can deduce that

$$d_{1,7}^1 : \mathbb{Z} \xrightarrow{x^2} \mathbb{Z}, \quad d_{2,5}^2 : \mathbb{Z} \xrightarrow{x^2} \mathbb{Z}, \quad d_{3,5}^3 : \mathbb{Z} \xrightarrow{x^0} \mathbb{Z},$$

from which it is clear that the spectral sequence must degenerate on the $E^2$-page. (See Figure 2 again.)

4. Universal Abelian Surface

In this section, I describe my recent paper [Lee20]. An abelian surface is an abelian variety of dimension 2. Over $\mathbb{C}$, all abelian varieties of dimension $g$ are of the form $\mathbb{C}^g/L$ for some $2g$-real dimensional lattice $L$. Conversely, there is a condition that guarantees that a lattice $L$ gives rise to an abelian variety, not merely a complex torus. The ideas involved generalize abstractly to abelian varieties over arbitrary fields, but we discuss the complex case here following the exposition of [HS02], [BvdGHZ08], and [Mil08].

**Definition 4.1.** Given a lattice $L \subseteq \mathbb{C}^{2g}$ which spans $\mathbb{C}^{2g}$ over $\mathbb{R}$, a polarization with respect to $L$ is Hermitian form $H \geq 0$ such that $H' := \Im(H)(H)^{\mathbb{R}}$ is integer-valued. In particular, $H'$ defines an alternating bilinear form $L \otimes L \to \mathbb{Z}$, which is nondegenerate if and only if $H$ is positive definite.

After extending $H'$ over the $\mathbb{R}$-span of $L$, the identity $H(x, y) = H'(ix, y) + iH'(x, y)$ shows that $H'$ completely determines $H$. By linear algebra, there exists some basis of $L$ such that

$$H'(v, w) = v \Lambda w^T, \quad \Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \quad E = \text{diag}(e_1, \ldots, e_g).$$

If $e_i = 1$ for all $i$, then $H$ is called a principal polarization.

From now on, we only consider the principal case so that $E = 1$. We want to determine a condition on lattices which are principally polarizable. Given a basis $\{v_1, \ldots, v_{2g}\}$ of a lattice $L$, define $\Omega$ to be the $(2g \times g)$-matrix, where the $k$th row is $v_k$ with respect to the standard basis of $\mathbb{C}^g$. This matrix $\Omega$ is called the period matrix.

One can always pick the basis of $\mathbb{C}^g$ so that the period matrix is given by $\Omega = (\tau \ 1)^T$ for some $\tau \in M(g \times g; \mathbb{C})$. For period matrices of this form, the following result of Riemann characterizes those that come from principally polarizable lattices:

**Theorem 4.2** (Riemann’s bilinear relations). Suppose $\Omega = (\tau \ 1)^T$. Then $\Omega$ determines a principally polarizable lattice if and only if

$$\tau^T = \tau, \quad \exists \tau > 0.$$

The associated polarization for $\tau \in \mathcal{H}_g$ is

$$H(x, y) = x \Im(\tau)^{-1}y^T.$$

With this characterization in hand, we define a parameter space of principally polarizable lattices in $\mathbb{C}^g$, i.e. of principally polarized complex abelian varieties of dimension $g$.

**Definition 4.3.** The Siegel upper half space of dimension $g$ is defined

$$\mathcal{H}_g := \{ \tau \in M(g \times g; \mathbb{C}) : \tau^T = \tau, \exists \tau > 0 \}.$$

Viewing these principally polarized lattices up to isomorphism corresponds to a change of basis of the lattice preserving $H' : \wedge^2 L \to \mathbb{Z}$, i.e. the change of basis has to preserve a symplectic form. For any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(4, \mathbb{Z})$, the change of basis of a lattice with period matrix $\Omega = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$ by $M$ corresponds to the action

$$\tau \mapsto (A\tau + B)(C\tau + D)^{-1}.$$

This defines an action of $\text{Sp}(4, \mathbb{Z})$ on $\mathcal{H}_g$, allowing the following definition.

**Definition-Proposition 4.4.** The coarse moduli space of principally polarized (complex) abelian varieties is

$$(A_g)^{an}_C := \text{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g.$$
Remark 4.5. The action of $\text{Sp}(4, \mathbb{Z})$ on $H_g$ is not free. In fact, every abelian variety has a nontrivial automorphism group. For example, $-I$ acts trivially on every point in $H_g$ inducing the inversion automorphism on the group structure of the corresponding abelian variety. However, the stabilizer of any point in $H_g$ is finite. Therefore, one can define

$$A_g(C) := [\text{Sp}(2g, \mathbb{Z}) \backslash H_g]$$

as an orbifold (or stack) quotient in order to obtain a fine moduli space over $C$.

The moduli space of principally polarized abelian varieties is, more generally, a Deligne–Mumford stack $A_g$, although intuitively, one should keep the orbifold picture in mind.

A natural object associated to $A_g$ is the universal abelian surface $X_g$, which is a family of abelian varieties over $A_g$. Complex analytically, the construction is the following:

For every point $\tau \in H_g$, the associated abelian variety is $A_\tau := C^g/L_\tau$, with lattice $L_\tau = \tau \mathbb{Z}^g + \mathbb{Z}^g$. The change of basis isomorphism $f_M : C^g \to C^g$ given by some $M \in \text{Sp}(2g, \mathbb{Z})$ acts as the $R$-linear extension of the corresponding map $v \mapsto Mv$ on $L_\tau$. Because this map preserves the lattice, it descends to a map $C^g/L_\tau \to C^g/L_\tau$.

There is a natural action of $\mathbb{Z}^{2g} \ltimes \text{Sp}(2g, \mathbb{Z})$ on $C^g \times H_g$, along with a $\text{Sp}(2g, \mathbb{Z})$-equivariant projection map to $H_g$. Therefore, the (orbifold or stack) quotient

$$X_g(C) := [(\mathbb{Z}^{2g} \ltimes \text{Sp}(2g, \mathbb{Z})) \backslash (C^g \times H_g)]$$

is the desired universal family over $A_g(C)$. The points of $X_g(C)$ correspond to pairs $(p, A)$ where $A$ is an abelian variety of dimension $g$ defined over $C$ and $p$ is a $C$-point of $A$. Forgetting the orbifold structure of the quotient above again yields the coarse moduli space of $X_g(C)$. Finally, $X_g$ is also more generally a Deligne–Mumford stack.

From now on, we restrict to the case of abelian surfaces ($g = 2$). The properties of the projection $\pi : X_2 \to A_2$ prove to be very useful in computing the cohomology of $X_2$, which is one of the main theorems of this section.

We establish some notation before stating the results. From now on, all cohomology will denote $\ell$-adic cohomology and we drop the subscripts to write $H^* (\mathcal{X}; \mathbf{V})$ in place of both $H^*_\ell(\mathcal{X}, \mathbf{V})$ and $H^*_\ell(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbf{V})$ for any $\ell$-adic local system $\mathbf{V}$ on $\mathcal{X}$ with $\ell$ coprime to $q$. (See [Lee20] Remark 2.4 for details justifying this notation.) For all $m \in \mathbb{N}$ and Galois representation $V$, denote $V^{\otimes m}$ by $mV$.

Theorem 4.6 ([Lee20 Theorem 1.1]). The étale cohomology of the universal abelian surface $X_2$ is given by

$$H^k(X_2; \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & k = 0 \\ 0 & k = 1, 3, k > 7 \\ 2\mathbf{Q}_\ell(-1) & k = 2 \\ 2\mathbf{Q}_\ell(-2) & k = 4 \\ \mathbf{Q}_\ell(-5) & k = 5 \\ \mathbf{Q}_\ell(-3) & k = 6 \end{cases}$$

up to semi-simplification.

Proof Sketch. A version of the usual Leray spectral sequence applies to the projection $\pi : X_2 \to A_2$. The local system $R^k \pi_* \mathbf{Q}_\ell$ on $A_2$ corresponds to the cohomology of the abelian surface $A = C^2/L$, as a representation of $\text{Sp}(4, \mathbb{Z})$ acting by change of basis maps with an action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Define $V_{1,0} := R^1 \pi_* \mathbf{Q}_\ell$; this is a local system with weight 1 whose corresponding $\text{Sp}(4, \mathbb{Z})$-representation is the standard one. This gives rise to local systems $V_{a,b}$ for any $a \geq b \geq 0$ via Weyl’s construction.
The local systems $R^k\pi_*\mathbb{Q}_\ell \cong H^k(A; \mathbb{Q}_\ell)$ are

$$H^k(A; \mathbb{Q}_\ell) \cong \bigwedge^k V_{1,0} \cong \begin{cases} \mathbb{Q}_\ell & k = 0 \\ V_{1,0} & k = 1 \\ \mathbb{Q}_\ell(-1) \oplus V_{1,1} & k = 2 \\ V_{1,0}(-1) & k = 3 \\ \mathbb{Q}_\ell(-2) & k = 4. \end{cases}$$

The Leray spectral sequence for $\pi : \mathcal{X}_2 \to A_2$ has the $E_2$-page

$$E_2^{p,q} = H^p(A_2; H^q(A; \mathbb{Q}_\ell))$$

for which all terms shown in Figure 4 can be determined using [Pet15, Theorem 2.1] and [LW85, Corollary 5.2.3].

At first glance, there is one potential nontrivial differential, $d_3 : E_3^{0,4} \to E_3^{3,2}$. However, a theorem of Deligne ([Del68]) implies that the Leray spectral sequence of projective morphisms degenerate on the $E_2$-page, and a version of this theorem applies to $\pi$. Alternatively, one can see directly that $d_3$ must be the zero map because $E_3^{0,4}$ and $E_3^{3,2}$ have different weights but $d_3$ respects weights. □

Consider $\mathcal{X}_2^n$, the $n$th fiber power of $\mathcal{X}_2$ over $A_2$ via the projection $\pi : \mathcal{X}_2 \to A_2$. The fiber over each point corresponding to an abelian surface $A$ in $A_2$ is the product $A^n$. The Künneth theorem simplifies the local system $H^q(A^n; \mathbb{Q}_\ell)$ for each $q$ and $n$. In fact [Lee20, Section 4] determines recursive formulas in $n$ for $H^q(A^n; \mathbb{Q}_\ell)$ as a $\text{Sp}(4, \mathbb{Z})$-representation.

These calculations and an analogous method as above allow the computation of the cohomology of $\mathcal{X}_2^n$ in low degrees for all $n$.

**Theorem 4.7** ([Lee20, Theorem 1.2]). For all $n$,

$$H^k(\mathcal{X}_2^n; \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & k = 0 \\ 0 & k = 1, 3 \\ \binom{n+1}{2} \mathbb{Q}_\ell(-1) & k = 2 \\ \binom{n(n+1)(n^2+n+2)}{n} \mathbb{Q}_\ell(-2) + \binom{n+1}{2} \mathbb{Q}_\ell(-4) & k = 4 \\ \binom{n+1}{2} \mathbb{Q}_\ell(-5) + \binom{n}{2} \mathbb{Q}_\ell(-4) & k = 5 \end{cases}$$

up to semi-simplification.

For fixed $n \geq 1$, these recursive formulas can be used to determine all of $H^*(\mathcal{X}_2^n; \mathbb{Q}_\ell)$ given enough information about certain (Siegel) modular forms. This computation is explicitly carried out for the case $n = 2$ in [Lee20, Section 4] as an example.
Theorem 4.8 ([Lee20 Theorem 4.17]).

\[ H^k(X_2^2; \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & k = 0 \\ 0 & k = 1, 3, k > 10 \\ 4\mathbb{Q}_\ell(-1) & k = 2 \\ 9\mathbb{Q}_\ell(-2) & k = 4 \\ 3\mathbb{Q}_\ell(-5) \oplus \mathbb{Q}_\ell(-4) & k = 5 \\ 9\mathbb{Q}_\ell(-3) & k = 6 \\ 3\mathbb{Q}_\ell(-5) \oplus 4\mathbb{Q}_\ell(-6) & k = 7 \\ 4\mathbb{Q}_\ell(-4) & k = 8 \\ 3\mathbb{Q}_\ell(-7) \oplus \mathbb{Q}_\ell(-6) & k = 9 \\ \mathbb{Q}_\ell(-5) & k = 10 \end{cases} \]

up to semi-simplification.

4.1. Arithmetic statistics. A classical theorem in topology is the Lefschetz fixed point theorem which relates the fixed points of some map \( \varphi : M \to N \) and the induced map \( \varphi^* \) on \( H^*(N) \to H^*(M) \). The analog of this theorem in algebraic geometry is the Grothendieck–Lefschetz trace formula which counts the number of fixed points of Frobenius acting on the \( \overline{\mathbb{F}}_q \)-points of a smooth variety defined over \( \mathbb{F}_q \). Because we are interested in the smooth Deligne–Mumford stack \( X_2 \) and its variants, we need a further generalization given by the Grothendieck–Lefschetz–Behrend trace formula which computes the groupoid cardinality \#\mathcal{X}(\mathbb{F}_q)\) for stacks \( \mathcal{X} \) smooth over \( \mathbb{F}_q \).

Theorem 4.9 (Grothendieck–Lefschetz–Behrend trace formula, [Beh93 Theorem 3.1.2]). Let \( \mathcal{X} \) be a smooth Deligne–Mumford stack of finite type and constant dimension over the finite field \( \mathbb{F}_q \) and let \( \mathcal{X}_{\mathbb{F}_q} \) be its base-change to \( \mathbb{F}_q \). Let \( \Phi_q \) be the arithmetic Frobenius acting on \( H^*(\mathcal{X}_{\mathbb{F}_q,\text{sm}}; \mathbb{Q}_\ell) \), which acts by multiplication by \( q \) on the cyclotomic character \( \mathbb{Q}_\ell(1) \). Then

\[
q^{\lim \mathcal{X}} \sum_{k \geq 0} (-1)^k (\text{tr} \Phi_q | H^k(\mathcal{X}_{\mathbb{F}_q,\text{sm}}; \mathbb{Q}_\ell)) = \sum_{\xi \in \mathcal{X}(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(\xi)} =: \#\mathcal{X}(\mathbb{F}_q).
\]

Remark 4.10. In the context of the stacks of interest in [Lee20], the cohomology theory of constructible sheaves on the smooth site coincides with that of the étale site. Therefore, we proceed with the computations below using the trace formula without making this distinction.

The fruits of the cohomological labor of the previous section then imply the following statistics immediately.

Theorem 4.11 ([Lee20 Theorems 6.6 and 6.7]).

\[
\#\mathcal{X}_2(\mathbb{F}_q) = q^5 + 2q^4 + 2q^3 + q^2 - 1,
\]

\[
\#\mathcal{X}_2^2(\mathbb{F}_q) = q^7 + 4q^6 + 9q^5 + 9q^4 + 3q^3 - 5q^2 - 5q - 3.
\]

For all \( n \geq 1 \),

\[
\#\mathcal{X}_2^n(\mathbb{F}_q) = q^{3+2n} + \left( \binom{n+1}{2} + 1 \right) q^{2+2n} + \left( \frac{n(n+1)(n^2+n+2)}{8} + \binom{n+1}{2} \right) q^{1+2n} + O(q^{2n}).
\]

The groupoid cardinality \( \#\mathcal{A}_2(\mathbb{F}_q) = q^3 + q^2 \) is well-known. Using this fact, we define a natural probability measure on \( \mathcal{A}_2(\mathbb{F}_q) \) for all finite fields \( \mathbb{F}_q \).

Definition 4.12. Let \( P \) be a probability measure on \( \mathcal{A}_2(\mathbb{F}_q) \) defined by

\[
P([A_0]) = \frac{1}{\#\mathcal{A}_2(\mathbb{F}_q) \# \text{Aut}_{\mathbb{F}_q}(A_0)} = \frac{1}{(q^3 + q^2) \# \text{Aut}_{\mathbb{F}_q}(A_0)}
\]

for each \( \mathbb{F}_q \)-isomorphism class \([A_0] \in \mathcal{A}_2(\mathbb{F}_q)\).
A simple orbit-stabilizer argument shows the following simplification of the expected value of certain random variables on $A_2(F_q)$.

**Lemma 4.13** ([Lee20, Lemma 6.8]). Let $\mathcal{X}$ be any stack considered in this document. For $Z$ the fiber in $\mathcal{X}$ over a point in $A_2$, the expected value of $\# Z(F_q)$ with respect to the probability measure defined above on $A_2(F_q)$ is

$$E[\# Z(F_q)] = \frac{\# \mathcal{X}(F_q)}{\# A_2(F_q)}.$$ 

As a corollary, we get a handle on all moments of $\# A(F_q)$ since for all abelian surfaces and $n \geq 1$, 

$$\# A^n(F_q) = (\# A(F_q))^n.$$ 

In particular, the results above give the exact expected value and second moment of $\# A(F_q)$ as well asymptotics on higher moments. The first and second moments then yield the exact variance of $\# A(F_q)$. Below is a summary of these computations.

**Corollary 4.14** ([Lee20, Corollaries 1.4, 1.5, 1.7]). The expected value of $\# A^n(F_q)$ for all $n \geq 1$ are

$$E[\# A(F_q)] = q^2 + q + 1 - \frac{1}{q^3 + q^2},$$

$$E[\# A^2(F_q)] = q^4 + 3q^3 + 6q^2 + 3q - \frac{5q^2 + 5q + 3}{q^3 + q^2},$$

$$E[\# A^n(F_q)] = q^{2n} + \binom{n+1}{2} q^{2n-1} + \frac{n(n+1)(n^2 + n + 2)}{8} q^{2n-2} + O(q^{2n-3}).$$

The variance of $\# A(F_q)$ is

$$\text{Var}(\# A(F_q)) = q^3 + 3q^2 + q - 1 - \frac{3q^2 + 3q + 1}{q^3 + q^2} - \frac{1}{(q^3 + q^2)^2}.$$ 

**References**


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