Finite-order mapping classes of del Pezzo surfaces

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Finite group actions on $M$

$$q : \text{Homeo}^+(M) \to \text{Mod}(M) := \pi_0(\text{Homeo}^+(M))$$

**Question ((Cyclic) Nielsen realization problem)**

For $g \in \text{Mod}(M)$ of order $n < \infty$, does there exist $f \in \text{Homeo}^+(M)$ of order $n$ such that $[f] = g$?

**Answer ($\dim \mathbb{R} 2$)**

Yes! (Nielsen '43) Also for $\dim \mathbb{C} 1$.

What about for higher dimensional complex manifolds?
Example: order 2 diffeomorphisms on $\mathbb{CP}^2$

$M = \mathbb{CP}^2$

Example

$\sigma : [X : Y : Z] \mapsto [-X : Y : Z]$ and $\sigma \in \text{PGL}_3(\mathbb{C})$ which is connected so $[\sigma] = \text{Id}$.

Example

$\tau : [X : Y : Z] \mapsto [\overline{X} : \overline{Y} : \overline{Z}]$

$\tau_* = -\text{Id} : H_2(M) \to H_2(M)$

so $[\tau] \neq \text{Id}$.
del Pezzo surfaces

**Definition**

A del Pezzo surface $M$ is a complex surface

$$M = \mathbb{CP}^1 \times \mathbb{CP}^1 \quad \text{or} \quad M = \text{Bl}_P \mathbb{CP}^2$$

where $P \subseteq \mathbb{CP}^2$ is a finite set of points in general position with $0 \leq |P| \leq 8$.

**Lemma**

As smooth manifolds, $\text{Bl}_P \mathbb{CP}^2$ is diffeomorphic to $M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with $n = |P|$.
A classical example – Geiser involution

\[ \gamma \in \text{Aut}(\text{Bl}_P \mathbb{C}\mathbb{P}^2) \quad P = \{7 \text{ points in general position}\} \]

Let \( q \in \mathbb{C}\mathbb{P}^2 - P \).

Set of cubic curves through \( P \cup \{ q \} = \mathbb{C}\mathbb{P}^1 \)
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\[ \gamma \in \text{Aut}(\text{Bl}_P \mathbb{CP}^2) \quad P = \{7 \text{ points in general position}\} \]

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Cayley–Bacharach

\[ \gamma(q) = q'. \]

\( \gamma \) extends to an order two diffeomorphism of \( \text{Bl}_P \mathbb{CP}^2 \cong \mathbb{CP}^2 \# 7\mathbb{CP}^2 \).
Corollary (Special case of Freedman ’82, Quinn ’86)

\[ \text{Mod}(M_n) \to \text{Aut}(H_2(M_n), Q_{M_n}) \cong O(1, n)(\mathbb{Z}) \]

\[ [f] \mapsto f_* : H_2(M_n) \to H_2(M_n) \]

is an isomorphism of groups.

\[ O^+(1, n)(\mathbb{Z}) \leq O^+(1, n)(\mathbb{R}) \cong \text{Isom}(\mathbb{H}^n) \]

Vinberg: Hyperbolic reflection groups! (2 \leq n \leq 9)
Two types of elements in $\text{Mod}(M_n)$

1. (reducible) $g$ preserves some

\[ M_n \cong M \# k\mathbb{CP}^2 \]

on the level of homology

\[ H_2(M_n) \cong H_2(M) \oplus H_2\left(\# k\mathbb{CP}^2\right) \]

2. (irreducible) $g$ does not preserve any such direct sum decomposition of $H_2(M_n)$. 

\[ M \quad \# k\mathbb{CP}^2 \]
A structure theorem for all involutions

Birational geometry: three classical involutions of Bl\(_P\) \(\mathbb{CP}^2\):

1. Geiser involutions,
2. Bertini involutions, and
3. de Jonquières involutions.

**Theorem (L. ’22)**

A mapping class \(g \in \text{Mod}^+(M_n)\) of order 2 is irreducible if and only if \(g\) is realized by a de Jonquières, Geiser, or Bertini involution on Bl\(_P\) \(\mathbb{CP}^2\) for some \(P \subseteq \mathbb{CP}^2\).
Making new actions out of old

\[ S_1 \subseteq \text{Fix}(f_1) \]

\[ S_2 \subseteq \text{Fix}(f_2) \]

\[ T_p M \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \]

\[ T_q \mathbb{C}P^2 \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \]
Making new actions out of old

$S_1 \subseteq \text{Fix}(f_1)$

$S_2 \subseteq \text{Fix}(f_2)$

"Equivariant connected sum"
Corollary

Any element $g \in \text{Mod}(M_n)$ of order 2 is realized by a diffeomorphism $f \in \text{Diff}^+(M_n)$ of order 2.
Nielsen realization for $M_2$

**Theorem (L. ’21)**

A finite subgroup $G \leq \text{Mod}(M_2)$ has a lift to $\text{Diff}^+(M_2) \leq \text{Homeo}^+(M_2)$ under $q : \text{Homeo}^+(M_2) \to \text{Mod}(M_2)$ if and only if $G$ is realizable by an equivariant connected sum.

**Corollary**

1. If $g \in \text{Mod}(M_2)$ has finite order $n$ then there exists $f \in \text{Diff}^+(M_2)$ with order $n$ such that $[f] = g \in \text{Mod}(M_2)$.
2. Some finite subgroups $G \leq \text{Mod}(M_2)$ have no lift to $\text{Diff}^+(M_2)$. 
Thank you!