A geometric proof of the circular maximal theorem

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1 Introduction

A well–known result of Bourgain [1] asserts that the circular maximal function

$$\mathcal{M}f(x) = \sup_{1<t<2} \int_{S^1} |f(x-ty)|\, d\sigma(y) \tag{1}$$

is bounded on $L^p(\mathbb{R}^2)$ for $p > 2$. Simple examples show that this fails for $p = 2$. In this note we derive Bourgain's result by geometric and combinatorial methods. In particular, we do not use the Fourier transform in any way. Our proof is based on a combinatorial argument from [6], which in turn uses Marstrand's three circle lemma [7], and a lemma involving two circles that seems to originate in [10]. The three circle lemma was used in [7] to prove the following result, which is a simple consequence of Bourgain's theorem:

Suppose a planar set E has the property: for every point in the plane, E contains some circle with that point as center. Then E has positive measure.

Thus we show here that Marstrand's lemma, in combination with other ideas, does indeed allow one to establish the stronger maximal function estimate. Furthermore, we demonstrate in section 4 how to obtain the entire known range of $L^p \to L^q$ estimates for the circular maximal function (which is optimal possibly up to endpoints), see [9] and [11], by using the methods from sections 2 and 3. One — perhaps significant — distinction from the techniques developed in [1], [8], [11], and [12], which involve the Fourier transform, is the fact that the methods presented here do not seem to yield estimates for the global maximal function

$$\overline{\mathcal{M}}f(x) = \sup_{0<t<\infty} \int_{S^1} |f(x-ty)|\, d\sigma(y).$$

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For the strong maximal function defined in terms of rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes it was shown in [4] that weak L^p bounds are equivalent to certain geometric properties of collections of rectangles. [4] is related to this paper in so far as inequalities for maximal functions are proved by working directly with the associated geometric families. Moreover, in [3] page 37, A. Córdoba posed the problem of finding a geometric proof of Stein's spherical maximal theorem [13] and he suggested that it might be possible to settle the two–dimensional case by studying families of annuli in the plane. Here it is shown that it is indeed possible to prove the correct bounds on the circular maximal function by a careful analysis of collections of annuli.

This paper is organized as follows. Proposition 1.1 illustrates how maximal function estimates can be reduced to counting problems involving large families of thin annuli in the plane. We do not use the full equivalence as stated in Proposition 1.1, but only the fact that multiplicity estimates imply suitable L^p bounds. However, it might be of interest to know that the multiplicity bounds are indeed natural. Given a large collection of annuli with δ–separated centers, section 2 establishes estimates on the total number of annuli that can intersect a typical one. It turns out that these inequalities are essential in the analysis of the circular maximal function. In sections 3 and 4 they are used in combination with the three circle lemma to prove Bourgain's theorem and the $L^p \to L^q$ bounds, respectively. In those sections the reader will find heuristic arguments which explain the underlying observations for the main results, i.e., Theorems 3.1 and 4.1. It is perhaps worth mentioning that we do not use the method of cell decomposition that was recently applied in [14] to prove a sharp maximal function estimate. The motivation for this paper was to adapt the method from [6], which was developed there for one–parameter families of circles, to the two–parameter setting of (1). It seems that the improvement over the method in [6] that was achieved in [14] using cells and the work required to pass from the one–parameter to the two–parameter case are of a different nature.

Definition 1.1 *Let $\delta > 0$ be an arbitrary but fixed small number. By $\mathcal{C}$ we shall always* *mean a family of circles with δ–separated centers lying in some fixed compact set of diameter $< \frac{1}{2}$. For our purposes we may assume that the circles in $\mathcal{C}$ are in general position, in particular, all radii are distinct. Let*

$$\begin{aligned} C &= C(x,r) = \{y \in \mathbb{R}^2 \ : \ |x-y| = r\} \\ C^\delta &= C^\delta(x,r) = \{y \in \mathbb{R}^2 \ : \ r-\delta < |x-y| < r+\delta\}. \end{aligned} \tag{2}$$

We shall always assume that $r \in (1,2)$. For any family of circles $\mathcal{C}$ the multiplicity function is defined as

$$\mu_\delta^{\mathcal{C}} = \sum_{\overline{C} \in \mathcal{C}} \chi_{\overline{C}^\delta}.$$

$\mathcal{M}_\delta$ will denote the following auxiliary maximal function:

$$\mathcal{M}_\delta f(x) = \sup_{1<r<2} \frac{1}{|C^\delta(x,r)|} \int_{C^\delta(x,r)} |f(y)|\, dy.$$

Finally, $a \lesssim b$ means $a \leq Ab$ for some absolute constant A, and similarly with $a \gtrsim b$ and $a \sim b$. Lebesgue measure will be denoted by $|\cdot|$ and we will use both $|\cdot|$ and card *interchangeably for the cardinality of a set.*

Proposition 1.1 *The following are equivalent:*

i. For every $p > 2$ there exists a constant $c(p)$ depending only on p so that

$$\|\mathcal{M}_\delta f\|_{L^p(\mathbb{R}^2)} \leq c(p)\|f\|_{L^p(\mathbb{R}^2)} \tag{3}$$

for all $f \in L^p(\mathbb{R}^2)$ and all $\delta > 0$.

ii. Given $\delta > 0$ and $\mathcal{C}$, a family of circles with δ–separated centers, and a small number $\rho > 0$, there exists $\mathcal{A} \subset \mathcal{C}$ with $|\mathcal{A}| > c_\rho^{-1}|\mathcal{C}|$ for some constant c_ρ depending only on ρ and so that

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}} > c_\rho \lambda^{-1-\rho}\delta^{-1}\}| < \lambda |C^\delta| \tag{4}$$

for all $C \in \mathcal{A}$ and all $0 < \lambda \leq 1$.

Proof: Assume the second statement. For this implication we follow [6]. In view of Marcinkiewicz's interpolation theorem it suffices to prove a restricted weak–type estimate for every $p > 2$. Fix such a p and let $\delta > 0$, $E \subset \mathbb{R}^2$ compact, and $0 < \lambda \leq 1$. Pick a δ–net $\{x_j\}$ in

$$\{x \in \mathbb{R}^2 \,:\, \mathcal{M}_\delta \chi_E(x) > \lambda\}$$

and let $\mathcal{C} = \{C(x_j, r_j)\}$ where $r_j \in (1,2)$ is chosen so that

$$|C^\delta(x_j, r_j) \cap E| > \lambda |C^\delta(x_j, r_j)|$$

for all j. Applying (ii) with $\rho = p - 2$ yields $\mathcal{A} \subset \mathcal{C}$ with property (4). Hence

$$|\{C^\delta \cap E \,:\, \mu_\delta^{\mathcal{A}} \leq c_\rho (\lambda/2)^{-1-\rho}\, \delta^{-1}\}| \geq \frac{\lambda}{2} |C^\delta|$$

for all $C \in \mathcal{A}$ and thus

$$\lambda \delta |\mathcal{A}| \lesssim \int_{\{E \,:\, \mu_\delta^{\mathcal{A}} \leq c_\rho (\lambda/2)^{-1-\rho}\, \delta^{-1}\}} \mu_\delta^{\mathcal{A}} \ \leq |E| c_\rho (\lambda/2)^{1-p} \delta^{-1}.$$

In view of $|\mathcal{A}| > c_\rho^{-1} |\mathcal{C}|$ this implies

$$\lambda (\delta^2 |\mathcal{C}|)^{\frac{1}{p}} \lesssim c_\rho^{\frac{2}{p}} |E|^{\frac{1}{p}}.$$

By our choice of $\{x_j\}$ we finally conclude that

$$\lambda |\{x \in \mathbb{R}^2 \,:\, \mathcal{M}_\delta \chi_E(x) > \lambda\}|^{\frac{1}{p}} \leq c(p) |E|^{\frac{1}{p}},$$

as desired.

To deduce the second statement from the first, we shall use an argument that seems to originate in [9], cf. Lemma 2.1. Fix $\rho > 0$ small. We claim that there exists c_ρ so that for half the circles $C \in \mathcal{C}$

$$|\{C^\delta \,:\, \mu_\delta^{\mathcal{C}} > c_\rho \lambda^{-1-\rho} \delta^{-1}\}| < \lambda |C^\delta|$$

for all $0 < \lambda \le 1$. Assume that this fails with a choice of c_ρ to be specified below. Then at least half the circles $C \in \mathcal{C}$ satisfy

$$|\{C^\delta \,:\, \mu_\delta^{\mathcal{C}} > c_\rho \lambda^{-1-\rho}\delta^{-1}\}| > \frac{1}{2}\lambda |C^\delta| \tag{5}$$

for some dyadic $\lambda = 2^{-j} \in (0,1]$ depending on C. Let $a_\rho = \sum_{j=0}^{\infty} 2^{-j\rho}$. We claim that there is $\mathcal{B} \subset \mathcal{C}$ satisfying $|\mathcal{B}| \ge \frac{1}{2} a_\rho^{-1} \bar{\lambda}^\rho |\mathcal{C}|$ with some dyadic $\bar{\lambda} \in (0,1]$ and so that (5) holds with $\lambda = \bar{\lambda}$ for all $C \in \mathcal{B}$. This is a simple application of the pigeon hole principle. Indeed, suppose our claim failed. Then

$$\begin{aligned} &\operatorname{card}(\{C \in \mathcal{C} \,:\, C \ \text{ satisfies (5) for some } \ \lambda = 2^{-j} \in (0,1]\}) \\ < &\sum_{\lambda = 2^{-j} \le 1} \frac{1}{2} a_\rho^{-1} \lambda^\rho |\mathcal{C}| = \frac{1}{2}|\mathcal{C}|, \end{aligned}$$

contrary to our assumption.

Now let

$$E = \{\mu_\delta^{\mathcal{C}} > c_\rho \bar{\lambda}^{-1-\rho}\delta^{-1}\}.$$

We distinguish two cases. Let $p = 1 + \sqrt{1+\rho}$.
<u>Case 1:</u> $|E| < (a_\rho\, c(p)^p\, 2^{p+1})^{-1}\ \bar{\lambda}^{p+\rho}\ |\mathcal{C}|\delta^2$

Applying (3) to $f = \chi_E$ yields

$$2^{-1-\frac{1}{p}} a_\rho^{-\frac{1}{p}}\ \bar{\lambda}^{1+\frac{\rho}{p}}\ \delta^{\frac{2}{p}} |\mathcal{C}|^{\frac{1}{p}} \le \frac{1}{2}\bar{\lambda}(\delta^2|\mathcal{B}|)^{\frac{1}{p}} \le c(p)|E|^{\frac{1}{p}},$$

which is a contradiction.
<u>Case 2:</u> $|E| \ge (a_\rho\, c(p)^p\, 2^{p+1})^{-1}\ \bar{\lambda}^{p+\rho}\ |\mathcal{C}|\delta^2$

In this case we use duality. Note that the dual inequality to (3) is

$$\|\sum_j a_j \chi_{C^\delta(y_j,\rho_j)}\|_{L^{p'}(\mathbb{R}^2)} \le c(p)\delta^{-1+\frac{2}{p'}}(\sum_j |a_j|^{p'})^{\frac{1}{p'}}$$

for all families $\{C(y_j,\rho_j)\}$ with δ–separated centers. We apply this to our family $\mathcal{C}$ with $a_j = 1$. Then

$$c_\rho \bar{\lambda}^{-1-\rho}\delta^{-1}|E|^{\frac{1}{p'}} \le \|\mu_\delta^{\mathcal{C}}\|_{L^{p'}(\mathbb{R}^2)} \le c(p)\delta^{-1+\frac{2}{p'}}|\mathcal{C}|^{\frac{1}{p'}}.$$

This contradicts our assumption on $|E|$ for large c_ρ, since p was chosen so that $(1+\rho)p' = p + \rho$. $\blacksquare$

We prove Bourgain's theorem, i.e., statement (i), by showing directly that a set $\mathcal{A} \subset \mathcal{C}$ as in the second statement exists, cf. Theorem 3.1.

2 The two circle lemma

The following simple geometric lemma is well–known, see [1], [6], [7], and [14]. We refer the reader to [9], Lemma 4.2 for a proof of the statement below. Let $C = C(x,r)$ and $\overline{C} = \overline{C}(\bar{x}, \bar{r})$. The notation

$$\Delta(C, \overline{C}) = \max(||x-\bar{x}| - |r-\bar{r}||, \delta), \quad d(C, \overline{C}) = |x-\bar{x}| + |r-\bar{r}|$$

will be used throughout. Note that $||x-\bar{x}| - |r-\bar{r}|| = 0$ if and only if the two circles are internally tangent. If $\Delta(C, \overline{C}) = \epsilon$ we say that C and $\overline{C}$ are ϵ–tangent. This means that the shortest distance between the intersection points of $C, \overline{C}$ with the line joining their centers is equal to ϵ.

Lemma 2.1 *Suppose $x, y \in \mathbb{R}^2$, $0 < |x-y| \leq \frac{1}{2}$, and $r, s \in (1,2)$, $0 < \delta < 1$. Then there is an absolute constant A_0 so that*

i. $C^\delta(x,r) \cap C^\delta(y,s)$ is contained in a δ–neighborhood of an arc on $C(x,r)$ of length $\leq A_0 \sqrt{\frac{\Delta}{|x-y|}}$ centered at the point $x - r\,\mathrm{sgn}(r-s)\frac{x-y}{|x-y|}$.

ii. the area of intersection satisfies

$$\left| C^\delta(x,r) \cap C^\delta(y,s) \right| \leq A_0 \frac{\delta^2}{\sqrt{\Delta |x-y|}}.$$

The second part of Lemma 2.1 shows that the angle of intersection of $C, \overline{C}$ is proportional to $\sqrt{\Delta(C, \overline{C})|x-\bar{x}|}$. This should indicate that it is important to know the size of

Δ and the distance of the centers of intersecting circles. For this reason we introduce the set

$$\mathcal{C}_{\epsilon t}^{C} = \{\overline{C} \in \mathcal{C} \ : \ C^\delta \cap \overline{C}^\delta \neq \emptyset, \ \epsilon - \delta \leq \Delta(C, \overline{C}) \leq 2\epsilon, \ t \leq |x - \bar{x}| \leq 2t\},$$

where $\epsilon, t \in [\delta, 1]$. We shall make frequent use of the following simple observations. Firstly, $\overline{C} \in \mathcal{C}_{\epsilon t}^{C}$ iff $C \in \mathcal{C}_{\epsilon t}^{\overline{C}}$. Secondly, let $y \in C^\delta \cap \overline{C}^\delta$, i.e. , $||x-y| - r| < \delta$ and $||\bar{x} - y| - \bar{r}| < \delta$. Then

$$\begin{aligned} |r - \bar{r}| &\leq ||x-y| - r| + ||\bar{x} - y| - \bar{r}| + ||x-y| - |\bar{x} - y|| \\ &< 2\delta + |x - \bar{x}| \leq 2(\delta + t) \leq 4t. \end{aligned} \tag{6}$$

In particular, $d(C, \overline{C}) \leq 6t$ and $\epsilon \leq 4t$ if $\mathcal{C}_{\epsilon t}^{C} \neq \emptyset$.

In the following paragraph we give a heuristic discussion of the results in this section. Using the Fourier transform one obtains the well–known estimate, see [1] and [2],

$$\|\mathcal{M}_\delta f\|_{L^2(\mathbb{R}^2)} \lesssim |\log \delta|^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2)}.$$

By the arguments in Proposition 1.1 this is equivalent to

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{C}} > |\log \delta|^b \lambda^{-1} \delta^{-1}\}| < \lambda\delta$$

for some constant b, most circles $C \in \mathcal{C}$, and all $\lambda \in (0,1]$. These estimates can improved. In fact, in Corollary 3.6 of [9] it was shown that there exists an absolute constant C_0 so that

$$\|\mathcal{M}_\delta f\|_{L^2(B(x_0,t))} \leq C_0 \, t^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2)}$$

for all $x_0 \in \mathbb{R}^2, 0 < t < 1$. By the second argument in the proof of Proposition 1.1 one concludes from this, see Lemma 3.7 in [9],

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{C}_{\epsilon t}^{C}} \gtrsim \ |\log \delta|^b \lambda^{-1} \delta^{-1} t\}| < \lambda\delta \tag{7}$$

for at least half the circles in $\mathcal{C}$, all $\lambda, \epsilon, t \in [\delta, 1]$, and a suitable constant b. This in turn implies that

$$|\mathcal{C}_{\epsilon t}^{C}| \lesssim \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} |\log \delta|^{b+1} \tag{8}$$

for at least half the circles $C \in \mathcal{C}$ and all $\epsilon, t \in [\delta, 1]$. Indeed, let

$$S_j = \{C^\delta \,:\, \mu_\delta^{\mathcal{C}_{\epsilon t}^C} \in [2^j, 2^{j+1})\}.$$

On the one hand, in view of (7), $|S_j| \lesssim |\log\delta|^b 2^{-j} t$. On the other hand, the area estimate in Lemma 2.1 implies, roughly speaking, that S_j splits into $\lesssim |S_j|/(\delta^2/\sqrt{\epsilon t})$ many rectangles of size $\delta \times \frac{\delta}{\sqrt{\epsilon t}}$ each of which is intersected by no more than 2^{j+1} many $\overline{C}^\delta \in \mathcal{C}_{\epsilon t}^C$. Thus

$$\mathrm{card}(\{\overline{C} \in \mathcal{C}_{\epsilon t}^C \,:\, \overline{C}^\delta \cap S_j \neq \emptyset\}) \lesssim \frac{|\log\delta|^b 2^{-j} t}{\delta^2/\sqrt{\epsilon t}} 2^{j+1} \sim |\log\delta|^b \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}}.$$

Since clearly

$$\mathcal{C}_{\epsilon t}^C \subset \bigcup_{1 \le 2^j \le \delta^{-2}} \{\overline{C} \in \mathcal{C}_{\epsilon t}^C \,:\, \overline{C}^\delta \cap S_j \neq \emptyset\},$$

(8) follows. It seems reasonable to conjecture that (7) and (8) should hold without the logarithmic factors. This would be optimal, as can be seen from the family $\mathcal{C}$ of circles with δ–separated centers in $B(0,t)$ which are ϵ–tangent to the unit circle. Indeed, if $C = C(x,r) \in \mathcal{C}$ with $t/2 < |x| < t$ then $\overline{C} \in \mathcal{C}_{\epsilon t}^C$ implies that $\bar{x}$ lies in a rectangle of size $\sim t \times \sqrt{\epsilon t}$ with axis $0x$. Moreover, note that any $C(x,r) \in \mathcal{C}$ with x close to 0 satisfies $|\mathcal{C}_{\epsilon t}^C| \sim \frac{t^2}{\delta^2}$. Hence it is necessary to pass to a suitable subfamily of $\mathcal{C}$ in order to obtain the $\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}}$–improvement in (8) over the trivial bound $|\mathcal{C}_{\epsilon t}^C| \lesssim \frac{t^2}{\delta^2}$. This improvement will be crucial in sections 3 and 4. The purpose of this section is to show that (7) and (8) hold with a factor of $\epsilon^{-\eta}$ instead of the logarithmic terms for any $\eta > 0$, see Proposition 2.1. In section 3 it will turn out that this loss of $\epsilon^{-\eta}$ can be compensated for by a factor $\lambda^{-\rho}$ for some small $\rho > 0$. In view of Proposition 1.1 this is exactly what one can afford to loose.

The following statement is the main ingredient for the two circle lemma, Lemma 2.5. It will be understood that $C_j = C_j(x_j, r_j)$ for $j = 0, 1, \ldots$.

Lemma 2.2 *Suppose* $C_2 \in \mathcal{C}_{\beta\tau}^{C_1}$, $t \ge 8\epsilon$, *and that* $\beta \ge 100\epsilon$. *Then*

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2} \frac{\epsilon}{\sqrt{\beta\tau}}.$$

<u>Proof:</u> Let $F(x,r) = (|x-x_1| - |r-r_1|, |x-x_2| - |r-r_2|)$ be defined on

$$\Omega = \{(x,r) \in \mathbb{R}^2 \times [1,2] \ : \ t \le |x-x_j| \le 2t, \ j = 1,2\}.$$

Suppose $(x,r) \in \Omega$ and let $e_i = \frac{x-x_i}{|x-x_i|}$ and $\sigma_j = \operatorname{sgn}(r-r_j)$. Then

$$DF(x,r) = \begin{pmatrix} e_1 & -\sigma_1 \\ e_2 & -\sigma_2 \end{pmatrix}$$

and thus $JF(x,r) \sim \sphericalangle(e_1\sigma_1, e_2\sigma_2) = \alpha$.

By $\sphericalangle$ we mean the angle $\in [0,\pi]$ and JF^2 denotes the sum of the squares of all 2×2 subdeterminants of DF. Suppose $(x,r) \in \Omega$ and $|F(x,r)| < 4\epsilon$. Then there exist r_j' so that $|r_j - r_j'| < 4\epsilon$ and

$$|x-x_j| = |r-r_j'| \quad \text{for} \quad j = 1,2.$$

Moreover, $|x-x_j| \ge t \ge 8\epsilon$ and $||r-r_j| - |x-x_j|| < 4\epsilon$ imply that $\operatorname{sgn}(r-r_j') = \sigma_j$. Thus

$$\begin{aligned} |x_1-x_2|^2 &= |x-x_1|^2 + |x-x_2|^2 - 2(x_1-x)\cdot(x_2-x) \\ &= |r-r_1'|^2 + |r-r_2'|^2 - 2\sigma_1\sigma_2|r-r_1'||r-r_2'| + \\ &\quad +2\sigma_1\sigma_2|x-x_1||x-x_2|(1-\cos\alpha) \\ &= |r_1'-r_2'|^2 + 2\sigma_1\sigma_2|x-x_1||x-x_2|(1-\cos\alpha), \end{aligned}$$

and consequently, in view of the definition of $\Delta(C_1,C_2)$ and (6)

$$t^2\alpha^2 \gtrsim \beta\tau - 50\tau\epsilon \gtrsim \beta\tau.$$

We conclude that

$$JF \gtrsim \frac{\sqrt{\beta\tau}}{t} \quad \text{on } \Omega \cap F^{-1}(B(0,4\epsilon)).$$

Changing variables, or more precisely, using the coarea formula, see Theorem 3.2.11 in [5], we obtain

$$\begin{aligned} \int_{B(0,4\epsilon)} \mathcal{H}^1(F^{-1}(y)\cap\Omega)\,dy &= \int_{\Omega\cap F^{-1}(B(0,4\epsilon))} JF(x,r)\,dxdr \\ \epsilon^2 t &\gtrsim \frac{\sqrt{\beta\tau}}{t}|\Omega\cap F^{-1}(B(0,4\epsilon))|. \end{aligned}$$

To bound the Hausdorff measure, note that $\mathrm{diam}(F^{-1}(y) \cap \Omega) \lesssim t$ provided $|y| \lesssim t$ and hence the length of the (algebraic) curve $F^{-1}(y) \restriction \Omega$ will also be bounded by t. This clearly implies that

$$|\mathrm{proj}_{\mathbb{R}^2}(\Omega \cap F^{-1}(B(0, 2\epsilon)))| \lesssim t^2 \frac{\epsilon}{\sqrt{\beta\tau}},$$

as desired. ■

Lemma 2.2 is sufficient for our purposes. However, we show below how to estimate $|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}|$ in those cases where Lemma 2.2 does not apply, cf. Lemma 2.5. This can be done using Lemma 2.3, which we shall use repeatedly in what follows. It is a quantitative version of the following simple observation: if $C_j = C(x_j, 3/4)$ are internally tangent to $C(0,1)$ for $j = 1,2$ with the points of tangency being far apart, then C_1 and C_2 intersect each other transversely. This fact is of course well–known and has been used in [1] and [14]. See [12] for a harder version in the context of variable coefficients.

Lemma 2.3 *Let C_0, C_1, C_2 be three circles so that $C_j \in \mathcal{C}_{\beta_j \tau_j}^{C_0}$ for $j = 1,2$. Assume*

$$\begin{aligned} \alpha &= \sphericalangle(\mathrm{sgn}(r_1 - r_0)(x_1 - x_0), \mathrm{sgn}(r_2 - r_0)(x_2 - x_0)) \\ &\geq A_0 \sqrt{\frac{(\beta_1 + \beta_2)(\tau_1 + \tau_2)}{\tau_1 \tau_2}} \end{aligned} \tag{9}$$

for some sufficiently large constant A_0. Then

$$\Delta(C_1, C_2) d(C_1, C_2) \sim \alpha^2 \tau_1 \tau_2$$

and thus, in particular,

$$\Delta(C_1, C_2) \geq 2(\beta_1 + \beta_2).$$

<u>Proof:</u> Let $\sigma_j = \mathrm{sgn}(r_j - r_0)$. Then

$$\begin{aligned} |x_1 - x_2|^2 &= |x_1 - x_0|^2 + |x_2 - x_0|^2 - 2(x_1 - x_0) \cdot (x_2 - x_0) \\ |r_1 - r_2|^2 &= |r_1 - r_0|^2 + |r_2 - r_0|^2 - 2(r_1 - r_0)(r_2 - r_0) \\ |x_1 - x_2|^2 - |r_1 - r_2|^2 &= |x_1 - x_0|^2 - |r_1 - r_0|^2 + |x_2 - x_0|^2 - |r_2 - r_0|^2 + \end{aligned}$$

$$+2\sigma_1\sigma_2(|r_1-r_0||r_2-r_0|-|x_1-x_0||x_2-x_0|)+$$
$$+2\sigma_1\sigma_2|x_1-x_0||x_2-x_0|(1-\cos\alpha).$$

This implies that (recall that $d(C_j, C_0) \le 6\tau_j$, see (6))

$$\begin{aligned}\Delta(C_1,C_2)d(C_1,C_2) &\sim \tau_1\tau_2\alpha^2 + O(\beta_1\tau_1+\beta_2\tau_2+\tau_1\beta_2+\tau_2\beta_1)\\ &= \tau_1\tau_2\alpha^2 + O((\beta_1+\beta_2)(\tau_1+\tau_2)) \sim \tau_1\tau_2\alpha^2\end{aligned}$$

where we have used (9) in the last step. Since $d(C_1, C_2) \le d(C_1, C_0) + d(C_0, C_2) \lesssim \tau_1 + \tau_2$ the lemma follows. ■

A_0 will denote the constants in Lemmas 2.1 and 2.3. Using Lemma 2.3 we can deal with the case $\beta \le 100\epsilon$ that was left open in Lemma 2.2. The intuition behind Lemma 2.4 is as follows. If C_1 and C_2 are tangent, then any circle $C \in \mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ has to intersect the arc of minimal length on C_1 that contains $C_1^\epsilon \cap C_2^\epsilon$.

Lemma 2.4 *Suppose $C_2 \in \mathcal{C}_{\beta\tau}^{C_1}$. Then*

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2}\sqrt{\frac{\epsilon+\beta}{\tau}}.$$

Proof: We may assume that $\tau \le 4t$. Indeed, let $C = C(x,r) \in \mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$. Then $\tau \le |x_1 - x_2| \le |x_1 - x| + |x - x_2| \le 4t$. Let

$$\gamma_0 = \sqrt{\frac{\beta+\epsilon}{\tau}} \sim \sqrt{\frac{(\beta+\epsilon)(t+\tau)}{t\tau}}.$$

Suppose $\overline{C} \in \mathcal{C}_{\epsilon t}^{C_1}$ satisfies $\min(\sphericalangle(\bar{x}-x_1, x_2-x_1), \sphericalangle(\bar{x}-x_1, x_1-x_2)) > A_0\gamma_0$.

We apply Lemma 2.3 with $C_0 = C_1$, $C_1 = C_2$, $C_2 = \overline{C}$, $\beta_1 = \beta$, $\beta_2 = \epsilon$, $\tau_1 = \tau$, and $\tau_2 = t$ to wit $\Delta(\overline{C}, C_2) \ge 2(\epsilon+\beta) > 2\epsilon$. In particular $\overline{C} \notin \mathcal{C}_{\epsilon t}^{C_2}$. We conclude that any $\overline{C} \in \mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ has to satisfy $\min(\sphericalangle(\bar{x}-x_1, x_2-x_1), \sphericalangle(\bar{x}-x_1, x_1-x_2)) \le A_0\gamma_0$.

In particular, the centers of all circles in $\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ are contained in a $4t \times 2t\, A_0\gamma_0$ rectangle centered at x_1 and thus $|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2}\gamma_0$, as claimed. ■

The two circle lemma now follows easily from Lemmas 2.2 and 2.4.

Lemma 2.5 *Suppose $C_2 \in \mathcal{C}_{\beta\tau}^{C_1}$. Then*

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2} \min(\sqrt{\frac{\epsilon}{\tau}}, \frac{\epsilon}{\sqrt{\beta\tau}}). \tag{10}$$

Proof: As before we may assume that $\tau \leq 4t$. Moreover, we may also assume that $8\epsilon \leq t$. Indeed, since $|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2}$ either

$$\sqrt{\frac{\epsilon}{\tau}} \leq \frac{1}{10} \quad \text{or} \quad \frac{\epsilon}{\sqrt{\beta\tau}} \leq \frac{1}{100}$$

without loss of generality. Hence, if $\beta \geq 100\epsilon$ we apply Lemma 2.2 to conclude

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2} \frac{\epsilon}{\sqrt{\beta\tau}}.$$

If on the other hand $\beta \leq 100\epsilon$, then (10) follows from Lemma 2.4. ∎

The following lemma is a simple technical statement that we shall use repeatedly. It is based on the observation that two circles $C_1, C_2 \in \mathcal{C}_{\epsilon t}^{C}$ will have to intersect provided $\operatorname{dist}(C^\delta \cap C_1^\delta, C^\delta \cap C_2^\delta)$ is sufficiently large (recall that all radii are $\in [1,2]$ and that the centers are no more than a distance $1/2$ from each other).

Lemma 2.6 *Let $C_1, C_2 \in \mathcal{C}_{\epsilon t}^{C}$ such that $\operatorname{sgn}(r - r_1) = \operatorname{sgn}(r - r_2)$, and $\sphericalangle(x_1 - x, x_2 - x) \geq A_0\sqrt{\frac{\epsilon}{t}}$. Then $C_1 \cap C_2 \neq \emptyset$.*

Proof : Consider the case $r > r_1, r_2$. We may assume that $x = 0$. By Lemma 2.1,

$$C^\epsilon \cap C_i^\epsilon \subset R_i,$$

an ϵ–neighborhood of an arc on C centered at $r\frac{x_i}{|x_i|}$, $i = 1, 2$, of length $A_0\sqrt{\frac{\epsilon}{t}}$. By our assumptions, $R_1 \cap R_2 = \emptyset$. In particular, $p_1 = x_1 + r_1\frac{x_1}{|x_1|} \in R_1 \subset \operatorname{exterior}(C_2)$ and $p_2 = x_2 + r_2\frac{x_2}{|x_2|} \in R_2 \subset \operatorname{exterior}(C_1)$. Since $r_i > 1 > |x_i|$ (recall that the centers lie in a set of diameter $< \frac{1}{2}$), $0 \in \operatorname{interior}(C_1) \cap \operatorname{interior}(C_2)$ and thus the segment $(0, p_2)$ intersects C_1 in an interior point of C_2, say q_1. Hence the arc p_1q_1 on C_1 intersects C_2. Finally, the case $r < r_1, r_2$ can be dealt with in a similar manner. ∎

In Lemma 2.7 we apply the two circle lemma in order to bound the total number of circles that can intersect a given one. This will be crucial in proving the multiplicity estimate (4). Lemma 2.7 states, roughly speaking, that an unwanted power of $\frac{t}{\epsilon}$ can be cut in half at the cost of eliminating half the circles.

Lemma 2.7 *Suppose that for some $1 \geq \alpha, \rho \geq 0$ and some constant A*

$$|\mathcal{C}_{\epsilon t}^{C}| \leq A \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \left(\frac{t}{\epsilon}\right)^{\alpha} t^{-\rho} \tag{11}$$

for all $C \in \mathcal{C}$, $\epsilon, t \in [\delta, 1]$.

Fix $0 < \nu \leq 1$ and assume $\nu + \rho \leq 1 + \alpha$. Then there exists $\mathcal{A} \subset \mathcal{C}$, $|\mathcal{A}| \geq \frac{1}{2}|\mathcal{C}|$ and a constant b_ν so that

$$|\mathcal{C}_{\epsilon t}^{C}| \leq (b_\nu A)^{\frac{1}{2}} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}+\nu} t^{-\frac{\rho}{2}-\frac{3\nu}{2}}$$

for all $C \in \mathcal{A}$, $\epsilon, t \in [\delta, 1]$.

Proof: Assume false. Then for at least half the circles C in $\mathcal{C}$

$$|\mathcal{C}_{\epsilon t}^{C}| \gtrsim (b_\nu A)^{\frac{1}{2}} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}+\nu} t^{-\frac{\rho}{2}-\frac{3\nu}{2}} \tag{12}$$

for some choice of dyadic $\epsilon, t \in [\delta, 1]$ depending on C. This will lead to a contradiction if b_ν is sufficiently large. The idea is as follows. Suppose (12) holds for a fixed choice of $\epsilon, t \in [\delta, 1]$ and for all $C \in \mathcal{B} \subset \mathcal{C}$ where $|\mathcal{B}| \geq \frac{1}{2}|\mathcal{C}|$. Consider the set

$$S_0 = \{(C, C_1, C_2) \,:\, C \in \mathcal{B},\;\; C_1, C_2 \in \mathcal{C}_{\epsilon t}^{C}\}.$$

Clearly,

$$\mathrm{card}(S_0) \geq |\mathcal{B}| \min_{C \in \mathcal{B}} |\mathcal{C}_{\epsilon t}^{C}|^2, \tag{13}$$

which in turn can be estimated by (12). To bound $\mathrm{card}(S_0)$ from above, we assume that the majority of $(C, C_1, C_2) \in S_0$ satisfy $C_2 \in \mathcal{C}_{\epsilon t}^{C_1}$. Using Lemma 2.3 we will see below that this is the most significant case. Now we count by choosing first C_1, then C_2, and finally C:

$$\mathrm{card}(S_0) \le \sum_{C_1 \in \mathcal{C}} \sum_{C_2 \in \mathcal{C}_{\epsilon t}^{C_1}} |\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}|. \tag{14}$$

The cardinality of the intersection is estimated by Lemma 2.5, whereas $|\mathcal{C}_{\epsilon t}^{C_1}|$ is controlled by our hypothesis (11). The reader will easily check that the bounds (13) and (14) obtained in this way agree if $\nu = 0$. The terms involving ν are of a technical nature. They arise because we apply the pigeon hole principle to make the above argument rigorous. The details are as follows.

Let $a_\nu = \sum_{j=0}^{\infty} 2^{-j\nu}$. We claim that for some (fixed) choice of ϵ, t (12) holds for at least $(8a_\nu^2)^{-1}\epsilon^\nu|\mathcal{C}|$ many circles C. This follows from a standard pigeon hole argument. Indeed, if our claim failed then

$$\begin{aligned} & \mathrm{card}(\{C \in \mathcal{C} \ : \ (12) \text{ holds for some } \ \epsilon = 2^{-j}, t = 2^{-k} \in [\delta, 1]\}) \\ < \ & \sum_{\epsilon = 2^{-j} \le 1} \ \sum_{\epsilon/4 \le t = 2^{-k} \le 1} (8a_\nu^2)^{-1} \left(\frac{\epsilon}{t}\right)^\nu t^\nu |\mathcal{C}| \\ < \ & (8a_\nu^2)^{-1} \sum_{k=0}^{\infty} \sum_{l=-2}^{\infty} 2^{-l\nu} 2^{-k\nu} |\mathcal{C}| \le \frac{1}{2}|\mathcal{C}|, \end{aligned}$$

contradicting our assumption. Now fix ϵ, t as in the claim and let $\mathcal{B}$ be the set of circles for which (12) holds with those values. Thus

$$|\mathcal{B}| \gtrsim a_\nu^{-2}\epsilon^\nu|\mathcal{C}|. \tag{15}$$

Define

$$S_0 = \{(C, C_1, C_2) \ : \ C \in \mathcal{B}, \ \ C_1, C_2 \in \mathcal{C}_{\epsilon t}^{C}, \ \ \mathrm{sgn}(r - r_1) = \mathrm{sgn}(r - r_2)\}. \tag{16}$$

Clearly, $|S_0| \ge \frac{1}{4}|\mathcal{B}| \ \min_{C \in \mathcal{B}} |\mathcal{C}_{\epsilon t}^{C}|^2$.

<u>Case 1:</u> The majority of $(C, C_1, C_2) \in S_0$ satisfy

$$\sphericalangle(x_1 - x, x_2 - x) \le A_0\sqrt{\frac{\epsilon}{t}}.$$

Let S_1 be the set of those triples. Then, in view of (12),

$$|S_1| \geq \frac{1}{8}|\mathcal{B}| \min_{C\in\mathcal{B}} |\mathcal{C}_{\epsilon t}^C|^2 \gtrsim |\mathcal{B}|\ b_\nu A\ \frac{\epsilon}{\delta}\left(\frac{t}{\delta}\right)^3\left(\frac{t}{\epsilon}\right)^{\alpha+2\nu} t^{-\rho-3\nu}. \tag{17}$$

On the other hand,

$$|S_1| \lesssim |\mathcal{B}| A\frac{\epsilon}{\delta}\left(\frac{t}{\delta}\right)^3\left(\frac{t}{\epsilon}\right)^{\alpha} t^{-\rho}. \tag{18}$$

Indeed,

$$\begin{aligned} |S_1| \ &\leq\ \sum_{C\in\mathcal{B}}\ \sum_{C_1\in\mathcal{C}_{\epsilon t}^C} \mathrm{card}(\{C_2\in\mathcal{C}_{\epsilon t}^C\ :\ \sphericalangle(x_1-x, x_2-x)\leq A_0\sqrt{\frac{\epsilon}{t}}\}) \qquad (19)\\ &\lesssim\ |\mathcal{B}|\ \max_{C\in\mathcal{B}} |\mathcal{C}_{\epsilon t}^C|\ A_0\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}\\ &\lesssim\ |\mathcal{B}|\ A\ \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}\left(\frac{t}{\epsilon}\right)^{\alpha} t^{-\rho}\ \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}} \end{aligned}$$

by (11). To bound the cardinality of the set in (19), simply observe that the centers x_2 will lie in a rectangle of size $\sim A_0\sqrt{\epsilon t}\times t$ centered at x.

Clearly, (17) and (18) contradict each other for large b_ν.

Case 2: The majority of $(C, C_1, C_2)\in S_0$ satisfy

$$\sphericalangle(x_1-x, x_2-x) > A_0\sqrt{\frac{\epsilon}{t}}. \tag{20}$$

Let

$$\begin{aligned} S_2\ =\ &\{(C, C_1, C_2)\ :\ C\in\mathcal{B},\ C_1, C_2\in\mathcal{C}_{\epsilon t}^C,\ \mathrm{sgn}(r_1-r)=\mathrm{sgn}(r_2-r),\ \sphericalangle(x_1-x,\\ &x_2-x) > A_0\sqrt{\frac{\epsilon}{t}},\ \beta-\delta\leq\Delta(C_1, C_2)\leq 2\beta,\ \tau\leq|x_1-x_2|\leq 2\tau\}. \end{aligned}$$

Here $\beta, \tau\in[\delta, 1]$ are chosen by applying the pigeon hole principle with weights $a_\nu^{-1}\beta^\nu$ and $a_\nu^{-1}\tau^\nu$, respectively so that

$$\mathrm{card}(S_2) \gtrsim a_\nu^{-2}(\beta\tau)^\nu\ \mathrm{card}(S_0) \gtrsim |\mathcal{B}| a_\nu^{-2}(\beta\tau)^\nu\ \min_{C\in\mathcal{B}} |\mathcal{C}_{\epsilon t}^C|^2. \tag{21}$$

By (20) and the intersecting circles lemma, Lemma 2.6, any $(C, C_1, C_2) \in S_2$ satisfies $C_2 \in \mathcal{C}_{\beta\tau}^{C_1}$. Thus Lemma 2.5 and (11) imply that

$$\begin{aligned}\mathrm{card}(S_2) &\le \sum_{C_1 \in \mathcal{C}} \sum_{C_2 \in \mathcal{C}_{\beta\tau}^{C_1}} |\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \\ &\lesssim |\mathcal{C}|\, A \left(\frac{\beta}{\delta}\right)^{\frac{1}{2}} \left(\frac{\tau}{\delta}\right)^{\frac{3}{2}} \left(\frac{\tau}{\beta}\right)^{\alpha} \tau^{-\rho}\, \frac{t^2}{\delta^2} \min\left(\sqrt{\frac{\epsilon}{\tau}}, \frac{\epsilon}{\sqrt{\beta\tau}}\right). \end{aligned} \tag{22}$$

Since $(C, C_1, C_2) \in S_2$ satisfy (20), Lemma 2.3 implies that $2\beta \ge \Delta(C_1, C_2) \ge 4\epsilon$. Moreover, $\tau \le |x_1 - x_2| \le |x_1 - x| + |x - x_2| \le 4t$. Now it is easy to see that (21) and (22) are incompatible. Indeed, first note that (22) is the same as

$$\mathrm{card}(S_2) \lesssim |\mathcal{C}|\, A\, \frac{\epsilon}{\delta} \frac{t^2 \tau}{\delta^3} \left(\frac{\tau}{\beta}\right)^{\alpha} \tau^{-\rho}.$$

On the other hand, (12), (15), and (21) imply

$$\begin{aligned}\mathrm{card}(S_2) &\gtrsim a_\nu^{-2} \epsilon^\nu |\mathcal{C}|\, a_\nu^{-2} (\beta\tau)^\nu\, \beta_\nu A\, \frac{\epsilon}{\delta} \left(\frac{t}{\delta}\right)^{3} \left(\frac{t}{\epsilon}\right)^{\alpha+2\nu} t^{-\rho-3\nu} \\ &\gtrsim a_\nu^{-4} b_\nu A\, |\mathcal{C}|\, \frac{\epsilon}{\delta} \frac{t^2 \tau}{\delta^3} \left(\frac{t}{\tau}\right)^{1+\alpha-\rho-\nu} \left(\frac{\beta}{\epsilon}\right)^{\alpha+\nu} \left(\frac{\tau}{\beta}\right)^{\alpha} \tau^{-\rho}.\end{aligned}$$

Since $1 + \alpha \ge \rho + \nu$, the upper and lower bound will contradict each other for large b_ν. ■

Proposition 2.1 is the main result of this section. Starting from the trivial bound

$$|\mathcal{C}_{\epsilon t}^{C}| \lesssim \frac{t^2}{\delta^2}, \tag{23}$$

we iterate Lemma 2.7 in order to get as close to the $\sqrt{\frac{\epsilon}{t}}$–improvement as possible, see the discussion following Lemma 2.1 above.

Proposition 2.1 *Let $\mathcal{C}$ be as above and let $\eta > 0$ be a small number. Then there exist a constant c_η, a subset $\mathcal{A} \subset \mathcal{C}$ so that $|\mathcal{A}| \ge c_\eta^{-1} |\mathcal{C}|$, and*

$$|\mathcal{A}_{\epsilon t}^{C}| \le c_\eta \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \epsilon^{-\eta} \tag{24}$$

for all $C \in \mathcal{A}$ and all $\epsilon, t \in [\delta, 1]$. In particular, every $C \in \mathcal{A}$ satisfies

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^C} > c_\eta^2 \, \lambda^{-1}\delta^{-1} t \, \epsilon^{-\eta}\}| < \lambda\delta \tag{25}$$

for all $\epsilon, t \in [\delta, 1]$, $0 < \lambda \leq 1$.

Proof: Let $\alpha_j = \frac{1}{2}\left(\frac{2}{3}\right)^j$ for $j = 0, 1, 2, \ldots$ and $\nu = \eta/6$. We claim that there exist $\mathcal{A}_j \subset \mathcal{C}$, $|\mathcal{A}_j| \geq 2^{-j}|\mathcal{C}|$ so that for all $C \in \mathcal{A}_j$ (b_ν being the constant from Lemma 2.7)

$$|(\mathcal{A}_j)_{\epsilon t}^C| \leq b_\nu \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \left(\frac{t}{\epsilon}\right)^{\alpha_j} t^{-\eta/2}$$

for all j satisfying $\alpha_j \geq 2\eta/3$.

This follows by induction using Lemma 2.7 with $\rho = 3\nu$. For $j = 0$ simply observe that

$$|\mathcal{C}_{\epsilon t}^C| \lesssim \frac{t^2}{\delta^2} \leq b_\nu \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \left(\frac{t}{\epsilon}\right)^{\alpha_0} t^{-\rho}$$

for all $C \in \mathcal{A}_0 = \mathcal{C}$. For the induction step note that

$$\frac{1}{2}\alpha_{j-1} + \nu \leq \frac{2}{3}\alpha_{j-1} = \alpha_j$$

since $\alpha_{j-1} = \frac{3}{2}\alpha_j \geq 6\nu = \eta$. Furthermore, $\rho/2 + 3\nu/2 = \rho = \eta/2$. Finally, we have $1+\alpha_{j-1} \geq 5\eta/6 = \nu+\rho$, which establishes our claim. Thus the first part of the proposition follows by letting $\mathcal{A} = \mathcal{A}_{j_0}$ where j_0 is maximal with $\alpha_{j_0} \geq 2\eta/3$.

Assume that the second part of the proposition is false. Then there is a circle $C \in \mathcal{A}$ and numbers ϵ, t, λ so that

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^C} > c_\eta^2 \, \lambda^{-1}\delta^{-1} t \, \epsilon^{-\eta}\}| > \lambda\delta.$$

This implies, in view of Lemma 2.1, that

$$c_\eta^2 \, t\epsilon^{-\eta} \lesssim \int_{C^\delta} \mu_\delta^{\mathcal{A}_{\epsilon t}^C} = \sum_{\overline{C} \in \mathcal{A}_{\epsilon t}^C} |C^\delta \cap \overline{C}^\delta| \lesssim |\mathcal{A}_{\epsilon t}^C| \frac{\delta^2}{\sqrt{\epsilon t}},$$

which contradicts (24) if c_η is large. ■

3 The three circle lemma

The following lemma is essentially Marstrand's three circle lemma, cf. Lemma 5.2 in [7]. It is a quantitative version of the following fact, known in incidence geometry as the circles of Apollonius:

Given three circles which are not internally tangent at a single point, there are at most two other circles that are internally tangent to the three given ones.

Lemma 3.1 *Fix circles C_1, C_2, C_3, some $\epsilon \in [\delta, 1]$, and positive numbers $\lambda_{12}, \lambda_{13}, \lambda_{23}$. Let*

$$S = \{x \in \mathbb{R}^2 \setminus \bigcup_{j=1}^{3} B(x_j, \epsilon) \ : \ \exists \, r \in (1,2) \ \text{ with } ||x_i - x| - |r_i - r|| < \epsilon$$
$$\text{for } i = 1,2,3 \text{ and } \quad |e_i(x,r) - e_j(x,r)| \sim \lambda_{ij} \ \text{ for } \ 1 \leq i < j \leq 3\}$$

where we have set $e_i(x,r) = \frac{x_i - x}{|x_i - x|}\text{sgn}(r_i - r)$. Then

$$|S| \lesssim \epsilon^2 (\lambda_{12}\lambda_{13}\lambda_{23})^{-1}.$$

For a proof of Lemma 3.1 the reader is referred to [7] or [9]. These references deal with the case $\lambda_{12} = \lambda_{13} = \lambda_{23}$, but the same arguments apply. [6] contains a version of Lemma 3.1 (with $\lambda_{12} = \lambda_{13} = \lambda_{23} \geq \sqrt{\frac{\epsilon}{t}}$) that gives further information on the set of circles under consideration and, moreover, applies to families of curves satisfying the cinematic curvature condition from [12].

The proof of Theorem 3.1 below follows a scheme that originates in [6]. The basic idea is to pass from the three circle lemma to bounds on the number of tangencies occurring in an arbitrarily large collection of circles by means of a simple result of extremal graph theory. Roughly speaking, this amounts to counting a suitable set of quadruples of circles in two different ways, as explained in the following paragraph. The Kolasa–Wolff argument splits naturally into two cases: Given a typical annulus C^δ, either the majority of intersections of C^δ with other annuli are concentrated on a small set of C^δ or they are spread out over an arc on C^δ of sufficient length. In view of the nondegeneracy condition in the circles of Apollonius, Marstrand's lemma can be applied only in the second case. However, Kolasa and Wolff observed that for one–parameter families $\mathcal{C}$ (they consider circles with δ–separated radii), it suffices to consider a single annulus in the first case. Roughly speaking, if their multiplicity estimate failed on a sufficiently small set of a fixed annulus C^δ, then the number of annuli intersecting C^δ would have to exceed δ^{-1}, which is the trivial bound on card($\mathcal{C}$), see [6] section 4 or [14] section 3 for details. One can show that this type of argument does not apply to the two–parameter families of circles that arise in the analysis of (1). More precisely, we will require stronger bounds than the ones given by (23). This was the original motivation for considering a two circle argument and it is also were Proposition 2.1 becomes important. Note that one cannot expect to use the two circle lemma in the one–parameter setting of [6] and [14]. Indeed, Lemma 2.5 estimates a neighborhood of a curve which might contain the centers of all circles in $\mathcal{C}$ if $|\mathcal{C}| < \delta^{-1}$. However, it turns out that the bounds (24) are also needed in the second case, i.e., that part of the argument involving Lemma 3.1.

Theorem 3.1 *Let $\mathcal{C}$ be as above and let $\rho > 0$ be a small number. Then there exists $\mathcal{A} \subset \mathcal{C}$ so that $|\mathcal{A}| > c_\rho^{-1}|\mathcal{C}|$ for some constant c_ρ and so that*

$$|\{C^\delta \,:\, \mu_\delta^{\mathcal{A}} > c_\rho \lambda^{-1-\rho}\delta^{-1}\}| \le \lambda\delta \tag{26}$$

for all $C \in \mathcal{A}$ and all $0 < \lambda \le 1$.

In view of Proposition 1.1 this result implies Bourgain's theorem.

The heuristics behind the proof of Theorem 3.1 are explained in this paragraph. We suppress any pigeon hole factors. Suppose that (26) fails with $\rho = 0$ in the following slightly stronger sense: for half the circles $C \in \mathcal{C}$

$$|\{C^\delta \,:\, \mu_\delta^{\mathcal{C}^C_{\epsilon t}} > \lambda^{-1}\delta^{-1}t\}| > \lambda\delta \tag{27}$$

for a fixed choice of ϵ and t, and $\lambda \ge 100A_0\sqrt{\frac{\epsilon}{t}}$. The factor t is motivated by (7). Following [6] we consider the set

$$Q \;=\; \{(C, C_1, C_2, C_3) \,:\, C \text{ satisfies } (27),\; C_1, C_2, C_3 \in \mathcal{C}^C_{\epsilon t},$$

$$\operatorname{dist}(C_i^\delta \cap C^\delta, C_j^\delta \cap C^\delta) > \frac{\lambda}{20} \ \text{ for } \ 1 \le i < j \le 3\}. \tag{28}$$

According to Lemma 2.1 the intersection $\overline{C}^\delta \cap C$ is contained in an arc on C^δ of length $\le A_0\sqrt{\frac{\epsilon}{t}}$ for any $\overline{C} \in \mathcal{C}^C_{\epsilon t}$. Therefore, choosing first C and then an arbitrary $C_1 \in \mathcal{C}^C_{\epsilon t}$ we see that $\lambda \ge 100A_0\sqrt{\frac{\epsilon}{t}}$ and (27) imply that there are $\sim |\mathcal{C}^C_{\epsilon t}|$ many choices of C_2, C_3 satisfying the distance assumption in (28). Hence

$$|Q| \gtrsim |\mathcal{C}| \min_{C\in\mathcal{C}} |\mathcal{C}^C_{\epsilon t}|^3.$$

Moreover, by the area estimate in Lemma 2.1,

$$\lambda\delta\,\lambda^{-1}\delta^{-1}t < \int_{C^\delta} \mu_\delta^{\mathcal{C}^C_{\epsilon t}} = \sum_{\overline{C}\in\mathcal{C}^C_{\epsilon t}} |C^\delta \cap \overline{C}^\delta| \ \lesssim \ |\mathcal{C}^C_{\epsilon t}| \frac{\delta^2}{\sqrt{\epsilon t}}$$

for any C satisfying (27). We conclude that

$$\operatorname{card}(Q) \gtrsim |\mathcal{C}| \left(\lambda^{-1}\delta^{-1}t\frac{\lambda\delta}{\delta^2/\sqrt{\epsilon t}}\right)^3 = |\mathcal{C}| \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}}\right)^3.$$

To find an upper bound on card(Q) we assume that the majority of $(C, C_1, C_2, C_3) \in Q$ satisfy $C_2 \in \mathcal{C}^{C_1}_{\epsilon t}$ and $C_3 \in \mathcal{C}^{C_2}_{\epsilon t}$. Using Lemma 2.3 one can show that this is indeed the most significant case, see the proof of Theorem 3.1 for details. Using (24) with $\eta = 0$ and Lemma 3.1 we now obtain, choosing first C_1, then C_2, C_3, and finally C (recall that $\lambda \ge \sqrt{\frac{\epsilon}{t}}$)

$$\operatorname{card}(Q) \lesssim \sum_{C_1\in\mathcal{C}} \sum_{C_2\in\mathcal{C}^{C_1}_{\epsilon t}} \sum_{C_3\in\mathcal{C}^{C_2}_{\epsilon t}} \frac{\epsilon^2}{\delta^2}\lambda^{-3} \lesssim |\mathcal{C}| \ \frac{\epsilon}{\delta} \left(\frac{t}{\delta}\right)^3 \frac{\epsilon^2}{\delta^2} \left(\frac{t}{\epsilon}\right)^{\frac{3}{2}},$$

which agrees with our lower bound. To apply Lemma 3.1 note that $C^\delta \cap C_j^\delta$ is contained in an arc of length $A_0\sqrt{\frac{\epsilon}{t}}$ on C^δ centered at $x + r\frac{x_i - x}{|x_i - x|}\operatorname{sgn}(r - r_i)$, see Lemma 2.1. Hence definition (28) implies that $|e_i(x,r) - e_j(x,r)| > \lambda/20$, as required.

We hope that the reader will not be sidetracked by the simple technicalities in the following proof. In order to make the heuristic argument above rigorous, we apply the pigeon hole principle several times. The various factors that arise by doing so as well as the factors $\epsilon^{-\eta}$ in Proposition 2.1 can then be controlled by $\lambda^{-\rho}$.

Proof of Theorem 3.1: Let $\eta = \rho/10$ and choose $\mathcal{A} \subset \mathcal{C}$, $|\mathcal{A}| > c_\eta^{-1}|\mathcal{C}|$ as in Proposition 2.1. Suppose (26) failed for half the circles in $\mathcal{A}$, i.e., for at least half the circles C in $\mathcal{A}$

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}} > c_\rho \lambda^{-1-\rho}\delta^{-1}\}| \geq \frac{1}{2}\lambda\delta, \tag{29}$$

with some dyadic λ depending on C. This will lead to a contradiction for sufficiently large c_ρ. We will apply the pigeon hole principle to obtain a bound of this type that holds for a large number of circles and a fixed choice of λ. Let $\nu = \rho/100$ and recall that $a_\nu = \sum_{k=0}^\infty 2^{-k\nu}$. We claim that there exists a dyadic $\bar\lambda$ so that

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}} > c_\rho \bar\lambda^{-1-\rho}\delta^{-1}\}| > \frac{1}{2}\bar\lambda\delta \tag{30}$$

for at least $\max(\frac{1}{2}a_\nu^{-1}\bar\lambda^\nu|\mathcal{A}|, 1)$ many circles $C \in \mathcal{A}$. For if not, then the number of circles C for which (29) holds is

$$< \sum_{\lambda=2^{-j}\leq 1} \frac{1}{2}a_\nu^{-1}\lambda^\nu \ |\mathcal{A}| = \frac{1}{2}|\mathcal{A}|,$$

contrary to our assumption. Since

$$\mu_\delta^{\mathcal{A}} \restriction C^\delta \leq \sum_{\epsilon/4\leq t\leq 1} \mu_\delta^{\mathcal{A}_{\epsilon t}^C},$$

where the sum is taken over dyadic $\epsilon, t \in [\delta, 1]$, and since

$$\sum_{\epsilon/4\leq t\leq 1} \epsilon^\nu = \sum_{\epsilon/4\leq t\leq 1} \left(\frac{\epsilon}{t}\right)^\nu t^\nu \leq 4a_\nu^2,$$

one easily sees that for each C satisfying (30) there are ϵ, t depending only on C so that

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^C} > c_\rho(4a_\nu^2)^{-1}\epsilon^\nu\bar\lambda^{-1-\rho}\delta^{-1}\}| > \frac{1}{2}(4a_\nu^2)^{-1}\epsilon^\nu\bar\lambda\delta. \tag{31}$$

Indeed, given an x in the set appearing on the left–hand side of (30), we can find ϵ, t depending on x so that

$$\mu_\delta^{\mathcal{A}_{\epsilon t}^C}(x) > c_\rho(4a_\nu^2)^{-1}\epsilon^\nu\bar\lambda^{-1-\rho}\delta^{-1}. \tag{32}$$

For if this were not so, sum as above to get a contradiction. For the same reason, (30) implies that for some ϵ, t depending on C the measure of all $x \in C^\delta$ satisfying (32) has to be $> \frac{1}{2}(4a_\nu^2)^{-1}\epsilon^\nu\bar{\lambda}\delta$. Finally, there are fixed values of ϵ an t so that (31) holds with those ϵ, t for at least $\max(\frac{1}{8}a_\nu^{-3}(\bar{\lambda}\epsilon)^\nu|\mathcal{A}|, 1)$ many circles. Let the set of those circles be denoted by $\mathcal{B}$. Setting $\lambda = \frac{1}{8}a_\nu^{-2}\epsilon^\nu\bar{\lambda}$, we conclude that there is a nonempty subset $\mathcal{B}$ of $\mathcal{A}$ satisfying

$$|\mathcal{B}| \gtrsim a_\nu^{-3}(\lambda\epsilon)^\nu|\mathcal{A}| \tag{33}$$

and fixed numbers $\lambda, \epsilon, t \in [\delta, 1]$ so that

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^C} \gtrsim c_\rho \, a_\nu^{-6}\epsilon^{3\nu}\lambda^{-1-\rho}\delta^{-1}\}| > \lambda\delta \tag{34}$$

for all $C \in \mathcal{B}$.

<u>Case 1:</u> $\lambda \le 100A_0\sqrt{\frac{\epsilon}{t}}$

Here A_0 is the constant from Lemmas 2.1 and 2.3. Then (34) implies that there is a circle $C \in \mathcal{A}$ so that

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^C} \gtrsim c_\rho \, a_\nu^{-6}\epsilon^{3\nu}\lambda^{-1}\delta^{-1}\left(\frac{t}{\epsilon}\right)^{\frac{\rho}{2}}\}| > \lambda\delta$$

and thus (recall $\eta = \rho/10$, $\nu = \rho/100$)

$$|\{C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^C} \gtrsim c_\rho a_\nu^{-6}\lambda^{-1}\delta^{-1}t^{\frac{\rho}{2}}\epsilon^{-\eta}\}| > \lambda\delta$$

which contradicts our choice of $\mathcal{A}$, see (25), provided c_ρ is large.

<u>Case 2:</u> $\lambda > 100A_0\sqrt{\frac{\epsilon}{t}}$

Fix any $C \in \mathcal{B}$ and let

$$\begin{aligned} Q^{(C)} \;\; &= \;\; \{(C_1, C_2, C_3) \ : \ C_1, C_2, C_3 \in \mathcal{A}_{\epsilon t}^C, \ \mathrm{dist}(C_i^\delta \cap C^\delta, C_j^\delta \cap C^\delta) > \frac{\lambda}{20} \\ &\quad\;\; \text{for} \ \ 1 \le i < j \le 3\}. \end{aligned} \tag{35}$$

We claim that

$$\mathrm{card}(Q^{(C)}) \gtrsim \left(c_\rho a_\nu^{-6}\epsilon^{3\nu}\lambda^{-\rho}\delta^{-1}\left(\frac{\epsilon t}{\delta^2}\right)^{\frac{1}{2}}\right)^3. \tag{36}$$

By Lemma 2.1, all $\overline{C} \in \mathcal{A}_{\epsilon t}^{C}$ satisfy $|\overline{C}^{\delta} \cap C^{\delta}| \sim \frac{\delta^2}{\sqrt{\epsilon t}}$ and hence (34) implies

$$\begin{aligned} c_\rho \, a_\nu^{-6} \epsilon^{3\nu} \lambda^{-\rho} &< \int_{C^\delta} \mu_\delta^{\mathcal{A}_{\epsilon t}^{C}} \lesssim |\mathcal{A}_{\epsilon t}^{C}| \frac{\delta^2}{\sqrt{\epsilon t}} \,, \quad \text{i.e.,} \\ |\mathcal{A}_{\epsilon t}^{C}| &\gtrsim c_\rho a_\nu^{-6} \lambda^{-\rho} \epsilon^{3\nu} \delta^{-1} \left(\frac{\epsilon t}{\delta^2}\right)^{\frac{1}{2}} . \end{aligned} \tag{37}$$

Now fix any $C_1, C_2 \in \mathcal{A}_{\epsilon t}^{C}$. Let R_1, R_2 be arcs of C^δ of thickness δ and length $\lambda/5$ centered at $e_j = x - r \,\mathrm{sgn}(r - r_j) \frac{x - x_j}{|x - x_j|}$ with $j = 1, 2$, respectively. Lemma 2.1 implies that any $\overline{C} \in \mathcal{A}_{\epsilon t}^{C}$ with the properties

$$\overline{C}^{\delta} \cap C^{\delta} \neq \emptyset, \quad R_j \cap \overline{C}^{\delta} = \emptyset \;\; \text{for} \quad j = 1, 2$$

satisfies

$$\mathrm{dist}(C^\delta \cap \overline{C}^{\delta}, C^\delta \cap C_j^\delta) > \frac{\lambda}{20} \;\; \text{for} \quad j = 1, 2.$$

Since (34) implies that

$$|\{C^\delta \setminus (R_1 \cup R_2) \, : \, \mu_\delta^{\mathcal{A}_{\epsilon t}^{C}} \gtrsim c_\rho a_\nu^{-6} \epsilon^{3\nu} \lambda^{-1-\rho} \delta^{-1}\}| > \lambda\delta/2$$

claim (36) follows from calculation (37) by choosing first C_1, then C_2, and finally C_3.

Next we assert that the set

$$\begin{aligned} Q_0 &= \{(C, C_1, C_2, C_3) \, : \, C \in \mathcal{B}, \; C_1, C_2, C_3 \in \mathcal{A}_{\epsilon t}^{C}, \; \mathrm{sgn}(r_1 - r) = \mathrm{sgn}(r_2 - r) \\ &\quad = \mathrm{sgn}(r_3 - r), \; \sphericalangle(x_i - x, x_j - x) > \lambda/20 \;\; \text{for} \;\; 1 \le i < j \le 3\} \end{aligned} \tag{38}$$

satisfies

$$\mathrm{card}(Q_0) \gtrsim |\mathcal{B}| \left(c_\rho a_\nu^{-6} \epsilon^{3\nu} \lambda^{-\rho} \delta^{-1} \left(\frac{\epsilon t}{\delta^2}\right)^{\frac{1}{2}} \right)^3 . \tag{39}$$

First note that at least half the circles $\overline{C} \in \mathcal{A}_{\epsilon t}^{C}$ satisfy either $\mathrm{sgn}(\bar{r} - r) > 0$ or $\mathrm{sgn}(\bar{r} - r) < 0$. Thus one can add the condition

$$\mathrm{sgn}(r_1 - r) = \mathrm{sgn}(r_2 - r) = \mathrm{sgn}(r_3 - r)$$

to the definition of $Q^{(C)}$ without violating (36). Furthermore, according to Lemma 2.1, the set $C^\delta \cap C_i^\delta$ is centered at $e_i = x - r\,\mathrm{sgn}(r - r_i)\frac{x-x_i}{|x-x_i|}$. Given any $(C_1, C_2, C_3) \in Q^{(C)}$ we therefore conclude from the conditions

$$\mathrm{dist}(C_i^\delta \cap C^\delta, C_j^\delta \cap C^\delta) > \lambda/20, \;\; \mathrm{sgn}(r - r_i) = \mathrm{sgn}(r - r_j)$$

for $1 \le i < j \le 3$ that

$$\sphericalangle(x_i - x, x_j - x) > \lambda/20,$$

and (39) follows from (36). We want to bound $\mathrm{card}(Q_0)$ from above using Proposition 2.1 and Lemma 3.1. In order to do so, we need to specify the relative positions of (C_1, C_2, C_3) for the majority of $(C, C_1, C_2, C_3) \in Q_0$. This will be accomplished by applying the pigeon hole principle to the variables $\sphericalangle(x_1 - x, x_2 - x), \sphericalangle(x_2 - x, x_3 - x), \sphericalangle(x_1 - x, x_3 - x)$ and $\Delta(C_1, C_2), \Delta(C_2, C_3), |x_1 - x_2|, |x_2 - x_3|$ with weights $a_\nu^{-1}\lambda_{12}^\nu,\ a_\nu^{-1}\lambda_{13}^\nu,\ a_\nu^{-1}\lambda_{23}^\nu$ and $a_\nu^{-1}\beta_1^\nu,\ a_\nu^{-1}\beta_2^\nu,\ a_\nu^{-1}\tau_1^\nu,\ a_\nu^{-1}\tau_2^\nu$, respectively (recall that $a_\nu = \sum_{j\ge 0} 2^{-j\nu}$). Indeed, the set

$$\begin{aligned} Q \;\; = \;\; & \{(C, C_1, C_2, C_3) \in Q_0 \;:\; \lambda_{ij} \le \sphericalangle(x_i - x, x_j - x) \le 2\lambda_{ij} \;\text{ for }\; 1 \le i < j \le 3, \\ & \beta_1 - \delta \le \Delta(C_1, C_2) \le 2\beta_1,\; \tau_1 \le |x_1 - x_2| \le 2\tau_1, \\ & \beta_2 - \delta \le \Delta(C_2, C_3) \le 2\beta_2,\; \tau_2 \le |x_2 - x_3| \le 2\tau_2\} \end{aligned} \tag{40}$$

satisfies

$$\mathrm{card}(Q) \ge a_\nu^{-7}(\lambda_{12}\lambda_{13}\lambda_{23}\beta_1\tau_1\beta_2\tau_2)^\nu \mathrm{card}(Q_0)$$

for a suitable choice of dyadic $\lambda_{12}, \lambda_{13}, \lambda_{23} \in [\lambda/20, 1]$ and dyadic $\beta_1, \tau_1, \beta_2, \tau_2 \in [\delta, 1]$. This is because any element of Q_0 has to satisfy the conditions in (40) for some choice of those parameters. Hence, in view of (39) and (33),

$$\begin{aligned} \mathrm{card}(Q) \;\; & \gtrsim \;\; |\mathcal{B}| c_\rho^3 a_\nu^{-25}(\beta_1\beta_2\tau_1\tau_2)^\nu \epsilon^{9\nu}\lambda^{3\nu-3\rho}\delta^{-3}\left(\frac{\epsilon t}{\delta^2}\right)^{\frac{3}{2}} \\ & \gtrsim \;\; |\mathcal{A}| c_\rho^3 a_\nu^{-28}(\beta_1\beta_2\tau_1\tau_2)^\nu \epsilon^{10\nu}\lambda^{4\nu-3\rho}\delta^{-3}\left(\frac{\epsilon t}{\delta^2}\right)^{\frac{3}{2}}. \end{aligned} \tag{41}$$

To bound $\mathrm{card}(Q)$ from above, first observe that $\lambda_{12}, \lambda_{23} \geq \lambda/20 > A_0\sqrt{\frac{\epsilon}{t}}$ implies by Lemma 2.6 that $C_2 \in \mathcal{A}^{C_1}_{\beta_1\tau_1}$ and $C_3 \in \mathcal{A}^{C_2}_{\beta_2\tau_2}$ for any $(C, C_1, C_2, C_3) \in Q$. Furthermore, in view of Lemma 2.3, the angles β_1, β_2 satisfy

$$\tau_1\beta_1 \sim \lambda_{12}^2 t^2, \quad \tau_2\beta_2 \sim \lambda_{23}^2 t^2. \tag{42}$$

Hence it follows from (24) of Proposition 2.1 and Lemma 3.1 that

$$\begin{aligned}
\mathrm{card}(Q) \ &\leq \ \sum_{C_1\in\mathcal{A}} \sum_{C_2\in\mathcal{A}^{C_1}_{\beta_1\tau_1}} \sum_{C_3\in\mathcal{A}^{C_2}_{\beta_2\tau_2}} |\{C \in \mathcal{A} \, : \, C \in \mathcal{A}^{C_j}_{\epsilon t} \ \text{ for } \ j = 1,2,3 \ \text{ and} \\
&\qquad \mathrm{sgn}(r_i - r) = \mathrm{sgn}(r_j - r), \ \sphericalangle(x_i - x, x_j - x) \sim \lambda_{ij} \ \text{ for } \ 1 \leq i < j \leq 3\}| \\
&\lesssim \ |\mathcal{A}| \, c_\eta \left(\frac{\beta_1}{\delta}\right)^{\frac{1}{2}} \left(\frac{\tau_1}{\delta}\right)^{\frac{3}{2}} \beta_1^{-\eta} \, c_\eta \left(\frac{\beta_2}{\delta}\right)^{\frac{1}{2}} \left(\frac{\tau_2}{\delta}\right)^{\frac{3}{2}} \beta_2^{-\eta} \, \frac{\epsilon^2}{\delta^2}(\lambda_{12}\lambda_{13}\lambda_{23})^{-1} \\
&\lesssim \ |\mathcal{A}| \, c_\eta^2 \, \beta_1^{-\eta}\beta_2^{-\eta} \left(\frac{t\epsilon}{\delta^2}\right)^2 \frac{\tau_1\tau_2}{\delta^2}\lambda^{-1} \\
&\lesssim \ |\mathcal{A}| \, c_\eta^2 \, \lambda^{4\nu-3\rho}\beta_1^{-\eta}\beta_2^{-\eta} \left(\frac{t}{\epsilon}\right)^{\frac{1}{2}+2\nu-\frac{3\rho}{2}} \left(\frac{t\epsilon}{\delta^2}\right)^2 \frac{\tau_1\tau_2}{\delta^2}.
\end{aligned} \tag{43}$$

It is easy to see that (41) and (43) are incompatible for large c_ρ because ν and η are small compared with ρ and since $\tau_1, \tau_2 \leq 4t$ (indeed, if $C_1, C_2 \in \mathcal{C}^C_{\epsilon t}$ then $\tau_1 \leq |x_1 - x_2| \leq |x_1 - x| + |x - x_2| \leq 4t$), and $\beta_1, \beta_2 \geq \epsilon$ (see (42) or Lemma 2.3). ■

4 $L^p \to L^q$ estimates

It was shown in [9] and [11] that $\mathcal{M} : L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$ provided $(\frac{1}{p}, \frac{1}{q})$ lies in the open triangle with vertices $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{2}{5}, \frac{1}{5})$ or on the half open segment $(0,0), (\frac{1}{2}, \frac{1}{2})$ (which corresponds to Bourgain's theorem). According to the examples in the introduction to [9] the triangle cannot be replaced by any strictly larger open set. However, the optimal inequalities on the shorter sides are unknown. The argument in [9] used both the combinatorial method from [6] and Fourier transform techniques, whereas [11] is based entirely on Fourier integral operator estimates. In this section we show how the purely geometric/combinatorial argument above yields the full range of $p \to q$ estimates just stated. Since the technical details are very similar to those in the previous sections, we shall be very brief.

By interpolation with (3) it suffices to show

$$\|\mathcal{M}f\|_{q,\infty} \lesssim \|f\|_{\frac{5}{2},1}$$

for every $q < 5$. Using the argument from the first part of the proof of Proposition 1.1 it is easy to see that this follows from the following theorem (with $q(1+\sigma) = 5$).

Theorem 4.1 *Given $\sigma > 0$ small, there exists $\mathcal{A} \subset \mathcal{C}$ with $|\mathcal{A}| > c_\sigma^{-1}|\mathcal{C}|$ for some constant c_σ depending only on σ and so that*

$$|\{C^\delta \,:\, \mu_\delta^{\mathcal{A}} > c_\sigma \lambda^{-\frac{3}{2}}\delta^{-1}(\delta^2|\mathcal{C}|)^{\frac{1}{2}(1-\sigma)}\}| < \lambda\delta \tag{44}$$

for all $C \in \mathcal{A}$, $0 < \lambda \leq 1$.

Before turning to the proof we give a heuristic discussion that parallels the one preceding Theorem 3.1. Set $\sigma = 0$ and assume that (44) fails in the following sense. For half the circles $C \in \mathcal{C}$

$$|\{C^\delta \,:\, \mu_\delta^{\mathcal{C}_{\epsilon t}^C} \geq \lambda^{-\frac{3}{2}}|\mathcal{C}|^{\frac{1}{2}}\}| > \lambda\delta \tag{45}$$

with a fixed choice of $\epsilon, t \in [\delta, 1]$ and some fixed $\lambda \geq A\sqrt{\frac{\epsilon}{t}}$ with A large. As before we consider the set

$$\begin{aligned} Q \;=\; &\{(C, C_1, C_2, C_3) \,:\, C \text{ satisfies } (45),\ C_1, C_2, C_3 \in \mathcal{C}_{\epsilon t}^C, \\ &\operatorname{dist}(C_i^\delta \cap C^\delta, C_j^\delta \cap C^\delta) > \frac{\lambda}{20} \text{ for } 1 \leq i < j \leq 3\}, \end{aligned}$$

which will again satisfy $\operatorname{card}(Q) \geq |\mathcal{C}| \min_{C\in\mathcal{C}} |\mathcal{C}_{\epsilon t}^C|^3$. Note that for any C satisfying (45)

$$\lambda\delta\, \lambda^{-\frac{3}{2}}|\mathcal{C}|^{\frac{1}{2}} \leq \int_{C^\delta} \mu_\delta^{\mathcal{C}_{\epsilon t}^C} = \sum_{\overline{C}\in\mathcal{C}_{\epsilon t}^C} |C^\delta \cap \overline{C}^\delta| \;\lesssim\; |\mathcal{C}_{\epsilon t}^C| \frac{\delta^2}{\sqrt{\epsilon t}}$$

and thus

$$\mathrm{card}(Q) \gtrsim |\mathcal{C}| \left(\lambda^{-\frac{3}{2}} |\mathcal{C}|^{\frac{1}{2}} \frac{\lambda\delta}{\delta^2/\sqrt{\epsilon t}} \right)^3 .$$

On the other hand, assuming as above that the majority of $(C, C_1, C_2, C_3) \in Q$ satisfy $C_2 \in \mathcal{C}_{\epsilon t}^{C_1}$ and $C_3 \in \mathcal{C}_{\epsilon t}^{C_2}$ (it will follow from Lemma 2.3 that this is the most significant case),

$$\mathrm{card}(Q) \lesssim \sum_{C_1 \in \mathcal{C}} \sum_{C_2 \in \mathcal{C}_{\epsilon t}^{C_1}} \sum_{C_3 \in \mathcal{C}_{\epsilon t}^{C_2}} \frac{\epsilon^2}{\delta^2} \lambda^{-3}.$$

In view of Proposition 2.1,

$$|\mathcal{C}_{\epsilon t}^{C_j}| \lesssim |\mathcal{C}|^{\frac{3}{4}} \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \right)^{\frac{1}{4}}$$

for $j = 1, 2$ and thus

$$\mathrm{card}(Q) \lesssim |\mathcal{C}|^{\frac{5}{2}} \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \frac{\epsilon^2}{\delta^2} \lambda^{-3}.$$

Comparing the upper and lower bounds yields $\lambda^{\frac{3}{2}} \lesssim \left(\frac{\epsilon}{t}\right)^{\frac{3}{4}}$, which contradicts our assumption on λ.

<u>Proof of Theorem 4.1:</u> Let $\eta = \sigma/20$, $\nu = \sigma/100$, and choose $\mathcal{A}$ as in Proposition 2.1. Assuming that (44) fails for at least half the circles in $\mathcal{A}$ we obtain as in the proof of Theorem 3.1 that for fixed $\lambda, \epsilon, t \in [\delta, 1]$

$$|\{C^\delta \; : \; \mu_\delta^{\mathcal{A}_{\epsilon t}^C} \gtrsim c_\sigma a_\nu^{-6} \epsilon^{3\nu} \lambda^{-\frac{3}{2}} \delta^{-1} (\delta^2 |\mathcal{C}|)^{\frac{1}{2}(1-\sigma)}\}| > \lambda\delta \tag{46}$$

for all $C \in \mathcal{B}$, where $|\mathcal{B}| \gtrsim a_\nu^{-3} (\lambda\epsilon)^\nu |\mathcal{A}|$.

<u>Case 1:</u> $\lambda \leq 100 A_0 \sqrt{\frac{\epsilon}{t}}$

On the one hand, using Lemma 2.1 and (46), we obtain

$$c_\sigma a_\nu^{-6} \epsilon^{3\nu} \lambda^{-\frac{1}{2}} (\delta^2 |\mathcal{C}|)^{\frac{1}{2}(1-\sigma)} \lesssim \int_{C^\delta} \mu_\delta^{\mathcal{A}_{\epsilon t}^C} \lesssim |\mathcal{A}_{\epsilon t}^C| \frac{\delta^2}{\sqrt{\epsilon t}}. \tag{47}$$

On the other hand, (24) implies that

$$|\mathcal{A}_{\epsilon t}^C| \lesssim |\mathcal{C}|^{\frac{1}{2}(1-\sigma)} \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{t}{\delta}\right)^{\frac{3}{2}} \epsilon^{-\eta} \right)^{\frac{1}{2}(1+\sigma)} .$$

As the reader will easily verify, this contradicts (47) for large c_σ since $\lambda \lesssim \sqrt{\frac{\epsilon}{t}}$ and since ν and η are small compared with σ.

Case 2: $\lambda > 100A_0\sqrt{\frac{\epsilon}{t}}$

With Q defined as in (40),

$$
\begin{aligned}
\operatorname{card}(Q) &\gtrsim a_\nu^{-7}(\lambda_{12}\lambda_{13}\lambda_{23}\beta_1\tau_1\beta_2\tau_2)^\nu \operatorname{card}(Q_0)\\
&\gtrsim |\mathcal{C}|\, c_\eta^{-1} a_\nu^{-3}(\lambda\epsilon)^\nu a_\nu^{-7}\, \lambda^{3\nu}(\beta_1\tau_1\beta_2\tau_2)^\nu \left(c_\sigma a_\nu^{-6}\epsilon^{3\nu}\lambda^{-\frac{1}{2}}\frac{\sqrt{\epsilon t}}{\delta^2}(\delta^2|\mathcal{C}|)^{\frac{1}{2}(1-\sigma)}\right)^3
\end{aligned}
$$

for a suitable choice of dyadic $\lambda_{12}, \lambda_{13}, \lambda_{23} \in [\lambda/20, 1]$ and dyadic $\beta_1, \tau_1, \beta_2, \tau_2 \in [\delta, 1]$. Indeed, this follows from the same reasoning as in the proof of Theorem 3.1 if one uses (46) instead of (34). On the other hand, applying Proposition 2.1 and Lemma 3.1 as in the upper bound (43) and using (42), i.e., $\tau_1\beta_1 \sim \lambda_{12}^2 t^2, \quad \tau_2\beta_2 \sim \lambda_{23}^2 t^2$, we conclude that

$$
\begin{aligned}
\operatorname{card}(Q) &\lesssim |\mathcal{C}| \left(c_\eta \left(\frac{\beta_1}{\delta}\right)^{\frac{1}{2}} \left(\frac{\tau_1}{\delta}\right)^{\frac{3}{2}} \beta_1^{-\eta}\right)^{\frac{1}{4}+\frac{3\sigma}{4}} |\mathcal{C}|^{\frac{3}{4}(1-\sigma)} \left(c_\eta \left(\frac{\beta_2}{\delta}\right)^{\frac{1}{2}} \left(\frac{\tau_2}{\delta}\right)^{\frac{3}{2}} \beta_2^{-\eta}\right)^{\frac{1}{4}+\frac{3\sigma}{4}}\\
&\quad |\mathcal{C}|^{\frac{3}{4}(1-\sigma)}\, \frac{\epsilon^2}{\delta^2}(\lambda_{12}\lambda_{13}\lambda_{23})^{-1}\\
&\lesssim \delta^{-3-3\sigma}|\mathcal{C}|^{1+\frac{3}{2}(1-\sigma)}(\lambda_{12}\lambda_{23}t^2)^{\frac{1}{4}+\frac{3\sigma}{4}}(\tau_1\tau_2\beta_1^{-\eta}\beta_2^{-\eta})^{\frac{1}{4}+\frac{3\sigma}{4}}\ \epsilon^2(\lambda_{12}\lambda_{13}\lambda_{23})^{-1}\\
&\lesssim |\mathcal{C}|\ \delta^{-6}\ (\delta^2|\mathcal{C}|)^{\frac{3}{2}(1-\sigma)}\ t^{\frac{1}{2}+\frac{3\sigma}{2}}(\tau_1\tau_2\beta_1^{-\eta}\beta_2^{-\eta})^{\frac{1}{4}+\frac{3\sigma}{4}}\ \lambda^{-\frac{3}{2}+4\nu}\ \left(\frac{t}{\epsilon}\right)^{\frac{1}{2}+2\nu-\frac{3\sigma}{4}}\ \epsilon^2,
\end{aligned}
$$

where we have used $\lambda_{12}, \lambda_{13}, \lambda_{23} \geq \lambda/20 \gtrsim \sqrt{\frac{\epsilon}{t}}$ in the last step. Since $\beta_1, \beta_2 \geq \epsilon$ and $\tau_1, \tau_2 \leq 4t$, and since ν and η are small compared with σ, the lower and upper estimate for $\operatorname{card}(Q)$ will contradict each other for large c_σ. ■

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