The Bernstein center.

Say G is a Lie Group / $\mathbb{R}$. (or $\mathbb{C}$)

The Lie Alg $\mathfrak{g}$, and universal enveloping algebra $U(\mathfrak{g})$ are obviously important associated structures; the center $Z(U(\mathfrak{g}))$ is, too. There are 3 interpretations.

1) $\exists$ an algebraic description of $Z(U(\mathfrak{g}))$ in terms of $\mathfrak{g}$

2) $\exists$ Harish-Chandra homomorphism:

$$Z(\mathcal{U}(\mathfrak{g})) \xrightarrow{\sim} S(\mathfrak{h})^{W}$$

$\downarrow$ Weyl group-invariant regular functions on symmetric alg. of Cartan algebra.

3) Invariant distributions on G supported at the identity.

There is also:

4) A categorical perspective.

Say A is an arbitrary (not-nec.-commutative!) algebra w/ identity. We may extract $Z(A)$ via the endomorphisms of the identity functor of the category of all A-algebras.

I.e.: $\forall\ M \in Ob(A\text{-Mod})$, an endo

$$\varphi_M : M \longrightarrow M \quad \text{s.t.}$$

$\forall$ A-morphisms, $f: M \longrightarrow N$, T.F.D.C.:

$$\begin{array}{ccc} M & \xrightarrow{\varphi_M} & M \\ f\downarrow & & \downarrow f \end{array}$$

$$N \xrightarrow[\varphi_N]{} N .$$

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A \\ 1 & \longmapsto & ? \end{array}$$ Must lie in center, determines whole of φ.

Thus $Z(U(\mathfrak{g})) = \mathrm{End}(\mathrm{Id}_{\mathfrak{g}\text{-mod}})$.

Bernstein's approach: apply this perspective to p-adic (or non-archimedean) groups.

Say G is a reductive k-group, k a non-arch local field, w/ integers $\mathcal{O}$. Let $\mathcal{S}(G)$ be the category of smooth reps of G; that is:

<u>Def.</u> Let $V/\mathbb{C}$ be a vector space, $G \longrightarrow \mathrm{Aut}_{\mathbb{C}} V$ a homomorphism. This is a <u>smooth</u> representation if $\forall\ v \in V$, $\mathrm{Stab}_G(v)$ is open in $G(k)$.

<u>Def.</u> The Bernstein Center $\mathfrak{Z}(G)$ is the algebra of endomorphisms of the identity functor of $\mathcal{S}(G)$.

Recall: this means that we must associate an endomorphism φ_V to each smooth rep V.

There are two realications of the Bernstein Center:

i) Let H represent an open, compact sub of G. An element of $\mathfrak{Z}(G)$ acts on the permutation representation $\mathbb{C}[G/H]$ (note: this is infinite, countable dimension). This corresponds to an element of the Hecke algebra $\mathcal{H}(G,H)$ — in fact the center of the Hecke algebra, $Z(G,H)$. Thus $\Phi \in \mathfrak{Z}(G)$ gives us a family of compatible elements in $Z(G,H)$ (w/ transition maps given by applying idempotents). I.e.

$$\mathfrak{Z}(G) \cong \varprojlim_{H} Z(G,H)$$

over H compact open subgroups of G.

ii) Description in terms of "essentially compact distributions." $\mathfrak{Z}(G)$ is the set of G-invariant (that's what makes it "central": conjugacy-invariance) distributions that are "essentially compact" — i.e. convolution w/ $f\,dg$, ($f \in C_c^\infty(G)$, dg a Haar measure) yields another distribution of the form $h\,dg$, $h \in C_c^\infty(G)$. [I.e. convolution preserves compact support]. Indeed: consider $\Phi \in \mathfrak{Z}(G)$'s action on $C_c^\infty(G)$. Let $z_\Phi = \Phi(C_c^\infty(G)) \in \mathrm{End}(C_c^\infty(G))$

Then we define the map:

$$\mathfrak{Z}(G) \longrightarrow \mathrm{Hom}(C_c^\infty(G), \mathbb{C})$$

$$\Phi \longmapsto \left\{ D_\Phi : f \longmapsto z_\Phi(f)\big|_e \right\}.$$

Observe that:

$$D_\Phi * f\,dg = z_\Phi(f)\,dg \in C_c^\infty(G)\,dg;$$

whence "essential compactness!"

iii) Via Algebraic Geometry. Let $\tilde{G}$ be the set of all equivalence classes of smooth irreps of G. It carries a natural topology.

Let $\Omega(G)$ denote the Hausdorffization of $\tilde{G}$. This becomes an algebraic variety (!), which the

"Bernstein theorem" describes explicitly. $\Omega(G)$ is often called the "Bernstein Variety." The Bernstein center $\mathfrak{Z}(G)$ is then the algebra of regular functions on $\Omega(G)$!

Before we discuss the structure of $\Omega(G)$, let us see why $\Phi \in \mathfrak{Z}(G)$ produces a function on it. Let $\pi \in \Omega(G)$.

$$\Phi \longmapsto \Phi(\pi) \in \mathrm{End}(\pi) \cong \mathbb{C}, \text{ by Schur.}$$

This gives us a function:

$$\Phi \longmapsto \Theta_\pi(\Phi) = \Phi(\pi), \text{ so:}$$

$$\mathfrak{Z}(G) \longrightarrow \{\mathrm{Fun}(\tilde{G}, \mathbb{C})\}.$$

Descriptions of Bernstein Variety.

G compact: $\tilde{G}$ is discrete as reps are fully reducible. $\Phi \in \mathfrak{Z}(G)$ (by Schur's lemma) is determined by a single scalar for each irrep π. The Bernstein center $\cong \mathbb{C}^{\tilde{G}}$.

G compact-mod-center: Let $\Psi(G) = \{$unramified characters of $G\}$ (i.e. those that kill the maximal compact). $G/_{\text{Maximal compact}}$ is a free Abelian group of finite type: $\Psi(G)$ is an algebraic torus over $\mathbb{C}$. Now: $\Psi(G)$ acts on $\tilde{G}$, via $\otimes$. The connected components of $\tilde{G}$ are precisely the $\Psi(G)$ orbits. Thus each component of $\tilde{G}$ carries ~~a~~ natural torus-quotient variety structure / $\mathbb{C}$. $\mathfrak{Z}(G)$ is the direct product (of rings) of regular functions on these orbits.

If G semisimple, Each supercuspidal representation (i.e. those whose matrix coefficients are compact-modulo-center), correspond to isolated points in $\tilde{G}$.

For G reductive, $\text{cusp}(G)$ is a disjoint union of quotients of $\Psi(G) = X^{\circ}(G)$. Let $\text{Cusp}(G) \subseteq \tilde{G}$ denote the subspace of cuspidals. It is open and closed in $\tilde{G}$. The corresponding ring of regular functions is $\mathfrak{Z}(G)_{\text{cusp}}$ is a factor ring of $\mathfrak{Z}(G)$.

In general:

$$\mathfrak{Z}(G) = \Big(\prod_M \mathfrak{Z}(M)_{\text{cusp}} \Big)^G$$

where M ranges over Levi factors in G. Thus $\mathfrak{Z}(G)$ is built up inductively via parabolic induction.

Connected components of $\tilde{G}$ are thus given by G-conjugacy classes of $\Psi(M)$ orbits in $\text{Cusp}(M)$.

Say G is split (i.e. has maximal torus $T \cong G_{m,k}^{\text{rk}\,G}$). Then the "most induced" reps will come from T. One component of $\tilde{G}$ will contain the trivial character of T. In $\mathfrak{Z}(G)$, this gives us $\mathbb{C}[X^\bullet(T)]^W$. Recall: an element in here will act on irreps to produce a scalar. This scalar will be 0 on components of $\tilde{G}$ <u>not</u> containing $\{\mathrm{Ind}_T^G \mathrm{triv}\}$. But this component does contain all unramified representations. For these, the scalar of $\theta_\varepsilon(\Phi)$, $\Phi \in \mathbb{C}[X^\bullet(T)]^W \subseteq \mathfrak{Z}(G)$ is the Satake Parameter:

[Recall Satake isomorphism:

$$\mathcal{H}_{sph}(G) := \mathbb{C}[\mathcal{O}\backslash G/\mathcal{O}] \xrightarrow{\sim} \mathcal{H}_T^W \cong \mathbb{C}[X^\bullet(T)]^W \cong \mathrm{Rep}[G^\vee]$$

The Satake parameter corr.

In fact this connected component actually consists of those reps w/ nontrivial fixed vector under an Iwahori subgroup... Thus center of Iwahori-Hecke Algebra is $\mathbb{C}[X(T)]^W$. Note similarity to Harish-Chandra isomorphism: this truly is a "group version" of center of universal enveloping algebra.

Consider $\hat{G} \subseteq \tilde{G}$ (no longer algebraic!) the unitary dual inside

$\tilde{G}$. Recall that if $f \in C_c^\infty(G)$, (π, V) is a rep of G, then $\exists$ $\pi(f) \in \mathrm{End}\, V$ defined via:

$$\int_G f(g)\,\pi(g)\,dg .$$

If π is a smooth unitary irrep, then $\pi(f)$ is trace class. Meanwhile $\hat{G}$ has a topology (the Fell Topology).

Theorem: (Plancherel Theorem). $\exists$ a measure $d\pi$ on $\hat{G}$, s.t. $\forall$ $f \in C_c^\infty(G)$ (and even f in an appropriately-defined Schwartz space),

$$\int_{\hat{G}} \mathrm{Tr}(\pi(f))\, d\pi = f(1). \quad \star$$

(Harish Chandra even supplies a formula for $d\pi$!)

Now: let $\Phi \in \mathfrak{Z}(G)$. Applying the formula to $z_\Phi(f)$, we get:

$$\int_{\hat{G}} \mathrm{Tr}(\pi(f))\, \Theta_\pi(z)\, d\pi = (z_\Phi * f)(1).$$

Recall that a smooth, unitary rep π is <u>tempered</u> if its matrix coefficients all lie in $L^{2+\varepsilon}(G)$ $\forall$ $\varepsilon > 0$. By assumption:

$$\hat{G}_{temp} \subseteq \hat{G}.$$

The tempered spectrum decomposes much like Bernstein spectrum: but union of $\Psi_u(M)$ orbits where $\Psi_u(M)$ = <u>unitary</u> unramified characters. These are a compact torus in $\Psi(M)$.

Claim. The Plancherel measure is supported on the tempered spectrum.

In other words, only tempered reps contribute to the $\star$.

Now $\Psi_u(M)$ and its homogeneous spaces have a natural Lebesgue measure. Harish-Chandra showed that this measure is absolutely cts. w.r.t. Plancherel measure — in fact, it is some rational function!

The Galois side:

Let $G = GL_n$, consider $\pi \longmapsto \sigma(\pi) \in$ Weil–Deligne rep.

π, π' are in same Bernstein component

$\iff$ $\sigma(\pi), \sigma(\pi')$ have isomorphic restriction to the inertia group.

π, π' are tempered iff the monodromy filtration of $\sigma(\pi)$ and $\sigma(\pi')$ are pure of weight 0; in this case they lie in the same tempered component $\iff$

$$\sigma(\pi)\Big|_{\text{Inertia}} = \sigma(\pi')\Big|_{\text{Inertia}} \quad \text{and} \quad N_{\sigma(\pi)} = N_{\sigma(\pi')}.$$

[Recall: Weil–Deligne rep (ρ, N) is a map

$\rho : W_K \longrightarrow \mathrm{End}(V)$ $\qquad W_K = v^{-1}\{\mathrm{Frob}^n\}_{n \in \mathbb{Z}}$

$\qquad v : \mathrm{Gal}(\bar{F}/K) \xrightarrow{v} \mathrm{Gal}(\bar{k}/k)$ residue map.

N : nilpotent $\in \mathrm{End}(V)$, s.t.

$\rho(\sigma) N \rho(\sigma)^{-1} = N \cdot q^{-v(\sigma)}$.].