Langlands Functoriality and the Jacquet-Langlands Correspondence

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1 Motivation: The Langlands Programme

The Langlands programme, broadly construed, has two main parts: reciprocity and functoriality.

1.1 Reciprocity

Reciprocity predicts relationships between different types of representations ("Galois" and "automorphic") attached to a given (connected reductive) group.

Langlands reciprocity for $G = GL_n(\mathbb{Q})$ predicts:

$$\left\{ \begin{matrix} \text{Cuspidal, algebraic, automorphic} \\ \text{representations of } GL_n(\mathbb{Q}) \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{Compatible families of geometric Galois} \\ \text{representations } \mathrm{Gal}(\overline{\mathbb{Q}}/Q) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}) \end{matrix} \right\}$$

A **cuspidal automorphic** representation is an irreducible subrepresentation π of $L^2(G(\mathbb{Q})G(\mathbb{A}))$ such that every $f \in \pi$ satisfies some vanishing condition. **Algebraic** is some "geometric" condition at the places at infinity. It generalizes the notion of "algebraic Hecke character", which is the $n = 1$ case.

Note that this quotient is locally compact, so we can define a (right) Haar measure on the quotient with respect to which integration makes sense.

On the Galois side, **compatible** means that we get a family of representations for every prime ℓ which are all related in some way, and **geometric** means that locally, the representation is unramified at all but finitely many places and it is de Rham at ℓ.

Example 1. The Tate modules of an elliptic curve for different ℓ are maps

$$\rho_\ell \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}_\ell}),$$

but as long as $p \nmid \ell$, $\operatorname{tr} \mathrm{Frob}_p = a_p$ and $\det \mathrm{Frob}_p = p$, independently of ℓ. Usually, the trace and the determinant together determine a two-dimensional matrix up to conjugaacy. There's a bit of a wrinkle here because these representations are over different fields, but nevertheless this suggests some sort of a relationship between the representations associated to different ℓ.

Let's say a little bit more about the quotient $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. For a Lie group H, the decomposition of $L^2(H)$ into irreducible representations looks very different when H is compact than when H is not. When H is compact, all the irreducible unitary representations are finite-dimensional, and the Peter-Weyl theorem gives a direct sum decomposition

$$L^2(H) \cong \hat{\bigoplus_{\pi \text{ unitary irrep}}} \pi^{\dim \pi}.$$

This should not be surprising, since compact groups are basically finite groups with more analysis.

However, if H is not compact, this picture breaks down completely! Unitary representations of non-compact G are *not* generally finite-dimensional, and there may be parts of the decomposition of $L^2(G)$ which are direct integrals instead of direct sums.

Example 2. A function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ has a Fourier decomposition

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y} dx.$$

Here, the characters $\chi_y(x) := e^{2\pi i x y}$ are the unitary representations of the locally compact abelian group $\mathbb{R}$.

Roughly, representations appearing in direct sums are known as **discrete series** representations, and those in direct integrals are known as **principal series** representations. There are also "limit of discrete series" representations, which I won't discuss much here.

Example 3. If $G = \mathrm{GL}_2$, the discrete series representations appearing in $L^2(G)$ are things we understand – they correspond to modular forms of weight ≥ 2! The limit of discrete series representations are forms of weight 1. The principal series representations correspond to Maass forms. For $G = \mathrm{GL}_2$, we can more or less prove reciprocity for discrete series representations.

1.2 Functoriality and the Jacquet-Langlands correspondence

Functoriality describes maps between representations that arise from maps between groups. Given a group G, we can associate to it an L-group, denoted ${}^L G$. For our purposes, the L-group is some group which depends only on the $\overline{\mathbb{Q}}$-points of G.

Langlands functoriality predicts that given (nice) groups G and G', maps

$$\phi \colon {}^L G \longrightarrow {}^L G'$$

should lift to maps sending automorphic representations of G to automorphic representations of G'. The Jacquet-Langlands correspondence is an example of Langlands functoriality for $G = \mathrm{GL}_2$, and was historically one of the motivating examples for functoriality in general.

Definition 4. A **form** of an algebraic group $G/\mathbb{Q}$ is another group $G'/\mathbb{Q}$ such that there exists a $\overline{\mathbb{Q}}$-isomorphism ϕ between G and G'.

Definition 5. Let G' be a form of G and $\phi \colon G \to G'$ a $\overline{\mathbb{Q}}$-isomorphism. The group G' is an **inner form** of G if the action of $\phi^{-1} \circ \sigma \circ \phi$ on G is an inner automorphism for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Exercise 6. Check that this condition is well defined i.e. that it doesn't depend on the choice of $\overline{\mathbb{Q}}$-isomorphism ϕ.

A group and its inner form will have the same L-group. If we believe functoriality, then the identity morphism on L-groups should induce a corresponding map of automorphic representations. This is exactly what Jacquet-Langlands correspondence gives us for $G = \mathrm{GL}_2$.

2 Quaternion algebras

Let's start with some basic definitions from noncommutative algebra. See [?, ?] for proofs and details. Throughout, no field has characteristic 2.

Definition 7. A left (right) **ideal** of a ring R is a left (right) submodule of R, i.e. a subset I such that $RI = I$ ($IR = I$). An ideal which is both a left and right ideal is said to be a **two-sided ideal**. The ring R is **simple** if it has no proper two-sided ideals.

Proposition 8. *The center of a simple ring is a field, so simple rings are simple algebras.*

Definition 9. A $\mathbb{Q}$-algebra is called **central** if its center is $\mathbb{Q}$. A quaternion algebra (over a field K) is a central simple K-algebra of dimension 4.

Example 10. Here are some examples of quaternion algebras.

- $M_2(K)$ is a 4-dimensional CSA for every K;
- The usual quaternions, $\mathbb{H}$, are a quaternion algebra over $\mathbb{R}$;

Proposition 11. *The K-algebra*

$$\left(\frac{a,b}{K}\right) := \{K + Ki + Kj + Kij | i^2 = a, j^2 = b, ij = -ji\}$$

is a quaternion algebra, and every quaternion algebra over K can be expressed in this form.

Example 12. Many common quaternion algebras can be expressed in this form.

- $M_2(K) = \left(\frac{1,1}{K}\right)$. To see this, take

$$i := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, j := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

- $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$.

Definition 13. The **reduced norm** of an element $x = x_0 + x_1 i + x_2 j + x_3 ij \in \left(\frac{a,b}{\mathbb{Q}}\right)$ is defined as

$$\mathrm{Nm}(x) := (x_0 + x_1 i + x_2 j + x_3 ij)(x_0 - x_1 i - x_2 j - x_3 ij) = x_0^2 + ax_1^2 + bx_2^2 - abx_3^2.$$

Proposition 14. *The reduced norm is multiplicative.*

Example 15. Here are some norms.

- The norm on $M_2(K)$ is the determinant.
- The norm on $\mathbb{H}$ is the usual norm on quaternions: $\mathrm{Nm}(x_0 + x_1 i + x_2 j + x_3 k) = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

Proposition 16. *Let K be a field.*

1. *If K is algebraically closed, the only quaternion algebra over K is $M_2(K)$.*
2. *If K is a local field, there are two quaternion algebras over K: $M_2(K)$, and a division algebra.*

Let $B/\mathbb{Q}$ be a quaternion algebra over the rationals (or any number field. Proposition 16 tells us that for any place v of $\mathbb{Q}$, $B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is either $M_2(\mathbb{Q}_v)$ or the unique division algebra above $\mathbb{Q}_v$.

Definition 17. We say that B is **split** at a place v if $B \otimes \mathbb{Q}_v \cong M_2(\mathbb{Q}_v)$, and **ramified** if $B \otimes \mathbb{Q}_v$ is a division algebra.

Proposition 18. *Let $B/\mathbb{Q}$ be a quaternion algebra.*

1. *The set S of places at which $B/\mathbb{Q}$ is ramified is finite;*
2. *The cardinality of S is even;*
3. *The set S determines B up to isomorphism.*

Definition 19. The product of the primes at which $B/\mathbb{Q}$ ramifies is called the **discriminant** of B.

Definition 20. A quaternion algebra is **definite** if it is ramified at infinity and **indefinite** otherwise.

Remark. The theory of quaternion algebras over $\mathbb{Q}$ parallels the theory of quadratic fields over $\mathbb{Q}$. The fact that the cardinality of S is even is more or less equivalent to quadratic reciprocity! Definite quaternion algebras are analogous to imaginary quadratic fields, and indefinite quaternion algebras to real quadratic fields.

Proposition 21. *Let $B/\mathbb{Q}$ be a quaternion algebra. Then, $B^\times$ is an inner form for* $\mathrm{GL}_2\mathbb{Q}$.

Proof. This follows from the first part of Proposition 16. □

The converse is actually also true, but I won't discuss it during this talk.

By Proposition 21, if we believe functoriality, then cuspidal automorphic forms for $B^\times$ ought to relate somehow to cuspidal automorphic forms for $\mathrm{GL}_2\mathbb{Q}$ (i.e. cusp newforms). Before we can state our theorem, we need a few more definitions.

Definition 22. An **order** of $B/\mathbb{Q}$ is a $\mathbb{Z}$-lattice of B which is also a ring. A **maximal order** of B is a maximal order.

Remark. It's not obvious that orders exist, because in this regime, the set of integral elements don't form a ring!

Proposition 23. *Maximal orders in $B/\mathbb{Q}_p$ are conjugate to $M_2(\mathbb{Z}_p)$.*

If B is split at a place, then $B \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{Q}_v \cong M_2(\mathbb{Q}_v)$. If $\mathcal{O}_0$ is a maximal order, then $\mathcal{O}_0 \hookrightarrow \mathcal{O}_0 \otimes_{\mathbb{Z}} \mathbb{Z}_v$. The latter is still an order: it's clearly a ring, and it's a lattice because as a $\mathbb{Z}$-module $\mathcal{O}_0 \cong \mathbb{Z}^n$.

Proposition 24. *Fix a positive integer N and quaternion algebra $B/\mathbb{Q}$. For any maximal order $\mathcal{O}_0(1)$, there is an injection*

$$\iota_N \colon \mathcal{O}_0(1) \hookrightarrow \mathcal{O}_0(1) \otimes \mathbb{Z}\mathbb{Z}_N \cong M_2(\mathbb{Z}_N)$$

In what follows, we fix p and write $\mathcal{O} := \mathcal{O}_0(N)$.

Definition 25. A right fractional ideal I of $\mathcal{O}$ is **invertible** if there exists a left fractional ideal J such that $JI = \mathcal{O}$. In this case, we write $I^-1 := J$.

Definition 26. Two right ideals I and J are in the same **ideal class** if there exists $x \in B^\times$ such that $I = xJ$.

The ideals I and J are in the same ideal class if and only if they are isomorphic as right $\mathcal{O}$-modules, so this is an equivalence relation.

Definition 27. The **right class set** of $\mathcal{O}$ is the set of ideal classes of invertible right ideals. We denote it by $\operatorname{Cl}\mathcal{O}$

This set turns out to be finite and independent of the choice of Eichler order (it depends only on the level).

2.1 Modular forms over a definite quaternion algebra

I've just subjected you to 20 minutes of noncommutative algebra. All of that was for what's coming up now.

Definition 28. Let $B/\mathbb{Q}$ be a quaternion algebra. A **quaternionic modular form** of weight 2 and level N over a definite quaternion algebra is a function

$$f\colon \operatorname{Cl}\mathcal{O}_0(N) \longrightarrow \mathbb{C}.$$

A **quaternionic cusp form** is a quaternionic modular form which is orthogonal (with respect to the usual Hermitian inner product on $\mathbb{C}^n$) to space of constant functions.

We write $M_2^B(N)$ for the space of modular forms and $S_2^B(N)$ for the space of cusp forms.

Theorem 29 (Eichler-Shimizu-Jacquet-Langlands, very special case)**.** *Let B be a quaternion algebra over $\mathbb{Q}$ with discriminant p. There is an isomorphism of Hecke modules*

$$S_2^B(1) \cong S_2(p),$$

where $S_2(p) := S_2(\Gamma_0(p))$.