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Displaced Lognormal Volatility Skews: Analysis and Applications to Stochastic Volatility Simulations

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Abstract We analyze the implied volatility skews generated by displaced lognormal diffusions. In particular, we prove the global monotonicity of implied volatility, and an at-the-money bound on the steepness of downward volatility skews, under displaced lognormal dynamics, which therefore cannot reproduce some features observed in equity markets. A variant, the displaced anti-lognormal, overcomes the steepness constraint, but its state space is bounded above and unbounded below.

In light of these limitations on what features the displaced (anti-)lognormal (DL) can model, we exploit the DL, not as a model, but as a control variate, to reduce variance in Monte Carlo simulations of the CEV and SABR local/stochastic volatility models.

For either use – as model, or as control variate – the DL's parameters require estimation. We find an explicit formula for the DL's short-expiry limiting volatility skew, which allows direct calibration of its parameters to volatility skews implied by market data or by other models.

Keywords displaced lognormal · displaced diffusion · implied volatility · control variate

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1 Introduction

Given an empirically-observed or model-generated price of a call or put, the implied volatility is by definition the volatility parameter for which the Black-Scholes formula recovers the given option price. Regardless of what process actually underlies the given option price, the implied volatility provides a canonical language or scale by which option prices are commonly quoted and compared. At any expiry, the volatility skew – meaning the implied volatility as a function of all strikes – captures the full risk-neutral underlying distribution at that expiry, and hence constitutes a natural framework to understand and to compare distributions.

In particular, comparison can occur between an empirically-observed volatility skew and a model-generated volatility skew, for the purpose of calibrating the model's parameters, or for the purpose of understanding what empirical features can or cannot be reproduced by the model. Comparison can also occur, between volatility skews generated by two different models, for the purpose of approximating the features of a more complex model, using a simpler model.

The lognormal (Black-Scholes 1973) model generates a flat implied volatility skew, which does not agree with the sloping skews observed empirically in equity, FX, and interest rate markets. *Displacing* the lognormal (Rubinstein 1983) does generate a sloping implied volatility skew. Marris (1999), Brigo and Mercurio (2002), Joshi and Rebonato (2003), and Svoboda-Greenwood (2009), have investigated the displaced lognormal (and extensions thereof), as a pricing model or as a analytical approximation to other models, motivated largely by applications to interest rate derivatives. In contrast, we draw motivation mainly from problems arising in equity markets, such as how to calibrate to volatility skews that slope downward more steeply than all displaced lognormal skews; and we intend to use the calibrated process less for its analytical pricing than for its applicability to Monte Carlo pricing.

First, we bound the level and slope of the implied volatility skews generated by displaced lognormal diffusions in various regimes (global, or at-the-money, or short-expiry). We prove, among other results, the global monotonicity of implied volatility, and an at-the-money upper bound on the absolute slope of downward volatility skews, under displaced lognormal dynamics, which therefore cannot model some features (non-monotonicity and a steep downward slope) observed in equity market volatility skews. A variant, the displaced anti-lognormal, overcomes the steepness constraint, but its state space is bounded above and unbounded below, unlike stock prices.

In light of these restrictions on what features the displaced (anti-)lognormal (DL) can model, we then exploit the DL, not as a model, but as a control variate, to reduce variance in Monte Carlo simulations of other models, such as the CEV and SABR local/stochastic volatility models. Numerical examples show significant reductions of variance.

For either use – as a model, or as a control variate – the DL's parameters require estimation. We find an explicit formula for the DL's short-expiry limiting volatility skew, which allows direct calibration of its parameters to volatility skews implied by market data or by other models.

2 Implied Volatility

We work under martingale measure, and we either assume zero interest rates, or stipulate that all prices are quoted as forward prices.

Our definition of the implied volatility skew will refer to the function C^{BS}, specified as follows.

Define $C^{BS} : \mathbb{R}_*^3 \times \mathbb{R}_+ \to \mathbb{R}$ and $P^{BS} : \mathbb{R}_*^3 \times \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_* := \mathbb{R}\backslash\{0\}$ and $\mathbb{R}_+ := (0,\infty)$, by

$$C^{BS}(x,k,\Sigma,T) := xN(d_+) - kN(d_-) \tag{2.1}$$

$$P^{BS}(x,k,\Sigma,T) := kN(-d_-) - xN(-d_+) \tag{2.2}$$

$$d_\pm := \frac{\log(x/k)}{\Sigma\sqrt{T}} \pm \frac{\Sigma\sqrt{T}}{2} \tag{2.3}$$

where N denotes the standard normal CDF. We may suppress the last argument (T) of C^{BS}.

Definition 1 (Implied volatility) Let X be a process with $X_0 > 0$. For all positive K,T such that

$$(X_0 - K)^+ < \mathbb{E}(X_T - K)^+ < X_0, \tag{2.4}$$

define the *implied volatility* of X at (K,T) to be the $\sigma_{\text{imp}} > 0$ such that

$$C^{BS}(X_0, K, \sigma_{\text{imp}}, T) = \mathbb{E}(X_T - K)^+. \tag{2.5}$$

Let us write $\sigma_{\text{imp}}^X(K,T)$ for this implied volatility, which is well-defined, because $C^{BS}(X_0,K,\cdot,T)$ is strictly increasing on $\mathbb{R}_+$, and has range $((X_0-K)^+, X_0)$.

We refer to the function $\sigma_{\text{imp}}^X(\cdot,T)$ as the implied *volatility skew* of X at expiry T.

3 Displaced Lognormal

Definition 2 A process S follows *displaced lognormal* dynamics, with displacement $\theta \in \mathbb{R}$, if

$$\mathrm{d}S_t = \sigma(S_t - \theta)\mathrm{d}W_t, \qquad S_0 > \theta,\ \sigma > 0, \tag{3.1}$$

where W is Brownian motion.

Thus $S-\theta$ is a driftless geometric Brownian motion with volatility σ, and the interval of points attainable by S is (θ,∞). If modeling a nonnegative underlying such as a stock price, this model for $\theta < 0$ will misprice deep-out-of-the-money puts, due to the possibility of $S_T < 0$. This model has further limitations, even for at-the-money contracts, as we will see later.

For $K > \theta$, a K-strike T-expiry European call option on a displaced lognormal S has price

$$\mathbb{E}(S_T - K)^+ = \mathbb{E}((S_T-\theta) - (K-\theta))^+ = C^{BS}(S_0-\theta, K-\theta, \sigma, T). \tag{3.2}$$

In general, payoffs invariant to parallel shifts of the S path and the contract parameters can be priced using Black-Scholes model valuation methods, but applied to displaced arguments.

3.1 Implied volatility

Let us apply Definition 1 to the case that $X = S$, a displaced lognormal. If $\theta \geq 0$ then (2.4) holds for all K,T positive. If $\theta < 0$, then the first inequality in (2.4) holds for all K,T positive, but the second inequality $\mathbb{E}(S_T - K)^+ < S_0$ may fail for small positive K, due to the nonzero probability that $S_T < 0$. We therefore take care to define the strike interval on which implied volatility exists. For displaced lognormal S, let

$$\mathbb{K}^S(T) := \{K > \theta^+ : C^{BS}(S_0 - \theta, K - \theta, \sigma, T) < S_0\} \tag{3.3}$$

which is a semi-infinite interval, by monotonicity of C^{BS} in K. For each $T > 0$ and each $K \in \mathbb{K}^S(T)$, equation (2.5) defines the implied volatility of S to be the σ_{imp} such that

$$C^{BS}(S_0, K, \sigma_{\text{imp}}, T) = C^{BS}(S_0 - \theta, K - \theta, \sigma, T). \tag{3.4}$$

To abbreviate the $\sigma^S_{\text{imp}}(K,T)$ and $\mathbb{K}^S(T)$ notations for displaced lognormal S, we will suppress the S superscript, and possibly also the T argument.

3.2 Global behavior

Theorem 1 establishes the following global properties of σ_{imp}: If $\theta < 0$ then σ_{imp} is everywhere strictly decreasing in K, and bounded below by σ. If $\theta > 0$ then σ_{imp} is everywhere strictly increasing in K, and bounded above by σ. In both cases, the global bounds are also asymptotes.

Theorem 1 (Global behavior) *Implied volatilities in the displaced lognormal model* (3.1) *have the following global properties.*

1. (Monotonicity in strike). *For all $T > 0$ and $K \in \mathbb{K}(T)$,*

$$\operatorname{sgn} \frac{\partial \sigma_{\text{imp}}}{\partial K}(K) = \operatorname{sgn} \theta. \tag{3.5}$$

2. (Upper/lower bound). *For all $T > 0$ and $K \in \mathbb{K}(T)$:*

$$\textit{If } \theta > 0 \textit{ then } \sigma_{\text{imp}}(K) < \sigma. \textit{ If } \theta < 0 \textit{ then } \sigma_{\text{imp}}(K) > \sigma. \tag{3.6}$$

3. (Sharpness of bound). *For all $T > 0$, we have $\sigma_{\text{imp}}(K) \to \sigma$ as $K \to \infty$.*
   Hence $\sup_{K \in \mathbb{K}(T)} \sigma_{\text{imp}}(K) = \sigma$ *if $\theta > 0$; and* $\inf_{K \in \mathbb{K}(T)} \sigma_{\text{imp}}(K) = \sigma$ *if $\theta < 0$.*

Proof Appendix A.

Brigo and Mercurio (2002) proves the $K = S_0$ case of (3.5, 3.6). Theorem 1 extends to all $K \in \mathbb{K}(T)$.

Remark 1 Empirical volatility skews are typically not monotonic over the entire range of strikes; a volatility skew which slopes downward in the central portion of the strike range will usually still turn upward at sufficiently large strikes. Theorem 1 proves that the displaced lognormal cannot reproduce this empirical feature.

3.3 At-the-money behavior

Theorems 2 and 3 focus on two different subsets of the (K,T) domain.

Theorem 2 examines the at-the-money strike $K = S_0$. Specifically, if $T > 0$ and $S_0 \in \mathbb{K}(T)$, then define the at-the-money implied volatility $\sigma_{\text{atm}}(T) := \sigma_{\text{imp}}(S_0,T)$, which may be abbreviated as σ_{atm}. We bound the level σ_{atm} and also the slope of σ_{imp} at-the-money. By "slope" we always mean $\partial \log \sigma_{\text{imp}}/\partial \log K$, the strike-elasticity of implied volatility.

Theorem 2 (At-the-money behavior) *At-the-money implied volatilties in the displaced lognormal model* (3.1) *have the following properties.*

1. (ATM level). *If* $T > 0$ *and* $S_0 \in \mathbb{K}(T)$ *then*

$$\begin{aligned} \sigma_{\text{atm}} &\geq \Big(1+\frac{|\theta|}{S_0}\Big)\sigma \quad \textit{if } \theta \leq 0; \\ \sigma_{\text{atm}} &\leq \Big(1-\frac{\theta}{S_0}\Big)\sigma \quad \textit{if } \theta \geq 0. \end{aligned} \tag{3.7}$$

2. (ATM slope). *If* $\theta < 0$ *and* $T > 0$ *and* $S_0 \in \mathbb{K}(T)$ *then*

$$\frac{1}{2}\frac{|\theta|}{|\theta|+S_0} \leq \left|\frac{\partial \log \sigma_{\text{imp}}}{\partial \log K}\right|_{K=S_0} = \frac{N(\sigma_{\text{atm}}\sqrt{T}/2) - N(\sigma\sqrt{T}/2)}{\phi(\sigma_{\text{atm}}\sqrt{T}/2)\sqrt{T}\sigma_{\text{atm}}} < \frac{1}{2}e^{\sigma_{\text{atm}}^2 T/8}. \tag{3.8}$$

Proof Appendix B.

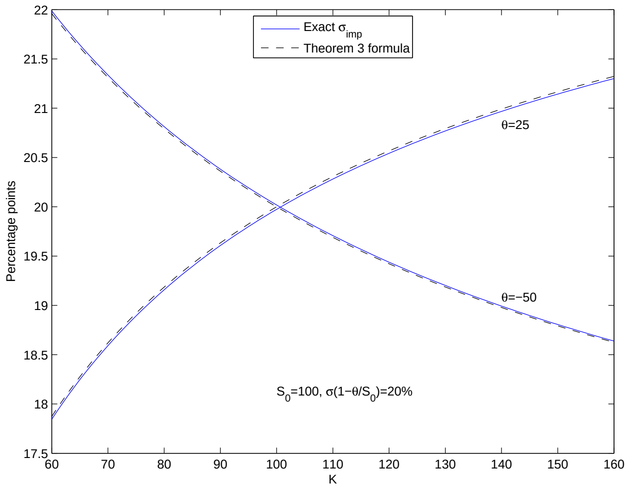
Remark 2 If $T \leq 1$ and $\sigma_{\text{atm}} \leq 100\%$, then $(1/2)e^{\sigma_{\text{atm}}^2 T/8} < 0.57$. Empirically, however, equity volatility skews typically slope downward more steeply than -0.57. Indeed, in S&P500 daily data from all dates (1996–2008) in the OptionMetrics database, the approximate[1] 3-month-expiry at-the-money slope $\partial \log \sigma_{\text{imp}}/\partial \log K$ is more negative than -1.29 on 90% of the days in the sample. Theorem 2 proves that the displaced lognormal cannot reproduce steepness of this magnitude.

3.4 Short-expiry behavior

Theorem 3 takes the short-expiry $T \downarrow 0$ limit of the implied volatility skew, and expresses the solution explicitly. The $K = S_0$ case is known (indeed Rebonato (2004) refines the $K = S_0$ formula, to address the case of T large). The contribution of Theorem 3 is to find and prove a short-expiry σ_{imp} formula valid for *every* strike $K > \theta^+$.

[1] Our source is the "Volatility Surface" data set, which contains volatility skews interpolated by OptionMetrics using kernel smoothing. We approximate the slope as $\log(\sigma_{\text{imp}}(K_1)/\sigma_{\text{imp}}(K_0))/\log(K_1/K_0)$, where K_1 is the strike of a 0.50-delta call, and K_0 is the strike of a 0.55-delta call, as computed by OptionMetrics.

Fig. 1 Exact σ_{imp} and Theorem 3 formula for $T = 1.0$.



Theorem 3 *For all $K > \theta^+$, in the displaced lognormal model* (3.1)*,*

$$\lim_{T\to 0} \sigma_{\text{imp}}(K,T) = \begin{cases} \dfrac{\sigma \log(S_0/K)}{\log((S_0-\theta)/(K-\theta))} & \text{if } K \neq S_0 \\ \sigma(1-\theta/S_0) & \text{if } K = S_0. \end{cases} \tag{3.9}$$

Proof Appendix C.

Remark 3 The formula in the right-hand side of (3.9) provides, moreover, a remarkably accurate approximation to $\sigma_{\text{imp}}(K,T)$ even for some T *not* close to 0. Figure 1 compares the Theorem 3 formula and the exact $\sigma_{\text{imp}}(K,T)$, at expiry $T = 1.0$.

For some expirations of moderate length, therefore, the $\sigma \log(S_0/K)/\log((S_0 - \theta)/(K-\theta))$ formula may still facilitate calibration of the displaced lognormal parameters (σ,θ) to an empirically observed volatility skew, or to a model-generated volatility skew.

4 Displaced anti-Lognormal

The previous section's results show that with $\theta < 0$, the displaced lognormal produces downward-sloping implied volatility, but not of steepness commensurate with typical equity options data – regardless of how large a negative value θ takes.

A related process, however, does generate arbitrarily large downward slopes.

Definition 3 A process S follows displaced *anti*-lognormal dynamics if

$$\mathrm{d}S_t = \sigma(S_t - \theta)\mathrm{d}W_t, \qquad 0 < S_0 < \theta,\ \sigma < 0, \tag{4.1}$$

where W is a Brownian motion.

Thus $\theta - S$ is a driftless geometric Brownian motion with volatility $-\sigma > 0$, and the interval of points attainable by S is $(-\infty, \theta)$.

Displaced anti-lognormal pricing calculations have the tractability of the displaced lognormal. For instance, to price a T-expiry call struck at $K < \theta$, on a displaced anti-lognormal S,

$$\begin{aligned}\mathbb{E}(S_T - K)^+ = \mathbb{E}(\theta - K - (\theta - S_T))^+ &= P^{BS}(\theta - S_0, \theta - K, -\sigma, T) \\ &= C^{BS}(S_0 - \theta, K - \theta, \sigma, T).\end{aligned} \tag{4.2}$$

So we have formally the same C^{BS} formula as in the displaced lognormal case. Here its first three arguments are negative, which presents no problem; the C^{BS} function is still well-defined by (2.1). An equivalent way to express the result, without negative arguments, is $C^{BS}(\theta - K, \theta - S_0, -\sigma, T)$.

Recognizing the similarities between the displaced lognormal and anti-lognormal, the following terminology groups them together:

Definition 4 (DL) A process S which satisfies either the displaced lognormal (3.1) or the displaced anti-lognormal (4.1) specification is said to be a *DL* process.

4.1 Implied volatility

Implied volatility for displaced *anti*-lognormal S is defined on the strike interval

$$\mathbb{K}^S(T) := \{K \in (0, \theta) : C^{BS}(S_0 - \theta, K - \theta, \sigma, T) < S_0\}, \tag{4.3}$$

similarly to the displaced lognormal definition (3.3). For $K \in \mathbb{K}^S(T)$ we have

$$\begin{aligned}S_0 > C^{BS}(S_0 - \theta, K - \theta, \sigma) &= P^{BS}(\theta - S_0, \theta - K, -\sigma) \\ &> ((\theta - K) - (\theta - S_0))^+ = (S_0 - K)^+,\end{aligned} \tag{4.4}$$

so (2.4) holds and $\sigma^S_{\text{imp}}(K,T)$ is thereby well-defined. To abbreviate the $\sigma^S_{\text{imp}}(K,T)$ and $\mathbb{K}^S(T)$ notations for displaced anti-lognormal S, we will suppress the S superscript, and possibly also the T argument.

The conclusion of Theorem 3 extends to all DL processes:

Theorem 4 (Short-expiry behavior) *For all $K > \theta^+$ in the displaced lognormal model* (3.1)*, as well as for all $K \in (0, \theta)$ in the displaced anti-lognormal model* (4.1)*, the conclusion* (3.9) *holds.*

Proof Appendix C.

The Theorem 4 conclusion, and its derivative with respect to $\log K$, yield the short-expiry limiting volatility skew's level and slope

$$\sigma_{\text{imp}}\bigg|_{T\downarrow 0,\ K=S_0} = \frac{\sigma(S_0 - \theta)}{S_0}, \qquad \frac{\partial \log \sigma_{\text{imp}}}{\partial \log K}\bigg|_{T\downarrow 0,\ K=S_0} = \frac{\theta}{2(S_0 - \theta)}. \tag{4.5}$$

This holds under all DL dynamics. The distinction is that under displaced lognormal dynamics, we have $\theta < S_0$, hence the short-expiry slope cannot be more negative than $-1/2$. Under displaced *anti*-lognormal dynamics, we have $S_0 < \theta$, hence (4.5) can produce arbitrarily steep negative slopes.

5 Calibration of DL to a given volatility skew level and slope

Whether one chooses to use the DL as a model, or as an approximation of another model, or (as we will) as a control variate for another model, in any case the DL parameters (σ,θ) require estimation/calibration. We use the Theorem 4 implications (4.5) to fit the DL parameters to a given implied volatility level and slope.

5.1 Calibration of DL

Given a short-expiry at-the-money skew level a and slope b (either from some model, or from direct empirical measurement), and given an underlying level S_0, there exists a DL process, with $S_0 = \mathsf{S}_0$ and parameters (θ,σ), such that the DL skew's short-expiry level and slope (4.5) match the given level a and slope b, provided that $b \neq -1/2$. Explicitly, we find

$$\sigma = a(1+2b), \qquad\qquad \theta = \frac{2b}{1+2b}S_0. \tag{5.1}$$

In the case of slope $b > -1/2$, the calibrated DL process is a displaced lognormal. In the case of slope $b < -1/2$, the calibrated DL process is a displaced anti-lognormal. The singular case of slope $b = -1/2$ can be matched by *normal* or "Bachelier" dynamics $\mathrm{d}S_t = aS_0\mathrm{d}W_t$. The DL and Bachelier models belong to the family $\mathrm{d}S_t = (\sigma S_t + A)\mathrm{d}W_t$, where $\sigma \neq 0$ in the case of DL, and $\sigma = 0$ in the singular case of Bachelier.

Remark 4 Although (4.5) is a short-expiry limit, Remark 3 indicates its accuracy at T of moderate length. Therefore (5.1) may still facilitate calibration of (σ,θ) to volatility skews (a,b) even at moderately long expiries.

5.2 CEV and SABR stochastic volatility models

For many local or stochastic volatility models, there exist explicit short maturity approximations of implied volatility, such as in Lewis (2000) and Berestycki et al. (2004), making it easy to calculate the implied volatility level a and slope b, and to calibrate DL parameters σ and θ via (5.1).

Two such models capable of generating realistically steep at-the-money implied volatility skews are the Constant Elasticity of Variance (CEV) model and the SABR model. In the CEV model (Cox 1996),

$$\mathrm{d}\mathsf{S}_t = \alpha \mathsf{S}_t^{\beta}\mathrm{d}\mathsf{W}_t, \qquad \mathsf{S}_0 > 0, \tag{5.2}$$

where $\beta \leq 1$, and absorption is imposed at $\mathsf{S} = 0$.

The CEV model can generate a steep downward implied volatility skew at-the-money. Indeed, by Berestycki et al. (2002) and Roper (2009), for all $K > 0$,

$$\lim_{T\to 0} \sigma_{\mathrm{imp}}^{\mathrm{CEV}}(K,T) = \begin{cases} \dfrac{\alpha(1-\beta)\log(\mathsf{S}_0/K)}{\mathsf{S}_0^{1-\beta} - K^{1-\beta}} & \text{if } K \neq S_0 \\ \alpha \mathsf{S}_0^{\beta-1} & \text{if } K = S_0. \end{cases} \tag{5.3}$$

Differentiating with respect to $\log K$, we have

$$\left.\frac{\partial \log \sigma_{\text{imp}}^{\text{CEV}}}{\partial \log K}\right|_{T\downarrow 0,\ K=\mathsf{S}_0} = \frac{\beta-1}{2}, \tag{5.4}$$

which can take arbitrarily large negative values.

The widely-used SABR model (Hagan et al. 2002) generalizes the CEV, by making the coefficient α stochastic, with volatility-of-volatility $\nu \geq 0$, and correlation $\rho \in [-1,1]$ between S and α:

$$\begin{aligned} \mathrm{d}\mathsf{S}_t &= \alpha_t \mathsf{S}_t^\beta \mathrm{d}\mathsf{W}_t, \qquad \mathsf{S}_0 > 0 \\ \mathrm{d}\alpha_t &= \nu \alpha_t \mathrm{d}\mathsf{B}_t, \qquad \alpha_0 > 0 \\ \mathrm{d}\mathsf{B}_t &= \rho \mathrm{d}\mathsf{W}_t + \sqrt{1-\rho^2}\mathrm{d}\mathsf{W}_t^* \end{aligned} \tag{5.5}$$

where W and W^* are independent Brownian motions, and absorption is imposed at $\mathsf{S} = 0$.

Taking $\nu = 0$ in the SABR model reduces to the CEV case.

According to a short-expiry approximation in Hagan et al. (2009, eq. 3.1a),

$$\sigma_{\text{imp}}^{\text{SABR}}(K,T) \approx \alpha_0 \mathsf{S}_0^{\beta-1}\left(1 + \left(\frac{\beta-1}{2} + \frac{\rho\nu}{2\alpha \mathsf{S}_0^{\beta-1}}\right)\log(K/\mathsf{S}_0)\right). \tag{5.6}$$

The approximation's slope at $K = \mathsf{S}_0$ is

$$\left.\frac{\partial \log \sigma_{\text{imp}}^{\text{SABR}}}{\partial \log K}\right|_{T\downarrow 0,\ K=\mathsf{S}_0} \approx \frac{\beta-1}{2} + \frac{\rho\nu}{2\alpha \mathsf{S}_0^{\beta-1}}, \tag{5.7}$$

reflecting the contributions to the SABR volatility skew, not just from the functional relationship between price levels and volatility, as expressed by β, but also from the correlation between price increments and volatility, as expressed by ρ.

5.3 Calibration of DL to CEV/SABR

For the SABR process, $a = \alpha_0 \mathsf{S}_0^{\beta-1}$, and b is given by (5.7), so we have

$$\sigma = \alpha_0 \beta \mathsf{S}_0^{\beta-1} + \rho\nu, \qquad\qquad \theta = \mathsf{S}_0 - \frac{\alpha_0 \mathsf{S}_0^\beta}{\alpha_0 \beta \mathsf{S}_0^{\beta-1} + \rho\nu}. \tag{5.8}$$

The $\nu = 0$ special case of (5.8) gives the DL parameters that match the CEV level and slope:

$$\sigma = \alpha\beta \mathsf{S}_0^{\beta-1}, \qquad\qquad \theta = \mathsf{S}_0(\beta-1)/\beta. \tag{5.9}$$

In the CEV case, Marris (1999) and Svoboda-Greenwood (2009) have previously investigated displaced lognormal approximation, by an approach which chooses parameters such that the displaced lognormal *instantaneous* volatility approximates the

CEV *instantaneous* volatility function $\mathsf{S} \mapsto \alpha \mathsf{S}^{\beta}$, in contrast to our approach which matches the *implied* volatility functions. Their approach arrived at the same result (5.9) as our approach, in the CEV case.

A distinction is that our implied volatility approach is intended to apply moreover to models, such as SABR, where instantaneous volatility varies not just as a function of S, but also other stochastic factors. The implied volatility skew reflects the dependence of volatility on the S level together with the other stochastic factors in the model, such as α in the SABR case.

6 Applications to Stochastic Volatility Simulations

Theorems 1 and 2 imply that the displaced lognormal is inconsistent with the steep downward slopes (Remark 2) and non-monotonicity (Remark 1) typical of stock market volatility skews. The displaced anti-lognormal, by (4.5), overcomes the steepness constraint, but introduces other drawbacks: its paths, which take values in $(-\infty, \theta)$, are bounded above and unbounded below – the opposite of the behavior desirable in a model of stock prices.

For these reasons, we do not generally advocate the DL to *model* stock price processes. Rather, we propose the DL to generate *control variates* to reduce variance in the Monte Carlo pricing of derivative contracts under commonly-used dynamics which do match the empirical at-the-money volatility skew.

Indeed, suppose the underlying S dynamics follow some specification that a modeler deems appropriate, such as the CEV or the SABR stochastic volatility model. Suppose the modeler intends to price a derivative contract for which the desired model lacks analytical pricing formulas, such as a discretely-monitored barrier option on the CEV/SABR process S. Let the contract's payoff Y be given by a specified function of the S path. In the absence of analytical solutions, consider the use of Monte Carlo simulation to estimate the price $\mathbb{E}\mathsf{Y}$. The basic Monte Carlo estimator is the sample average

$$\hat{C} := \frac{1}{M} \sum_m \mathsf{Y}_m \tag{6.1}$$

where the simulations $\mathsf{Y}_1, \dots \mathsf{Y}_M$ are iid as Y. To improve accuracy, in the sense of reducing variance, let us apply the control variate technique, where the control comes from a DL process calibrated by (5.1).

There exist, of course, other variance reduction methods, combinable with a DL control variate. We do not investigate them here; rather we maintain focus on the DL, with the intent of illustrating how much variance reduction the DL control brings by itself, and with the understanding that further improvement can come from applying the DL control in concert with other techniques.

6.1 DL as a control variate

The control variate estimator of $\mathbb{E}\mathsf{Y}$, using a control Y, where Y has a known expectation $C := \mathbb{E}Y$ and a known simulation methodology, is defined by

$$\hat{C}^{\mathrm{cv}} := \frac{1}{M}\sum_m \left(\mathsf{Y}_m - \beta Y_m + \beta C\right), \tag{6.2}$$

where the simulated pairs $(\mathsf{Y}_1, Y_1), \ldots, (\mathsf{Y}_M, Y_M)$ are iid as (Y, Y). Good choices of Y have large correlation $\rho_{\mathsf{Y},Y}$ with Y, because increasing $|\rho_{\mathsf{Y},Y}|$ decreases the estimator's variance. Specifically,

$$\operatorname{Var}\hat{C}^{\mathrm{cv}} = (1 - \rho^2_{\mathsf{Y},Y}) \operatorname{Var}\hat{C} \tag{6.3}$$

for the optimal choice of the β coefficient, namely $\beta = \operatorname{Cov}(\mathsf{Y}, Y)/\operatorname{Var}(Y)$, which may also be estimated by simulation. For further details see, for instance, Boyle et al. (1997).

Because the payoff Y is a specified function of the S path, we choose Y to be that same payoff function applied to the S path, where S follows a DL process driven by a Brownian motion that also drives S. Aiming to produce high correlation $\rho_{\mathsf{Y},Y}$, we choose the S process parameters by taking $S_0 = \mathsf{S}_0$ and applying (5.1) to find (θ, σ) such that the short-expiry at-the-money volatility skews implied by S and by S agree in both level and slope.

The suitability of the DL process S to serve in this role stems from a confluence of flexibility and tractability; the DL is potentially flexible enough to generate significant correlation between Y and Y (by linking the parameters of S and S, as discussed above), and yet potentially tractable enough to allow analytic evaluation of $\mathbb{E}Y$ and unbiased simulation of Y, as discussed below.

For shift-invariant contracts (including barriers and lookbacks), exact evaluation of $C = \mathbb{E}Y$ under DL dynamics is just as easy as under Black-Scholes dynamics; more precisely, if the contract's payoff is invariant to parallel shifts of the underlying price path and the contract parameters (such as strike and barrier level), then Black-Scholes model valuation methods, applied to shifted arguments, produce the contract's DL valuation. If, moreover, we can simulate the exact distribution of Y – which is often the case, because S is a transformed Gaussian – then Y can serve as a control variate that reduces variance without introducing any bias.

6.2 Example: Discretely sampled barrier option under CEV/SABR dynamics

To take a concrete example, consider a discretely sampled barrier option on S, which follows CEV (5.2) or SABR (5.5) dynamics. In particular, let the contract be a down-and-out call with expiry T, barrier H, strike K, sampling dates $t_1 < t_2 < \cdots < T_N = T$, and payoff

$$\mathsf{Y} := (\mathsf{S}_T - K)^+ \mathbf{1}(\min_n \mathsf{S}_{t_n} > H). \tag{6.4}$$

Analytical solutions exist for continuous barriers in the CEV model (Davydov-Linetsky 2001), but not for discrete barriers, nor for the SABR model, so we turn to Monte Carlo simulation.

To generate a control variate, we apply the same payoff function to a DL process, driven by the same Brownian motion $W = \mathsf{W}$. More precisely,

$$\begin{aligned} Y &:= (S_T - K)^+ \mathbf{1}(\min_n S_{t_n} > H) \\ \mathrm{d}S_t &= \sigma(S_t - \theta)\mathrm{d}W_t, \end{aligned} \tag{6.5}$$

where $S_0 = \mathsf{S}_0$, and (σ, θ) are calibrated by (5.8) in the SABR case, or (5.9) in the CEV case.

This Y is easily simulated without bias, and the value of $C = \mathbb{E}Y$ can be computed by shifting any of the fast and exact (up to numerical truncation/quadrature error) solutions for discrete barrier option prices in the Gaussian framework, such as Broadie and Yamamoto (2005), or in the Lévy framework, such as Petrella and Kou (2004) or Feng and Linetsky (2008).

An alternative to (6.5) is to choose instead a continuously-monitored control

$$Y^* := (S_T - K)^+ \mathbf{1}(\min_{t \in [0,T]} S_t > H). \tag{6.6}$$

The control expectation $\mathbb{E}Y^*$ has a simple exact formula, and the control Y^* can be simulated without bias, using Brownian bridge techniques of Beaglehole et al. (1997).

6.3 Numerical results: Discretely sampled barrier option

Our experiments simulate the payoff (6.4), where

$$K = S_0 = 100,\ H = 95,\ T \in \{2/12, 4/12\},\ N = 252 \times T. \tag{6.7}$$

Table 1 Percentage reduction of variance, using DL control for CEV

	$T = 2$ months				$T = 4$ months		
β	$\alpha\mathsf{S}_0^{\beta-1}$			β	$\alpha\mathsf{S}_0^{\beta-1}$		
	0.15	0.20	0.25		0.15	0.20	0.25
-0.5	99.99%	99.98%	99.95%	-0.5	99.96%	99.94%	99.91%
-1.0	99.97%	99.97%	99.88%	-1.0	99.96%	99.94%	99.82%
-1.5	99.97%	99.92%	99.81%	-1.5	99.93%	99.79%	99.62%

Payoff: (6.4) with (6.7). Control: (6.5) with (5.9).

Table 2 Percentage reduction of variance, using DL control for SABR

	$T = 2$ months				$T = 4$ months		
ρ	ν			ρ	ν		
	0.2	0.4	0.6		0.2	0.4	0.6
-0.4	99.12%	97.58%	95.48%	-0.4	98.07%	94.91%	91.26%
-0.6	99.30%	97.95%	96.28%	-0.6	98.40%	95.78%	92.72%
-0.8	99.47%	98.66%	97.52%	-0.8	98.84%	97.23%	95.07%

S process parameters: $\beta = 0.2$, $\alpha_0 S_0^{\beta-1} = 20\%$.
Payoff: (6.4) with (6.7). Control: (6.5) with (5.8).

In the CEV case, shown in Table 1, we take $\beta \in \{-0.5, -1.0, -1.5\}$, with α such that $\alpha S_0^{\beta-1} \in \{0.15, 0.20, 0.25\}$, chosen to approximate Hirsa-Courtadon-Madan's (2003) estimates of S&P500 CEV parameters; our β is what they denote as $\beta+1$. We use the control (6.5), where (θ, σ) are tuned to the CEV process by (5.9).

In the SABR case, we take $\beta = 0.2$, with α_0 such that $\alpha_0 S_0^{\beta-1} = 0.2$, with an array of choices for (ν, ρ). We use the control (6.5), where (θ, σ) are tuned to the SABR process by (5.8).

Given these parameter values[2], we then simulate 100000 paths. Tables 1 and 2 report the estimated "percentage reduction of variance"

$$100\% - \frac{\widehat{\mathrm{Var}}(\hat{C}^{\mathrm{cv}})}{\widehat{\mathrm{Var}}(\hat{C})}, \tag{6.8}$$

where each $\widehat{\mathrm{Var}}$ is the scaled sample variance of the summands in (6.1) and (6.2) respectively. Equivalently, the percentage reduction of variance equals the "R-squared" of an OLS regression of the CEV/SABR barrier-option payoff Y on the DL control payoff Y.

7 Conclusion

By establishing properties shared by all displaced lognormal volatility skews, Theorems 1 and 2, in effect, exhibit limitations on what phenomena the displaced lognormal can faithfully model. In particular, the displaced lognormal's everywhere-monotonic skews (Theorem 1) and its state space $(\theta, \infty) \neq \mathbb{R}_+$ may be drawbacks, when pricing contracts having sensitivity to the tail behavior of an underlying process whose true state space is $\mathbb{R}_+$ or whose true volatility skew is non-monotonic. The slope-constrained skew (Theorem 2) of the displaced lognormal may be a drawback, when modeling markets, such as typical equity markets, which exhibit downward skews of steepness greater than the Theorem 2 upper bound. The displaced *anti*-lognormal overcomes the slope constraint, but it imposes bounds from above on S, while allowing unbounded negative S.

We therefore do not endorse the DL as a *model* of equity markets, but we do propose the DL as a *control variate* to improve the accuracy of Monte Carlo pricing under alternative dynamics (such as CEV and SABR) that do model equity prices. The DL is effective as a control variate, because it is simple enough to admit both unbiased simulation and explicit pricing formulas for many contracts, yet flexible

[2] These parameters generate notable variability in the (annualized) integrated variance on $[0, T]$, defined as the random variable

$$\frac{1}{T}\int_0^T \left(\alpha_t S_t^{\beta-1}\right)^2 \mathrm{d}t \in [0, \infty]$$

in the SABR case and (with $\alpha_t := \alpha$) in the CEV case. For instance, in the SABR model with $\rho = -0.6$ and $\nu = 0.4$, the *interquartile interval*, meaning the (25th percentile, 75th percentile) of integrated variance on $[0, T]$, expressed as a proportion of the initial instantaneous variance, is $(0.85, 1.18)$ at $T = 2$ months, and $(0.79, 1.25)$ at $T = 4$ months. In the CEV model with $\beta = -1.0$ and $\alpha S_0^{\beta-1} = 0.20$, the interquartile interval is $(0.89, 1.15)$ at $T = 2$ months, and $(0.85, 1.22)$ at $T = 4$ months, according to our simulations.

enough to generate high correlations and hence significant variance reduction with respect to the CEV/SABR dynamics of Table 1.

Toward either purpose – as a model, or as a computational device on behalf of another model – the DL calibrates easily to a given volatility skew level and slope, via the limiting implied volatility formula of Theorem 3 and 4.

A Appendix: Proof of Theorem 1

Define for all $K > \theta^+$

$$d_2(K) := \frac{\log[(S_0-\theta)/(K-\theta)]}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}.$$

Let $\underline{K}^S(T) := \inf \mathbb{K}^S(T)$. Our notation may suppress the S or T. Note that $\mathbb{K} = (\underline{K}, \infty)$.

If $\theta \geq 0$ then $\underline{K} = \theta$. If $\theta < 0$ then $\underline{K} > 0$ because $C^{BS}(S_0-\theta, 0-\theta, \sigma) > S_0 - \theta + \theta = S_0$.

Proposition 1 *For all $T > 0$ and $K \in \mathbb{K}(T)$:*
If $\theta > 0$ then $\sigma_{\text{imp}}(K) < \sigma$. If $\theta < 0$ then $\sigma_{\text{imp}}(K) > \sigma$.

Proof For all $K > \theta$ and $S_0 > \theta$,

$$\frac{d}{d\theta} C^{BS}(S_0-\theta, K-\theta, \sigma) = -\frac{\partial C^{BS}}{\partial S} - \frac{\partial C^{BS}}{\partial K} = -N(d_2 + \sigma\sqrt{T}) + N(d_2) < 0 \tag{A.1}$$

so, according as $\theta \gtrless 0$, we have $C^{BS}(S_0-\theta, K-\theta, \sigma) \lessgtr C^{BS}(S_0, K, \sigma)$. By (3.4),

$$\sigma_{\text{imp}}(K) \lessgtr \sigma. \tag{A.2}$$

for all $K \in \mathbb{K}$.

Proposition 2 *For all $T > 0$, we have $\sigma_{\text{imp}}(K) \to \sigma$ as $K \to \infty$.*
Hence $\sup_{K\in\mathbb{K}(T)} \sigma_{\text{imp}}(K) = \sigma$ if $\theta > 0$; and $\inf_{K\in\mathbb{K}(T)} \sigma_{\text{imp}}(K) = \sigma$ if $\theta < 0$.

Proof We prove for $\theta > 0$. The proof for $\theta < 0$ is similar.

By Proposition 1, $\sigma_{\text{imp}}(K) < \sigma$ for all $K \in \mathbb{K}$. So it suffices to show that for all $\delta > 0$, there exists k such that for all $K > k$,

$$\sigma - \delta < \sigma_{\text{imp}}(K)$$

or equivalently

$$C^{BS}(S_0, K, \sigma-\delta) < C^{BS}(S_0, K, \sigma_{\text{imp}}(K)) = C^{BS}(S_0-\theta, K-\theta, \sigma).$$

Hence it suffices that for K large enough,

$$0 < M(K) := C^{BS}(S_0-\theta, K-\theta, \sigma) - C^{BS}(S_0, K, \sigma-\delta), \tag{A.3}$$

which we verify as follows. For all $K > \theta^+$,

$$\frac{\partial M}{\partial K} = -N\left(\frac{\log[(S_0-\theta)/(K-\theta)]}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right) + N\left(\frac{\log(S_0/K)}{(\sigma-\delta)\sqrt{T}} - \frac{(\sigma-\delta)\sqrt{T}}{2}\right). \tag{A.4}$$

So

$$\operatorname{sgn}\frac{\partial M}{\partial K} = \operatorname{sgn}\left[\frac{\log(S_0/K)}{(\sigma-\delta)\sqrt{T}} - \frac{\log[(S_0-\theta)/(K-\theta)]}{\sigma\sqrt{T}} + \frac{\delta\sqrt{T}}{2}\right] \tag{A.5}$$

$$= \operatorname{sgn}\left[\frac{\log(S_0/K)^{\sigma} - \log[(S_0-\theta)/(K-\theta)]^{\sigma-\delta}}{(\sigma-\delta)\sigma\sqrt{T}} + \frac{\delta\sqrt{T}}{2}\right] \tag{A.6}$$

$$= \operatorname{sgn}\left[\frac{\log[(S_0\times(K-\theta))/(K\times(S_0-\theta))]^{\sigma} - \log[(K-\theta)/(S_0-\theta)]^{\delta}}{(\sigma-\delta)\sigma\sqrt{T}} + \frac{\delta\sqrt{T}}{2}\right]. \tag{A.7}$$

The part inside the sgn in (A.7) approaches $-\infty$ as $K \to \infty$.

So for K sufficiently large, $\partial M/\partial K(K) < 0$.

Moreover $\lim_{K\to\infty} M(K) = 0 - 0 = 0$.

Therefore $M(K) > 0$ for K sufficiently large, which proves (A.3).

Lemma 1 *For all $K \in \mathbb{K}$,*

$$\operatorname{sgn}\frac{\partial\sigma_{\text{imp}}}{\partial K}(K) = \operatorname{sgn} F(\sigma_{\text{imp}}(K), K)$$

where $F : \mathbb{R} \times (\theta^+, \infty) \to \mathbb{R}$ is defined by

$$F(y, K) := -y^2 T/2 - y\sqrt{T} d_2(K) + \log(S_0/K).$$

Proof Taking the K derivative of (3.4),

$$\frac{\partial C^{BS}}{\partial K}(S_0, K, \sigma_{\text{imp}}(K)) + \frac{\partial C^{BS}}{\partial \sigma}(S_0, K, \sigma_{\text{imp}}(K))\frac{\partial\sigma_{\text{imp}}}{\partial K}(K) = \frac{\partial C^{BS}}{\partial K}(S_0 - \theta, K - \theta, \sigma). \tag{A.8}$$

Therefore

$$\begin{aligned}
\operatorname{sgn}\frac{\partial\sigma_{\text{imp}}}{\partial K}(K) &= \operatorname{sgn}\left[\frac{\partial C^{BS}}{\partial K}(S_0 - \theta, K - \theta, \sigma) - \frac{\partial C^{BS}}{\partial K}(S_0, K, \sigma_{\text{imp}}(K))\right] \\
&= \operatorname{sgn}\left[\left(\frac{\log(S_0/K)}{\sigma_{\text{imp}}\sqrt{T}} - \frac{\sigma_{\text{imp}}\sqrt{T}}{2}\right) - d_2(K)\right] \\
&= \operatorname{sgn}\left[-\sigma_{\text{imp}}^2(K)T/2 - \sigma_{\text{imp}}(K)\sqrt{T}d_2(K) + \log(S_0/K)\right] \\
&= \operatorname{sgn} F(\sigma_{\text{imp}}(K), K),
\end{aligned}$$

as claimed.

Lemma 2 *There exists $K \in \mathbb{K}$ such that*

$$\operatorname{sgn}\frac{\partial\sigma_{\text{imp}}}{\partial K}(K) = \operatorname{sgn}\theta.$$

Proof If $\theta = 0$ then this is obvious.

If $\theta < 0$ then as $K \downarrow \underline{K}$ we have $C^{BS}(S_0 - \theta, K - \theta, \sigma) \to S_0$ hence $\sigma_{\text{imp}} \to \infty$. So there exists $K > \underline{K}$ such that $\partial\sigma_{\text{imp}}/\partial K(K) < 0$, as claimed.

If $\theta > 0$ then the at-the-money strike $K = S_0$ satisfies the conclusion, because

$$\operatorname{sgn}\frac{\partial\sigma_{\text{imp}}}{\partial K}(S_0) = \operatorname{sgn}\left[-\sigma_{\text{imp}}(S_0)\sqrt{T}/2 + \sigma\sqrt{T}/2\right] > 0$$

by Proposition 1.

Proposition 3 *For all $T > 0$ and $K \in \mathbb{K}(T)$,*

$$\operatorname{sgn}\frac{\partial\sigma_{\text{imp}}}{\partial K}(K) = \operatorname{sgn}\theta. \tag{A.9}$$

Proof If $\theta = 0$ then this is obvious. Otherwise, by Lemma 2 the conclusion holds for at least one $K \in \mathbb{K}$. By continuity of $\frac{\partial\sigma_{\text{imp}}}{\partial K}$ on $\mathbb{K}$, the conclusion will hold for all $K \in \mathbb{K}$, if we can show that $\partial\sigma_{\text{imp}}/\partial K \neq 0$ on $\mathbb{K}$, or equivalently (by Lemma 1) that $F(\sigma_{\text{imp}}(K), K) \neq 0$ for all $K \in \mathbb{K}$. The remaining lemmas complete the proof, by verifying that $F(\sigma_{\text{imp}}(K), K) \neq 0$.

For $K > \theta^+$ define

$$\Delta(K) := d_2^2(K) + 2\log(S_0/K) \tag{A.10}$$

$$h(K) := d_2(K) + \sigma\sqrt{T}\frac{K-\theta}{K}. \tag{A.11}$$

Let

$$\mathbb{D} := \{K > \theta^+ : \Delta(K) \geq 0\},$$

and for all $K \in \mathbb{D}$, define

$$y_\pm(K) := \frac{-d_2(K) \pm \sqrt{\Delta(K)}}{\sqrt{T}}. \tag{A.12}$$

Lemma 3 *For $(y,K) \in \mathbb{R} \times \mathbb{K}$, we have $F(y,K) = 0$ if and only if $K \in \mathbb{D}$ and $y = y_\pm(K)$.*

Proof If $K \notin \mathbb{D}$ then $F(\cdot,K)$ is a quadratic with no real roots. If $K \in \mathbb{D}$ then $F(\cdot,K)$ has roots $y = y_\pm(K)$.

Lemma 4 *For all $K \in \mathbb{D}$ define*

$$H_\pm(K) := S_0 N(\pm\sqrt{\Delta(K)}) - (S_0-\theta)N(d_2(K)+\sigma\sqrt{T}) - \theta N(d_2(K))$$

Then for all $K \in \mathbb{D} \cap \mathbb{K}$,

$$H_\pm(K) = C^{BS}(y_\pm(K)) - C^{BS}(\sigma_{\text{imp}}(K))$$

where $C^{BS}(\cdot)$ is shorthand for $C^{BS}(S_0,K,\cdot)$.

Hence $y_+(K) \gtrless \sigma_{\text{imp}}(K)$ if $H_+(K) \gtrless 0$. Likewise $y_-(K) \gtrless \sigma_{\text{imp}}(K)$ if $H_-(K) \gtrless 0$.

Proof Using $\log(S_0/K)/(y_\pm(K)\sqrt{T}) - y_\pm(K)\sqrt{T}/2 = d_2(K)$ and (3.4), we have

$$\begin{aligned}
C^{BS}(y_\pm(K)) - C^{BS}(\sigma_{\text{imp}}) &= \left[S_0 N\left(\frac{\log(S_0/K)}{y_\pm(K)\sqrt{T}} + \frac{y_\pm(K)\sqrt{T}}{2}\right) - KN(d_2(K))\right] \\
&\quad - \left[(S_0-\theta)N(d_2(K)+\sigma\sqrt{T}) - (K-\theta)N(d_2(K))\right] \\
&= S_0 N(d_2(K)+y_\pm(K)\sqrt{T}) - (S_0-\theta)N(d_2(K)+\sigma\sqrt{T}) - \theta N(d_2(K)) \\
&= H_\pm(K)
\end{aligned}$$

The remaining conclusion is by monotonicity of C^{BS}.

Lemma 5 *If $\theta \gtrless 0$ then for all $K \in \mathbb{D}$ we have $\sqrt{\Delta(K)} \gtrless |h(K)|$.*

Proof Consider only the case $\theta > 0$. The proof for $\theta < 0$ is similar.

We need to show that for all $K \in \mathbb{D}$,

$$d_2^2(K) + 2\log(S_0/K) > d_2^2(K) + \left(\frac{K-\theta}{K}\right)^2 \sigma^2 T + 2d_2(K)\sigma\sqrt{T}\frac{K-\theta}{K}$$

or equivalently that

$$\left(\left(\frac{K-\theta}{K}\right)^2 - \left(\frac{K-\theta}{K}\right)\right)\sigma^2 T + 2\frac{K-\theta}{K}\log[(S_0-\theta)/(K-\theta)] - 2\log(S_0/K) < 0. \tag{A.13}$$

The first term is negative, because $0 < (K-\theta)/K < 1$; so it suffices to show that for all $K > \theta$,

$$L(K) := \frac{K-\theta}{K}\log[(S_0-\theta)/(K-\theta)] - \log(S_0/K) \leq 0.$$

This is verified by $L(S_0) = 0$ and $\partial L/\partial K = (\theta/K^2)\log[(S_0-\theta)/(K-\theta)] \lessgtr 0$ for $K \gtrless S_0$.

Lemma 6 *For all $K > \theta^+$:*

- *If $\theta > 0$ then $K \in \mathbb{D}$ and $\frac{\partial H_-}{\partial K} > 0,\ \frac{\partial H_+}{\partial K} > 0$.*
- *If $\theta < 0, h(K) > 0$ then $\frac{\partial \Delta}{\partial K}(K) < 0$. If moreover $K \in \mathbb{D}$ then $\frac{\partial H_-}{\partial K}(K) > 0,\ \frac{\partial H_+}{\partial K}(K) < 0$.*
- *If $\theta < 0, h(K) < 0$ then $\frac{\partial \Delta}{\partial K}(K) > 0$. If moreover $K \in \mathbb{D}$ then $\frac{\partial H_-}{\partial K}(K) < 0,\ \frac{\partial H_+}{\partial K}(K) > 0$.*

Proof The Δ conclusions hold because

$$\frac{\partial \Delta}{\partial K}(K) = -\frac{2h(K)}{(K-\theta)\sigma\sqrt{T}}.$$

If $\theta > 0$, the $K \in \mathbb{D}$ conclusion clearly holds for $K \in (S_0, \infty)$, and also holds for $K \in (\theta, S_0]$ because

$$\log[(S_0-\theta)/(K-\theta)]/(\sigma\sqrt{T}) - \sigma\sqrt{T}/2 < \log(S_0/K)/(\sigma\sqrt{T}) - \sigma\sqrt{T}/2 < 0$$

implies

$$\Delta(K) > (\log(S_0/K)/(\sigma\sqrt{T}) - \sigma\sqrt{T}/2)^2 + 2\log(S_0/K) = (\log(S_0/K)/(\sigma\sqrt{T}) + \sigma\sqrt{T}/2)^2 \geq 0.$$

Lastly, in all cases, the $H_\pm$ conclusions hold because of

$$\frac{\partial H_\pm}{\partial K}(K) = \frac{Ke^{-d_2^2(K)/2}}{(K-\theta)\sigma\sqrt{T}\sqrt{\Delta(K)}}\left(\sqrt{\Delta(K)} \mp h(K)\right)$$

and Lemma 5.

Lemma 7 *If $\theta > 0$ then for all $K \in \mathbb{K} = (\theta, \infty)$, we have $K \in \mathbb{D}$ and*

$$y_1(K) < \sigma_{\text{imp}}(K) < y_2(K).$$

Proof By Lemma 6 we have $K \in \mathbb{D}$ and $y_\pm(K)$ are well-defined.

To prove $\sigma_{\text{imp}} < y_+$, note that as $K \downarrow \theta$ we have $d_2(K) \to \infty$ hence $H_+(K) \to 0$. Moreover, $\partial H_+/\partial K > 0$ on $\mathbb{K}$, by Lemma 6. So on $\mathbb{K}$ we have $H_+ > 0$, hence $\sigma_{\text{imp}} < y_+$ by Lemma 4.

To prove $y_- < \sigma_{\text{imp}}$, note that as $K \to \infty$ we have $d_2(K) \to -\infty$ hence $H_-(K) \to 0$. Moreover, $\partial H_-/\partial K > 0$ on $\mathbb{K}$, by Lemma 6. So on $\mathbb{K}$ we have $H_- < 0$, hence $y_- < \sigma_{\text{imp}}$ by Lemma 4.

Lemma 8 *If $\theta < 0$ then for each $K \in \mathbb{K} \cap \mathbb{D}$ we have*

$$\sigma_{\text{imp}}(K) \notin [y_-(K), y_+(K)].$$

Proof Because $\theta < 0$, it is clear that h is decreasing on $(0, \infty)$.

Because $K \in \mathbb{D}$, we have $h(K) \neq 0$ by Lemma 5.

If $h(K) < 0$, then for all $k > K$ we have $h(k) < 0$ and $\partial\Delta/\partial K(k) > 0$ by Lemma 6. So for all $k > K$ we have $k \in \mathbb{D}$ and $\frac{\partial H_-}{\partial K}(k) < 0$ by Lemma 6. Moreover $\lim_{k\to\infty} H_-(k) = 0$. Therefore $H_-(K) > 0$, hence $\sigma_{\text{imp}}(K) < y_-(K)$ by Lemma 4.

If $h(K) > 0$, then for all $k \in (0, K)$ we have $h(k) > 0$ and $\partial\Delta/\partial K(k) < 0$ by Lemma 6. So for all $k \in (0, K)$ we have $k \in \mathbb{D}$ and $\frac{\partial H_+}{\partial K}(k) < 0$ by Lemma 6. Moreover, as $k \downarrow 0$, we have $\Delta(k) \to \infty$, hence

$$\begin{aligned} H_+(k) &\to S_0 - (S_0-\theta)N\left(\frac{\log[(S_0-\theta)/(-\theta)]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) - \theta N\left(\frac{\log[(S_0-\theta)/(-\theta)]}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right) \\ &= S_0 - C^{BS}(S_0-\theta, -\theta, \sigma) < S_0 - (S_0-\theta+\theta) = 0. \end{aligned}$$

Therefore $H_+(K) < 0$, hence $y_+(K) < \sigma_{\text{imp}}(K)$ by Lemma 4.

Lemma 9 *For all $K \in \mathbb{K}$ we have $F(\sigma_{\text{imp}}(K), K) \neq 0$.*

Proof Combine Lemmas 7, 8, and 3.

B Appendix: Proof of Theorem 2

Proof (of Theorem 2.1) Taking $K = S_0$ in (3.4), we have

$$C^{BS}(S_0, S_0, \sigma_{\text{atm}}) = C^{BS}(S_0 - \theta, S_0 - \theta, \sigma), \tag{B.1}$$

hence

$$S_0[2N(\sigma_{\text{atm}}\sqrt{T}/2) - 1] = (S_0 - \theta)[2N(\sigma\sqrt{T}/2) - 1], \tag{B.2}$$

and

$$\begin{aligned} N(\sigma_{\text{atm}}\sqrt{T}/2) &= \frac{S_0 - \theta}{2S_0}[2N(\sigma\sqrt{T}/2) - 1] + \frac{1}{2} \\ &= (1 - \frac{\theta}{S_0})N(\sigma\sqrt{T}/2) + \frac{\theta}{S_0}N(0) \le N((1 - \frac{\theta}{S_0})d_1) \end{aligned} \tag{B.3}$$

for $\theta \ge 0$, because $N(x)$ is concave on $x \ge 0$. Monotonicity of N implies $\sigma_{\text{atm}} \le (1 - \theta/S_0)\sigma$.
Similarly, for $\theta \le 0$, we have $\sigma_{\text{atm}} \ge (1 - \theta/S_0)\sigma$.

Proof (of Theorem 2.2) Using (A.8), we have

$$\left|\frac{\partial \log \sigma_{\text{imp}}}{\partial \log K}\right|_{K=S_0} = \left|\frac{S_0}{\sigma_{\text{imp}}}\frac{\partial \sigma_{\text{imp}}}{\partial K}\right|_{K=S_0} = \frac{N(\sigma_{\text{atm}}\sqrt{T}/2) - N(\sigma\sqrt{T}/2)}{\phi(\sigma_{\text{atm}}\sqrt{T}/2)\sigma_{\text{atm}}\sqrt{T}}, \tag{B.4}$$

because $\theta < 0$ implies $\sigma_{\text{atm}} > \sigma$. By concavity of N on $[0, \infty)$,

$$\phi(\sigma_{\text{atm}}\sqrt{T}/2) \le \frac{N(\sigma_{\text{atm}}\sqrt{T}/2) - N(\sigma\sqrt{T}/2)}{\sigma_{\text{atm}}\sqrt{T}/2 - \sigma\sqrt{T}/2} \le \phi(\sigma\sqrt{T}/2). \tag{B.5}$$

Combining (B.4), (B.5), and Theorem 2.1 produces the lower bound

$$\left|\frac{\partial \log \sigma_{\text{imp}}}{\partial \log K}\right|_{K=S_0} \ge \frac{\sigma_{\text{atm}} - \sigma}{2\sigma_{\text{atm}}} \ge \frac{1}{2}\Big(1 - \frac{1}{1 - \theta/S_0}\Big) = \frac{|\theta|}{2(S_0 + |\theta|)}. \tag{B.6}$$

Combining (B.4) and (B.5) produces the upper bound

$$\left|\frac{\partial \log \sigma_{\text{imp}}}{\partial \log K}\right|_{K=S_0} \le \frac{\sigma_{\text{atm}} - \sigma}{2\sigma_{\text{atm}}} \times \frac{\phi(\sigma\sqrt{T}/2)}{\phi(\sigma_{\text{atm}}\sqrt{T}/2)} \le \frac{1}{2}e^{\sigma_{\text{atm}}^2 T/8}. \tag{B.7}$$

as claimed.

C Appendix: Proof of Theorems 3 and 4

The notation $A(T) \sim B(T)$ means that $A(T)/B(T) \to 1$ as $T \downarrow 0$.

Proof (of Theorem 3) For all $K > \theta^+$, we have $C^{BS}(S_0 - \theta, K - \theta, \sigma, T) \downarrow (S_0 - K)^+$ as $T \downarrow 0$, so for all T sufficiently small, $C^{BS}(S_0 - \theta, K - \theta, \sigma, T) < S_0$ hence $K \in \mathbb{K}^S(T)$. Applying Roper-Rutkowski (2007) Proposition 5.1 to the displaced lognormal model, we have, as $T \downarrow 0$,

$$\sigma_{\text{imp}}(K, T) \sim \frac{|\log(S_0/K)|}{\sqrt{-2T\log(C^{BS}(S_0 - \theta, K - \theta, \sigma, T) - (S_0 - K)^+)}}. \tag{C.1}$$

On the other hand, applying it to the Black-Scholes model, we have, as $T \downarrow 0$,

$$\sigma \sim \frac{|\log((S_0 - \theta)/(K - \theta))|}{\sqrt{-2T\log(C^{BS}(S_0 - \theta, K - \theta, \sigma, T) - (S_0 - K)^+)}}. \tag{C.2}$$

Therefore

$$\sigma_{\text{imp}}(K, T) \sim \frac{|\sigma||\log(S_0/K)|}{|\log((S_0 - \theta)/(K - \theta))|} = \frac{\sigma\log(S_0/K)}{\log((S_0 - \theta)/(K - \theta))} \tag{C.3}$$

as $T \downarrow 0$.

Proof (of Theorem 4) For all $K \in (0,\theta)$, we have $C^{BS}(\theta-K,\theta-S_0,-\sigma,T) \downarrow (S_0-K)^+$ as $T \downarrow 0$, so for all T sufficiently small, $C^{BS}(\theta-K,\theta-S_0,-\sigma,T) < S_0$ hence $K \in \mathbb{K}^S(T)$. Applying Roper-Rutkowski (2007) Proposition 5.1 to the displaced anti-lognormal model, we have, as $T \downarrow 0$,

$$\sigma_{\text{imp}}(K,T) \sim \frac{|\log(S_0/K)|}{\sqrt{-2T\log(C^{BS}(\theta-K,\theta-S_0,-\sigma,T)-(S_0-K)^+)}}. \tag{C.4}$$

On the other hand, applying it to the Black-Scholes model, we have, as $T \downarrow 0$,

$$-\sigma \sim \frac{|\log((\theta-K)/(\theta-S_0))|}{\sqrt{-2T\log(C^{BS}(\theta-K,\theta-S_0,-\sigma,T)-(S_0-K)^+)}}. \tag{C.5}$$

Therefore (C.3) holds.

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References

1. Beaglehole, D., Dybvig, P., Zhou, G.: Going to extremes: Correcting simulation bias in exotic option valuation. Financial Anal. J. 53(1), 62-68 (1997)
2. Berestycki, H., Busca, J., Florent, I.: Asymptotics and calibration of local volatility models. Quant. Finance 2, 61-69 (2002)
3. Berestycki, H., Busca, J., Florent, I.: Computing the implied volatility in stochastic volatility models. Commun. Pure Appl. Math. 57, 1352-1373 (2004)
4. Black, F., Scholes, M.: The pricing of options and corporate liabilities. J. Political Econ. 81, 637-659 (1973)
5. Boyle, P., Broadie, M., Glasserman, P.: Monte Carlo methods for security pricing. J. Econ. Dyn. Control 21, 1267-1321 (1997)
6. Brigo, D., Mercurio, F.: Displaced and mixture diffusions for analytically-tractable smile models. Mathematical Finance – Bachelier Congress 2000, 151-174. Springer, (2002)
7. Broadie, M., Yamamoto, Y.: A double-exponential fast Gauss transform algorithm for pricing discrete path-dependent options. Oper. Res. 53(5), 764-779 (2005)
8. Cox, J.: The constant elasticity of variance option pricing model. J. Portf. Manag. 23, 15-17 (1996)
9. Davydov, D., Linetsky, V.: Pricing and hedging path-dependent options under the cev process. Manag. Sci. 47(7), 949-965 (2001)
10. Feng, L., Linetsky, V.: Pricing discretely monitored barrier options and defaultable bonds in Lévy process models. Math. Finance 18(3), 337-384 (2008)
11. Hagan, P., Kumar, D., Lesniewski, A., Woodward, D.: Managing smile risk. Wilmott, 1(8), 84-108 (2002)
12. Hirsa, A., Courtadon, G., Madan, D.: The effect of model risk on the valuation of barrier options. J. Risk Finance 4, 47-55 (2003)
13. Joshi, M., Rebonato, R.: A displaced-diffusion stochastic volatility LIBOR market model: motivation, definition and implementation. Quant. Finance 3, 458-469 (2003)
14. Lewis, A.: Option Valuation under Stochastic Volatility. Finance Press (2000)
15. Marris, D.: Financial option pricing and skewed volatility. MPhil Thesis, Statistical Laboratory, University of Cambridge (1999)
16. Petrella, G., Kou, S.: Numerical pricing of discrete barrier and lookback options via Laplace transforms. J. Comput. Finance 8, 1-37 (2004)
17. Rebonato, R. Volatility and Correlation. John Wiley & Sons, 2nd edition (2004)
18. Roper, M.: Implied volatility: General properties and asymptotics. PhD Thesis, University of New South Wales (2009)
19. Roper, M., Rutkowski, M.: A note on the behaviour of the Black-Scholes implied volatility close to expiry. University of New South Wales (2007)
20. Rubinstein, M.: Displaced diffusion option pricing. J. Finance 38(1), 213-217 (1983)
21. Svoboda-Greenwood, S.: Displaced diffusion as an approximation of the constant elasticity of variance. Appl. Math. Finance (2009) Forthcoming.