LOCAL VOLUMES, EQUISINGULARITY AND GENERALIZED SMOOTHABILITY

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Abstract. We introduce the local volume of a relatively very ample invertible sheaf as an invariant of equisingularity by determining its change across families. We apply this result to give numerical control of Whitney–Thom (differential) equisingularity for families of isolated complex analytic singularities. The characterization of the vanishing of the local volume gives rise to the class of deficient conormal (dc) singularities. We introduce a notion of generalized smoothability by considering the class of singularities that deform to dc singularities. Using Whitney stratifications and the functoriality properties of conormal spaces we show that fibers of conormal spaces are preserved under transverse maps. Then by Thom’s transversality, the structure theorems of Hilbert–Burch and Buchsbaum–Eisenbud, we show that all smoothable singularities of dimension at least 2, Cohen–Macaulay codimension 2, Gorenstein codimension 3, and determinantal singularities deform to dc singularities.

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1. INTRODUCTION

A principal goal of equisingularity theory is to decide if two germs of sets or maps look alike in some sense. In general, this is a hard problem, but if the germs are part of a family, then it is somewhat easier to predict when the members of the family are the “same” [Z71]. The conditions that will guarantee this “similarity” depend on the total space of the family. Nevertheless, we would like to control these conditions by fiberwise dependent numerical invariants.

The main notion of equsingularity theory that will be addressed in this work is that of Whitney–Thom equisingularity, also known as differential equisingularity. We will use its algebro-geometric formulation and interpret the problem of finding numerical control for it as a problem of intersection theory. In turn, variations of this problem appear in resolutions

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of singularities, equiresolutions, numerical control of flatness and existence of simultaneous canonical models. For most part we will work in fairly general setup: the schemes considered will be of finite type over an arbitrary field. The applications of our results though will be in the complex analytic category via a classical translation.

First, let's fix some notation. Let $X$ and $Y$ be affine reduced schemes of finite type over a field $k$. Assume $X$ is equidimensional and $Y$ is regular and integral of dimension one. Suppose $h: X \to Y$ is a morphism with equidimensional fibers of positive dimension. Let $S$ be a subscheme of $X$ that is proper over $Y$ such that $S_y$ is nowhere dense in $X_y$ for each closed point $y$. Let $C$ be an equidimensional reduced scheme projective over $X$ such that the structure morphism $c: C \to X$ maps each irreducible component of $C$ to an irreducible component of $X$. Set $D := c^{-1}S$ and $\dim C := r + 1$.

Fix a closed point $y_0$ in $Y$. Denote by $D_{\text{vert}}$ the union of components of $D$ of top dimension $r$ that map to $y_0$ under $h \circ c$. Our goal is to control numerically the presence of $D_{\text{vert}}$.

In our applications to equisingularity, $S$ is a subscheme supported over the singular locus of a complex analytic variety $X$, $Y$ is a smooth subspace of $X$, and $X$ is viewed as the total space of a family obtained by a transverse retraction from $X$ to $Y$. The scheme $C$ is a conormal modification of $X$. Then the condition for $X$ to be equisingular along $Y$ in a neighborhood of $y_0$ is expressed by asking $D_{\text{vert}}$ to be empty.

The first case where one can use basic intersection theory to control numerically the presence of $D_{\text{vert}}$ is when $C := \text{Bl}_S X$. In this case $D$ is the exceptional divisor of the blowup. For each $y$ denote by $C(y)$ the blowup of $X_y$ by $S_y$ and by $D(y)$ the exceptional divisor. Set $l := c_1\mathcal{O}_C(1)$ and $l_y := c_1\mathcal{O}_C(y)(1)$. Let $U$ be a small enough affine neighborhood of $y_0$. Then we have the following Excess-Degree Formula (EDF):

$$\int S_y \vert l_y^{-1}[D(y)] \vert - \int S_y \vert l_y^{-1}[D(y)] \vert = \int S_y \vert l [D_{\text{vert}}] \vert$$

for $y \in U - \{y_0\}$. Suppose $\mathcal{O}_C(1)$ is ample on $D_{\text{vert}}$. Then $D_{\text{vert}}$ is empty if and only if the intersection number $\int S_y \vert l_y^{-1}[D(y)] \vert$ is constant for each closed $y \in Y$. A typical situation when $\mathcal{O}_C(1)$ is ample is when $S$ is finite over $Y$, for example.

What happens if $D$ is more generally a Weil divisor in $C$, and if $C$ is not birational to $X$? If $\dim X = 2$ we show that the EDF still applies. In general, to obtain a formula similar to the EDF, one needs to introduce a volume-type invariant that generalizes the top self-intersection number of a Cartier divisor. The volume of a line bundle is an invariant studied extensively by birational geometers (cf. [Laz04, Chp. 2] for definitions, basic properties and constructions). Its local counterpart the local volume was introduced by Fulger [Ful13]. The local volume’s algebraic analogue was studied by [UV11].

From now on assume $S$ is finite over $Y$ and for simplicity of the exposition assume that $k$ is the residue field of each closed point $y$ and the points in $S_y$. Let $\mathcal{L}$ be an invertible very ample sheaf on $C$ relative to $X$. Let $\mathcal{A} := \bigoplus_{n \geq 0} \Gamma(C, \mathcal{L}^\otimes n)$ be the ring of sections of $\mathcal{L}$. Denote by $\mathcal{A}_n$ the $n$th graded piece of $\mathcal{A}$. Inspired by [Ful13], [UV11] and [ELMNP], for each closed point $y \in Y$ we define the restricted local volume of $\mathcal{L}$ at $S_y$ as

$$\text{vol}_{\mathcal{L}}(y) := \limsup_{n \to \infty} \frac{r!}{n^r} \dim_k H^0(\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y).$$

The existence of the volumes as a limit has been a topic of extensive research (cf. [Cut13] and Kaveh and Khovanskii [KK12]). The local volume might be an irrational number as shown by Cutkosky, Herzog and Srinivasan [CHS10].
One of the main result of the paper is the following relation which we refer simply as the Local Volume Formula (LVF):

\[ \text{vol}_{C_{w_0}}(\mathcal{L}) - \text{vol}_{C_y}(\mathcal{L}) = \int_{S_{w_0}} t'[D_{\text{vert}}]. \]

Particular instances of the LVF and the EDF were known before as results of Hironaka, Schickhoff (see Rmk. 2.6 in [Lip82] and [Hir70] for a related result) and Teissier [TS1] for the Hilbert–Samuel multiplicity, Kleiman and Gaffney [GK99], and Gaffney ([Gaff04] and [Gaff08]) for the Buchsbaum–Rim multiplicity. For related work in the projective setting see [Kol15] and [HMX18] Sct. 4.

A direct consequence of the proof of the LVF is the following result: D is flat over Y if and only if \( \dim_k H^1_X((\mathcal{A}_n) \otimes \mathbb{C}_Y, k(y)) \) is constant for \( n \gg 0 \). Thus we can control the scheme structure of D, not only the components of top dimension. This consequence of the proof of the LVF should be seen as the local counterpart to the classical result of Hironaka that says that a family of projective schemes over an integral base is flat if and only if the Hilbert polynomials of the fibers remain the same (see [Hir58] or Thm. 9.9 in Chapter III of [Har77]). An important local analogue of Hironaka’s flatness result appears in [Hir64] in regard to what Hironaka calls normal flatness, which is the flatness of the normal cone of a smooth subvariety of a smooth ambient variety. Hironaka controls normal flatness using the Hilbert–Samuel function. His result can be derived as a consequence of the proof of the LVF.

How do we control numerically the presence of \( D_{\text{vert}} \) when Y is integral of arbitrary dimension? In the case of the EDF we do this with the help of Grothendieck’s connectedness theorem and Zariski upper semi-continuity of intersection numbers. In the setup of the LVF the major obstacle is to show that the volume is constant over Zariski open dense subset of the base. From now on assume \( k = \mathbb{C} \). View \( X \to Y \) as a part of a larger family \( X \to W \), where \( W \) is integral and \( Y \subset W \). Assume that \( S \) and \( C \) are defined in the same way for \( X \).

We say that \( \text{vol}_{C_w}(\mathcal{L}) \) is stable if it is constant for \( w \) in a Zariski open dense subset of \( W \). In this case we say that \( W \) is a good base space.

Following [GR16] we show that we can control the presence of vertical components of \( D \) for \( X \to Y \) by computing the restricted local volumes from generic one-parameter families connecting \( X_{y_0} \) and \( X_y \) to fibers \( X_w \) where the local volume stabilizes. This gives an extension of the LVF to good base spaces of arbitrary positive dimension.

The extension of the LVF to the complex analytic setting is obtained in a standard fashion (see Scts. 1–3 in Chp. II of Moonen’s appendix in [HIO88]). The main application of the LVF in this work is obtaining numerical control for Whitney–Thom (differential) equisingularity. Let’s review briefly its definition and some of its applications. By a classical result of Whitney every complex analytic variety \( V \) can be partitioned into a locally finite family of submanifolds, called strata, so that any pair of incident strata gives rise to a family of singularities obtained as the fibers of a retraction to the lower dimensional stratum and the total space of the family satisfies certain geometric compatibility conditions.

More precisely, let \( X \subset V \) be a complex analytic variety and \( Y \) a smooth subvariety of \( X \) of dimension \( k \) such that \( X - Y \) is smooth and \( (X - Y, Y) \) is a pair of strata. Embed \( X \) in \( \mathbb{C}^{n+k} \) in such a way that so that \( Y \) is linear subspace through the origin \( 0 \) of dimension \( k \). We say \( H \) is a tangent hyperplane at \( x \in X - Y \) if \( H \) is a hyperplane in \( \mathbb{C}^{n+k} \) that contains the tangent space \( T_xX \). Let \( (x_i) \) be a sequence of points from \( X - Y \) and \( (y_i) \) be a sequence of points from \( Y \) both converging to \( 0 \). Suppose that the sequence of secants \( (x_i y_i) \) has limit \( l \) and the sequence of tangent hyperplanes \( \{T_{x_i}X\} \) has limit \( T \). We say that \( X \) is Whitney
equisingular along $Y$ at 0, or that the pair of strata $(X - Y, Y)$ satisfies Whitney conditions at 0, if $I \subset T$.

The Whitney stratifications play an important role in the classification of differentiable maps ([M73], [M76], cf. [Gal93]), in D-module theory and the solution of the Riemann–Hilbert problem [LM83], and in the Goresky–McPherson theory of intersection homology [GM80], to mention few.

Let $f$ be a function on $X$ of constant rank off $Y$. The relative form of Whitney conditions, called the $W_f$ condition, is defined in the same way as the Whitney condition for the pair $(X - Y, Y)$ by replacing the tangent hyperplanes that contain $T_x X$ by tangent hyperplanes that contain the tangent space $T_x, f^{-1} f x_i$ to the level surface $f^{-1} f x_i$, where each $x_i$ is a smooth point of $f^{-1} f x_i$.

Denote by $X_y$ the fiber of a transverse projection to $Y$ and set $f_y := f|X_y$. Then the Thom-Mather second isotopy lemma yields the following result: If $W_f$ holds, then there exists a homeomorphism $q$ from $X_0 \times Y$ onto $X$ such that $f q = f_0 \times 1_Y$. Hence, for all $y \in Y$ close enough to 0 the pairs $X_y, f_y$ are topologically the same.

Our goal now is to find numerical invariants depending on the fibers $X_y, f_y$ and use the LVF to show that their constancy across $Y$ is equivalent to $W_f$. As $W_f$ involves limits of tangent hyperplanes and secant lines, it’s natural to expect that the invariants should be defined in terms of conormal spaces. Define the conormal variety $C(X, f)$ (see Sect. 5) relative to $f$ as the closure in $X \times \mathbb{P}^{n+k-1}$ of the set of pairs $(x, H)$ such that $x$ is a smooth point of the level set $f^{-1} f x$ and $H$ is a tangent hyperplane to $f^{-1} f x$. Let $c_{X,f} : C(X, f) \to X$ be structure morphism. Consider the blowup of $C(X, f)$ with center $c_{X,f}^{-1}(Y)$. Teissier shows that $W_f$ holds at 0 if and only if the exceptional divisor of the blowup does not have vertical components of top dimension. To control the presence of vertical components with invariants depending on the fibers $X_y, f_y$ one needs to replace $C(X, f)$ with its relative version. Define the relative conormal variety $C_{rel}(X, f)$ of $X$ relative to $h : X \to Y$ in the same way but this time requiring that $H$ contains a parallel to $Y$. This replacement is achieved by Teissier’s Principle of Specialization of Integral Dependence.

Define the conormal variety $C_{rel}(X)$ of $X$ relative to $h : X \to Y$ to be the closure in $X \times \mathbb{P}^{n+k-1}$ of the set of pairs $(x, H)$ where $x$ is a smooth point of $X$ and $H$ is a tangent hyperplane at $x$ containing a parallel to $Y$. Denote by $J_{rel}(X)$ the Jacobian module, and by $J_{rel}(X, f)$ the augmented Jacobian module of $X$ and $f$ with respect to the fiber coordinates. Both modules are contained in a free module $\mathcal{F}$. Then $C_{rel}(X, f) = \text{Proj}(R(J_{rel}(X, f)))$ where $R(J_{rel}(X, f))$ is the Rees algebra of $J_{rel}(X, f)$. Denote by $C$ the blowup of $C_{rel}(X, f)$ with center the inverse image of $Y$. Then $C \cong \text{Proj}(R(m_Y J_{rel}(X, f)))$ where $m_Y$ is the ideal of $Y$ in $\mathcal{O}_X$.

Assume $X_y$ and $f_y$ have isolated singularities at $y$. To align with previously used notation we denote the restricted local volume corresponding to $C_y$ by $\varepsilon(m_y J_{rel}(X,Y))(y)$ where $m_y$ is the ideal of $y$ in $\mathcal{O}_{X_y}$ because of its connection to the $\varepsilon$-multiplicity (see [KUV] and [UV1]). Denote the volume corresponding to $C_{rel}(X)_y$ by $\varepsilon(J_{rel}(X))(y)$.

Assume $X \to Y$ can be read from the family $X \to W$, where $W$ is a component of the miniversal base space of $X_0$ and $X'$ is the total deformation space of $X_0$ over $W$, such that $\varepsilon(J_{rel}(X', f))(w)$ is stable for generic $w \in W$ and generic deformation $f$ of $f$. By generic specialization the stability condition means that the local volume $\varepsilon(J(X_w, f_w))$ is constant for generic $w$ where $J(X_w, f_w)$ is the augmented Jacobian module of $X_w, f_w$. As an application of the LVF we show that $(X - Y, Y)$ satisfies $W_f$ if and only if

$$y \to \varepsilon(m_y J_{rel}(X, f))(y)$$
is constant along $Y$. Thus owing to Grauert’s theorem [Gra72], in the case of volume stability, to each isolated singularity $X_0$ we can associated finitely many invariants, each associated with an irreducible component of the base space of miniversal deformations of $X_0$, which control all equisingular deformations of $X_0$.

We prove a similar statement for Thom’s $A_f$ condition, which is a relative stratification condition for the study of functions and mappings on stratified sets. It plays an important role in Thom’s second isotopy theorem, and provides a transversality condition in the development of the Milnor fibration.

Our result for $W_f$ was first proved for isolated hypersurface singularities by Teissier [T72] using the Hilbert-Samuel multiplicity. Gaffney ([Gaf92] and [Gaf96] based on ideas conceived in [Gaf93]) and Gaffney and Kleiman ([GK99]) treated the case of isolated complete-intersection singularities using the Buchsbaum–Rim (BR) multiplicity. More recently, Gaffney and Rangachev [GR16] addressed the case of families of isolated determinantal singularities using Gaffney’s Multiplicity-Polar Theorem for the relative BR multiplicity. In all this cases the base space $W$ of miniversal deformations of $X_0$ is smooth, $X_w$ is smooth for generic $w$, and $\varepsilon(J_{rel}(X, f)(w)) = 0$ which as we show in Rmk. 6.2 is equivalent to $\varepsilon(J(X_w)) = 0$.

Thus it is natural to consider the class of isolated singularities $X_0$ for which $\varepsilon(J(X_w)) = 0$ for generic $w$, which is the simplest instance of volume stability. Let $c_w : C(X_w) \to X_w$ be the conormal space of $X_w$ and $S_w$ the singular locus of $X_w$. We show that $\varepsilon(J(X_w)) = 0$ if and only if $\text{codim}(c_w^{-1}S_w, C(X_w)) \geq 2$.

In other words the volume vanishes if and only if the fibers of $C(X_w)$ over the singular points of $X_w$ are of dimension less than expected. We call such $X_w$ deficient conormal (dc) singularity. We show that the dc property is independent of the embedding of $X_w$ in affine space and stable under infinitesimal deformations. Furthermore, if $X_w$ is smooth of dimension at least 2, then $X_w$ is a dc singularity. Thus the class of singularities that admit deformations to dc singularities is a natural generalization of the class of smoothable singularities. Finally, we show that Cohen–Macaulay codimension 2, Gorenstein codimension 3, and determinantal singularities belong to this class. This allows us to propose a notion of generalized smoothability by considering all singularities deforming to dc singularities.

The paper is organized as follows. In Sct. 2 we prove the EDF. Based on Ramanujam’s interpretation of the Hilbert–Samuel multiplicity we recover and generalize formulas due to Teissier for the Hilbert–Samuel multiplicity and Gaffney’s Multiplicity-Polar Theorem. Our approach relies on basic intersection theory as developed by Fulton in [Ful84], Grothendieck connectedness theorem and the geometric theory of the multiplicity of pairs of standard graded algebras developed by Kleiman and Thorup in [KT94]. We also discuss some applications of the EDF to the deformation theory of singularities.

In Sct. 3 we compute the local volume using a Noether normalization-type result of the author (Prp. 2.6 in [Ran19a]) that shows that every reduced standard graded algebra $\mathcal{A}$ admits a homogeneous embedding in a standard graded algebra $\mathcal{B}$ that behaves like a polynomial ring. We show that in the computation for the local volume we can replace $H^1(\mathcal{A}_n)$ by $H^0(\mathcal{B}_n/\mathcal{A}_n)$ which is much more manageable. The section culminates with a result characterizing the vanishing of the local volume. The main technical ingredient here are results of the author ([Ran19a] and [Ran18]) about the structure of $\text{Ass}_A/B_n/A_n$. We pay special attention to the case of Rees algebras of modules, in which case the local volume was introduced by [UV11] under the name of epsilon multiplicity. Its relevance to equisingularity was discovered by Kleiman, Ulrich and Validashti [KUV]. Their work served as an inspiration for our work.
In Sct. 4 we prove the LVF. We show that the local volume specializes generically with passage to the fibers using a result of the author about the finiteness of \(\text{Ass}_A(B_n/A_n)\) (see Thm. 1.1 (ii) in [Ran19a]). We show how to extend the LVF to the case when \(\dim Y > 1\) under the assumption of volume stability using a covering argument due to Gaffney and the author [GR19] and by computing the local volumes through one-parameter generic deformations. We prove a general version of Teissier’s Principle of Specialization of Integral Dependence.

In Sct. 5 we review the algebraic and analytic formulations of Whitney–Thom equisingularity (\(W_f\) and Thom’s \(A_f\) conditions) and we state the necessary results about integral closures of modules and conormal geometry needed for their numerical characterization. In Sct. 6 we show how to characterize numerically \(W_f\) and \(A_f\) using the LVF and following the approach pioneered by Teissier and developed and simplified by Gaffney and Kleiman.

In Sct. 7 we introduce the notion of deficient conormal (dc) singularities. We give examples coming from affine cones over projective varieties having duals with positive defect. We show that the dc property is intrinsic using a result of Teissier about polar varieties and that is stable under infinitesimal deformations using the LVF. We show that Cohen–Macaulay codimension 2, Gorensten codimension 3, and determinantal singularities admit deformations to dc singularities. The proof is based on five key results.

The first one shows that the fibers of conormal spaces pullback set-theoretically under transverse morphisms by using stratification theory, the Lagrangian and functoriality nature of conormal spaces and a recent result of Gaffney and the author (see Thm. 3.1 in [GR19]).

The second set of results are structure theorems due to Hilbert and Burch, and Buchsbaum and Eisenbud that tells us that the classes of singularities under consideration can be obtained as pullbacks of generic determinantal and Pfaffian singularities by holomorphic maps between complex affine spaces. The third result due to Buchweitz allows us to deform our classes of singularities by deforming these holomorphic maps.

The fourth result is a version of Thom’s transversality in the complex analytic case due to Trivedi [Tr13] that allows us to deform the holomorphic maps so that the generic fibers of these maps are transverse to the strata of Whitney stratification of the generic singularities. Then generic deformations of our singularities will be pullbacks of transverse maps to the generic singularities. The fifth set of results due to Gaffney and the author, and Lakshmibai and Singh is a computation that shows that the generic singularities are dc. Combining all these results we obtain that the generic deformations of the given classes of singularities are dc.

Inspired by a result of Kollár and Kovács [KK18] we show that affine cones over normally embedded abelian varieties of dimension at least 2 cannot be deformed to dc singularities. We finish the section with showing how to compute the restricted local volume associated with the conormal space of an isolated nonsmoothable Cohen–Macaulay codimension 2 singularity.

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2. The Excess–Degree Formula and Applications

In this section we prove the EDF and generalize and recover multiplicity-polar results by Gaffney for the Buchsbaum–Rim multiplicity and by Teissier for the Hilbert–Samuel multiplicity.

The following problem is at the heart of equisingularity theory. Let $g: X \to Y$ be a morphism of schemes of finite type over a field $k$ with equidimensional fibers of positive dimension $r$. Assume $X$ is equidimensional and $Y$ is regular. Let $S$ be a closed subscheme of $X$ proper over $Y$ such that for each point $y \in Y$ the fiber $S_y$ is nowhere dense in $X_y$. Denote by $C$ the blowup of $X$ with center $S$, by $D$ the exceptional divisor, and by $c$ the blowup map $c: C \to X$. For each $y \in Y$ denote by $C(y)$ the blowup of $X_y$ with center $S_y$ and by $D(y)$ the exceptional divisor. Let $y_0$ be a closed point of $Y$. We can ask: when do we have an equality of fundamental cycles

$$[C_{y_0}] = [C(y_0)]?$$

It’s not hard to see that (1) holds if and only if the fiber of the exceptional divisor $D_{y_0}$ is equidimensional. More precisely, we would like to find a numerical invariant depending solely on $D(y)$ whose constancy across $Y$ guarantees (1) for each closed point $y_0 \in Y$.

Set $l := c_1\mathcal{O}_C(1)$ and $l_y := c_1\mathcal{O}_{C(y)}(1)$ for each $y \in Y$. Denote by $D_{\text{vert}}$ the union of the components of $D$ that surject onto $y_0$. For a closed point $y_0 \in Y$ and affine neighborhood $U$ of $y_0$ denote by $U'$ the punctured neighborhood $U - \{y_0\}$. A partial answer to our question is given by the following result.

**Theorem 2.1** (Excess–Degree Formula). The following holds.

(i) Assume $Y$ is integral and regular of dimension one and $y_0$ is a closed point in $Y$. Let $U$ be a small enough affine neighborhood of $y_0$ such that $C_{y_0}$ is principal Cartier divisor in $C$ after base change $U \to Y$ and $C$ is flat over $U'$. Then we have the following Excess–Degree Formula (EDF):

$$\int_{S_{y_0}'} l_y^{-1}[D(y_0)] - \int_{S_y'} l_y^{-1}[D(y)] = \int_{S_{y_0}} l'[D_{\text{vert}}]$$

for $y \in U'$.

(ii) Assume $Y$ is regular of arbitrary positive dimension and $\mathcal{O}_C(1)$ is ample on $D_{\text{vert}}$. Then (1) holds if and only if the intersection number $\int_{Z_y} l_y^{-1}[D(y)]$ is constant for each closed $y \in Y$.

We prove a slightly stronger statement, allowing $S_{y_0}$ to be dense in $X_{y_0}$ which is needed sometimes for applications to equisingularity theory. Part (i) is interesting in its own right: it provides a powerful tool in deformation theory as shown by Gaffney [Gaf08]. For instance, consider a function $f$ defined on a complex affine space with an isolated critical point. Then a generic perturbation of $f$ is a function with only Morse critical points, and the number of such points is the Milnor number which is the multiplicity of the Jacobian ideal of $f$. In turn, this number is the degree of the exceptional divisor of the blowup of the ambient complex space by the Jacobian ideal of $f$.

As another example, let $(X, x_0) \to (Y, y_0)$ be a smoothing of an isolated hypersurface $(X_{y_0}, x_0)$ singularity, defined as the zero locus in $\mathbb{C}^n$ of a polynomial $f$, and $S$ is the subscheme defined by the partials of $f$ with respect to the fiber coordinates. In the EDF we get that the first left-hand side term is equal to the Hilbert–Samuel multiplicity of the Jacobian ideal in $\mathcal{O}_{X_{y_0}, x_0}$, whereas the second term is zero because $X_y$ is smooth. By conservation of number
argument, the right-hand side of the EDF is equal to the intersection multiplicity of the relative polar curve associated with the total space of the family with a generic fiber. In turn, this intersection multiplicity is the sum of the Milnor number of \((X_{y_0}, x_0)\) and the Milnor number of a generic hyperplane slice of \((X_{y_0}, x_0)\).

As a first application of Thm. 2.5(i) we generalize and recover multiplicity-polar formulas by Gaffney and Teissier for pairs of modules and ideals of finite colength in a local ring respectively. We work more generally with a pair of standard graded algebras inspired by [KT94]. Our approach yields global versions of their results valid over an arbitrary field.

**Proof.** Preserve the setup from the introduction. For each \(k\) denote by \(Z_k(C)\) the group of \(k\)-cycles on \(C\) and by \(A_k(C)\) the group of \(k\)-cycles modulo rational equivalence. For each \(y \in Y\) consider the refined Gysin homomorphism \(i^!_y: A_k(C) \to A_{k-1}(C_y)\) defined from the fiber square

\[
\begin{array}{ccc}
C_y & \longrightarrow & C \\
\downarrow & & \downarrow \\
\{y\} & \longrightarrow & Y
\end{array}
\]

(see Sct. 6.2 in [Ful84]). For a \(k\)-cycle class \(Z\) on \(C\) set \(Z_y := i^!_y(Z)\). Finally, denote by \(D_{\text{hor}}\) the union of the irreducible components of \(D\) that surject onto \(Y\).

First, note that \(C(y_0)\) and \(C_{y_0}\) are isomorphic over points \(x \in X_{y_0}\) with \(x \notin S_{y_0}\). Also, the irreducible components of \(C(y_0)\) surject onto those of \(X_{y_0}\). But \(S_{y_0}\) is nowhere dense in \(X_{y_0}\) by hypothesis. Hence the irreducible components of \(C_{y_0}\) are either vertical components of \(D\) or components that surject onto the irreducible components of \(X_{y_0}\). Thus we have

\[
(C(y_0)) - [C_{y_0}] = - [D_{\text{vert}}] \text{ in } Z_r(C).
\]

Second, \(D \cdot [C_{y_0}] = C_{y_0} \cdot [D]\) because \(C_{y_0}\) is Cartier. Write \([D] = [D_{\text{hor}}] + [D_{\text{vert}}]\). Then \(C_{y_0} \cdot [D] = C_{y_0} \cdot [D_{\text{hor}}]\) in \(Z_{r-1}(C)\) because \(C_{y_0}\) is principal in \(C\), so \(C_{y_0} \cdot [D_{\text{vert}}] = 0\) (see Rmk. 2.3 in [Ful84]). Thus, \(C_{y_0} \cdot [D] = [D_{\text{hor}}]_{y_0}\). Hence \(D \cdot [C_{y_0}] = [D_{\text{hor}}]_{y_0}\).

Additionally, observe that \(D \cdot [C(y_0)] = [D(y_0)]\). Hence by intersecting each term in (2) with \(D\) we get

\[
[D(y_0)] - [D_{\text{hor}}]_{y_0} = -D \cdot [D_{\text{vert}}] \text{ in } A_{r-1}(C)
\]

or equivalently

\[
[D(y_0)] - [D_{\text{hor}}]_{y_0} = t[D_{\text{vert}}]
\]

as \(D\) is dual to \(O_C(1)\). Because \(O_C(1)\) restricts to \(O_{C(y_0)}(1)\) on \(C(y_0)\) then \(t^{-1}[D(y_0)] = l^{r-1}_y[D(y_0)]\). Apply \(l^{-1}\) to both sides of (3) to get

\[
l^{r-1}_y[D(y_0)] - l^{r-1}_y[D_{\text{hor}}]_{y_0} = l^{r-1}_y[D_{\text{vert}}] \text{ in } A_0(C).
\]

Next apply Prop. 10. 2 in [Ful84] to the 1-cycle \(l^{r-1}_y[D_{\text{hor}}]\) in \(D\) and the map \(D \to Y\) which is proper because it is composition of two proper maps: \(D \to S\) and \(S \to Y\). We have

\[
\int_{S_{y_0}} (l^{r-1}_y[D_{\text{hor}}])_{y_0} = \int_{S_y} (l^{r-1}_y[D_{\text{hor}}])_y.
\]

Note that \([D] = [D(y)]\) because by assumption \(C\) is flat over \(U\) so \(C_y = C(y)\) as schemes. Then by Prop. 10.1 (d) in [Ful84] about specialization of Chern classes, we have
Let $y'$ be a regular curve passing through $y_0$. Denote by $C(Y')$ the blowup of $X \times_Y Y'$ with center the image of $S$ in $X \times_Y Y'$. Set $C_{Y'} = C \times_Y Y'$. Denote by $D(Y')$ and $D_{Y'}$ the corresponding exceptional divisors.

Suppose

$$\int_{S_{y_0}} l_y^{-1}[D(y_0)] \neq \int_{S_y} l_y^{-1}[D(y)].$$

Then by (i) the exceptional divisor $D(Y')$ has an irreducible component $D_1(Y')$ that maps to $y_0$. Because $C(Y')$ is equidimensional of dimension $r + 1$, then $D(Y')$ is equidimensional of dimension $r$. But $D_1(Y')$ is a closed subscheme of $C_{y_0}$ of dimension $r$ supported over $S_{y_0}$ and the exceptional divisor $D(y_0)$ is of dimension $r - 1$. Hence $[C_{y_0}] \neq [C(y_0)]$.

Next, suppose $[C_{y_0}] = [C(y_0)]$. Assume $\dim Y = 2$. The general case follows by induction on the dimension of $Y$. We have $\dim D_{y_0} = r + k$ with $k = 0$ or 1. Select $Y'$ in an open neighborhood $U$ of $y_0$ in $Y$ defined as the subscheme of zeroes of a regular element $h_1$ from $C_{Y', y_0}$ such that $D_y$ is equidimensional for closed points $y \in U$ with $y \neq y_0$.

Suppose $k = 0$. In this case $D_{\text{vert}}$ surjects onto a curve in $Y$. Replace $C(Y')$ and $C_{Y'}$ by affine neighborhoods. Because $C_{Y'} = C \cap H_1$, then $C_{Y'}$ is equidimensional of dimension $d + 1$. In this $[C_{Y'}] = [C(Y')]$ and $(D_{\text{vert}}(y_0)$ is a subscheme of both $C_{Y'}$ and $C(Y')$ of dimension $d$. Then by (i) applied to $C(Y')$ we have

$$\int_{S_{y_0}} l_y^{-1}[D(y_0)] \neq \int_{S_y} l_y^{-1}[D(y)].$$

Suppose $k = 1$. In this case $D_{\text{vert}}$ surjects onto $y_0$. Because $C_{Y'} = D_{\text{vert}} \cup C(Y')$ and $C_{Y'}$ is defined by a single equation in $C$, then by Grothendieck connectedness theorem it follows that there exists an irreducible component of $D_{\text{vert}}$ such that its intersection with $C(Y')$ is of dimension $d$. Hence the exceptional divisor of $C(Y')$ has a vertical component. As above, an application of (i) shows that there is a jump in the intersection number of the fibers of the exceptional divisors of $C(y)$ for $y \in Y'$. The proof of Theorem 2.3 is now complete. □

Below we prove strengthening of Thm. 2.5 (i). We show that the assumption that $S_{y_0}$ is nowhere dense in $X_{y_0}$ in part (i) can be substantially relaxed which is what’s needed for applications to equisingularity theory.

**Corollary 2.2.** Let $X'_{y_0}$ be the union of those components of $X_{y_0}$ that are not irreducible components of $S$. Set $S'_{y_0} = S_{y_0} \times_{X_{y_0}} X'_{y_0}$. Denote by $C'(y_0)$ the blowup of $X'_{y_0}$ with center $S'_{y_0}$ and denote by $D'(y_0)$ its exceptional divisor. Then the EDF remains valid after replacing $D(y_0)$ with $D'(y_0)$.

**Proof.** Note that $C_{y_0}$ and $C'(y_0)$ are isomorphic over points $x \in X_{y_0}$ with $x \notin S_{y_0}$. Therefore the cycle $[C(y_0)] - [C_{y_0}]$ is supported over the irreducible components of $S$ that are also
irreducible components of $X_{y_0}$. Hence, once again $[C(y_0)] - [C_{y_0}] = -[D_{\text{vert}}]$. Additionally, observe that $D \cdot [C'(y_0)] = [D'(y_0)]$ because $D \times_C C'(y_0) = D'(y_0)$. The rest of the proof goes through unchanged. 

In equisingularity theory we apply Cor. 2.2 with $C$ the blowup of the relative conormal space of a family $(x,x_0) \to (Y,y_0)$ with center the singular locus of $X$. In this setting $X'_{y_0}$ is the conormal space of $X_{y_0}$. So the intersection numbers appearing on the left-hand side of the EDF depend only on the fibers.

For results of similar flavor to Thm. 2.5 see Kollár [Kol15]. As a first application of Theorem 2.5 (i) we recover a result by Teissier for the Hilbert–Samuel multiplicity.

Preserve the setup of Thm. 2.5. Assume $Z$ is finite over $Y$. Following Fulton (see [Ram73], and §4.3 and Ex. 4.3.4 in [Ful84]) define the Hilbert–Samuel multiplicity

$$e(S_y, X_y) := \int_{S_y} t_y^{-1}[D(y)].$$

**Theorem 2.3** (Teissier, Prp. 3.1 in [T72] and Rmk. 5.1.1 in [T81]). We have

$$e(S_{y_0}, X_{y_0}) - e(S_y, X_y) = \deg(D_{\text{vert}})$$

where $\deg(D_{\text{vert}}) := \int t'[D_{\text{vert}}]$.

**Proof.** Follows immediately from Theorem 2.5 (i). 

Let $X \to Y$ be a morphism of schemes of finite type over a field $k$ with equidimensional fibers. Assume $Y$ is integral and regular of dimension one. Consider two graded sheaves of $\mathcal{O}_X$-algebras $G' := \bigoplus G'_n$ and $G := \bigoplus G_n$ with $G'_0 = G_0 = \mathcal{O}_X$ such that $G'$ and $G$ are locally generated by $G'_1$ and $G_1$ as $\mathcal{O}_X$-algebras. Assume $G'_1$ and $G_1$ are coherent $\mathcal{O}_X$-modules and $G' \subset G$. Set $P := \text{Proj}(G')$ and $Q := \text{Proj}(G)$. Denote by $Z := \mathbb{V}(G'_1)$ the variety of the ideal sheaf in $G$ generated by $G'_1$. Assume $Z$ is proper over $Y$ and nowhere dense in $Q$. Let $C$ be the blowup of $Q$ with center $Z$ and let $D$ be its exceptional divisor. Denote by $c_p$ and $c_q$ the projections of $C$ to $P$ and $Q$ respectively. Set $L' := \mathcal{O}_P(1)$ and $l' := c_1(L')$, and $L := \mathcal{O}_Q(1)$ and $l := c_1(L)$.

Further, assume $G$ is contained in $\text{Sym}(\mathcal{F})$ where $\mathcal{F}$ is a locally free coherent sheaf on $X$. Denote by $\mathcal{F}_y$ the induced sheaf on $X_y$ and by $G'(y)$ and $G(y)$ the images of $G'$ and $G$ in $\text{Sym}(\mathcal{F}_y)$. Set $P(y) := \text{Proj}(G'(y))$ and $Q(y) := \text{Proj}(G(y))$. Consider the following diagram

$$\begin{array}{ccc}
C(y) & \xrightarrow{c_q} & Q(y) \\
\downarrow c_p & & \downarrow \\
P(y) & \longrightarrow & X_y
\end{array}$$

where $C(y)$ is the blowup of $Q(y)$ with center $Z \times_Q Q(y)$ and $D(y)$ is the corresponding exceptional divisor. Set $L'_y := \mathcal{O}_{P(y)}(1)$ and $L_y := \mathcal{O}_{Q(y)}(1)$ and $l'_y := c_1(L'_y)$ and $l_y := c_1(L_y)$. Assume $P(y)$ and $Q(y)$ are equidimensional of dimension $r$. Define the generalized Buchsbaum–Rim multiplicity (see Sect. 5 in [KT94])

$$e(G'(y), G(y)) := \sum_{i=0}^{r-1} \int_{Z_y} (c'_{y} l'_y)^{r-i-1}(c''_{y} l_y)^i[D(y)].$$

where $c'_{y} l'_y = c_1(c'_{y} L'_y)$ and $c''_{y} l_y = c_1(c''_{y} L_y)$. Finally, denote by $D^P_{\text{vert}}$ and $D^Q_{\text{vert}}$ the projections of $D_{\text{vert}}$ to $P$ and $Q$. 


Theorem 2.4. There exists an affine neighborhood $U$ of $y_0$ in $Y$ such that
\begin{equation}
 e(G'(y_0), G(y_0)) - e(G'(y), G(y)) = l^r[D_{\text{vert}}] - l^r[D_{\text{vert}}^Q].
\end{equation}
for each closed point $y \in U - \{y_0\}$.

Proof. By Cor. 2.2 applied to the family $Q \to Y$ with $S := Z$ and by (3) we get
\begin{equation}
 (c_p^*l_p^r)^{r-i-1}(c_q^*l_q^i)[D_{\text{hor}}]|_{y_0} = -(c_q^*l_q^i)^{r-1}(c_q^*l_q^i)^{D}[D_{\text{vert}}].
\end{equation}
By (5) and (6) we get
\begin{equation}
 \int_{Z_{y_0}} (c_p^*l_p^r)^{r-i-1}(c_q^*l_q^i)^{[D_{\text{hor}}]}|_{y_0} = \int_{Z_y} (c_p^*l_p^r)^{r-i-1}(c_q^*l_q^i)^{[D]}[y].
\end{equation}
By a result of Kleiman and Thorup (see Prp. 2.2 in [KT94]) we have
\begin{equation}
 \mathcal{O}_C(D) = c_p^*\mathcal{L} \otimes c_q^*\mathcal{L}^{r-1}.
\end{equation}
Then (11) yields
\begin{equation}
 \sum_{i=1}^{r-1} -(c_q^*l_q^i)^{r-i-1}(c_q^*l_q^i)^{D}[D_{\text{vert}}] = (c_q^*l_q^i)^{r}[D_{\text{vert}}] - (c_q^*l_q^i)^{r}[D_{\text{vert}}].
\end{equation}
Summing over $i$ in (9) and plugging (12) and (10) in (9) we get
\begin{equation}
 e(G'(y_0), G(y_0)) - e(G'(y), G(y)) = -(b_p^*l_p^r)^{D}[D_{\text{vert}}] - (b_q^*l_q^r)^{D}[D_{\text{vert}}].
\end{equation}
Applying the projection formula (see Prp. 2.5 (c) in [Ful84]) to each term of the right-hand side of the last equality we get (8). \hfill \Box

Suppose $X$ and $Y$ are local with closed points $x_0$ and $y_0$ respectively and suppose the fibers of $X \to Y$ are equidimensional of positive dimension $d$. Let $\mathcal{M} \subset \mathcal{N} \subset \mathcal{F}$ be $\mathcal{O}_X$-modules such that $\mathcal{F}$ is free and $\mathcal{M}$ and $\mathcal{N}$ are free of constant rank $e$ at the generic point of each irreducible component of $X$. Let $\mathcal{R}(\mathcal{M})$ and $\mathcal{R}(\mathcal{N})$ be the Rees algebras of $\mathcal{M}$ and $\mathcal{N}$ respectively. These are defined as the subalgebras of $\text{Sym}(\mathcal{F})$ generated in degree one by the generators of each of the two modules.

Now work in the setup of Thm. 2.4. Assume $Z$ is finite over $Y$. For each $y \in Y$ define the Buchsbaum–Rim multiplicity $e(\mathcal{M}(y), \mathcal{N}(y))$ as in (7) with $G = \mathcal{R}(\mathcal{M})$ and $G = \mathcal{R}(\mathcal{N})$. Here we turn results of Kleiman and Thorup (see Sect. 5 in [KT94]) into a definition. In the original treatment of Buchsbaum and Rim [BR64], they define $e(\mathcal{M}(y), \mathcal{N}(y))$ as the normalized leading coefficient of the length $\lambda(\mathcal{N}^i(y)/\mathcal{M}^i(y))$ which is a polynomial of degree $r := d + e - 1$ for $i$ large enough, where $N^i(y)$ and $M^i(y)$ are the $i$th graded components of the respective Rees algebras, assuming that $\text{Supp}(\mathcal{M}/\mathcal{N})$ is finite over $Y$.

Suppose $X$ and $Y$ are of finite type over an algebraically closed field of characteristic 0. Assume $X$ is generically reduced and $Y$ is regular of arbitrary dimension. Let $c_\mathcal{M}$ be the structure map $c_\mathcal{M} : \text{Proj}(\mathcal{R}(\mathcal{M})) \to X$. Let $C(\mathcal{M})$ be the nonfree locus of $\mathcal{M}$ in $X$. Consider the composition of maps
\begin{equation}
 c_\mathcal{M}^{-1}(C(\mathcal{M})) \hookrightarrow X \times \mathbb{P}^{g(\mathcal{M})-1} \overset{\text{Pr}_2}{\to} \mathbb{P}^{g(\mathcal{M})-1}
\end{equation}
where $g(\mathcal{M})$ is the number of a generating set for $\mathcal{M}$ as an $\mathcal{O}_X$-module. As $\mathcal{M}$ is of generic rank $e$, by the Kleiman Transversality Theorem [Kle74], the intersection of $c_\mathcal{M}^{-1}(C(\mathcal{M}))$ with a general plane $H_r$ from $\mathbb{P}^{g(\mathcal{M})-1}$ of codimension $r$, is of dimension at most $\dim Y - 1$. Therefore, for a generic $y \in Y$ the fiber over $y$ of the projection $\Gamma_d(\mathcal{M})$ of $\text{Proj}(\mathcal{R}(\mathcal{M})) \cap H_{d+g-1}$ to $X$ consists of the same number of points, each of them appearing with multiplicity one because
$\mathcal{X}$ is reduced, and at which the rank of $\mathcal{M}$ is maximal. Denote this number by $\text{mult}_Y \Gamma_d(\mathcal{M})$. Similarly, define $\text{mult}_Y \Gamma_d(\mathcal{N})$. Set $Z := \text{Supp}_X(\mathcal{N}/\mathcal{M})$ where $\overline{\mathcal{N}}$ and $\mathcal{M}$ are the integral closure of the two modules in $\mathcal{F}$. Thm. 2.4 yields Gaffney’s Multiplicity-Polar Theorem.

**Corollary 2.5.** (Gaffney, [Gaf04]) Suppose $Z$ is proper over $Y$. Then

$$e(M(y_0), \mathcal{N}(y_0)) - e(M(y), \mathcal{N}(y)) = \text{mult}_Y \Gamma_d(\mathcal{M}) - \text{mult}_Y \Gamma_d(\mathcal{N}).$$

**Proof.** By a conservation of number, for generic curve $Y'$ passing through $y_0$, the degrees on the right-hand side of (8) are equal to $\text{mult}_Y \Gamma_d(\mathcal{M})$ and $\text{mult}_Y \Gamma_d(\mathcal{N})$ respectively. Then the statement follows from (8). \qed

### 3. Computing and Vanishing of Local Volumes

Let $(R, \mathfrak{m})$ be a reduced Noetherian local equidimensional ring of dimension at least 2. Let $\mathcal{A} := \oplus_{i=0}^\infty \mathcal{A}_i$ be a reduced equidimensional standard graded $R$-algebra of dimension $r$. Denote by $\lambda_R(-)$ the length function. In this section we will be interested in computing

$$\varepsilon(\mathcal{A}) := \limsup_{n \to \infty} \frac{r!}{n^r} \lambda_R(H^1_m(\mathcal{A}_n)).$$

In our applications to geometry, $R$ will be essentially of finite type over a field, $\mathcal{L}$ will be an invertible very ample sheaf on $\text{Proj}(\mathcal{A})$ relative to $\text{Spec}(R)$. Then (13) applied with $\mathcal{A}$ the coordinate ring of $\text{Proj}(\mathcal{A})$ with respect to the embedding induced by $\mathcal{L}$, is the local volume of $\mathcal{L}$.

When $\mathcal{A}$ is the Rees algebra of a torsion-free finitely generated $R$-module $\mathcal{M}$ (see [EHU03] for a definition), then (13) is called the epsilon multiplicity of $\mathcal{M}$. It is denoted by $\varepsilon(\mathcal{M})$ (see [UV11] and the remark below).

Often it’s preferable to work with an $H^0_m$ instead of $H^1_m$. For example, we can do this if we assume that the minimal primes of $\mathcal{A}$ contract to minimal primes of $R$. Set $\mathcal{X} := \text{Spec}(R)$. Let $x_0$ be the closed point of $\mathcal{X}$. Then $\mathcal{A}$ viewed as $\mathcal{O}_{\mathcal{X},x_0}$-module is torsion free. Set $U := \mathcal{X} - x_0$. For each $n$ consider the standard exact sequence (see [Gr67])

$$0 \to H^0_0(\mathcal{X}, \mathcal{A}_n) \to H^0(\mathcal{X}, \mathcal{A}_n) \to H^0(U, \mathcal{A}_n) \to H^1_{x_0}(\mathcal{X}, \mathcal{A}_n) \to H^1(\mathcal{X}, \mathcal{A}_n),$$

where $\mathcal{A}_n$ is the associated sheaf of $\mathcal{A}_n$. The two extreme terms vanish because $\mathcal{A}$ is torsion free and $\mathcal{X}$ is affine. Thus we can compute $H^1_{x_0}(\mathcal{A}_n)$ as $H^0(U, \mathcal{A}_n)/\mathcal{A}_n$ (see Ex. 3.3 (b) Chp. III in [Har77]).

The existence of (13) as a limit has been established in some cases by Cutkosky (cf. [Cut15]) based on ideas of Kaveh and Khovanskii [KK12], and Okounkov [Oko03] and by Fulger [Ful13].

Assume $\mathcal{B}$ is a graded $R$-algebra, not necessarily finitely generated, such that $\mathcal{A} \subset \mathcal{B}$ is a homogeneous inclusion and

$$\lim_{n \to \infty} \frac{r!}{n^r} \lambda_R(H^1_m(\mathcal{B}_n)) = 0$$

for $i = 1$ and $2$. Then an exact sequence of local cohomology yields

$$\limsup_{n \to \infty} \frac{r!}{n^r} \lambda_R(H^1_m(\mathcal{A}_n)) = \limsup_{n \to \infty} \frac{r!}{n^r} \lambda_R(H^0_m(\mathcal{B}_n/\mathcal{A}_n)).$$

The reason we will make use of (15) is that $H^0_m(\mathcal{B}_n/\mathcal{A}_n)$ is more manageable than $H^1_m(\mathcal{A}_n)$ when studying the behavior of (13) in families. Below we list several instances when the representation (15) is possible. Denote by $c_A$ and by $c_B$ the structure morphisms from $\text{Proj}(\mathcal{A})$ and $\text{Proj}(\mathcal{B})$ to $\text{Spec}(R)$, respectively.
Proposition 3.1. Assume that \((R, \mathfrak{m})\) is a local Noetherian equidimensional ring with \(\dim R \geq 2\). The following holds:

(i) Assume \(\text{depth}(R) \geq 2\). Suppose the minimal primes of \(A\) contract to minimal primes of \(R\). Then (15) holds with \(\mathcal{B}_n := \mathcal{A}_n^{**}\), where \(\mathcal{A}_n^{**}\) is the double dual of the \(R\)-module \(\mathcal{A}_n\).

(ii) Assume that \(R\) is Nagata and \(\mathcal{B}\) is reduced, equidimensional of dimension \(r\). Suppose that \(\text{codim} c_{\mathcal{B}}^{-1}(\mathfrak{m}) \geq 2\) in \(\text{Proj}(\mathcal{B})\) and the minimal primes of \(\mathcal{B}\) contract to minimal primes of \(R\). Then (15) holds.

(iii) Assume \(R\) is essentially of finite type over a field. Let \(\mathcal{M}\) be an \(R\)-module free of rank \(e\) at locally at each minimal prime of \(R\). Then \(\mathcal{M}\) admits an embedding into an \(R\)-free module \(\mathcal{F}\) of rank \(e\) such that (15) holds with \(\mathcal{A} := \mathcal{R}(\mathcal{M})\) and \(\mathcal{B} := \text{Sym}(\mathcal{F})\). Moreover, the right-hand side of (15) is in fact a limit.

Proof. Consider (i). Because \(\mathcal{A}\) is reduced and its minimal primes contract to minimal primes of \(R\), then \(\mathcal{B}\) and \(\mathcal{A}\) are torsion free. Because \(X\) is reduced, the natural map \(\mathcal{A}_n \to \mathcal{A}_n^{**}\) is injection by a straightforward generalization of [Stks Tag 0AV0]. But \(\mathcal{A}_n^{**}\) is reflexive, so any 2-regular sequence from \(\mathfrak{m}\) lifts to a 2-regular sequence of \(\mathcal{A}_n^{**}\) (see [Stks Tag 0AV5]). Thus, \(H_{m,n}^i(\mathcal{B}_n) = 0\) for \(i = 0, 1\) by Thm. 3.8 in [Gr67].

Consider (ii). Observe that the total ring of fractions \(Q(\mathcal{B})\) of \(\mathcal{B}\) is a graded ring. Denote by \(\overline{\mathcal{B}}\) the integral closure of \(\mathcal{B}\) in \(Q(\mathcal{B})\). Then \(\overline{\mathcal{B}}\) is graded by Prp. 2.3.5 in [SH06]. Because \(\mathcal{B}\) is reduced and is of finite type over \(R\), which is Nagata, then \(\overline{\mathcal{B}}\) is module-finite over \(\mathcal{B}\) by [Stks Tag 03GH]. Thus \(H_{m,n}^0(\mathcal{B}/\mathcal{B})\) is a finite \(\mathfrak{m}\mathcal{B}\)-module for some positive \(l\). This implies that \(\lambda_R(H_{m,n}^0(\mathcal{B}_n/\mathcal{B}_n))\) is a polynomial of degree at most \(\dim c_{\mathcal{B}}^{-1}(\mathfrak{m})\). Because \(\mathcal{B}\) is equidimensional of dimension \(r\), and the minimal primes of \(\mathcal{B}\) contract to minimal primes of \(R\), then \(\dim c_{\mathcal{B}}^{-1}(\mathfrak{m}) \leq r - 1\). Hence

\[
\limsup_{n \to \infty} \frac{r!^l}{n^r} \lambda_R(H_{m,n}^0(\overline{\mathcal{B}}_n/\mathcal{B}_n)) = 0.
\]

Consider the sequence

\[
H_{m,n}^0(\overline{\mathcal{B}}_n/\mathcal{B}_n) \to H_{m,n}^1(\mathcal{B}_n) \to H_{m,n}^1(\overline{\mathcal{B}}_n).
\]

Because \(\overline{\mathcal{B}}\) and \(\mathcal{B}\) are generically equal, each minimal prime of the former ring contracts to a minimal prime of the latter. Select \(x_1 \in \mathfrak{m}\) that avoids the minimal primes of \(R\). Hence \(x_1\) avoids the minimal primes of \(\overline{\mathcal{B}}\). Because \(\overline{\mathcal{B}}\) is normal, it follows that the associated primes of the ideal \((x_1)\) generated by \(x_1\) in \(\overline{\mathcal{B}}\) are of height one. But codim \(c_{\overline{\mathcal{B}}}^{-1}(\mathfrak{m}) \geq 2\) in \(\text{Proj}(\overline{\mathcal{B}})\), so codim \(c_{\mathcal{B}}^{-1}(\mathfrak{m}) \geq 2\) in \(\text{Proj}(\mathcal{B})\). Therefore, none of the associated primes of \((x_1)\) in \(\overline{\mathcal{B}}\) contracts to \(\mathfrak{m}\). Hence by prime avoidance we can select \(x_2 \in \mathfrak{m}\) such that \(x_2\) is a nonzero divisor of \(\overline{\mathcal{B}}/x_1\overline{\mathcal{B}}\). In this way we have constructed a 2-regular sequence of \(\overline{\mathcal{B}}_n\) for each \(n\). Thus \(H_{m,n}^i(\overline{\mathcal{B}}_n) = 0\) for \(i = 0, 1\) by Thm. 3.8 in [Gr67]. Finally, (17) and (16) yield (15).

Consider (iii). Note that the Rees algebra of \(\mathcal{R}(\mathcal{M})\) is reduced because \(X\) is reduced. Thus \(\mathcal{R}(\mathcal{M}) \to \mathcal{R}(\mathcal{M}) \otimes Q(\mathcal{R})\) where \(Q(\mathcal{R})\) is the total ring of fractions of \(\mathcal{R}\). Let \(p_1, \ldots, p_q\) be the minimal primes of \(\mathcal{R}\). By hypothesis \(\mathcal{M}_{p_i} = e\) for each for each minimal prime \(p_i\). Then there exists \(e\) generic combinations of the generators of \(\mathcal{M}\) that generate \(\mathcal{M}_{p_i}\) for each \(i\). Consider the module \(\mathcal{F}'\) generated by these elements in \(\mathcal{M} \otimes Q(\mathcal{R})\). Scale the generators of \(\mathcal{F}'\) with elements from \(Q(\mathcal{R})\) in such a way that all of the remaining generators of \(\mathcal{M}\) can be expressed as a linear combination of the scaled generators \(\mathcal{F}'\) with coefficients in \(\mathcal{R}\). Denote by \(\mathcal{F}\) the \(R\) module generated by the scaled generators of \(\mathcal{F}'\). Then \(\mathcal{F}\) is free of rank \(e\). Let \(\mathcal{F}^n\) be the \(n\)th symmetric power of \(\mathcal{F}\) and \(\mathcal{M}^n\) be the \(n\)th homogeneous component of the subalgebra of
Sym(\mathcal{F}) generated by \mathcal{M}. Consider the exact sequence
\begin{equation}
H^n_m(F^n) \rightarrow H^n_m(F^n/\mathcal{M}^n) \rightarrow H^1_m(\mathcal{M}^n) \rightarrow H^1_m(F^n).
\end{equation}
Because \(X\) is reduced of positive dimension and \(F^n\) is free, then \(H^n_m(F^n) = 0\). Because \(F^n\) is free of rank \((e+n-1)\), then \(H^1_m(F^n)\) is equal to the direct sum of \((e+n-1)\) copies of \(H^1_m(R)\).

Because \(R\) is finitely generated, then \(R\) is a homomorphic image of a regular ring. Therefore \(\dim R \geq 2\), Grothendieck’s finiteness theorem (see Exposé VIII, Corollaire 2.3 in \cite{Gr68}) implies that \(H^1_m(R)\) is finitely generated. Hence \(\lambda_R(H^1_m(F^n)) = O(n^{e-1})\). Recall that by Prp. 5.1 (2) in \cite{Ran18} we have \(\dim R(\mathcal{M}) = \dim X + e - 1\). Because \(R\) is of positive dimension, then \(r > e - 1\), so \(\lim_{n \rightarrow \infty} \frac{r!}{n^r} \lambda_R(H^1_m(F^n)) = 0\) which proves (15) for \(\mathcal{A} := R(\mathcal{M})\) and \(\mathcal{B} := \text{Sym}(\mathcal{F})\).

Note that \(R\) is analytically unramified because \(R\) is reduced and of finite type over a field. So by Theorem 3.2 in \cite{Cut15}
\[
\limsup_{n \rightarrow \infty} \frac{r!}{n^r} \lambda_R(H^1_m(F^n/\mathcal{M}^n))
\]
exists as a limit. \hfill \Box

The next proposition guarantees the existence of a \(\mathcal{B}\) satisfying the hypothesis of Prp. 3.1 (ii). The proof is based on Noether normalization.

**Proposition 3.2** (Prp. 2.6 in \cite{Ran19a}). Suppose \(R\) is a reduced equidimensional universally catenary Noetherian ring of positive dimension or an infinite field. Assume \(\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}_i\) is a reduced equidimensional standard graded algebra over \(R\). Assume that the minimal primes of \(\mathcal{A}\) contract to minimal primes of \(R\). Then there exists a standard graded \(R\)-algebra \(\mathcal{B} = \bigoplus_{i=0}^{\infty} \mathcal{B}_i\) such that

(i) \(\mathcal{B}\) is a birational extension of \(\mathcal{A}\), and the inclusion \(\mathcal{A} \subset \mathcal{B}\) is homogeneous;

(ii) For each prime \(\mathfrak{p}\) in \(\mathcal{A}\) the minimal primes of \(\mathfrak{p}\mathcal{B}\) are of height at least \(\text{ht}(\mathfrak{p}/\mathfrak{p}_{\text{min}})\) where \(\mathfrak{p}_{\text{min}}\) is a minimal prime of \(\mathfrak{p}\) contained in \(\mathfrak{p}\).

The next proposition is key to proving our vanishing result for \(\varepsilon(\mathcal{A})\). It’s essentially the content of the two main results of \cite{Ran19a}. Let \(\mathcal{A} \subset \mathcal{B}\) be commutative rings with identity. Denote by \(\overline{\mathcal{A}}\) the integral closure of \(\mathcal{A}\) in \(\mathcal{B}\).

**Proposition 3.3.** Let \((R, m)\) be a local Noetherian equidimensional ring. Let \(\mathcal{A} \subset \mathcal{B}\) be finitely generated \(R\)-algebras.

(i) Suppose \(R\) is universally catenary and the minimal primes of \(\mathcal{B}\) contract to minimal primes of \(\mathcal{A}\). If \(m \in \text{Ass}_R(\mathcal{B}/\overline{\mathcal{A}})\), then \(\text{codim} \mathcal{B}_i^{-1}(m) \leq 1\).

(ii) Suppose \(\text{codim} \mathcal{B}_i^{-1}(m) \geq 2\). If \(\text{codim} \mathcal{A}_i^{-1}(m) \leq 1\), then \(m \in \text{Ass}_R(\mathcal{B}/\overline{\mathcal{A}})\).

**Proof.** By the second part of Prp. 2.1 in \cite{Ran19a} or \cite{Stks} Tag 05DZ it’s enough to consider primes in \(\text{Ass}_A(\mathcal{B}/\overline{\mathcal{A}})\) contracting to \(m\). Then part (i) follows from Thm. 1.1 (iii) in \cite{Ran19a}. Part (ii) follows from Thm. 1.2 \cite{Ran19a}. \hfill \Box

The following theorem characterizes completely the vanishing of \(\varepsilon(\mathcal{A})\). The result is the main inspiration for introducing the class of deficient conormal singularities in Sct. 7.

**Theorem 3.4.** Let \((R, m)\) be a reduced Noetherian local equidimensional ring with \(\dim R \geq 2\). Assume \(R\) is Nagata and universally catenary. Assume \(\mathcal{A}\) is a reduced standard graded \(R\)-algebra such that its minimal primes contract to minimal primes of \(R\). Then
\begin{equation}
\varepsilon(\mathcal{A}) = 0
\end{equation}
if and only if $\text{codim } c_A^{-1}(m) \geq 2$.

Proof. By Prp. 3.2 applied with $p = m$ there exists a standard graded $R$-algebra $B$ such that $A \subset B$ is a homogeneous inclusion, and $B$ satisfies the hypothesis of Prp. 3.1 (ii) because $R$ is local and $\dim R \geq 2$. Then by Prp. 3.1 (ii)

$$\varepsilon(A) = \limsup_{n \to \infty} \frac{1}{n^r} \lambda_R(H^0_{x_0}(B_n/A_n)).$$

Assume $\text{codim } c_A^{-1}(m) \geq 2$. Denote by $\mathcal{A}_n$ the integral closure of $A_n$ in $B_n$. Note that $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$ with $\mathcal{A}_0 = R$ by Prp. 2.3.5 in [SH06].

By Prp. 3.3 (i) $m \notin \text{Ass}_R(B_n/\mathcal{A}_n)$. Thus $H^0_m(B_n/\mathcal{A}_n) \subset \mathcal{A}_n/A_n$ for each $n$. But $R$ is Nagata. Then so is $B$. Furthermore, $B$ is reduced, because $A$ is. So, $\mathcal{A}$ is module-finite over $\mathcal{A}$. Therefore, by repeating the argument from the proof of Prp. 3.1 (ii) we obtain $\mathcal{A}$.

Conversely, assume that (19) holds. Suppose $b$ is an element from $B_k$ for some $k$ such that there exists positive $l$ with $m^l b_k \in \mathcal{A}_k$ and $b_k \notin \mathcal{A}_k$. Set $G' := \bigoplus_{k=1}^{\infty} \mathcal{A}_k$ and let $G$ be the algebra generated by $G$ and $b_k$. By shift in degrees assume $G'$ and $G$ are standard graded. Then by Cor. 5.10 in [KT94]

$$\lambda_R(G_n/G'_n) := e(G', G)n^r/r! + \cdots$$

where the Buchsbaum–Rim multiplicity $e(G', G)$ is a positive integer. Thus

$$\lim_{n \to \infty} \frac{1}{(nk)^r} \lambda_R(H^0_m(B_{nk}/\mathcal{A}_{nk})) \geq \lim_{n \to \infty} \frac{1}{n^r} \lambda_R(G_n/G'_n) = e(G', G) > 0$$

which contradicts our assumption. Hence $m \notin \bigcup_{n=1}^{\infty} \text{Ass}_R(B_n/\mathcal{A}_n)$ which reasons of grading is equivalent to $m \notin \bigcup_{n=1}^{\infty} \text{Ass}_R(B/\mathcal{A})$. So by Prp. 3.3 (ii) $\text{codim } c_A^{-1}(m) \geq 2$ in $\text{Proj}(A)$. □

The forward direction of Thm. 3.4 was proved for the epsilon multiplicity $\varepsilon(A|B)$ with the additional hypothesis that $B$ is $R$-flat by Ulrich and Validashti (see Thm. 4.2 in [UV11]). We record two observations about primary decomposition and local cohomology that will be used in the proof of the local volume formula in the next section. Their statements and proofs can be found in Chapter 18 of [AK12] for example.

**Proposition 3.5.** Let $X$ be an affine Noetherian scheme and let $S$ be a subscheme of $X$ with an ideal sheaf $\mathcal{I}_S$ in $\mathcal{O}_X$. Assume $\mathcal{M}$ and $\mathcal{F}$ are coherent $\mathcal{O}_X$-modules with $\mathcal{M} \subset \mathcal{F}$. Let $\mathcal{M} = \bigcap \mathcal{M}_i$ be a primary decomposition of $\mathcal{M}$ in $\mathcal{F}$ with $x_i = \text{Ass}_X(\mathcal{F}/\mathcal{M}_i)$. The following holds

$$H^0_S(\mathcal{F}/\mathcal{M}) = (\bigcap_{j \neq i \in S}\mathcal{M}_j)/\mathcal{M}.$$  

**Proposition 3.6.** Let $X$ be an affine Noetherian scheme, and let $x$ be a point in $X$. Consider the nested chain of coherent $\mathcal{O}_X$-modules $\mathcal{M} \subset \mathcal{N} \subset \mathcal{F}$. Assume that $\mathcal{M}_z = \mathcal{N}_z$ for every point $z \notin \{x\}$. Then

$$\mathcal{N}/\mathcal{M} \subset H^0_x(\mathcal{F}/\mathcal{M}).$$

Furthermore, $H^0_x(\mathcal{F}/\mathcal{M})$ is the largest submodule of $\mathcal{F}/\mathcal{M}$ equal to $\mathcal{M}$ locally off $\{x\}$.

Sometimes it will be more convenient to write the local cohomology in terms of saturations. Recall that $H^0_S(\mathcal{F}/\mathcal{M}) = \bigcup_{n=0}^{\infty} (\mathcal{M} :_{\mathcal{F}} \mathcal{I}^n_S)/\mathcal{M}$ and $H^0_x(\mathcal{F}/\mathcal{M}) = \bigcup_{n=0}^{\infty} (\mathcal{M} :_{\mathcal{F}} \mathcal{I}^n_x)/\mathcal{M}$.

**Remark 3.7.** We conclude this section with some historical background for the $\varepsilon$ invariant discussed above. Assume that $\mathcal{M}$ is an $R$-module embedded in a free module $\mathcal{F} := R^d$ such that $\mathcal{M}$ is free of rank $e$ at each minimal prime of $R$. Set $r := d + e - 1$. Inspired by work of
Cutkosky and collaborators (cf. [CHS10]), Ulrich and Validashti defined the $\varepsilon$-multiplicity of $M$ as

$$\varepsilon(M) := \limsup_{n \to \infty} \frac{r_1}{n^r} \lambda_R(H^0_m(F^n/M^n)).$$

They proved that $\varepsilon(M)$ satisfies the Rees criterion for integral dependence [UV7] and together with Kleiman established [KUV] that it is upper semi-continuous. These two properties allowed them to prove a generalization of Teissier’s Principle of Specialization of Integral Dependence. Thus, it seemed that $\varepsilon(M)$ was the right generalization of the Buchsbaum–Rim multiplicity for applications to equisingularity. However, Kleiman found the following counterexample (see Ex. 4.5 on p. 46 of [BGG80]): a family $(X,0) \to (\mathbb{C},0)$ of germs of curves in 4-space with parametrization given by

$$x = t^4, y = t^6 + 2ut^7, z = t^{11}, w = t^{13} + ut^{14}$$

where $u$ is the parameter of the family. Eliminating $t$ gives a one-parameter family of curves defined by 6 equations. The $\delta$ invariant equals 12 and the multiplicity of the fibers is 4 for any value of $u$. So the family is Whitney equisingular by Thm. III. 3 on p. 23 in [BGG80].

On the other hand, using Singular, Kleiman computed that $\varepsilon(M(u)) = 21$ if $u = 0$, but $\varepsilon(M(u)) = 19$ for generic $u$ where in this case $M(u)$ is the Jacobian module of the fiber over $u$ (for a definition see Sect. [1]) embedded in a free module $F(u)$ of rank 6 given by the 6 defining equations for the family. Kleiman’s computation is based on an observation of Ulrich who showed that $\varepsilon(M(u))$ can be computed as the multiplicity of the ideal of maximal minors of the presentation matrix of $M(u)$ corresponding to the embedding of $M(u)$ in $F(u)$.

However, if we instead embed $M$ in a free module $F$ of rank 3 as we did in Prp. 3.1 (iii), and evaluate $\varepsilon(M(u))$ on the fibers, we get the Buchsbaum–Rim multiplicity $e(M(u), F(u))$, because $M(u)$ and $F(u)$ are generically equal, so $F(u)/M(u)$ is supported over the origin. An easy computations shows that $e(M(u), F(u)) = 19$ for each $u$ as predicted by Cor. [2.5]

This example shows that one should use the homogeneous embedding of $A$ constructed in Prp. 3.2. Also, the example shows that the right invariant for equisingularity is not the local volume of the restrictions of the invertible sheaf $L$ but the restricted local volume of $L$.

4. The Local Volume Formula and volume stability

In this section we prove the Local Volume Formula (LVF) and consider its extension to the case $\dim Y > 1$ under the assumption of volume stability. As an application we prove a general version of Teissier’s Principle of Specialization of Integral Dependence.

Let $X$ and $Y$ be affine reduced schemes of finite type over a field $k$. Assume $X$ is equidimensional and $Y$ is regular and integral of dimension one. Suppose $h: X \to Y$ is a morphism with equidimensional fibers of positive dimension. Let $S$ be a subscheme of $X$ that is finite over $Y$. Assume that $k$ is the residue field of each closed point $y$ and the points in $S_y$. Let $C$ be an equidimensional reduced scheme projective over $X$ such that the structure morphism $c: C \to X$ maps each irreducible component of $C$ to an irreducible component of $X$. Set $D := c^{-1}S$ and $\dim C = r + 1$. Fix a closed point $y_0$ in $Y$. Denote by $D_{\text{vert}}$ the union of components of $D$ that maps to $y_0$ under $h \circ c$.

Let $L$ be an invertible very ample sheaf on $C$ relative to $X$. Let $A := \oplus_{n \geq 0} \Gamma(C, L^{\otimes n})$ be the ring of sections of $L$. Denote by $A_n$ the $n$th graded piece of $A$. Recall that restricted local volume of $L$ at $S_y$ is defined as

$$\text{vol}_{C_y}(L) := \limsup_{n \to \infty} \frac{r_1}{n^r} \dim_k H^1_S(A_n) \otimes_{O_Y} k(y).$$
In the definition of the restricted local volume we can assume that $A$ is the coordinate ring of $C$ as the $n$th graded component of the coordinate ring and $\Gamma(C, \mathcal{L}^\otimes n)$ coincide for $n \gg 0$ (cf. Chp. II Ex. 5.9 in [Har77]).

Let $B$ be a standard graded algebra containing $A$ satisfying the properties listed in Prp. 3.2. Denote by $\overline{A}$ the integral closure of $A$ in $B$. Set $B(y) := B \otimes_{\mathcal{O}_y} k(y)$ and $A_n(y) := B_n \otimes_{\mathcal{O}_y} k(y)$. Denote the image of $A$ in $B(y)$ and $A_n$ in $B_n(y)$ by $A(y)$ and $A_n(y)$, respectively. We say that the restricted volume specializes with passage to the fiber $X_y$ if

$$\text{vol}_{C_y}(\mathcal{L}) := \limsup_{n \to \infty} \frac{1}{n^r} \dim_k H^1_{S_y}(A_n(y)).$$

**Theorem 4.1** (Local Volume Formula). The following holds.

(i) Suppose $\dim X = 2$ and $S := \text{Supp}_{\mathcal{O}_X}(B/\overline{A})$. Then there exists an affine neighborhood $U$ of $y_0$ in $Y$ such that

$$e(A(y_0), B(y_0)) - e(A(y), B(y)) = \int_{S_{y_0}} l^r[D_{\text{vert}}]$$

for each $y \in U - \{y_0\}$.

(ii) Suppose $\dim X \geq 3$. Then there exists an affine neighborhood $U$ of $y_0$ in $Y$ such that

$$\text{vol}_{C_{y_0}}(\mathcal{L}) - \text{vol}_{C_y}(\mathcal{L}) = \int_{S_{y_0}} l^r[D_{\text{vert}}]$$

for each $y \in U - \{y_0\}$ and the restricted local volume specializes with passage to $X_y$.

**Proof**

Consider (i). Denote by $c_B : \text{Proj}(B) \to X$ the structure morphism. Because $B$ is a birational extension of $A$, it follows that the reducible components of $\text{Proj}(B)$ surject onto those of $X$. By Prp. 3.2 (ii) and the fact that $X$ and $X_y$ are equidimensional, and $Y$ is regular, it follows that the irreducible components of $\text{Proj}(B(y_0))$ surject onto those of $X_{y_0}$. So no irreducible components of $\text{Proj}(B(y_0))$ are supported over $S_{y_0}$. As $B(y_0)$ is integral over $A(y_0)$ locally of $S_{y_0}$, it follows that $c^{-1}X_{y_0} = \text{Proj}(A(y_0)) \cup D_{\text{vert}}$ (cf. Prp. 4.4). Applying Thm. 2.4, we get the desired result.

Consider (ii). We break the proof of the LVF into several parts. Because $\dim X \geq 3$ using the analysis in Prp. 3.1 we can replace in the LVF $H^1_B(A_n)$ by $H^1_{S_{y_0}}(B_n/A_n)$. In Proposition 4.2 for each $n$ we relate the change of the dimension of $H^1_{S_{y_0}}(B_n/A_n) \otimes \mathcal{O}_y k(y_0)$ as $y$ “moves” from $y_0$ to a generic $y$ in the punctured neighborhood $U' = U - \{y_0\}$, to the dimension of $H^1_{S_{y_0}}(B_n/A_n) \otimes \mathcal{O}_y k(y)$. Here the main algebraic tool is related to the well-known fact that a torsion-free module over a principal ideal domain is free. Then we relate the limit of normalized vector space dimensions of $H^0_{S_{y_0}}(B_n/A_n) \otimes \mathcal{O}_y k(y_0)$ to the degree of the cycle $[c^{-1}(S_{y_0})]_r$ using two reductions. The first relates the dimensions of $H^0_{S_{y_0}}(B_n/A_n) \otimes \mathcal{O}_y k(y_0)$ to the dimension of the $n$th graded piece of the ideal defining the residual scheme in $c^{-1}X_{y_0}$ to the union of components that surject onto $S_{y_0}$. The second one relates the limit of normalized vector space dimensions of the graded pieces of the ideal of the residual scheme to $\int_{S_{y_0}} l^r[D_{\text{vert}}]$. Finally, we prove that the formation of the “generic” limit term in the LVF specializes with passage to generic fibers using Prp. 2.1 or Thm. 1.1 (ii) in [Ran19a].

Let $P_n$ and $N_n$ be submodules of $B_n$ such that

$$H^0_B(B_n/A_n) = N_n/A_n \text{ and } H^0_{S_{y_0}}(B_n/A_n) = P_n/A_n.$$
Let \( V \) be an affine neighborhood in \( X \) containing \( S_{y_0} \). Denote by \( A[V] \) and \( A[U] \) the homogeneous coordinate rings of \( V \) and \( U \). Note that the support of each of the quotients \( N_n/\mathcal{P}_n \), \( \tilde{N}_n/\mathcal{A}_n \) and \( \mathcal{P}_n/\mathcal{A}_n \) is in \( S \). Because \( S \) is finite over \( Y \), the direct image of each of these quotient by \( h \) is a coherent \( \mathcal{O}_Y \)-module.

**Flatness**

**Proposition 4.2.** Assume \( U \) is small enough so that for each \( n \), the associated points of \( \mathcal{B}_n/\mathcal{A}_n \) viewed as \( A[V] \)-module map to \( y_0 \) or the generic point of \( U \). Then

\[
\dim_k (N_n/\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y_0) - \dim_k (N_n/\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y) = \dim_k (\mathcal{P}_n/\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y_0).
\]

**Proof.** By Prp. 2.1 in [Ran19a] there are finitely many points in \( \bigcup_{n=0}^{\infty} \text{Ass}_X(\mathcal{B}_n/\mathcal{A}_n) \). Each of them maps to a closed point in \( Y \), or the generic point of \( Y \). Thus we can select \( U \) so that the only associated points of \( \mathcal{B}_n/\mathcal{A}_n \) are those that map to \( y_0 \) or to the generic point of \( U \). Therefore, the only associated points of \( N_n/\mathcal{A}_n \) viewed now as \( A[U] \)-module are \( y_0 \) or the generic point of \( U \).

Consider the nested chain of modules

\[
\mathcal{A}_n \subset \mathcal{P}_n \subset \mathcal{N}_n
\]

Form the exact sequence

\[
0 \longrightarrow \mathcal{P}_n/\mathcal{A}_n \longrightarrow \mathcal{N}_n/\mathcal{A}_n \longrightarrow \mathcal{N}_n/\mathcal{P}_n \longrightarrow 0
\]

(21)

\[
0 \longrightarrow \mathcal{P}_n/\mathcal{A}_n \longrightarrow \mathcal{N}_n/\mathcal{A}_n \longrightarrow \mathcal{N}_n/\mathcal{P}_n \longrightarrow 0
\]

where the vertical maps \( \mu \) are multiplication by the ideal \( \mathfrak{m}_{y_0} \) of \( y_0 \). Since \( \mu \) is injective on \( \mathcal{N}_n/\mathcal{P}_n \), then the Snake Lemma yields

\[
0 \longrightarrow \mathcal{P}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y_0) \longrightarrow \mathcal{N}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y_0) \longrightarrow \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y_0) \longrightarrow 0.
\]

Therefore,

\[
\dim_k \mathcal{N}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y_0) = \dim_k \mathcal{P}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y_0) + \dim_k \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y_0).
\]

Now let \( y \in Y \) be a point from \( U' \). Then

\[
\mathcal{P}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y) \longrightarrow \mathcal{N}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y) \longrightarrow \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y) \longrightarrow 0.
\]

However, \( \mathcal{P}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y) = 0 \) as the support of \( \mathcal{P}_n/\mathcal{A}_n \) is \( S_{y_0} \). Thus,

\[
\mathcal{N}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y) \cong \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y).
\]

But the only associated point of the \( A[U] \)-module \( \mathcal{N}_n/\mathcal{P}_n \) is the generic point of \( U \), hence \( \mathcal{N}_n/\mathcal{P}_n \) is torsion-free \( \mathcal{O}_{Y,y} \)-module for each closed point \( y \in U \). Because \( \mathcal{O}_{Y,y} \) is a DVR, and \( Y \) is reduced, then \( \mathcal{N}_n/\mathcal{P}_n \) is locally free. Hence \( \dim_k \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y) = \dim_k \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y_0) \) (cf. Ex. 5.8 in Chp. II in [Har77]). So,

\[
\dim_k \mathcal{N}_n/\mathcal{A}_n \otimes_{\mathcal{O}_Y} k(y) = \dim_k \mathcal{N}_n/\mathcal{P}_n \otimes_{\mathcal{O}_Y} k(y_0).
\]

Subtracting (25) from (23) we get the desired result. \( \square \)

Let \( x \in S_{y_0} \). Define \( \mathcal{P}_n(x) \) so that \( H^0_x(\mathcal{B}_n/\mathcal{A}_n) = \mathcal{P}_n(x)/\mathcal{A}_n \). Clearly,

\[
\dim_k (\mathcal{P}_n/\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y_0) = \sum_{x \in S_{y_0}} \dim_k (\mathcal{P}_n(x)/\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y_0).\]
Let $D_{vert}(x)$ be the union of components of $D_{vert}$ that map to $x$. We will show that
\[
\lim_{n \to \infty} \frac{r!}{n^r} \dim_k(\mathcal{P}_n(x)/\mathcal{A}_n) \otimes \mathcal{O}_y \ k(y_0) = \int_x l'[D_{vert}(x)].
\]
Thus we can assume that $S_{y_0}$ consists of a single point $x_0$.

**Key Isomorphism**

The next proposition provides a key isomorphism that allows to connect $\mathcal{P}_n/\mathcal{A}_n$ to the ideal of the residual scheme to the union of components of dimension $r$ in $c^{-1}(x_0)$. Let $t \in m_{y_0} - m_{y_0}^2$ be a uniformizing parameter of $\mathcal{O}_{Y,y_0}$. Identify $t$ with its image $h \# t$ in $m_{x_0}$. Replace $\mathcal{P}_n$ and $\mathcal{A}_n$ by their localizations at $x_0$. This does not affect the length of $\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_y \ k(y_0)$.

**Proposition 4.3.** We have
\[
\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_y \ k(y_0) \simeq (t\mathcal{P}_n \cap \mathcal{A}_n)/t\mathcal{A}_n
\]
as $k(y_0)$-vector spaces.

**Proof.** To begin, observe that both quotients $\mathcal{P}_n/\mathcal{A}_n/t(\mathcal{P}_n/\mathcal{A}_n)$ and $(t\mathcal{P}_n \cap \mathcal{A}_n)/t\mathcal{A}_n$ are supported over $x_0$. Because $k(y_0)$ is contained in $\mathcal{O}_{X,x_0}$ and equals the residue field of $x_0$, then two the quotients are $k(y_0)$-vector spaces of finite dimension. Throughout the proof we will use repeatedly the fact that $t$ is a nonzero divisor of $\mathcal{O}_{X,x_0}$ and $\mathcal{B}$. Indeed, $X$ is reduced and $X$ and the fibers $X \to Y$ are equidimensional. So there is no component of $X$ supported over $X_{y_0}$. Also, $\text{Proj}(\mathcal{B})$ is reduced and its components surject onto those of $X$. Thus $t$ is a nonzero divisor of $\mathcal{B}$.

For each $b$ in $\mathcal{P}_n$ define $\text{ord}_i(f)$ to be the smallest integer such that $t^{\text{ord}_i(b)} b \in \mathcal{A}_n$. Set
\[
s_i := \max\{\text{ord}_i(b) | b \in \mathcal{P}_n\} \text{ and } \mathcal{P}_n^{(i)} := (\mathcal{A}_n :_{\mathcal{P}_n} t^i)
\]
for $i = 1, \ldots, s_n$. Note that $\mathcal{P}_n^{(s_n)} = \mathcal{P}_n$. For each $b \in \mathcal{P}_n$ denote by $\hat{b}$ the image of $b$ in $\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_{Y,y_0} k(y_0)$ and by $t^{\text{ord}_i(b)} b$ the image of $t^{\text{ord}_i(b)} b$ in $(t\mathcal{P}_n) \cap \mathcal{A}_n/t\mathcal{A}_n$. For each $\alpha \in \mathcal{O}_{Y,y_0}$ denote by $\overline{\alpha}$ its image in $k(y_0)$.

Let $\langle b_1, \ldots, b_l \rangle$ be a basis for the image of $\mathcal{P}_n^{(1)}$ in $\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_{Y,y_0} k(y_0)$. Extend it to a basis for the image of $\mathcal{P}_n^{(2)}$ in $\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_{Y,y_0} k(y_0)$ and so on until we get a basis $\langle b_1, \ldots, \hat{b}_i \rangle$ for the image of $\mathcal{P}_n$ in $\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_{Y,y_0} k(y_0)$. Then the image $\hat{b}$ of each $b \in \mathcal{P}_n$ can be written as
\[
\sum_{i=1}^l \overline{\alpha}_i b_i \text{ such that } \text{ord}_i(b_j) \leq \text{ord}_i(b) \text{ for each } j \text{ with } \overline{\alpha}_j \neq 0.
\]
Moreover,
\[
\text{ord}_i(b) = \max\{\text{ord}_i(b_j) \text{ with } \overline{\alpha}_j \neq 0\}.
\]
Next we claim that each $b \in \mathcal{P}_n$ can be written as
\[
b = \sum_{i=1}^l \alpha_i b_i + a
\]
where $a \in \mathcal{A}_n$ and $\alpha_i \in \mathcal{O}_{Y,y_0} + t\mathcal{O}_{X,x_0}$. Consider the composition of maps
\[
\phi_n : \mathcal{P}_n \to \mathcal{P}_n/\mathcal{A}_n \to \mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_{Y,y_0} k(y_0).
\]
The kernel of $\phi_n$ is $\mathcal{A}_n + t\mathcal{P}_n$. Because $\mathcal{P}_n/\mathcal{A}_n$ is a coherent $\mathcal{O}_{Y,y_0}$-module we can write
\[
b = \sum_{i=1}^l \alpha_i^{(1)} b_i + t \mathcal{P}_n^{(1)}
\]
for \( \alpha_i^{(1)} \in \mathcal{O}_{Y,y} + t\mathcal{O}_{X,x_0} \) and \( p_{n}^{(1)} \in \mathcal{P}_n \). Repeating this process for \( p_{n}^{(1)} \) we get
\[
p_{n}^{(1)} = \sum_{i=1}^{l} \alpha_i^{(2)} b_i + tp_{n}^{(2)}.
\]
Substitute \( p_{n}^{(1)} \) in \((28)\) with the expression above. Continuing this process \( s_n \) times we get \((27)\) with \( \alpha_i = \alpha_i^{(1)} \) and \( \alpha_i^{(2)} \) and \( \cdots \) and \( t^{n-1} \alpha_i^{(n)} \) and \( a = t^n p_{n}^{(s_n)} \).

Construct a \( k(y_0) \)-linear map \( \psi_n \) between \( \mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_{Y,y_0} k(y_0) \) and \( (t\mathcal{P}_n \cap \mathcal{A}_n)/t\mathcal{A}_n \) such that
\[
\sum_{i=1}^{l} \alpha_i \tilde{b}_i \rightarrow \sum_{i=1}^{l} \alpha_i \tilde{t}^{i} \bar{b}_i
\]
where \( s = \text{ord}_t(\sum_{i=1}^{l} \alpha_i b_i) \). Let \( b = \sum_{i=1}^{l} \alpha_i b_i \) and \( b' = \sum_{i=1}^{l} \beta_i b_i \) be elements in \( \mathcal{P}_n \) with the same images in \( \mathcal{P}_n/\mathcal{M}_n \otimes \mathcal{O}_{Y,y_0} k(y_0) \), i.e. \( \bar{\alpha_i} = \bar{\beta}_i \) for each \( i \). Then \( s = \text{ord}_t(b) = \text{ord}_t(b') \) because \( \bar{\alpha_i} = 0 \) if and only if \( \bar{\beta}_i = 0 \). Therefore,
\[
\psi_n(b) = \sum_{i=1}^{l} \text{ord}_t \tilde{b}_i = \sum_{i=1}^{l} \bar{\beta}_i \tilde{b}_i = \psi_n(b')
\]
which proves that \( \psi_n \) is well-defined. Next, observe that each element of \( t\mathcal{P}_n \cap \mathcal{A}_n \) is of the form \( t^i b \) where \( s = \text{ord}_t(b) \geq 1 \) and \( b \in \mathcal{P}_n \). So, let \( t^i b \in (t\mathcal{P}_n \cap \mathcal{A}_n)/t\mathcal{A}_n \) for \( b \in \mathcal{P}_n \). By \((27)\) we can write
\[
b = \sum_{i=1}^{l} \alpha_i b_i + a
\]
Then \( \psi_n(b) = \sum_{i=1}^{l} \bar{\alpha_i} \tilde{t}^{i} \tilde{b}_i = \tilde{t}^i \bar{b}_i \), because \( t^i a \in t\mathcal{A}_n \), so \( \psi_n \) is surjective. Suppose that for \( \hat{b} = \sum_{i=1}^{l} \bar{\alpha_i} \tilde{t}^{i} \tilde{b}_i \) we have \( \psi_n(\hat{b}) = 0 \). Then \( t^{\text{ord}_t(b)} b \in t\mathcal{A}_n \). If \( \text{ord}_t(b) = 0 \), then \( \bar{\alpha_i} = 0 \), so \( \hat{b} = 0 \). If \( \text{ord}_t(b) > 0 \), then \( t^{\text{ord}_t(b) - 1} b \in \mathcal{A}_n \), because \( t \) is a nonzero divisor in \( \mathcal{B} \). This contradicts the minimality of \( \text{ord}_t(b) \). Hence \( \psi_n \) is injective and therefore, an isomorphism between \( k(y_0) \)-vector spaces.

\[\square\]

**The Residual Scheme**

Our next proposition identifies the ideal of the residual scheme to the union of components of \( c^{-1}(X_{y_0}) \) that surject onto \( x_0 \). In what follows we will replace \( \mathcal{B} \) with its integral closure in its total ring of fractions and redefine \( \mathcal{P}_n \) and \( \mathcal{N}_n \). By the analysis in Prp. 3.1(ii) applied to the family setting, the terms participating in the LVF will be unaffected.

Let \( V \) be the union of the components of \( c^{-1}(X_{y_0}) \) whose support over \( X_{y_0} \) is \( x_0 \). Let \( W := (c^{-1}(X_{y_0}) - V)^{-} \) be the residual scheme of \( V \) in \( c^{-1}(X_{y_0}) \). As the problem is local at \( x_0 \), assume \( X_{y_0} \) is local with closed point \( x_0 \). Denote by \( A[C_{y_0}] \) the homogeneous coordinate ring of \( c^{-1}(X_{y_0}) \). Set \( \mathcal{P} := \mathcal{O}_{X,x_0} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 + \cdots \)

**Proposition 4.4.** The ideal of \( W \) in \( A[C_{y_0}] \) is
\[
\mathcal{I}_W = (t\mathcal{P} \cap \mathcal{A})/t\mathcal{A}.
\]

**Proof.** Since \( A[C_{y_0}] = \mathcal{A}/t\mathcal{A} \), then by Prp. 3.5
\[
\mathcal{I}_W = \bigcup_{i=0}^{\infty} (t\mathcal{A} : \mathcal{A} m_{x_0}^{i})/t\mathcal{A}
\]
where $m_{x_0}$ is the ideal of $x_0$ in $O_{X,x_0}$. We claim that
\[
\bigcup_{i=0}^{\infty} (tA :_{A} m_{x_0}^i) = tP \cap A.
\]
Indeed, let $tp \in tP \cap A$ where $p \in P$. Let $j \gg 0$ such that $m_{x_0}^j p \in A$. Then $m_{x_0}^j tp \in tA$ and hence $tp \in \bigcup_{i=0}^{\infty} (tA :_{A} m_{x_0}^i)$. As in the proof of Prp. 3.1 (ii) we can find $j$ such that $t, \delta$ is a 2-regular sequence for $B$.

Let $q$ be an element from $\bigcup_{i=0}^{\infty} (tA :_{A} m_{x_0}^i)$. For $j \gg 0$ we have $m_{x_0}^j q \in tA$ so we can write $\delta^j q = ta$ for $a \in A$. Consider the last identity as an identity in $B$. Because $\delta$ is a nonzero divisor modulo $t$, then $q/t \in B$. We claim that $q/t \in P$. Indeed, because $m_{x_0}^j q/t \in A$ we have $m_{x_0}^j q/t \in A$ which yields $q/t \in P$. Hence $q \in tP \cap A$. □

Let $X := \text{Spec}(R)$ be an affine Noetherian scheme of finite type over a field $K$. Let $C$ be a subscheme of $X \times \mathbb{P}_K^n$, where $u$ is a positive integer. Denote the homogeneous coordinate ring of $C$ by $A[C]$. It is a graded ring with respect to the coordinates of $\mathbb{P}_K^n$. Denote its $n$th graded piece by $A[C]_n$. Let $W$ be a closed subscheme of $C$ and denote its ideal in $A[C]$ by $I_W$. Set $(I_W)_n = A[C]_n \cap I_W$.

Let $pr_1$ be the projection from $X \times \mathbb{P}_K^n$ onto the first factor. Let $x$ be a closed point of $X$. Set $V := pr_1^{-1}(x) \cap C$ and $r := \dim V$. Let $l := c_1 O_C(1)$ and let $|V|$, be the dimension $r$ part of the fundamental cycle of $V$. Finally, define $\deg[V]_r := \int |V|.$

**Proposition 4.5.** Assume $C = V \cup W$ where $W$ is a closed subscheme of $X \times \mathbb{P}_K^n$ with $\dim(V \cap W) < r$. Then $(I_W)_n$ is a finite-dimensional $K$-vector space for each $n$, and
\[
\lim_{n \to \infty} \frac{r!}{n^r} \dim_K(I_W)_n = \deg[V]_r.
\]

**Proof.** Because $I_V \cap I_W = 0$ in $A[C]$, then as $R$-modules
\[
(I_W)_n \simeq ((I_W)_n + (I_V)_n)/(I_V)_n
\]
for each $n$. Set $A[V] = A[C]/I_V$. Since the residue field of $x$ is $K$, it follows that $A[V]_n$ is finite-dimensional $K$-vector space. The inclusion $((I_W)_n + (I_V)_n)/(I_V)_n \subset A[V]_n$ shows that $(I_W)_n + (I_V)_n)/(I_V)_n$ is finite dimensional $K$-vector space as well. But $R$ is a $K$-algebra because $X$ is of finite type over $K$ by assumption. Thus, by (29) the $R$-module $(I_W)_n$ is a finite-dimensional $K$-vector space and
\[
\dim_K(I_W)_n = \dim_K((I_W)_n + (I_V)_n)/(I_V)_n.
\]
Next, consider the exact sequence
\[
0 \to (I_W + I_V)/I_V \to A[V] \to A[V \cap W] \to 0.
\]
where $A[V \cap W] = A[C]/I_{V \cap W}$. As $\dim(V \cap W) < r$, we get
\[
\lim_{n \to \infty} \frac{r!}{n^r} \dim_K A[V \cap W]_n = 0.
\]
Thus
\[
\lim_{n \to \infty} \frac{r!}{n^r} \dim_K((I_W)_n + (I_V)_n)/(I_V)_n = \lim_{n \to \infty} \frac{r!}{n^r} \dim_K A[V]_n.
\]
But
\[
\lim_{n \to \infty} \frac{r!}{n^r} \dim_K A[V]_n = \deg[V]_r
\]
Combining (30), (31) and (32) we get the desired result. □

Proposition 4.6. We have
\[
\lim_{n \to \infty} \frac{n!}{n^n} \dim_k \mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y_0) = \int_{S_{y_0}} \nu'[D_{\text{vert}}].
\]

Proof. By Proposition 4.3
\[
\dim_k \mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y_0) = \dim_k (t\mathcal{P}_n \cap \mathcal{A}_n)/t\mathcal{A}_n.
\]
Adopt the setup of Proposition 4.4. By Proposition 4.4 we have
\[
\mathcal{P}_n/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y_0) \simeq (\mathcal{I}_W)_n \text{ as } k(y_0)-\text{vector spaces. Finally, Proposition 4.5 applied to } X = X_{y_0}, x = x_0, K = k(x_0) \text{ and } C = c^{-1}(X_{y_0}) \text{ along with the assumption that the residue fields of } y_0 \text{ and the points in } S_{y_0}
\]
are the same, give us the desired equality. □

Specialization to the generic fiber

Proposition 4.7. Let \( U \) be a sufficiently small neighborhood of \( y_0 \). Then
\[
H^0_S(B_n/\mathcal{A}_n) \otimes \mathcal{O}_Y \ k(y) = H^0_{S_y}(\mathcal{I}_S \mathcal{O}_Y \ k(y))
\]
for every \( y \in U' \) and each \( n \).

Proof. As usual identify \( H^0_S(B_n/\mathcal{A}_n) \) with \( \mathcal{N}_n/\mathcal{A}_n \). We want to show that
\[
\mathcal{N}_n/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y) = H^0_{S_y}(\mathcal{I}_S \mathcal{O}_Y \ k(y))
\]
for every \( y \) close enough to \( y_0 \). First, because the support of \( \mathcal{N}_n/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y) \) is in \( S_y \), then by Proposition 3.6 it follows that \( \mathcal{N}_n/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y) \) surjects onto \( U \) or onto \( y_0 \). Then further shrink \( U \) if necessary, so that each \( \{z\} \) of dimension one that is not a component of \( S \) intersects \( S' \) at \( x_0 \) only. Set \( S' = \text{Supp}_X(N_n'/\mathcal{A}_n) \). The support of \( N_n'/\mathcal{A}_n \otimes \mathcal{O}_Y \ k(y) \) is \( S_y' \). Also, the support of \( H^0_{S_y'}(\mathcal{I}_S \mathcal{O}_Y \ k(y)) \) is \( S_y \). This forces \( S_y' = S_y \). Therefore, the union of components of \( S \) that do not surject onto \( x_0 \) is of dimension one and hence equal to \( S \). By Prp. 3.6 \( N_n \) is the maximal submodule of \( B_n \) that contains \( \mathcal{A}_n \) with \( \text{Supp}(N_n/\mathcal{A}_n) = S \). Thus
\[
(N_n'/\mathcal{A}_n) \otimes \mathcal{O}_Y \ k(y) = 0.
\]

Remark 4.8. Assume \( \mathcal{A} \) is the Rees algebra of a module \( \mathcal{M} \) contained in a free module \( \mathcal{F} \), and \( S : = \text{Sym}(\mathcal{F}) \). Observe that \( \mathcal{O}_{X_S} \) is analytically unramified because \( X_S \) is reduced and of finite type over a field. Also, by a result of Kleiman, Ulrich and Validashti [KUV]
\[
\lim_{n \to \infty} \frac{r!}{n^n} \dim_k(s_i) H^0_{S_y}(\mathcal{F}^n/\mathcal{M}^n \otimes \mathcal{O}_Y \ k(y)) < \infty.
\]
Then by Thm. 3.2 in [Cut15] (33) exists as a limit. The existence of
\[
\limsup_{n\to\infty} \frac{r!}{n^r} \dim_k H^0_{s_y}(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y)
\]
as a limit for each closed point \(y \in Y\) where \(s_y\) is any point in the fiber of \(S\) over \(y\) will be discussed elsewhere. A closer inspection of Cutkosky’s treatment [Cut15] reveals that the existence of the limits as observed by Dao and Smirnov (see p. 4 in [Cut15]).

**Completing the proof of the LVF**

Now we are in position to prove Theorem 4.1. Proposition 4.6 imply that
\[
(34) \lim \frac{r!}{n^r} \dim_k H^0_{s_y}(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y)
\]
exists and is equal to \(\deg[c^{-1}(x_0)]_r\). By Prp. 4.7 and by our hypothesis
\[
\limsup_{n\to\infty} \frac{r!}{n^r} \dim_k H^0_S(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y)
\]
is finite. Therefore, by Proposition 4.2
\[
\limsup_{n\to\infty} \frac{r!}{n^r} \dim_k H^0_S(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y) - \limsup_{n\to\infty} \frac{r!}{n^r} \dim_k H^0_S(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y) = \int_{S_{y_0}} l'[D_{\text{vert}}]
\]
which completes the proof.

We conclude this section with two immediate applications of Theorem 4.1.

**Proposition 4.9.** The restricted local volume
\[
\lim_{n\to\infty} \frac{r!}{n^r} \dim_k H^0_S(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y)
\]
is constant over a punctured neighborhood \(U'\) of \(y_0\).

**Proof.** Indeed, both \(\limsup_{n\to\infty} \frac{r!}{n^r} \dim_k H^0_{s_y}(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y_0)\) and \(\int_{S_{y_0}} l'[D_{\text{vert}}]\) remain constant as we move \(y\) around a punctured neighborhood of \(y_0\). \(\Box\)

The following is an immediate corollary from the proof of the LVF.

**Corollary 4.10.** In the setup of Thm. 4.1 the scheme \(c^{-1}S\) is flat over \(Y\) if and only if
\[
\dim_k H^0_S(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y) \text{ is constant over } Y \text{ for } n \gg 0.
\]

**Proof.** We have \(\dim_k H^0_S(\mathcal{B}_n/A_n) \otimes_{O_Y} k(y) \text{ is constant over } Y \text{ for } n \gg 0 \iff (I_W)_n = 0 \text{ for } n \gg 0 \iff V \text{ is empty iff } c^{-1}S \text{ is flat (see Prp. 9.7 in [Har77]).}\( \Box\)

We would like to remark that in the setting \(A := R(M)\) and \(B := \text{Sym}(F)\), the “generic” term of the left-hand side of the LVF exists as a limit as remarked above. The right-hand side is a limit, too. Then so is “special” term of the left-hand side. This observation will be important for applications to equisingularity theory.

**Remark 4.11.** In [Ran19d] we give a Fujita-type version of the LVF by replacing local cohomology with intersection numbers. For each closed point \(y\) denote by \(\mathcal{A}_n(y)\) and \(\mathcal{N}_n(y)\) the images of \(\mathcal{A}_n\) and \(\mathcal{N}_n\) in \(B_n \otimes_{O_Y} k(y)\). For each \(n\) denote by \(\langle \mathcal{A}_n\rangle\) and \(\langle \mathcal{N}_n\rangle\) the subalgebras of \(B\) generated by \(\mathcal{A}_n\) and \(\mathcal{N}_n\). Denote by \(c_{\mathcal{A}_n} : \text{Proj}(\langle \mathcal{A}_n\rangle) \to X\) and \(c_{\mathcal{N}_n} : \text{Proj}(\langle \mathcal{N}_n\rangle) \to X\)
the corresponding structure morphisms. Denote the union of components of \( c^{-1}_{A_n} S \) and \( c^{-1}_{X_n} S \) supported over \( S_{y_0} \) by \( D_{\text{vert}}^{A_n} \) and \( D_{\text{vert}}^{X_n} \) respectively. Applying Thm. 2.4 we get
\[
e(A_n(y_0), N_n(y_0)) - e(A_n(y), N_n(y)) = \int_{S_{y_0}} \nu[D_{\text{vert}}^{A_n}] - \int_{S_{y_0}} \nu[D_{\text{vert}}^{X_n}].
\]
Because \( \langle A_n \rangle \) is the nth Veronese subalgebra of \( A \) we have
\[
\int_{S_{y_0}} \nu[D_{\text{vert}}^{A_n}] = n^r \int_{S_{y_0}} \nu[D_{\text{vert}}].
\]
In \cite{Ran19d} we prove that
\[
\limsup_{n \to \infty} \frac{r!}{n^r} \int_{S_{y_0}} \nu[D_{\text{vert}}^{X_n}] = 0.
\]
This is easy to see when \( \bigoplus_{r=0}^{\infty} N_i \) is finitely generated, for example. Let \( I_{S_{y_0}} \) be the ideal of \( S_{y_0} \) in \( O_X \). By Prp. 3.5 and Prp. 2.1 in \cite{Ran19a} one can find an element \( x \in I_{S_{y_0}} \) such that \( x \) is not a zero divisor of \( B_i/N_i \) for each \( i \). Then \( \bigoplus_{r=0}^{\infty} N_i/x \bigoplus_{r=0}^{\infty} N_i \) injects into \( B/xB \). By Prp. 3.2 and the assumption \( \dim X \geq 2 \) we can find \( y \in I_{S_{y_0}} \) which avoids the minimal primes of \( B/xB \) and hence those of \( \bigoplus_{r=0}^{\infty} N_i/x \bigoplus_{r=0}^{\infty} N_i \). Denote by \( c_X : \text{Proj}(\bigoplus_{r=0}^{\infty} N_i) \to X \) the structure morphism. Then \( \text{codim}(c_X^{-1} S_{y_0}, \text{Proj}(\bigoplus_{r=0}^{\infty} N_i)) \geq 2 \). But \( \bigoplus_{r=0}^{\infty} N_i \) is birational extension of \( A \) because \( B \) is. So \( \bigoplus_{r=0}^{\infty} N_i \) is of pure dimension \( r \). Also, \( D_{\text{vert}}^{X_n} \) is a subscheme of \( c_X^{-1} S_{y_0} \). Thus \( \dim D_{\text{vert}}^{X_n} < r \), which implies that all terms in \( (37) \) vanish for \( n \) sufficiently large.

Divide both sides of \( (35) \) by \( n^r \). Then by \( (36) \) and \( (37) \), and by taking limit superiors we obtain
\[
\limsup_{n \to \infty} \frac{e(A_n(y_0), N_n(y_0))}{n^r} - \limsup_{n \to \infty} \frac{e(A_n(y), N_n(y))}{n^r} = \int_{S_{y_0}} \nu[D_{\text{vert}}].
\]

Volume Stability

Let \( X \to Y \) be a family of reduced complex analytic spaces such that \( X \) is equidimensional, \( Y \) smooth of dimension one, and the fibers \( X_y \) are equidimensional of positive dimension. Let \( y_0 \) be a closed point in \( Y \). Assume \( Y \) is contained in a smooth complex analytic space \( W \) and
\[
X \to W
\]
is an equidimensional reduced family with equidimensional fibers of positive dimension such that \( X_{y_0} = X_{y_0} \). Assume \( S(W) \) is a subspace of \( \mathcal{X} \) finite over \( W \). Let \( C(W) \) be reduced, equidimensional and projective space over \( X \) such that its irreducible components surject onto irreducible components of \( X \). Let \( \mathcal{L} \) be an invertible very ample sheaf on \( C(W) \) relative to \( X \). Set \( \mathcal{A} := \bigoplus_{n \geq 0} \Gamma(C(W), \mathcal{L}^n) \). For each closed point \( w \) define the restricted local volume of \( \mathcal{L} \) at \( S(W)_w \) as
\[
\text{vol}_{C(W)_w}(\mathcal{L}) := \limsup_{n \to \infty} \frac{r!}{n^r} \dim_k H^1_{S(W)}(A_n) \otimes_{\mathcal{O}_w} k(w).
\]
Set \( B(w) := \mathcal{B} \otimes_{\mathcal{O}_w} k(w) \). For each \( n \) denote the image of \( A_n \) in \( B_n(w) \) by \( A_n(w) \) We say that the restricted local volume specializes with passage to the fiber \( X_w \) if
\[
\text{vol}_{C(W)_w}(\mathcal{L}) := \limsup_{n \to \infty} \frac{r!}{n^r} \dim_k H^1_{S(W)_w}(A_n(w)).
\]
Replacing $Y$ by $W$ in Prop. 4.7 we get that there exists a Zariski open subset $U$ of $W$ such that the restricted local volume specializes with passage to $X_w$ for each $w \in U$.

**Definition 4.12.** We say $W$ is a good base space for $\text{vol}_{C(W)}(\mathcal{L})$ if there exists a Zariski open dense subset $U$ of $W$ such that $\text{vol}_{C(W)}(\mathcal{L})$ is constant for all $w \in U$. In this case we say $\text{vol}_{C(W)}(\mathcal{L})$ is stable.

Let $B$ be a birational extension of $A$ satisfying the properties listed in Prop. 3.2. Set $B(Y) := B \otimes_{O_X} O_X$. Let $\mathcal{A}(Y)$ be the image of $\mathcal{A}$ in $B(Y)$. Set $C := \text{Proj}(\mathcal{A}(Y))$ and $S := S(W) \cap X$. Denote by $D_{\text{vert}}$ the union of components in the inverse image of $S$ in $C$ supported over $y_0$.

Let $W_1$ and $W_2$ be generic curves in $W$ passing through $y_0$ and a generic point $y$ in $Y$ close enough to $y_0$. Consider the families $\mathcal{X}(W_i) := \mathcal{X} \times_W W_i$. Let $\mathcal{A}(W_i)$ be the image of $\mathcal{A}$ in $B \otimes_{O_X} O_{X_i(W_i)}$. Set $C(W_i) := \text{Proj}(\mathcal{A}(W_i))$ and $S(W_i) := S \cap W_i$. Denote by $D(W_1)_{\text{vert}}$ the union of components of the inverse image of $S(W_1)$ supported in $C(W_1)$ over $S(W)_{y_0}$, and by $D(W_2)_{\text{vert}}$ the union of components of the inverse image of $S(W_2)$ supported in $C(W_2)$ over $S(W)_y$.

**Theorem 4.13.** Suppose $W$ is a good base space for $\text{vol}_{C(W)}(\mathcal{L})$. Assume that the set of closed points $x$ in $\mathcal{X}$ with $\dim c^{-1}x \geq r$ is contained in $S(W)$. Then we can replace $\text{vol}_{C(W)}(\mathcal{L})$ and $\text{vol}_{C(W)}(\mathcal{L})$ in the LVF applied to $X \to Y$, without violating its validity, by the restricted local volumes $\text{vol}_{C(W_1)}(\mathcal{L})$ and $\text{vol}_{C(W_2)}(\mathcal{L})$, i.e.

\[
\text{vol}_{C(W_1)}(\mathcal{L}) - \text{vol}_{C(W_2)}(\mathcal{L}) = \int_{S_{y_0}} l^r[D_{\text{vert}}].
\]

**Proof.** Select $W_1$ and $W_2$ generic enough so that $W_1 - \{y_0\}$ and $W_2 - \{y\}$ lie in the Zariski open subset $U$ over which the volume $\text{vol}_{C(W)}(\mathcal{L})$ is stable and specializes with passage to $X_w$. Replace $W_i$ by $W_i \cap U$. Let $w_1$ and $w_2$ be points from $W_1$ and $W_2$ close enough to $y_0$ and $y$, respectively. Apply the LVF to the families $\mathcal{X}(W_i) \to W_i$ to get

\[
\text{vol}_{C(W_1)}(\mathcal{L}) - \text{vol}_{C(W_1)}(\mathcal{L}) = \int_{S_{y_0}} l^r[D(W_1)_{\text{vert}}]
\]

and

\[
\text{vol}_{C(W_2)}(\mathcal{L}) - \text{vol}_{C(W_2)}(\mathcal{L}) = \int_{S_y} l^r[D(W_2)_{\text{vert}}].
\]

Because of volume stability and specialization $\text{vol}_{C(W_1)}(\mathcal{L}) = \text{vol}_{C(W_2)}(\mathcal{L})$. Subtracting the second identity from the first we get

\[
\text{vol}_{C(W_1)}(\mathcal{L}) - \text{vol}_{C(W_2)}(\mathcal{L}) = \int_{S_{y_0}} l^r[D(W_1)_{\text{vert}}] - \int_{S_y} l^r[D(W_2)_{\text{vert}}].
\]

Our goal is to show that the right-hand side (39) equals $\int_{S_{y_0}} l^r[D_{\text{vert}}]$. To do this we interpret each of the three intersection numbers as the local degrees of a covering of $W$.

Let $W(y_0)$ be a small enough neighborhood of $W$ around $y_0$. Let $y \in W(y_0) \cap Y$ be a generic point of $Y$ to be specified later, and let $W(y)$ be a small enough neighborhood of $y$ such that $W(y) \subset W(y_0)$. Let $c: C(W) \to \mathcal{X}$ be the structure morphism. Let $C(W) \hookrightarrow \mathcal{X} \times \mathbb{P}^k$ be the embedding of $C$ induced by $\mathcal{L}$.

Let $H_r$ be a general plane in $\mathbb{P}^k$ of codimension $r$, with finitely many genericity conditions to be specified later. Set $\Gamma(C(W)) := C(W) \cap H_r$. By Kleiman’s transversality theorem (see Thm. 2 (ii) and Rmk. 7 in [Kle74]), $\Gamma(C(W))$ is reduced of pure dimension equal to
dim \( W \). Furthermore, the intersection of \( e^{-1}(S(W)) \) with a general plane \( H_r \) is of dimension at most \( \dim W - 1 \). Therefore, for generic \( w \in W(y_0) \) the fiber \( \Gamma(C)_w \) consists of the same number of points, each of them appearing with multiplicity one, and none of them lying in \( e^{-1}S(W) \). Set \( \deg_{W(y_0)} \Gamma(C(W)) := \text{Card}(\Gamma(C)_w) \). Define \( \Gamma(C(W_1)) := C(W_1) \cap H_r \) and set \( \deg_{W_1(y_0)} \Gamma(C(W_1)) := \text{Card}(\Gamma(C(W_1))_{w_1}) \) for generic \( w_1 \in W_1 \), where \( W_1(y_0) \) is a small enough neighborhood of \( y_0 \) in \( W_1 \). Then for a generic \( W_1 \) we have \( \Gamma(C(W))_{w_1} = \Gamma(C(W_1))_{w_1} \) for generic \( w_1 \in W_1 \). Because by assumption the only points in \( X_{y_0} \) over which the fiber of \( C \) is of dimension at least \( r \) are contained in \( S_{y_0} \) and by conservation of number we obtain

\[
\deg_{W(y_0)} \Gamma(C(W)) = \deg_{W_1(y_0)} \Gamma(C(W_1)) = \int_{S_{y_0}} \iota^* [D(W_1)]_{\text{vert}}. 
\]

In the same way define \( \Gamma(C) = C \cap H_r \) and \( \deg_Y \Gamma(C) \) where \( H_r \) is generic enough so that the cover \( \Gamma(C) \to Y \) is unramified for generic \( y \in U := W(w_0) \cap Y \) and \( \Gamma(C) \cap S = \emptyset \). By conservation of number

\[
\deg_Y \Gamma(C) = \int_{S_y} \iota^*[D_{\text{vert}}].
\]

Set \( \Gamma(C(W_2)) := C(W_2) \cap H_r \) and \( \deg_{W_2(y)} \Gamma(C(W_2)) := \text{Card}(\Gamma(C(W_2))_{w}) \) where \( W_2(y) \) is small enough neighborhood of \( y \) in \( W_2 \) and \( w \in W_2 \) is generic.

We claim that

\[
\deg_{W_2(y)} \Gamma(C(W_2)) = \int_{S_y} \iota^*[D(W_2)]_{\text{vert}}.
\]

Note \( \int_{S_y} (H_r \cap e^{-1} S_{y_0}) = \int_{S_y} \iota^*[D(W_2)]_{\text{vert}} \) because the codimension of \( H_r \) is right. Denote by \( s_1, s_2, \ldots, s_q \) to be the points in \( S_y \) such that \( \dim e^{-1}s_i \geq r \). Because \( W_2 \) is generic and because the cover the cover \( \Gamma(C(W)) \to (W, w_0) \) is unramified for generic \( w \in W(y) \), we get that the branches passing through \( s_i \) form \( \Gamma(C(W_2)) \).

The following relation (see Thm. 2.8 [GR16]) shows that the presence of \( \deg_Y \Gamma(C) \) is controlled by \( \deg_{W(y_0)} \Gamma(C(W)) \) and \( \deg_{W_2(y)} \Gamma(C(W_2)) \). We claim

\[
(40) \quad \deg_{W(y_0)} \Gamma(C(W)) - \deg_{W_2(y)} \Gamma(C(W_2)) = \deg_Y \Gamma(C).
\]

Recall that \( H_r \) was chosen so that it produces the covers \( \Gamma(C(W)) \to (W, w_0) \) and \( \Gamma(C) \to (Y, y_0) \). Observe that for generic \( w \in W(y) \) close enough to \( y \) the degree of the cover \( \Gamma(C(W)) \to (W, w_0) \) is \( \deg_{W(y_0)} \Gamma(C(W)) \). Over \( y \) some of these branches merge at the \( s_i \)'s and their number is \( \deg_{W_2(y)} \Gamma(C(W_2)) \) as shown above. The rest of the branches intersect \( X_y \) at points away from \( S(W) \). Their number is \( \deg_Y \Gamma(C) \) by our choice of \( H_r \). The proof of the theorem is now complete. \( \square \)

The figure summarizes the proof of [40] where for convenience it’s assumed that there exists a section \( \sigma : W \to \mathcal{X} \) and \( W \) is identified with \( \sigma(W) \).

In complex analytic singularity theory we apply Theorem 4.13 [40] in the following setting: \( X, x_0 \to Y, y_0 \) is a family of isolated singularities, \( W \) is a larger deformation base space containing \( Y \), usually taken to be a component of the miniversal base space of \( X_{y_0}, C(W) \) is the relative conormal space of \( \mathcal{X} \to W \), and \( S(W) \) is the singular locus of \( \mathcal{X} \). Assume that \( W \) is a good deformation space. Thus for \( X_{y_0} \) and \( X_y \) for \( y \in Y \) close enough to \( y_0 \), we can associate unique nonnegative real numbers depending solely on \( W \) such that \( \int_{S_{y_0}} [D_{\text{vert}}] \) vanishes if and only if the corresponding restricted volumes associated with one-parameter generic deformations of \( X_{y_0} \) and \( X_y \) in \( W \) are the same.
We finish this section by obtaining a numerical control of Teissier’s Principle of Specialization of Integral Dependence. Preserve the setup of Thm. 4.13. When we write $\text{vol}_{C_y}(L)$ we really mean the restricted local volume computed through the family $\mathcal{X} \times_W W'$ with special fiber $X_y$ where $W'$ is a generic smooth curve in $W$ passing through $y$. As usual denote by $A(Y)$ the integral closure of $A(Y)$ in $B(Y)$.

**Theorem 4.14 (Principle of Specialization of Integral Dependence).** Assume $W$ is a good base space for $\text{vol}_{C_y}(W)$. Let $\mathcal{L} \subset B(\mathcal{O}_Y)$ be such that $\text{Supp}(\mathcal{A}(\mathcal{O}_X)/(\mathcal{A}(\mathcal{O}_Y))) \subset S(W)$. Assume that there exists a Zariski open set $U$ in $Y$ such that for each $y \in U$ the image of $g$ in $B \otimes O_Y/k(y)$ is integral over $A(y)$. If $\text{vol}_{C_y}(L)$ is constant, then $g$ is integral over $A(Y)$.

**Proof.** By Thm. 4.13, constancy of $\text{vol}_{C_y}(W)$ implies that $\dim c^{-1}S_y < r$ for each $y$. Set $Z := \text{Supp}(\mathcal{O}_X/(\mathcal{A}(\mathcal{O}_Y)))$. By Lem. 1.2 in [GK99] there exists a smaller Zariski subset $U'$ of $Y$ such that $(g, A)$ is integral over $A$. Thus $Z$ is proper closed subset of $S$. In particular, $\dim Z < \dim S = \dim Y$. Let $y_0 \in Y$ be such that $\dim c^{-1}S_{y_0}$ is maximal. Because $c^{-1}Z \subset c^{-1}S$, by upper semi-continuity of fiber dimension we have

$$\dim c^{-1}Z \leq \dim c^{-1}S_{y_0} + \dim Z \leq r - 1 + \dim Y - 1 = \dim C - 2.$$

Then by Cor. 10.7 in [KT94] (cf. Thm. 4.1 in [SUV01] or Thm. 1.1 (iii) in [Ran19a]) $Z$ is empty, and thus $g$ is integral over $A(Y)$. \qed

5. **Whitney–Thom conditions, Jacobian modules, conormal spaces and integral dependence**

First we review the analytic and algebro-geometric formulations of the Whitney conditions and their connection to integral closure of modules. For a thorough overview of the history of differential equisingularity theory we refer the reader to the masterful treatments of Kleiman [K99], Gaffney and Massey [GaM99] and Teissier [T75]. For a more recent account see [TF18].

Let $X \subset \mathbb{C}^{n+k}$ be a equidimensional complex analytic space, and let $Y$ be a smooth subspace of $X$ of dimension $k$, so that $X - Y$ is smooth. Choose an embedding of $(X,0)$ in $\mathbb{C}^n \times \mathbb{C}^k$, so that $(Y,0)$ is represented by $0 \times V$, where $V$ is an open neighborhood of 0 in $\mathbb{C}^k$. Let $pr : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^k$ be the projection. Set $h := pr|_X$. View $X$ as the total space of the family $h : (X,0) \to (Y,0)$. For each closed point $y \in Y$ set $X_y := X \cap \text{pr}^{-1}(y)$. Whitney [Wh64] introduced the following conditions on the regularity of the triple $(X - Y, Y, 0)$.

**Whitney Condition A.** Let $(x_i)$ be a sequence of points from $X - Y$ that converges to 0, and suppose that the sequence $\{T_{x_i}X\}$ of tangent hyperplanes has a limit $T$ in the corresponding Grassmannian. Then $T_0Y \subset T$. 
Whitney Condition B. Let \((x_i)\) be a sequence of points from \(X - Y\) and let \((y_i)\) be a sequence of points from \(Y\) both converging to 0. Suppose that the sequence of secants \((\pi_i y_i)\) has limit \(l\) and the sequence of tangents \(\{T_{x_i} X\}\) has limit \(T\). Then \(l \subseteq T\).

Whitney conjectured that these conditions would ensure the existence of a homeomorphism \(h\) from \(X_0 \times Y\) onto \(X\). In other words the family \(X_y\) is \emph{topologically equisingular}. The conjecture is nowadays known as the Thom–Mather [Tm69] first isotopy theorem. Thom and Mather proved it by considering constant tangent vector field to \(Y\) and lifting it carefully to \(X\) such that the lift is integrable. The integral provides a continuous flow on \(X\) which is \(C^\infty\) away from \(Y\).

Hironaka [Hir70] introduced an intrinsic modified version of the Whitney conditions. We say that \(X\) satisfies the strict Whitney condition A if the distance from \(Y\) to the tangent space \(T_x X\) approaches 0 as \(x\) approaches 0 along any analytic path \(\phi : (C, 0) \to (X, 0)\) such that \(\phi(u) \subset X - Y\) for any \(u \neq 0\), i.e.

\[
\text{dist}(Y, T_x X) \leq c \text{ dist}(x, Y)^e, 
\]

holds where the exponent \(e\) and the constant \(c\) depend on the path \(\phi\). Hironaka proved that if \((X - Y, Y, 0)\) satisfies both Whitney condition A and B, then it satisfies the strict Whitney condition A with exponent \(e\) independent of the path \(\phi\). The strict Whitney condition with exponent \(e = 1\) and \(c\) independent of the path \(\phi\) is called Verdier’s W condition. Verdier [V76] showed that his condition implies Whitney A and B. Teissier (Thm. 1.2 in Chap. V of [T81]) showed that in the complex-analytic case \(W\) is in fact equivalent to Whitney A and B.

Let \(f\) be a germ of complex analytic function \(f : C^{n+k} \to (C, 0)\). Identify \(f\) with its restriction to \((X, 0)\). Assume \(f\) is of constant rank off \(Y\). In (41) replace \(T_x X\) by \(T_x f^{-1} f x\). The “relative” form of the Whitney condition A, called Thom’s \(A_f\) condition, holds at 0 if (41) is satisfied. The “relative” form of the Whitney condition B, called the \(W_f\) condition, holds at 0 if (41) is satisfied with \(e = 1\) and \(c\) independent of the path \(\phi\).

Let \(U\) be an open set of \(C^{n+k}\) containing a representative of \((X, 0)\). The differential \(df\) of \(f\) defines an embedding of \(X\) in \(C^{n+k} \times C^{n+k}\) by the graph map. Let \(x_1, \ldots, x_{n+k}\) be coordinates on \(U\) and \(w_1, \ldots, w_{n+k}\) be the cotangent coordinates. Consider the blowup of \(T_X U\) along the image of the graph map. It is the blowup of \(T_X U\) by the ideal \((w_1 - \frac{\partial f}{\partial x_1}, \ldots, w_{n+k} - \frac{\partial f}{\partial x_{n+k}})\) in \(T_X U\). We denote this blowup by \(\text{Bl}_{\text{im}} df T_X^* U\). Thus, the blowup is contained in \(X \times C^{n+k} \times P^{n+k-1}\). Denote the exceptional divisor of this blowup by \(E_f\). Denote the projection on \(X\) of this exceptional divisor to \(X\) by \(\Sigma(f)\) and call it the \emph{critical locus} of \(f\).

Define the the absolute conormal space \(C(X, f)\) as the closure in \(X \times P^{n+k-1}\) of the set of pairs \((x, H)\) such that \(x\) is a point in \(X - \Sigma(f)\) and \(H\) is a hyperplane tangent at \(x\) to the level hypersurface \(f^{-1} f x\). Teissier inspired by some work of Hironaka, considered the following normal-conormal diagram

\[
\begin{align*}
\text{Bl}_{c^{-1}(Y)} C(X, f) &\xrightarrow{a'} C(X, f) \quad \downarrow c' \\
\text{Bl}_Y X &\xrightarrow{a} X 
\end{align*}
\]

Set \(\xi := c \circ a'\). Teissier [T81] in the case of Whitney B, and Henry, Merle and Sabbah [HMS84] in the case of \(W_f\) obtained the following equivalence for the Whitney conditions.

\textbf{Theorem 5.1 (Teissier).} Preserving the setup from above, the following conditions are equivalent:
(i) The triple \((X, Y, Y, 0)\) satisfies \(W_f\).

(ii) Let \(\xi_Y : D_Y \to Y\) be the restriction of \(\xi\) to the exceptional divisor \(D_Y\) of \(\text{Bl}_{c-1}(Y)C(X, f)\). Then \(\xi_Y\) is equidimensional and \(\dim \xi_Y^{-1}(0) = n - 2\).

Here is a way to see the relation between (i) and (ii) in the Whitney B case. Observe that set-theoretically each point of \(P\)

Observe that locally the sheaf \(\text{Image}(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)\) has the form \((y, l, H)\) where \(y\) is point of \(Y\), \(l\) belongs to \((\mathbb{P}(C_Y X))_y\) and \(H \in c^{-1}(y)\). Then the condition is equivalent to \(l \in H\). The triple \((X, Y, Y, 0)\) satisfies \(W_f\) if \((D_Y)^\text{red}\) is contained in the incidence correspondence of \(\mathbb{P}(m_Y/\langle m_Y \rangle^2) \times Y \mathbb{P}((\langle m_Y \rangle/\langle m_Y \rangle^2)^*)\). Then a simple dimension count yields (ii).

Next we provide a concrete algebraic description of the conormal variety \(C(X)\) using the Jacobian module of \(X\). Suppose \(X\) is reduced. Let \(X\) be defined by the vanishing of some analytic functions \(f_1, \ldots, f_p\) on a Euclidean neighborhood of \(0\) in \(\mathbb{C}^{n+k}\). Consider the following exact sequence

\[
I/I^2 \rightarrow \Omega_{\mathbb{C}^{n+k}}^{1,0} \rightarrow \Omega_X^{1,0} \rightarrow 0
\]

where \(I\) is the ideal of \(X\) in \(O_{\mathbb{C}^{n+k},0}\) and the map \(\delta\) sends a function \(f\) vanishing on \(X\) to its differential \(df\). Dualizing we obtain the following nested sequence of torsion-free sheaves:

\[
\text{Image}(\delta^*) \subset (\text{Image} \delta^*)^* \subset (I/I^2)^*.
\]

Observe that locally the sheaf \(\text{Image}(\delta^*)\) can be viewed as the column space of the \((\text{absolute})\) Jacobian module of \(X\) which we denote by \(J(X)\). It’s a module contained in the free module \(O_{X,0}^p\). Define the Rees algebra \(R(J(X))\) to be the subalgebra of \(\text{Sym}(O_{X,0}^p)\) generated by the generators of \(J(X)\) (for a general definition see [EHU03]). Define the conormal space \(C(X)\) of \(X\) to be the closure in \(X \times \mathbb{P}^{n+k-1}\) of the set of pairs \((x, H)\) where \(x\) is a smooth point of \(X\) and \(H\) is a tangent hyperplane at \(x\). Algebraically,

\[
C(X) = \text{Proj}(R(J(X))).
\]

To see that equality holds observe that both sides are equal over \(X - Y\) to the set of pairs \((x, H)\) where \(H\) is a tangent hyperplane to the simple point \(x\). The left side is the closure of this set, and so is the right side simply because the Rees algebra is by construction a subalgebra of the symmetric algebra \(\text{Sym}(O_{X,0}^p)\). Similarly, one sees that \(C(X, f) = \text{Proj}(R(J(X, f)))\) where \(J(X, f)\) is the augmented Jacobian module with the partials of \(f\).

Consider the family setup \(h : (X, 0) \rightarrow (Y, 0)\) from the beginning of the section. Assume each \(f_i\) is analytic function on two sets of variables: the fiber variables \(\{x_1, \ldots, x_n\}\) and the parameter variables \(\{y_1, \ldots, y_k\}\). Form the augmented Jacobian matrix of \(X\) and \(f\) with respect to the fiber variables:

\[
\begin{pmatrix}
\partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\
\vdots & \ddots & \vdots \\
\partial f_p/\partial x_1 & \cdots & \partial f_p/\partial x_n \\
\partial f/\partial x_1 & \cdots & \partial f/\partial x_n
\end{pmatrix}.
\]

The column space of this matrix generates a module over the local ring \(O_{X,0}\), which we denote by

\[
J_{\text{rel}}(X, f) := JM_x(f_1, \ldots, f_p; f) \subset O_{X,0}^{p+1}
\]

and call the relative Jacobian module of \(X\) and \(f\). Given \(y \in Y\) form the image of the module \(JM_x(f_1, \ldots, f_p; f)\) in the free module \(O_{X_y}^{p+1}\):

\[
J(X_y, f_y) := JM_x(f_1, \ldots, f_p; f)|_{X_y} \subset O_{X_y}^{p}.
\]
Obviously, $J(X_y, f_y)$ is the augmented Jacobian module of $(X_y, f_y)$. Analogously to our discussion above we have $C(X_y, f_y) = \text{Proj}(R(J(X_y, f_y)))$. Define the relative conormal space $C_{\text{rel}}(X, f) := \text{Proj}(R(J_{\text{rel}}(X, f)))$.

Dropping the row with partials of $f$ in $J_{\text{rel}}(X, f)$ we obtain the relative Jacobian module $J_{\text{rel}}(X)$. Then the relative conormal space $C_{\text{rel}}(X) = \text{Proj}(R(J_{\text{rel}}(X)))$.

The next ingredient we need in the proof of Thm. 6.1 is the relation between $W_f$ and integral dependence of modules. Keep the setup from above. Let $F$ be a free module, $M$ be a submodule of $F$, and let $g$ be an element from $F$. Assume $X$ is reduced. Form the Rees algebra $R(M)$ of $M$. It is a subalgebra of the symmetric algebra $\text{Sym}(F)$ (see Prp. 1.8 in [EHU03]). View $g$ as a degree one element in $\text{Sym}(F)$. We say that $g$ is integrally dependent on $M$ if it satisfies an equation of integral dependence,

$$g^u + m_1 g^{u-1} + \cdots + m_u = 0$$

where $u \geq 1$ and each $m_i$ belongs to the $i$th homogeneous component of $R(M)$. The following well-known criteria tell us that we can check integral dependence on curves or by considering speeds of vanishing.

(Valuative criterion) An element $g \in F$ is integrally dependent on $M$ if for any map germ $\phi : (C, 0) \to (X, 0)$ the following holds

$$\phi^* g \in \phi^* M \subset \phi^* F.$$ 

(Analytic criterion) A necessary condition for $g \in F$ to be integrally dependent on $M$ is that for any finite set of generators $m_i$ of $M$, there exists an Euclidean neighborhood $U$ of 0 in $X$ and a constant $c$ such that

$$|g(x)| \leq c \max |m_i(x)|$$

for each $x \in U$. Conversely, it suffices that this inequality holds for a finite generating set of $M$.

Now for each $j = 1, \ldots, k$ let $g_j$ be the column vector

$$g_j := \begin{pmatrix} \frac{\partial f_1}{\partial y_j} \\ \vdots \\ \frac{\partial f_p}{\partial y_j} \\ \frac{\partial f}{\partial y_j} \end{pmatrix}.$$ 

Denote the ideal of $Y$ in $O_{X,0}$ by $m_Y$. The following result (see Thm. 2.5 in [Gaf92] and Prp. 2.3 in [GK99]) characterizes the Whitney conditions by the integral dependence of $g$ on the module $m_Y M$. It is the same result for $W_f$ was obtained by Henry, Merle and Sabbah (see Thm. 5.1 in [HMS84]).
Therefore, the proof of Theorem 5.2 is reduced to show that the closed set in $Y$ where the integral dependence of $g_j$ on $m_Y J_x(X, f)$ fails is empty. The following fundamental result is a geometric criterion for the existence of this closed set.

Let $\mathcal{M}$ and $\mathcal{N}$ be two $O_X, 0$ modules with $\mathcal{M} \neq \mathcal{N}$ such that $\mathcal{M} \subset \mathcal{N} \subset O_X^0$. Assume that $X$ is equidimensional and $\mathcal{M}$ and $\mathcal{N}$ are generically equal and free of constant rank. Denote by $c_{\mathcal{M}}: \text{Proj}(R(\mathcal{M})) \to X$ the structure morphism. Since $\mathcal{M}$ is generically free of constant rank on a dense open subset of $X$, then $\text{Proj}(R(\mathcal{M}))$ is equidimensional. Let $Z$ be the closed set where $\mathcal{N}$ is not integral over $\mathcal{M}$ and let $E := c_{\mathcal{M}}^{-1} \mathcal{Z}$. The following is a result of Kleiman and Thorup ([KT94] and [KT01]). For a generalization of this result see Thm. 6.1 in [Ran18].

**Theorem 5.3** (Kleiman–Thorup). If $\mathcal{N}$ is not integral over $\mathcal{M}$, then $E$ has codimension 1.

### 6. Numerical control of Whitney–Thom equisingularity

Preserve the setup from the previous section. In this section we apply the LVF to obtain numerical control for Whitney–Thom equisingularity (the $W_f$ and $A_f$ conditions) for $(X, 0) \to (Y, 0)$ and a function $f: X \to \mathbb{C}$. By this we mean finding invariants that depend only on the fibers $X_y, f_y$ whose constancy across $Y$ is equivalent to Whitney–Thom equisingularity.

The goal is to come up with finitely many invariants associated with each $X_y$ which control equisingularity for all families having $X_y$ as a member. When $X_y$ are isolated singularities this can be done thanks to Grauert’s theorem [Gra2] as the base space of miniversal deformations of each $X_y$ is finite. Then we can view $(X, 0) \to (Y, 0)$ as a subfamily of $(\mathcal{X}, 0) \to (W, 0)$ such that $X_0 = X_0, W$ is a component of the miniversal base space of $X_0$ and $X$ is the total deformation space of $X_0$ over $W$. In the case the volume is stable over $W$, there is a unique invariant associated with each $X_y$ that controls equisingularity for all families with irreducible bases that are subspaces of $W$. All deformations considered below are embedded deformations of $X_0 \subset \mathbb{C}^n$.

In the setup of Thm. 5.1 set $C := B_{c^{-1}(Y)} C(X, f)$ Algebraically,

$$C := \text{Proj}(R(m_Y J_{rel}(X, f)))$$

where $m_Y$ is the ideal of $Y$ in $O_X$. Let $\Sigma(f)$ be the critical set of $f$ - the union of singular sets of the various level hypersurfaces. Set $Q := X \cap f^{-1} 0$ and $Q_y := X_y \cap f^{-1} 0$.

Assume $X$ is reduced and equidimensional. Embed $J_{rel}(X, f)$ in a free module $F$ of the same rank (see Ppr. 3.3 (iii)). To align with previously used notation we denote the restricted local volume corresponding to $C_y$ by $\varepsilon(m_y J_{rel}(X, f))(y)$ where $m_y$ is the ideal of $y$ in $O_{X_y}$ and $J_{rel}(X, f)(y)$ is the image of the relative Jacobian module in $F(y)$ because of its connection to the $\varepsilon$-multiplicity (see [KUV] and [UV11]). Denote the restricted local volume corresponding to $C_{rel}(X, f)$ by $\varepsilon(J_{rel}(X, f))(y)$ and the one corresponding to $C_{rel}(X)_y$ by $\varepsilon(J_{rel}(X))(y)$. We use Thm. 4.13 and Thm. 4.14 to obtain a numerical characterization of $W_f$.

**Theorem 6.1.** Assume $X$ is reduced and equidimensional and $X_y$ is equidimensional for each $y$. Suppose $X_y$ and $Q_y$ are isolated singularities. Suppose $h: X \to Y$ is a subfamily of $X \to W$, where $W$ is a component of the miniversal base space of $X_0$ and $X$ is the total deformation space of $X_0$ over $W$. Let $f$ be a generic deformation of $f$ over $X$. Suppose that $\varepsilon(J_{rel}(X, f))(w)$ is stable. The following holds:

(i) Suppose $\Sigma(f) = Y$. Assume $\varepsilon(m_y J_{rel}(X, f))(y)$ is independent of $y$. Then the union of the singular points of $f_y$ is $Y$ and the pair $(X - Y, Y)$ satisfies $W_f$.

(ii) Suppose $\Sigma(f)$ is equal to $Y$ or is empty and the pair $(X - Y, Y)$ satisfies $W_f$. Then $\varepsilon(m_y J_{rel}(X, f))(y)$ is independent of $y$.  


Corollary 6.3. The pairs $X, Y$ are topologically equisingular. It readily yields the following corollary to Thm. 1.3, then we claim that $\varepsilon(J_{rel}(X, f)) (w) = 0$. In particular, $\varepsilon(J_{rel}(X, f)) (w)$ is stable. To see this choose a deformation $\tilde{f} := f + \sum_{i=1}^{\dim W} \alpha_i w_i$, where $w_i$ are coordinates for $(W, 0)$ and $\alpha_i$ are generic constants so that the hyperplane determined by $\grad (\tilde{f})$ at $s$ is not part of $C(X)_s$ for generic $w$ and $s \in S(W)_w$. Then by Thm. 2.2 in GR19 $C(X, \tilde{f})_s$ is the join of $df(s)$ and $C(X)_s$. But $C(X)_s$ is not of maximal dimension because $X_w$ is deficient conormal. Because $\dim C(X, \tilde{f}) = \dim C(X) + 1$, it follows that $C(X, \tilde{f})_s$ is not of maximal dimension either. Thus, $\varepsilon(J_{rel}(X, f)) (w) = 0$. The existence of Zariski open subset of $W$ where the restricted local volume vanishes follows from upper semi-continuity of fiber dimension.

Denote by $X^i$ the section of $X$ by a general linear space of codimension $i$ in $\mathbb{C}^{n+k}$ containing $Y$. Set $f^i := f | X^i$. A fundamental result of Lé and Teissier LT88 states that $W_f$ holds if and only if $X^i_y, f^i_y$ are topologically equisingular. It readily yields the following corollary to Thm. 6.1.

Corollary 6.3. The pairs $X^i_y, f^i_y$ are topologically equisingular if and only if $\varepsilon(m_y J_{rel}(X, f)) (y)$ is constant.

Next we turn our attention to providing a numerical characterization for Thom’s $A_f$ condition.

Definition 6.4. We say that $A_f$ condition holds for the pair $X_{\text{sm}}, Y$ at $0$ if $f(Y) = 0$ and $Y$ lies in every hyperplane obtained as a limit of tangent hyperplanes to a level hypersurface at a point $x \in X_{\text{sm}}$ as $x$ approaches $0$.

The $A_f$ condition is known to hold generically along $Y$ by a result of Hironaka Hir76, Thm. 2, pg. 247]. We review briefly the connection between the theory of integral closure of
modules and Thom’s $A_f$ condition. Recall that given a submodule $M$ of a free $O_{X,0}$ module $F$, we say that $u \in F$ is strictly dependent on $M$ and we write $u \in M^\dag$, if for all analytic path germs $\phi : (\mathbb{C},0) \to (X,0)$, $\phi^* u$ is contained in $\phi^*(M)m_1$, where $m_1$ is the maximal ideal of $O_{X,0}$.

Denote by $c_{(X,f)} : \text{Proj}(C(X,f)) \to X$ the structure morphism and by $C(Y)$ the conormal space of $Y$. The following result expresses the $A_f$ condition in terms of strict dependence.

**Proposition 6.5.** Assume $f(Y) = 0$. The following are equivalent

(i) The $A_f$ condition holds for the pair $X_{\text{sm}}, Y$ at $0$.
(ii) $c_{(X,f)}(Y) \subset C(Y)$.
(iii) $g_j \in J(X,f)^\dag$ for all $j = 1, \ldots, k$.

**Proof.** The equivalence of i) and ii) is obvious; the equivalence of i) and iii) is Lemma 5.1 of [GK99].

A similar result holds for the Whitney $A$ condition. The condition we need for our main result is a much weaker version of Whitney $A$.

**Definition 6.6.** We say that $(X,0) \to (Y,0)$ satisfies the infinitesimal Whitney $A$ fiber condition at $0$ if the image of each $g_j$ in $F(y)$ is in $J(X_0)^\dag$ for $j = 1, \ldots, k$, where of $J(X_0)$ is the Jacobian module of $X_0$.

**Theorem 6.7.** Assume $X$ is reduced and equidimensional and $X_y$ is equidimensional for each $y$. Suppose $X_y$ and $Q_y$ are isolated singularities. Suppose $h : X \to Y$ is a subfamily of $\mathcal{X} \to W$, where $W$ is a component of the universal base space of $X_0$ and $X$ is the total deformation space of $X_0$ over $W$. Let $\tilde{f}$ be a generic deformation of $f$ over $\mathcal{X}$. Suppose that $\varepsilon(J_{\text{rel}}(X, \tilde{f}))(u)$ is stable and that the infinitesimal fiber condition holds in case $f \notin m_Y^2$. The following holds:

(i) Suppose $\Sigma(f) = Y$. Suppose $\varepsilon(J_{\text{rel}}(X,f))(y)$ is independent of $y$. Then the union of the singular points of $f_y$ is $Y$ and the pair $(X - Y, Y)$ satisfies $A_f$.
(ii) Suppose $\Sigma(f)$ is empty and the pair $(X - Y, Y)$ satisfies $A_f$. Then $\varepsilon(J_{\text{rel}}(X,f))(y)$ is independent of $y$.

**Proof.** Let $K$ be the submodule of $F$ generated by $g_1, \ldots, g_k$. Then $K + J_{\text{rel}}(X,f) = J(X,f)$. By Prp. 6.5 if $A_f$ holds, then

$$J(X,f) \circ \phi = K \circ \phi + J_{\text{rel}}(X,f) \circ \phi \subset m_1(J(X,f) \circ \phi) + J_{\text{rel}}(X,f) \circ \phi$$

for any analytic path $\phi : (\mathbb{C},0) \to (X,0)$. By Nakayama’s lemma we get $J(X,f) \circ \phi = J_{\text{rel}}(X,f) \circ \phi$. By the valuative criterion for integral dependence we get that $J(X,f) \subset J_{\text{rel}}(X,f)$, where $J_{\text{rel}}(X,f)$ is the integral closure of $J_{\text{rel}}(X,f)$ in $F$, if $A_f$ holds. In other words, if $A_f$ holds then exists a finite map $C(X,f) \to C_{\text{rel}}(X,f)$.

Consider (i). By Hironaka’s genericity result [Hir76] there exists a Zariski open subset $U$ of $Y$ such that $A_f$ condition holds at all points in $U$. Then by selecting analytic paths in $U$ and the discussion above we obtain that $J(X,f)$ and $J_{\text{rel}}(X,f)$ have the same integral closure over $U$. Because $\varepsilon(J_{\text{rel}}(X,f))(y)$ is constant on $y$ by Thm. 4.13 applied with $C(W) := C(X,f)$ and $S(W)$ the singular locus of $X$, and $g := g_j$ for $1 \leq j \leq k$, it follows that each $g_j$ is integral over $J_{\text{rel}}(X,f)$. Thus, $J(X,f) \subset J_{\text{rel}}(X,f)$ and so there exists a finite map $C(X,f) \to C_{\text{rel}}(X,f)$. Because $J_{\text{rel}}(X,f)$ is generated by $n$ elements dim $C_{\text{rel}}(X,f)_0 < n$. Thus dim $C(X,f)_0 < n$. Then by Thm. 4.4 in [Gr19] it follows that the pair $(X - Y, Y)$ satisfies $A_f$. As in the proof of Thm. 6.1 we obtain that union of singular points of $f_y$ is $Y$. 


Consider (ii). By Prp. 6.5 \( c^{-1}(X,f)(Y) \subset C(Y) \). But \( \dim C(Y)_0 = n - 1 \). Also, by the discussion above \( C(X,f) \to C_{\text{rel}}(X,f) \) is finite. Thus \( \dim C_{\text{rel}}(X,f)_0 < n \), so the LVF yields the constancy of \( \varepsilon(J_{\text{rel}}(X,f))(y) \).

7. DEFICIENT CONORMAL SINGULARITIES AND GENERALIZED SMOOTHABILITY

In this section we define the class of deficient conormal (dc) singularities. We show that the dc property is intrinsic and stable under infinitesimal deformations. We show that the fibers of conormal spaces behave well under pullbacks of transverse holomorphic maps between affine spaces. Then using Thom’s transversality we show that determinantal and Pfaffian singularities deform to dc singularities. As a corollary we obtain that the generic deformations of Cohen–Macaulay codimension 2 and Gorenstein codimension 3 singularities are dc. Inspired by a recent result of Kollár and Kovács and following work of Schlessinger, we show that the affine cones over normally embedded abelian varieties of dimension at least 2 do not admit infinitesimal deformations to dc singularities. We finish the section with computing the restricted local volume associated with the conormal space of a nonsmoothable Cohen–Macaulay codimension 2 singularity.

Let \( Z \subset \mathbb{C}^N \) be a reduced complex analytic variety. As before define the conormal space \( C(Z) \) of \( Z \) as the closure in \( Z \times \mathbb{P}^{N-1} \) of the set of pairs \((z,H)\) where \( z \) is a smooth point of \( Z \) and \( H \) is a tangent hyperplane at \( z \), i.e. a hyperplane containing \( T_{Z,z} \). Denote by \( c \) the structure map \( c : C(Z) \to Z \).

Definition 7.1. We say \( Z \) is deficient conormal (dc) at \( z_0 \in Z \) if
\[
\text{codim}(c^{-1}z_0, C(Z)) \geq 2.
\]

We say \( Z \) is dc if \( Z \) is dc at each of its points.

Example 7.2. If \( z_0 \) is a smooth point of \( Z \) with \( Z \) equidimensional and \( \dim Z \geq 2 \), then \( Z \) is dc at \( z_0 \). If \( Z \) is local complete intersection and dc at \( z_0 \), then by Lem. 5.7 in [GaM99] \( Z \) is smooth at \( z_0 \).

If \( Z \) is the affine cone over a smooth projective variety \( V \subset \mathbb{P}^{N-1} \) with positive defect (or degenerate dual) (see [Ein86] and [Ein85] for examples and classifications), then \( Z \) is dc at the vertex of the cone. To see this it’s enough to show that the fiber of the conormal of \( Z \) over the vertex is equal set-theoretically to the dual of the base (not requiring \( V \) to be smooth). An easy computation shows that \( H \) is a tangent hyperplane at a point \( v \) in \( V_\text{sm} \) if and only if \( H \) is tangent to each point of \( Z \) lying on the line \( 0v \) and different from the origin. If \( z_1, z_2, \ldots \) are points from \( Z \) converging to the origin in \( \mathbb{C}^{N+1} \), and \( H_1, H_2, \ldots \) are the corresponding tangent hyperplanes, then \( H_i \) are tangent hyperplanes at points of \( V \), and thus their limit belongs to the dual of \( V \). Conversely, if \( H \) is in the dual of \( V \), say \( H \) is a limit of tangent hyperplanes \( H_1, H_2, \ldots \) at smooth points \( v_1, v_2, \ldots \) then \( H \) is a limit of tangent hyperplanes at any sequence of points from the lines \( 0v_1, 0v_2, \ldots \), and it belongs to the fiber \( C(Z) \) over the origin.

It can be shown that many of the examples of rigid singularities considered by Schlessinger [Sch73] like fans (eg. two planes meeting at a point in \( \mathbb{C}^4 \)), quotient singularities of dimension at least 2, etc. are dc.

Set \( \mathcal{L} := \mathcal{O}_C(1) \). The next result shows that the dc property is characterized by the vanishing of the local volume of \( \mathcal{L} \). Recall that the local volume \( \text{vol}_C(\mathcal{L}) \) at \( z \) is given by the
epsilon multiplicity of the Jacobian module $J_z(Z)$ of $Z$ at $z$:
\[
\varepsilon(J_z(Z)) := \limsup_{n \to \infty} \frac{r!}{n!} \dim H^0_{\mathfrak{m}_z}(\mathcal{F}^n/J(Z)^n)
\]
where $\mathcal{F}$ is a free module containing $J(Z)$ having the same generic rank, $\mathcal{F}^n = \text{Sym}^n(\mathcal{F})$ and $J(Z) := \text{Sym}^n(J(Z))/(O_Z$-torsion).

**Proposition 7.3.** Suppose $Z$ is equidimensional with $\dim Z \geq 2$. Then $\varepsilon(J_z(Z)) = 0$ if and only if $Z$ is dc at $z$.

**Proof.** Follows immediately from Thm. 3.4.

Our next result shows that the dc property is intrinsic for $X$. Suppose we have two reduced complex analytic equidimensional varieties $Z_1 \subset \mathbb{C}^{n_1}$ and $Z_2 \subset \mathbb{C}^{n_2}$. Denote the two conormal spaces corresponding to each of the theses germs by $C(Z_1)$ and $C(Z_2)$. Denote by $c_i : C(Z_i) \to Z_i$ for $i = 1, 2$ the corresponding structure morphisms.

**Proposition 7.4.** Suppose that there exists an analytic isomorphism $\phi : Z_1 \to Z_2$ such that $\phi(z_1) = z_2$. Then $\text{codim} c_1^{-1}(z_1) = l$ in $C(Z_1)$ for some positive integer $l$ if and only if $\text{codim} c_2^{-1}(z_2) = l$ in $C(Z_2)$. In particular, $Z_1$ is dc at $z_1$ if and only if $Z_2$ is dc at $z_2$.

**Proof.** Let $Z \subset \mathbb{C}^N$ be a reduced equidimensional complex analytic variety. Let $z \in Z$. Consider the germ $(Z, z)$. Let’s recall the construction of the local polar varieties of $(Z, z)$ as developed by Teissier in Chapter IV of [T81]. Consider the following diagram

\[
\begin{array}{ccc}
C(Z) & \xrightarrow{i} & \mathbb{P}^{N-1} \\
\downarrow c & & \downarrow \lambda \\
Z & \xrightarrow{pr_2} & \mathbb{P}^{N-1}
\end{array}
\]

Assume $(Z, z)$ is a germ of pure dimension $d$ and of codimension $e$ in $\mathbb{C}^N$. Let $H_{e+l-1}$ be a general plane in $\mathbb{P}^{N-1}$ of codimension $e + l - 1$. Define the local polar variety $\Gamma_l(Z, H_{e+l-1})$ with respect to $H_{e+l-1}$ as $c(\lambda^{-1}(H_{e+l-1}))$. Then $\Gamma_l(Z, H_{e+l-1})$ is either empty or of pure codimension $l$. Teissier showed (see Thm. 3.1 in Chapter IV of [T81]) that for sufficiently general $H_{e+l-1}$ the multiplicity of $\Gamma_l(Z, H_{e+l-1})$ at $z$ depends only on the analytic type of $(Z, z)$. Often we suppress $H_{e+l-1}$ from the notation and we simply write $\Gamma_l$ because in the applications we will be interested mostly in the existence of the polar varieties of appropriate codimension. Finally, observe that $\Gamma_{d-l+1}(Z)$ is empty if and only if $\text{codim} c^{-1}(z) \geq l$ in $C(Z)$.

Apply Teissier’s result to $Z_1$ and $Z_2$. Because $\text{codim} c_1^{-1}(z_1) = l$ in $C(Z_1)$, then the multiplicity at $z_1$ of polar varieties of $(Z_1, z_1)$ of dimension $d - l + 1$ is zero. But then, by Teissier’s result, so is the multiplicity at $z_2$ of the polar varieties of $(Z_2, z_2)$ of dimension $d - l + 1$. Hence $\text{codim} c_2^{-1}(z_2) = l$ in $C_2(Z_2)$.

Next we show that the dc property is stable under infinitesimal deformations.

**Proposition 7.5.** Suppose $X \to Y$ is family with equidimensional fibers and equidimensional reduced total space $X$. Assume that $Y$ is smooth of dimension one and $X_{y_0}$ has an isolated dc singularity at a point $x_0$. Then $X_y$ is a dc singularity for each $y$ closed enough to $y_0$.

**Proof.** Let $J_{\text{rel}}(X)$ be the relative Jacobian module of $X \to Y$, and let $\mathcal{F}$ be a free module that contains it. Set $\mathcal{F}(y) := \mathcal{F} \otimes_{O_Y} k(y)$ for each point $y \in Y$. The image of $J_{\text{rel}}(X)$ in $\mathcal{F}(y)$
is the Jacobian module \( J(X_y) \) of \( X_y \). Let \( S \) be the singular locus of \( X \). By replacing \( X \) with a small enough neighborhood of \( x_0 \) we can assume that \( S \) is finite over \( Y \). Then

\[
H_S(\mathcal{F}^n/J_{\text{rel}}(X)^n) \otimes_{\mathcal{O}_Y} k(y_0) \hookrightarrow H_{S_{y_0}}(\mathcal{F}(y_0)^n/J(X_{y_0})^n).
\]

Thus \( \varepsilon(J_{\text{rel}}(X))(y_0) \leq \varepsilon(J(X_{y_0})) \). Because \( X_{y_0} \) is dc at each point of \( S_{y_0} \), then \( \varepsilon(J(X_{y_0})) = 0 \) and so \( \varepsilon(J_{\text{rel}}(X))(y_0) = 0 \). Applying the LVF and noting that the intersection number on its right-hand side is nonnegative, we obtain \( \varepsilon(J_{\text{rel}}(X))(y) = 0 \). By Prp. 4.7 \( \varepsilon(J_{\text{rel}}(X))(y) = \varepsilon(J(y)) \) for \( y \) close enough to \( y_0 \). Thus \( \varepsilon(J(y)) = 0 \). So by Thm. 3.4 it follows that \( X_y \) is a dc singularity. \( \square \)

The next three results are the key ingredients in the proof of the main theorem of this section. The first one shows that the fibers of conormal spaces are set-theoretically functorial under transverse maps. The second result due to Trivedi allows us to deform holomorphic maps in such a way so that the deformations become transverse to a predetermined collection of submanifolds in the target space. The third result due to Buchweitz allows us to obtain deformations of singularities obtained from pullbacks of holomorphic maps.

**Definition 7.6.** Let \( M \) and \( N \) be complex manifolds. Let \( g : M \rightarrow N \) be a smooth map. We say that \( g \) is transverse to a submanifold \( S \) of \( N \) at a point \( m \in M \) and we denote it by \( g \pitchfork_m S \), if either \( g(m) \notin S \) or \( g(m) \in S \) and \( Dg_m(T_m M) + T_{g(m)} S = T_{g(m)} N \). If \( V = \bigsqcup_{i=1}^q V_i \) is a stratification of \( V \subset N \) then by \( g \pitchfork_K V \) we mean that \( g \) is transverse to each stratum \( V_i \) at all points in \( K \subset M \).

Suppose \( (V, 0) \subset (\mathbb{C}^m, 0) \) is a complex analytic subvariety. We say that \( V = \bigsqcup_{i=1}^q V_i \) is Whitney A (respectively Whitney B) stratification if pairs of nearby strata satisfy Whitney condition A (respectively Whitney B). The existence of these stratifications was established by Whitney [Wh64] as discussed in Sct. 5. Recall that a Whitney B stratification is a Whitney A stratification.

**Theorem 7.7.** Let \( g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^m, 0) \) be a holomorphic map. Let \( (X, 0) \subset (\mathbb{C}^n, 0) \) and \( (V, 0) \subset (\mathbb{C}^m, 0) \) be equidimensional reduced complex analytic varieties with \( g^{-1} V = X \) and \( \text{codim} X = \text{codim} V \). Assume \( g^{-1} V' \) is an irreducible component of \( X \) for each irreducible component \( V' \) of \( V \). Assume that \( g \pitchfork_0 V \) where \( V \) is given a Whitney B stratification. Then set-theoretically

\[
C(X)_x \simeq C(V)_0
\]

for each \( x \in g^{-1}(0) \) where the isomorphism is provided by the dual of the differential \( dg \).

**Proof.** Let \( (V, 0) = \bigsqcup_{i=1}^q V_i \) be a Whitney B stratification of \( (V, 0) \). Suppose \( V_i \) is the smooth locus of \( V \) and \( 0 \in V_2 \). First we show that there exists a neighborhood \( U \) of \( 0 \) in \( (\mathbb{C}^m, 0) \) such that \( g \) is transverse to each point in \( U \cap V_i \) for \( i = 1, \ldots, q \). Note that if \( f \) is transverse at \( U \cap V_2 \) for sufficiently small \( U \) by openness of transversality. We will show that \( g \) is transverse to \( U \cap V_1 \) for \( U \) sufficiently small (the proof for the rest of the strata is the same). Suppose \( x_1, x_2, \ldots \) is a sequence of points in \( X \) converging to \( x \) with \( x \in g^{-1}(0) \) such that \( g \) fails to be transverse at \( g(x_1), g(x_2), \ldots \). Then \( Dg_{T_{x_j}} \mathbb{C}^n + T_{g(x_j)} V_1 \neq T_{g(x_j)} \mathbb{C}^m \) for \( j = 1, 2, \ldots \). Assume that as \( x_j \rightarrow x \) the linear spaces \( T_{g(x_j)} V_1 \rightarrow T \). Because \( V_1, V_2 \) satisfies Whitney condition A at 0 it follows that \( T_0 V_2 \subset T \). This would imply that \( Dg_{T_{x_j}} \mathbb{C}^n + T_0 V_2 \neq T_0 \mathbb{C}^m \) which contradicts our assumption \( g \pitchfork_0 V_1 \). Replace \( V_i \) by \( U \cap V_i \) for each \( i \) where \( U \) is a sufficiently small neighborhood of the origin in \( \mathbb{C}^m \). Because \( g \) is transverse to \( V_i \), then \( \bigsqcup_{i=1}^q g^{-1}(V_i) \) is a Whitney B stratification of \( V \) (see [Sc03, pg. 257]).
Set $X_1 := g^{-1}V_1$ and $X_2 := g^{-1}V_2$. Let $T(X_2)$ be a transversal of $X_2$ through $x$. Because $g$ is transverse at $V_2$, it follows from the local diffeomorphism theorem that after $V_2$ is replaced by a small enough neighborhood around 0 that $T(V_2) := g(T(X_2))$ is transversal of $V_2$ through 0 and it is diffeomorphic to $T(X_2)$. Let $X_x$ the fiber over $x$ of the transverse retraction from $X$ to $X_2$ induced by $T(X_2)$. Similarly, denote by $V_0$ the fiber over 0 of a transverse retraction from $V$ to $V_0$ induced by $T(V_2)$. Because $X_1, X_2$ satisfies Whitney B at the origin by Thm. 3.1 in [GR19] set-theoretically, $C(X)_x = C(X_x)_x$ where the conormal spaces are taken in $\mathbb{C}^n$. Similarly, $C(V)_0 = C(V_0)_0$. By hypothesis we can arrange $T(V_2)$ such that the set in $V_0$ where the preimage of $g$ is empty does not contain a component of $V_0$. Thus $g((X_0)_{sm})$ is dense in $V_0$. Because $g$ is a diffeomorphism between $T(X_2)$ and $T(V_2)$ it follows that $g$ is a bijection between $(X_0)_{sm}$ and $g((X_0)_{sm})$, and the last set is open. We will show that $C(X_x)_x$ and $C(V)_0$ are isomorphic. Consider the diagram where the right square is cartesian:

$$
\begin{array}{ccc}
T^*\mathbb{C}^n & \xrightarrow{(dg)^*} & g^*T^*\mathbb{C}^m \\
\downarrow \pi_n & & \downarrow \pi \\
\mathbb{C}^n & \xrightarrow{\text{Id}} & \mathbb{C}^n \\
\end{array}
$$

Set $\Lambda := g^{-1}(C(V_0))$. Let $\omega_n$ and $\omega_m$ be the Liouville 1-forms on $T^*\mathbb{C}^n$ and $T^*\mathbb{C}^m$ respectively. Because the pullbacks of these forms to $g^*T^*\mathbb{C}^m$ are the same, then $\omega_n$ vanishes on all smooth points of $(dg)^*(\Lambda)$. By Lem. 4.3.1 $(dg)^*(\Lambda)$ is closed. By Prp. 2.2 in [TF18] $(dg)^*(\Lambda)$ equals the conormal of its image under $\pi_n$, i.e. $(dg)^*(\Lambda) = C(X_x)$. Thus $C(V)_0$ surjects onto $C(X_x)_x$. By transversality the restriction of $(dg)^*$ to $g^{-1}_n(T_{V_2}\mathbb{C}^m)$ is injective for each $i = 1, \ldots, q$ (see [Sc03] pg. 255). Because the fiber over $x$ of $\Lambda$ is contained in $g^{-1}_n(T_{V_2}\mathbb{C}^m)$ it follows that $(dg)^* : C(V)_0 \simeq C(X_x)_x$.

Denote by $\mathcal{H}(M, N)$ the complete metric space of holomorphic maps between two complex manifold $M$ and $N$ with the weak topology induced by the weak topology of $C^\infty(M, N)$. The following result due to Trivedi is a generalization of Thom’s classical transversality result [Tim69] to the complex analytic setting.

**Theorem 7.8** (Trivedi [Ti13], Thm. 2.1 and Thm. 3.1). Let $M$ be a Stein manifold and $N$ be an Oka manifold. Let $V$ be a complex analytic variety in $N$ and let $V = \bigcup_{i=1}^k V_i$ be a Whitney A stratification. Then for any compact subset $K$ in $M$, the set of maps $\{ g \in \mathcal{H}(M, N) : g \pitchfork_K V_i \}$ is open and dense in $\mathcal{H}(M, N)$.

Next we record a key algebraic result due to Buchweitz which allows to obtain flat deformations of varieties obtained from pullbacks of holomorphic maps between complex affine spaces by deforming the components of these maps.

**Theorem 7.9** (Buchweitz, 4.3.4 in [Bu81]). Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a holomorphic map. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ and $(V, 0) \subset (\mathbb{C}^m, 0)$ be complex analytic varieties with $\text{codim} X = \text{codim} V$ and $g^{-1}V = X$. Assume $(V, 0)$ is Cohen–Macaulay. Then for every unfolding $\tilde{F} : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^m, 0)$ the family $F^{-1}(V) \to (\mathbb{C}^k, 0)$ is a flat deformation of $X$.

Consider the map $F : (\mathbb{C}^n, 0) \to \text{Hom}(\mathbb{C}^l, \mathbb{C}^{l+s})$ given by $l+s$ by $l$ matrix $M_X$ with entries complex analytic functions on a neighborhood of the origin in $\mathbb{C}^n$. We say $X$ is **determinantal of type** $(l+s, l, u)$ if $X$ consists of the points for which $\text{rk}(M_X) < u$ and $X$ is of codimension $(l+s-u+1)(l-u+1)$ in $\mathbb{C}^n$. Denote by $\Sigma^u$ the closed subset of $\text{Hom}(\mathbb{C}^l, \mathbb{C}^{l+s})$ consisting of linear maps of rank less than $i$. Then $X = F^{-1}\Sigma^u$. 

**LOCAL VOLUMES, EQUISINGULARITY, AND GENERALIZED SMOOTHABILITY**
An isolated determinantal singularity $X$ of type $(l + s, l, u)$ is smoothable if $\dim X \leq 2l + s - 2u + 3$ [Wah81, Thm. 6.2]. By the Hilbert–Burch theorem [E95, Sct. 20.4] Cohen–Macaulay codimension 2 varieties are determinantal of type $(l, l + 1, l)$. In particular, they are smoothable up to dimension 3 ([Sc77], cf. pg. 19-20 in [Art76]) and nonsmoothable in dimension 4 and higher unless they are complete intersections (M. Zach, priv. comm., 2019).

We say $X$ is Pfaffian if it’s defined by the $2u \times 2u$ Pfaffians of a skew-symmetric matrix. Isolated Pfaffian singularities of a $(2l + 1) \times (2l + 1)$ skew-symmetric matrix are smoothable if $\dim X \leq 4(l - u) + 6$ [Wah81] Thm. 6.3. Gorenstein codimension 3 singularities are Pfaffian with $u = l$ by the structure theorem of Buchsbaum and Eisenbud [BE77] Thm. 2.1.

The following theorem, which combines theorems 7.7, 7.8 and 7.9 is the main result of this section.

**Theorem 7.10.** Let $F : (C^n, 0) \to (C^m, 0)$ be a holomorphic map. Let $(X, 0) \subset (C^n, 0)$ and $(V, 0) \subset (C^m, 0)$ be reduced complex analytic varieties with $\text{codim} X = \text{codim} V$ and $g^{-1}V = X$. Assume $(V, 0)$ is Cohen–Macaulay dc singularity. Then there exists an embedded deformation $\mathcal{X} \to W$ of $X$ such that $X_w$ is dc for generic $w$. In particular, determinantal, with the exception of the case $u = 2$ and $s = 0$, and Pfaffian singularities deform to dc singularities.

**Proof.** Let $V = \bigcup_{i=1}^n V_i$ be a Whitney B stratification. Deforming the entries of $F$ by generic linear forms on $(C^n, 0)$ by Thm. 7.9 we obtain an unfolding $\tilde{F}$ of $F$ such that $\tilde{F}^{-1}V \to (C^m, 0)$ is a flat deformation of $X$. Set $\mathcal{X} := \tilde{F}^{-1}V$ and $W := (C^m, 0)$. By Thm. 7.8 $\tilde{F}_w \circ_0 V$ for generic $w$ as complex affine spaces are Oka and Stein. By Thm. 7.7 $X_w$ is dc singularity.

Each determinantal or Pfaffian variety $X$ is obtained as $F^{-1}V$ where $V$ is the generic determinantal or Pfaffian singularity of appropriate sizes. Note that $V$ is irreducible and Cohen–Macaulay by the result of Eagon and Hochster [EH71] in the determinantal case, and by a result of Kleppe and Laksov [KL80] in the Pfaffian case. In the determinantal case $\text{codim}(C(V)_0, C(V)) = (u - 1)(s + u - 1) - \text{Prp. 2.6 in [GR16]}$ which is greater or equal to 2 unless $u = 2$ and $s = 0$.

Suppose $V$ is the generic Pfaffian variety of $(2l + 1) \times (2l + 1)$ skew-symmetric matrices given by the $2u \times 2u$ Pfaffians (the case of $2l \times 2l$ skew-symmetric matrices is analogous). Then $\text{codim}(C(V)_0, C(V)) = (2l + 1)(2l)/2 - (u - 1)(2l(2l + 2(2l - u + 1))$ by Thm. 5.9 in [LS17].

Because $u$ is at most $l$, then $2l + 1 > \sqrt{4l^2 + 1}/2 + 2(u - 1)$. Thus $\text{codim}(C(V)_0, C(V)) \geq 2$. □

Suppose $X$ is isolated determinantal singularity with $u = 2$ and $s = 0$. Then $n \leq l^2$. If $n < l^2$, then $\tilde{F}_w$ will miss the singular locus of $V$ which is the origin. Thus by the implicit function theorem $X_w := \tilde{F}_w^{-1}V$ is smooth. Suppose $n = l^2$ and $X = V$. Then $X$ is the affine cone over the Segre embedding of $\mathbb{P}^{l-1} \times \mathbb{P}^{l-1}$ in $\mathbb{P}^{l^2 - 1}$. By Prp. 2.6 in [GR16] $C(X)_0$ is the hypersurface in $\mathbb{P}^{l^2 - 1}$ cut out by the determinant of the generic $l \times l$ matrix. All the determinantal deformations of $X$ are isomorphic to $X$; hence not dc.

An immediate consequence of the structure theorems of Hilbert–Burch and Buchsbaum–Eisenbud is that their versal deformation spaces are smooth (see pg. 67–68 and pg. 76 in [Har10]). Thus we have following corollary.

**Corollary 7.11.** Suppose $X$ is Cohen–Macaulay codimension 2 or Gorenstein codimension 3 singularity. Then the generic deformation of $X$ is dc singularity.

Next we give an example of isolated singularities that do not deform to dc singularities.

**Example 7.12.** Let $Z$ be abelian variety and $\mathcal{L}$ an ample line bundle on $Z$. Then $\mathcal{L}^3$ gives a projectively normal embedding $Z \hookrightarrow \mathbb{P}^{N - 1} = \mathbb{P} H^0(\mathcal{L})$ where $N = \dim H^0(Z, \mathcal{L})$(see [Koi76]).
Denote by $\Theta$ the tangent sheaf of $Z$. Then $H^1(\Theta(i)) = 0$ for each $i$ because the tangent bundle of any group variety is trivial. Also, $H^1(O_Z(i)) = 0$ for each $i$ by the vanishing theorem in [Mum70, pg.150]. Thus $Z$ is strongly rigid [Sch73, pg.155], which by [Sch73, Thm. 2, pg.159] implies that the versal deformation space of $C_Z$ is isomorphic to the versal deformation space of $Z$ in $\mathbb{P}^{N-1}$; every deformation of $C_Z$ is conical. In particular, $C_Z$ is not smoothable (this was proved without the hypothesis of projective normality by Kollár and Kovács in [KK18]).

Let $C_Z \to Y$ be a deformation of $C_Z$ with $(C_Z)_{y_0} = C_Z$ induced by a deformation $Z \to Y$ of $Z$ with $Z \subset Y \times \mathbb{P}^{N-1}$ and $Z_{y_0} = Z$ for some closed point $y_0 \in Y$. To show that the fiber of the conormal over the vertex of the cone ($C_Z)_y = C_Z_y$ is of maximal dimension for each $y$, by the correspondence established in Ex. 7.2, it is enough to show that the dual of $Z_y$ is a hypersurface in the dual of $\mathbb{P}^{N-1}$, i.e. $\text{def}(Z_y) = 0$. Suppose $\text{def}(Z) = r \geq 1$. Then by Thm. 1.8 (i) in [Tev05] $Z$ is ruled by projective subspaces of dimension $r$. But that’s impossible because any morphism from $\mathbb{P}^1$ to a group variety is constant. Thus $\text{def}(Z) = 0$. We claim that $\text{def}(Z_y) = 0$ for each $y$ close enough to $y_0$.

Let $e$ be the identity element in $Z$. Because $Z$ is smooth, after $Y$ is replaced by sufficiently small neighborhood of $y_0$, then there exists $Y' \subset Z$ passing through $e$ such that $r: Y' \to Y$ is étale. Consider the family $Z \times Y Y' \to Y'$ with the section $y' \to (y', y')$ for $y' \in Y'$. Then for each $y'$ the fiber over $y'$ is isomorphic $Z_y(y')$. By Thm. 6.14 in [MFS2] $Z \times Y Y'$ is abelian; hence, there are no projective spaces contained in it and thus $Z_y$ is not ruled. Therefore, $C_Z_y$ is not a dc singularity.

As mentioned before all Cohen–Macaulay codimension 2 singularities of dimension at most 3 are smoothable. The result is sharp because the cone in $\mathbb{C}^6$ over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$, which is a Cohen–Macaulay codimension 2 singularity, is rigid and hence not smoothable (this is the first example of a nonsmoothable singularity due to Thom). In fact, in dimension 4 and higher Cohen–Macaulay codimension 2 singularities are nonsmoothable unless they are complete intersections. Next we consider a class of 4-dimensional isolated nonsmoothable Cohen–Macaulay codimension 2 singularities suggested to me by T. Gaffney. The generic deformations of this isolated singularity are singular but dc by Cor. 7.11. For this class of singularities the restricted local volume associated with the the relative conormal spaces takes particularly nice form - it’s a sum of a Buchsbaum–Rim multiplicity and a polar multiplicity.

**Example 7.13.** To each polynomial $h(w, x, y)$ which defines an isolated singularity in $(\mathbb{C}^3, 0)$ associate the Cohen–Macaulay codimension 2 singularity $X_h$ in $(\mathbb{C}^6, 0)$ defined by the vanishing of the 2 by 2 minors of the following matrix

$$
\begin{pmatrix}
  u & x \\
  v & y \\
  h(w, x, y) & z
\end{pmatrix}
$$

(44)

where we view the presentation matrix above as a map $F_h : (\mathbb{C}^6, 0) \to \text{Hom}(\mathbb{C}^2, \mathbb{C}^3)$. Observe that $X_h$ is not smoothable (see p. 19–20 in [Art76]). Note that $X_h = F_h^{-1}(\Sigma^2)$. A one-parameter deformation $\mathcal{X}_h$ of $X_h$ with fibers $\mathcal{X}_h(t)$ is obtained by perturbing the entries of the presentation matrix (44). Then by Cor. 7.11 the generic fiber $\mathcal{X}_h(t)$ is a dc singularity.

Apply Gaffney’s Cor. 2.3 to the pair of modules: the relative Jacobian module $J_{rel}(\mathcal{X}_h)$, and the normal module of $\mathcal{X}_h$ which in this case is $F_h' (J(\Sigma^2))$, the pullback of the Jacobian module of $\Sigma^2$ (see Prp. 2.11 in [GRu16] and the discussion preceeding Prp. 2.4 in [GR16]). Observe that $J_{rel}(\mathcal{X}_h)$ specializes to the Jacobian module $J(\mathcal{X}_h(t))$ of each fiber $\mathcal{X}_h(t)$. Because $\mathcal{X}_h(t)$
is a dc singularity for generic $t$, the module $F^*_h(J(\Sigma^2))$ is integrally dependent on $J(X_h(t))$. So, for generic $t$ the Buchsbaum–Rim multiplicity vanishes:

$$e(J(X_h(t), F^*_h(J(\Sigma^2))) = 0.$$ 

For dimension reasons the codimension 4 polar variety $\Gamma_4(\Sigma^2)$ is empty (see the discussion that follows Prp. 2.14 in \cite{GR16}). Hence, by Thm. 2.5 in \cite{GR16}, $\Gamma_4(F^*_h(J(\Sigma^2)))$ is empty. Thus the MPT yields $e(J(X_h), F^*_h(J(\Sigma^2))) = m_4(X_h)$ where $m_4(X_h)$ is the degree of the polar curve of $X_h$. Applying the LVF with the observation that the generic term on the left-hand side vanishes by Prp. 7.3, we get

$$\varepsilon(J_{rel}(X_h))(0) = e(J(X_h), F^*_h(J(\Sigma^2))).$$

Thus we reduced the problem of computing the restricted local volume to computing a relative Buchsbaum–Rim multiplicity, which as defined in Sect. 3 can be computed as a sum of intersection numbers of the blowup of $\text{Proj}(\mathcal{R}(F^*_h(J(\Sigma^2))))$ with center the ideal generated by $J(X_h)$. First we need to find $\text{Proj}(\mathcal{R}(F^*_h(J(\Sigma^2))))$. Recall that

$$B_{X_h} := \{(x, t_1, t_2) | x \in X_h, t_1 \in \mathbb{P}(\text{Ker}(M_{X_h}(x))), t_2 \in \mathbb{P}(\text{Ker}(M'_{X_h}(x)))\}$$

which sits inside $X_h \times \mathbb{P}^1 \times \mathbb{P}^2$ is set-theoretically equal to $\text{Proj}(\mathcal{R}(F^*_h(J(\Sigma^2))))$ by Thm. 3.7 in \cite{GR16}. The morphism between $X_h \times \mathbb{P}^1 \times \mathbb{P}^2$ and $X_h \times \mathbb{P}(\text{Hom}(\mathbb{C}^2, \mathbb{C}^3))$ is given by

$$(x, [S_1, S_2], [T_1, T_2, T_3]) \mapsto \left(x, \begin{bmatrix} S_1T_1 & S_2T_1 \\ S_1T_2 & S_2T_2 \\ S_1T_3 & S_2T_3 \end{bmatrix} \right).$$

An easy computation shows that $B_{X_h}$ is cut out locally at the chart $[1, s], [1, t_1, t_2]$ from $\mathbb{C}[x, y, w, s, t_1, t_2]$ by $u + sx = 0, v + sy = 0, f + sz = 0, t_1x + t_2y + z = 0$. Hence $B_{X_h}$ is a complete intersection. Then $\text{Proj}(\mathcal{R}(F^*_h(J(\Sigma^2))))$ set-theoretically is a complete intersection. But it is generically reduced because $X_h$ is reduced. So, $\text{Proj}(\mathcal{R}(F^*_h(J(\Sigma^2))))$ is reduced. Thus

$$\text{Proj}(\mathcal{R}(F^*_h(J(\Sigma^2)))) \simeq \mathbb{C}(x, y, w)[s, t_1, t_2]/(f - s(t_1x + t_2y)),$$

where $\mathbb{C}(x, y, w)$ is localized at the origin. Our next task is to compute the ideal induced by $J(X_h)$ in $\mathcal{R}(F^*_h(J(\Sigma^2)))$. Set

$$G : \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \rightarrow \mathbb{C}$$

such that $G^{-1}(0) = \Sigma^2$. From the chain rule $D(G \circ F) = (D(G) \circ F) \circ D(F)$ it follows that the ideal $I_j$ generated by $J(X_h(0))$ in $\mathcal{R}(F^*_h(J(\Sigma^2)))$ is generated by $D(F)$. An easy computation shows that the generators for $I_j$ are $t_1, t_2, s_1 + x, s_2 + y, s, f_w$. Thus by \cite{GR16}, $\forall I_j = \text{Spec}(\mathbb{C}[x, y, w]/(f, J(f)))$ where $J(f)$ is the Jacobian ideal of $f$. Therefore, the computation of the Buchsbaum–Rim multiplicity reduces to computing the Hilbert–Samuel multiplicity of $I_j$ in $\mathbb{C}(x, y, w)[s, t_1, t_2]/(f - s(t_1x + t_2y))$. The latter ring is Cohen–Macaulay of dimension 5. Therefore, if $I'_j$ is a reduction of $I_j$, i.e. an ideal generated by 5 generic $\mathbb{C}$-linear combinations of the generators, then the Hilbert–Samuel multiplicity $e(I_j)$ of $I_j$ is equal to $\dim \mathbb{C} C(x, y, w)[s, t_1, t_2]/(I'_j, f - s(t_1x + t_2y))$. For generators of $I'_j$ we can choose $t_1, t_2, s$ and 2 generic linear combinations of $f_x, f_y$ and $f_w$. Thus,

$$e(I_j) = \dim \mathbb{C}[x, y, w]/(f, \alpha_1 f_x + \alpha_2 f_y + \alpha_3 f_w, \beta_1 f_x + \beta_2 f_y + \beta_3 f_w)$$

for generic $\alpha_i$ and $\beta_i$. Hence $e(I_j)$ is equal to the Hilbert–Samuel multiplicity $e(J(f))$ of $J(f)$ in $\mathbb{C}[x, y, w]$. Finally,

$$\varepsilon(J_{rel}(X_h))(0) = \mu(f) + \mu(f \cap H)$$
where $\mu(f)$ is the Milnor number of $f$ and $\mu(f \cap H)$ is the Milnor number of $\mathbb{V}(f) \cap H$ for a generic hyperplane section $H$ in $(\mathbb{C}^3, 0)$.

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