A VALUATION THEOREM FOR NOETHERIAN RINGS

ANTONI RANGACHEV

Abstract. Let \( A \subset B \) be integral domains. Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Denote by \( \overline{A} \) the integral closure of \( A \) in \( B \). We show that \( \overline{A} \) is determined by finitely many unique discrete valuation rings. Our result generalizes Rees’ classical valuation theorem for ideals. We obtain a variant of Zariski’s main theorem and then we give a simple proof of his result.

1. Introduction

Let \( A \subset B \) be integral domains. Denote the integral closure of \( A \) in \( B \) by \( \overline{A} \). Suppose there exist valuation rings \( V_1, \ldots, V_r \) in \( \text{Frac}(\overline{A}) \) such that

\[
\overline{A} = \cap_{i=1}^r V_i \cap B,
\]

where the intersection takes place in \( \text{Frac}(B) \). We say that (1) is a valuation decomposition of \( \overline{A} \). We say the decomposition is irredundant or minimal if dropping any \( V_i \) violates (1). The main result of this paper is the following valuation theorem.

**Theorem 1.1.** Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Then either \( \text{Ass}_{A}(B/A) = \{0\} \), or \( \overline{A} = B \), or there exist unique discrete valuation rings \( V_1, \ldots, V_r \) in \( \text{Frac}(A) \) such that \( \overline{A} = \cap_{i=1}^r V_i \cap B \) is minimal. Furthermore, if \( A \) is locally formally equidimensional, then each \( V_i \) is a divisorial valuation ring with respect to a Noetherian subring of \( A \).

It’s well-known that \( \overline{A} \) may fail to be Noetherian [SH06, Ex. 4.10]. The proof of Thm. 1.1 rests upon three key observations. First, we show that \( \overline{A} \) is generically Noetherian. Then we use this to prove that \( \text{Ass}_{\overline{A}}(B/\overline{A}) = \{0\} \), or \( \overline{A} = B \), or there exist unique discrete valuation rings \( V_1, \ldots, V_r \) in \( \text{Frac}(\overline{A}) \) such that \( \overline{A} = \cap_{i=1}^r V_i \cap B \) is minimal. Furthermore, if \( A \) is locally formally equidimensional, then each \( V_i \) is a divisorial valuation ring with respect to a Noetherian subring of \( \overline{A} \).

Let \( R \) be a Noetherian domain. Suppose \( A = \bigoplus_{i=0}^\infty A_i \subset B = \bigoplus_{i=0}^\infty B_i \) is a homogeneous inclusion of graded Noetherian domains with \( A_0 = B_0 = R \). Suppose \( B \) is a finitely generated \( A \)-algebra. For each \( n \) denote by \( \overline{A}_n \) the integral closure of \( A_n \) in \( B_n \). It’s the \( R \)-module consisting of all elements in \( B_n \) that are integral over \( A \). For the discrete valuations \( V_i \) in Thm. 1.1 set \( \overline{V}_i := V_i \cap \text{Frac}(R) \). Define \( A_n V_i \cap B_n \) to be the set of elements in \( B_n \) that map to \( A_n \overline{V}_i \) as a submodule of \( B_n \). The following is a corollary to our main result.

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Corollary 1.2. Suppose $\mathcal{A}V = \mathcal{A}V$ for each valuation $V$ in $\text{Frac}(R)$. Then either $\mathcal{A} = \mathcal{B}$, or $\text{Ass}_\mathcal{A}(\mathcal{B}/\mathcal{A}) = \{(0)\}$, or 

\[(2) \quad \mathcal{A}_n = \cap_{i=1}^r \mathcal{A}_i \cap \mathcal{B}_n \]

for each $n$. Furthermore, if $V_i \neq \text{Frac}(R)$ for $i = 1, \ldots, r$, then (2) is minimal and the $V_i$s in (2) are unique.

Let $I$ be an ideal in a Noetherian domain $R$. Let $t$ be a variable. The graded algebra \( R[t] := R \oplus It \oplus I^2t^2 \oplus \cdots \) is called the Rees algebra of $I$. It’s contained in the polynomial ring \( R[t] := R \oplus Rt \oplus R^2t^2 \oplus \cdots \). For each $n$ denote by $T^n$ the integral closure of $I^n$ in $R$. Set $\mathcal{A} := R[t]$ and $\mathcal{B} := R[t]$ in Cor. 1.2. Note that for each valuation ring $V$ in $\text{Frac}(R)$ we have $\mathcal{A}V = V[t]$ or $\mathcal{A}V = V[at]$ where $IV = (a)$ for some $a \in I$. Thus $\mathcal{A}V$ is integrally closed, and so $\mathcal{A}V = \mathcal{A}V$. Cor. 1.2 recovers a classical result due to Rees [R56].

Corollary 1.3. [Rees’ valuation theorem] Let $R$ be a Noetherian domain and $I$ be a nonzero ideal in $R$. There exists unique discrete valuations $V_1, \ldots, V_r$ in the field of fractions of $R$ such that $T^n = \cap_{i=1}^r I^n V_i \cap R$ for each $n$.

In the setting of Cor. 1.2 assume additionally that $R$ is locally formally equidimensional. We can give a geometric interpretation of the centers of the $V_i$s in $R$ using Chevalley’s constructability result as follows. Consider the structure map $c : \text{Proj}(\mathcal{A}) \to \text{Spec}(R)$. For each integer $l \geq 0$ set 

\[ S(l) := \{p \in \text{Spec}(R) : \dim \text{Proj}(\mathcal{A} \otimes R k(p)) \geq l\} \]

By Chevalley’s [EGAIV, Thm. 13.1.3 and Cor. 13.1.5] $S(l)$ is closed in $\text{Spec}(R)$. For $i = 1, \ldots, r$ denote by $m_i$ the center of $V_i$ in $R$. Set $e := \dim \text{Proj}(\mathcal{A} \otimes R \text{Frac}(R))$.

Theorem 1.4. Suppose $R$ is locally formally equidimensional. If $\text{ht}(m_i) > 1$ for some $i$, then $m_i$ is a minimal prime of $S(\text{ht}(m_i) + e - 1)$.

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2. Proofs

The proof of Thm. 1.1 is based on three key propositions.

Proposition 2.1. Suppose $\mathcal{A} \subset \mathcal{B}$ are integral domains. Suppose $\mathcal{A}$ is Noetherian and $\mathcal{B}$ is a finitely generated $\mathcal{A}$-algebra. Then there exists $f \in \mathcal{A}$ such that $\mathcal{A}_f$ is Noetherian.

Proof. Denote by $E$ the algebraic closure of $\text{Frac}(\mathcal{A})$ in $\text{Frac}(\mathcal{B})$. By Zariski’s lemma $E$ is a finite field extension of $\text{Frac}(\mathcal{A})$. Because $E = \text{Frac}(\mathcal{A})$, there exist $f_1, \ldots, f_k \in \mathcal{A}$ such that $E = \text{Frac}(\mathcal{A})(f_1, \ldots, f_k)$. Set $\mathcal{A}' := \mathcal{A}[f_1, \ldots, f_k]$. Then $\mathcal{A}'$ is Noetherian and $\text{Frac}(\mathcal{A}') = \text{Frac}(\mathcal{A})$. By [Ran20, Prp. 2.1] $\text{Ass}_{\mathcal{A}'}(\mathcal{B}/\mathcal{A}')$ is finite. But $\text{Ass}_{\mathcal{A}'}(\mathcal{A}/\mathcal{A'}) \subset \text{Ass}_{\mathcal{A}'}(\mathcal{B}/\mathcal{A'})$. So $\text{Ass}_{\mathcal{A}'}(\mathcal{A}/\mathcal{A'})$ is finite, too. Select $f \in \mathcal{A}'$ from the intersection of all minimal primes in $\text{Ass}_{\mathcal{A}'}(\mathcal{A}/\mathcal{A'})$. Then $\mathcal{A}'_f = \mathcal{A}_f$; hence $\mathcal{A}_f$ is Noetherian.

The next proposition strengthens [Ran20, Thm. 1.1 (ii)] in the domain case.

Proposition 2.2. Suppose $\mathcal{A} \subset \mathcal{B}$ are integral domains. Suppose $\mathcal{A}$ is Noetherian and $\mathcal{B}$ is a finitely generated $\mathcal{A}$-algebra. Then $\text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A})$ and $\text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A})$ are finite.
Proof. By Prp. 2.1 there exists \( f \in \mathcal{A} \) such that \( \mathcal{A}_f \) is Noetherian. Let \( q \in \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \). If \( f \not\in q \), then \( q \in \text{Ass}_{\mathcal{A}[f]}(B/f \mathcal{A}) \). The last set is finite by [Ran20]. Suppose \( f \in q \). As before, denote by \( E \) the algebraic closure of \( \text{Frac}(\mathcal{A}) \) in \( \text{Frac}(B) \). It’s a finite field extension of \( \text{Frac}(\mathcal{A}) \). Denote by \( L \) the integral closure of \( \mathcal{A} \) in \( E \). By the Mori–Nagata Theorem \( L \) is a Krull domain ([Bour75, Prp. 12, pg. 209] and [SH06, Ex. 4.5]). But \( L \) is also the integral closure of \( \mathcal{A} \) in its field of fractions. Let \( q' \) be a prime in \( L \) that contracts to \( q \). We have \( \mathcal{A}_q \subset L_{q'} \). By Thm. 1.1 (i) \( \mathcal{A}_q \) is a DVR. As \( \mathcal{A} \) and \( L \) have the same field of fractions, \( \mathcal{A}_q = L_{q'} \). Thus \( \text{ht}(q') = 1 \). Because \( L \) is a Krull domain, there are finitely many height one prime ideals in \( L \) containing \( f \). Thus there are finitely many \( q \in \text{Ass}_{\mathcal{A}[f]}(B/\mathcal{A}) \) containing \( f \). This proves the finiteness of \( \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \). Alternatively, apply directly [Ran20, Thm. 1.1 (ii)] for \( \mathcal{A}' \) and \( B \) noting that \( \mathcal{A} \) and \( \mathcal{A}' \) have the same integral closure in \( B \).

Let \( p \in \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \). If \( f \not\in p \), then \( p \) is a contraction from a prime in \( \text{Ass}_{\mathcal{A}[f]}(B/f \mathcal{A}) \) which is finite by [Ran20, Prp. 2.1]. If \( f \in p \), then the proof of [Ran20, Thm. 1.1 (ii)] shows that \( p \in \text{Ass}_{\mathcal{A}}(A/f \mathcal{A}) \) which is finite because \( A \) is Noetherian. The proof is now complete. \( \square \)

Proposition 2.3. Suppose \( \mathcal{A} \subset B \) are integral domains. Suppose \( \mathcal{A} \) is Noetherian and \( B \) is a finitely generated \( \mathcal{A} \)-algebra. Let \( b \in B \) be such that \( J := (\mathcal{A} :_\mathcal{A} b) \) is a nonunit ideal in \( \mathcal{A} \). Then the minimal primes of \( J \) are in \( \text{Ass}_{\mathcal{A}[f]}(B/\mathcal{A}) \).

Proof. If \( J = (0) \), then clearly \( J \in \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \). Suppose \( J \neq (0) \). Select a nonzero \( h \in J \). Then \( J := ((h) :_\mathcal{A} hb) \). Thus the minimal primes of \( J \) are among the minimal primes of \( (h) \) each of which is of height one. Denote by \( L \) the integral closure of \( \mathcal{A} \) in \( \text{Frac}(\mathcal{A}) \). Because \( L \) is a Krull domain, then there are finitely many minimal primes of \( hL \). But \( L \) is integral over \( \mathcal{A} \). So by incomparability each minimal prime of \( (h) \) is a contraction of a prime of height one in \( L \) which has to be a minimal prime of \( hL \). Therefore, \( (h) \) has finitely many minimal primes, and so does \( J \).

Denote by \( q_1, \ldots, q_l \) the minimal primes of \( J \). First, we want to show that for each \( 1 \leq i \leq l \) there exists a positive integer \( s_i \) such that \( q_i^{s_i} \subset J\mathcal{A}_{q_i} \). We proceed as in the proof of [Ran20, Thm. 1.1 (i)]. Set \( p_i := q_i \cap \mathcal{A} \). We can assume that \( \mathcal{A} \) is local at \( p_i \). Let \( \mathcal{A} \) be the completion of \( \mathcal{A} \) with respect to \( p_i \). Set \( \mathcal{A}' := \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \) and \( B' := B \otimes_{\mathcal{A}} \mathcal{A} \). Replace \( \mathcal{A}', \mathcal{A} \) and \( B' \) by their reduced structures. Because \( \mathcal{A} \) is a reduced complete local ring and \( B' \) is a finitely generated \( \mathcal{A} \)-algebra, then by [Stks, Tag 03GH] \( \mathcal{A}' \) is module-finite over \( \mathcal{A} \). In particular, \( \mathcal{A}' \) is Noetherian. Clearly, \( J\mathcal{A}' \) is primary to \( q_i \mathcal{A}' \). Thus there exists \( s_i \) such that \( q_i^{s_i} \mathcal{A}' \subset J\mathcal{A}' \). Hence \( q_i^{s_i} \subset J\mathcal{A}_q \) by [ALM69, Prp. 3.14] applied for \( (\mathcal{A}, b)/\mathcal{A} \).

Assume that the \( s_i \) defined above are the minimal possible. Fix \( 1 \leq j \leq l \). For each \( i \neq j \) by prime avoidance we can select \( c_i \in q_i^{s_i} \) and \( c_i \not\in q_j \). Let \( c_j \in q_j^{s_j-1} \) with \( c_j \not\in J\mathcal{A}_q \). Set \( c := c_1 \cdots c_l \). Then \( q_j = (\mathcal{A} :_\mathcal{A} cb) \) and thus \( q_j \in \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \). \( \square \)

Proof of Theorem 1.1

We can proceed with the proof of Thm. 1.1. Suppose \( \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \neq \{(0)\} \) and \( \mathcal{A} \neq \mathcal{B} \). Then by Prp. 2.2 \( \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \) contains finitely many nonzero prime ideals which we denote by \( q_1, \ldots, q_r \). By [Ran20, Thm. 1.1 (i)] \( V_i := \mathcal{A}_q \) is a DVR for each \( i = 1, \ldots, r \). Obviously, \( \mathcal{A} \subset \cap_{i=1}^r V_i \cap \mathcal{B} \). Let \( b = x/y \in \cap_{i=1}^r V_i \cap \mathcal{B} \). Set \( J := (y\mathcal{A} :_\mathcal{A} x) \). We have \( JV_i = V_i \) for each \( i \). Thus \( J \not\subset q_i \) for each \( i \). But \( Jb \in \mathcal{A} \) and \( J \neq (0) \). So by Prp. 2.3 if \( J \) is a nonunit ideal, its
minimal primes are among $q_1, \ldots, q_r$ which is impossible. Thus $J$ has to be the unit ideal, which implies that $\theta \in \overline{A}$.

To prove minimality of the valuation decomposition, suppose we can drop $V_j$ in (1) for some $j$. Then localizing both sides of (1) at $q_j$ we obtain that $\overline{A}_{q_j} = B_{q_j}$ which contradicts with $q_j \in \text{Supp}(B/\overline{A})$. Thus (1) is minimal. We are left with proving the uniqueness of the $V_i$s. Suppose
\[
\overline{A} = \bigcap_{i=1}^s V_j' \cap B
\]
is a minimal discrete valuation decomposition. Set $B' := \text{Frac}(\overline{A}) \cap B$. We have $\overline{A}_{q_i} \neq B'_{q_i}$ for each $i = 1, \ldots, r$. As the intersection in (3) takes place in $B'$ we can replace in it $B$ by $B'$. Localizing both sides of (3) at $q_1$ we obtain that there exists a valuation $V_1'$ such that $(V_1')_{q_1} \neq \text{Frac}(\overline{A})$. But $V_1 = \overline{A}_{q_1} \subset (V_1')_{q_1}$. Thus $V_1 = (V_1')_{q_1}$. Also, $V_1' \subset (V_1')_{q_1}$, and so $V_1' = (V_1')_{q_1}$. Continuing this process we obtain that each $V_i$ appears in (3). As (3) is minimal, we obtain that $s = r$ and after possibly renumbering we get $V_i = V_i'$ for $i = 1, \ldots, r$.

Let $A'$ be the module-finite $A$-algebra defined in the proof of of Prp. 2.1. Suppose $A$ is locally formally equidimensional. Then so is $A'$. Note that $\text{Frac}(A') = \text{Frac}(A)$. Denote by $m_{V_i}$ the maximal ideal of $V_i$. Set $p_i := m_{V_i} \cap A'$. By Cohen’s dimension inequality (see [SH06, Thm. B.2.5])
\[\text{tr. deg}_{\kappa(p_i)}(m_{V_i}) \leq \text{ht}(p_i) - 1.\]
Because $p_i = q_i \cap A'$, then by [Ran20, Thm. 1.1 (iii)] we get $\text{ht}(p_i) = 1$. Therefore, $\text{tr. deg}_{\kappa(p_i)}(m_{V_i}) = \text{ht}(p_i) - 1 = 0$. Hence each $V_i$ is a divisorial valuation ring in $\text{Frac}(\overline{A})$. $\square$

**Remark 2.4.** As it’s well-known, an integrally closed domain equals the intersection of all valuation rings in its field of fractions that contain it. From here one derives set-theoretically that $\overline{A} = \bigcap \mathcal{V} \cap B$ where the intersection is taken over all valuation rings in $\text{Frac}(\overline{A})$ that contain the integral closure of $\overline{A}$ in $\text{Frac}(\overline{A})$. Because $A'$ and $\overline{A}$ have the same integral closure and $A'$ is Noetherian (see the proof of Prp. 2.1), then in the intersection we can take only DVRs. Thus, the real contribution of Thm. 1.1 is that under the additional hypothesis that $B$ is a finitely generated $A$-algebra, one can take finitely many uniquely determined DVRs each of which is a localization of $\overline{A}$ at a height one prime ideal.

**Proof of Cor. 1.2 and Cor. 1.3**

Suppose $\text{Ass}\, (B/\overline{A}) \neq \{(0)\}$ and $\overline{A} \neq B$. Because $AV = \overline{A}V$ and $\overline{A} \subset AV$ we get $\overline{A} \subset AV$. Thus for each $i = 1, \ldots, r$
\[\overline{A} \subset AV_i \subset V_i.\]
But $AV_i = \bigoplus_{j=0}^\infty A_j V_i$. Also, by Thm. 1.1 $\overline{A} = \bigcap_{i=1} V_i \cap B$. Thus, set-theoretically $\overline{A} = \bigcap_{i=1} V_i \cap B$. Set $K := \text{Frac}(R)$. Suppose $V_i \neq K$ for each $i = 1, \ldots, r$. Showing that the decomposition is minimal and unique is done in the same way as in Thm. 1.1. Here we will show just the uniqueness of the valuations. Suppose there exist DVRs $V_1', \ldots, V_s'$ in $K$ such that
\[\overline{A} = \bigcap_{i=1} V_i' \cap B\]
is minimal. As in the proof of Thm. 1.1 we can assume that $\text{Frac}(B) = \text{Frac}(\overline{A})$. If $V_j' = K$ for some $j$, then because (4) is minimal we get $s = 1$ and $V_1' = K$. So $\overline{A} = AK \cap B$. Localizing at $q_1$ we obtain that $V_1 = (AK)_{q_1}$. But $K \subset (AK)_{q_1}$. Thus $V_1 = K$, a contradiction. Therefore, $V_j' \neq K$ for each $j$. Again, by localizing (4) at $q_1$, we get that there is a $j$ such that
\( \mathcal{V}_i = (AV_j')q_i \). But \( V_i = \mathcal{V}_i \cap \text{Frac}(R) \) and \( V_j' \in (AV_j')q_i \cap \text{Frac}(R) \). Because \( V_i \not\subseteq \text{Frac}(R) \), then \( V_i = V_j' \). Thus \( r = s \) by minimality and after possible renumbering \( V_i = V_j' \) for each \( i = 1, \ldots, r \).

Consider Cor. 1.3 Apply Cor. 1.3 with \( A := R[t] \) and \( B := R[t] \). In the introduction we proved that \( R[t]V = R[t]V \) for each valuation \( V \) in \( K \). What remains to be shown is that \( V_i \not\subseteq K \) for each \( i \). Indeed, the prime ideals in \( R[t] \) are contractions of extensions of prime ideals of \( R \) to \( R[t] \). Thus \( \text{ht}(q_i \cap R) \geq 1 \) and so \( V_i \not\subseteq K \) for each \( i \) otherwise \( q_i R[t]q_i \) is a unit ideal which is impossible. \( \square \)

A version of Cor. 1.2 for Rees algebras of modules is proved by Rees in \([R87, \text{Thm. 1.7}]\).

**Proof of Theorem 1.4**

First, we show that \( m_i \in S(\text{ht}(m_i) + e - 1) \). Recall that the maximal ideal of \( \mathcal{V}_i \) contracts to \( q_i \) in \( \mathbb{A} \). Set \( p_i := q_i \cap A \). Then by \([Ran20, \text{Thm. 1.1 (iii)}]\) \( \text{ht}(p_i) = 1 \). Consider the map \( \text{Proj}(\mathcal{A}_m) \to \text{Spec}(R_m) \). It’s closed, surjective and of finite type. By the dimension formula \([\text{Stks, Tag 02JX}]\) \( \dim \text{Proj}(\mathcal{A}_m) = \text{ht}(m_i) + e \). But \( \mathcal{A}_m \) is a local formally equidimensional ring. Because \( \text{ht}(p_i, \mathcal{A}_m) = 1 \), by \([SH06, \text{Lem. B.4.2}]\) \( \dim \text{Proj}(A \otimes k(m_i)) = \text{ht}(m_i) + e - 1 \). Thus \( m_i \in S(\text{ht}(m_i) + e - 1) \).

Next, suppose there exists a prime \( n_i \) in \( R \) with \( n_i \subseteq m_i \) and \( n_i \in S(\text{ht}(m_i) + e - 1) \). Then \( \dim \text{Proj}(\mathcal{A} \otimes k(n_i)) \geq \text{ht}(m_i) + e - 1 \). Note that \( n_i \neq (0) \) for otherwise \( n_i \subseteq S(e) \) which forces \( \text{ht}(m_i) = 1 \), a contradiction. Because \( \dim \text{Proj}(\mathcal{A}_n) = \text{ht}(n_i) + e \) then \( \dim \text{Proj}(A \otimes k(n_i)) \leq \text{ht}(n_i) + e - 1 \). Therefore,

\[
\text{ht}(m_i) + e - 1 \leq \text{ht}(n_i) + e - 1.
\]

But \( n_i \subseteq m_i \). Thus \( n_i = m_i \) and \( m_i \) is a minimal prime in \( S(\text{ht}(m_i) + e - 1) \). This completes the proof of Thm. 1.4. \( \square \)

Thm. 1.4 generalizes \([Ran18, \text{Thm. 7.8}]\), which is a result for Rees algebras of modules. To see that Thm. 1.4 is sharp, let \( (R, m) \) be a Noetherian regular local ring of dimension at least \( 2 \), and let \( h \in m \) be an irreducible element. Let \( B \) be the polynomial ring \( R[y_1, \ldots, y_{e+1}] \) for some \( e \geq 0 \). Set \( A := R[y_1, \ldots, y_{e+1}] \). Thus \( A \) is a polynomial subring of \( B \). It is normal because \( R \) is regular. In the setup of Cor. 1.2 there is only one \( \mathcal{V}_i = A_{q_i} \), where \( q_i = hA \). Note that \( S(k) = S(e) \) for all \( k \geq 0 \) because \( A \) is a polynomial ring over \( R \) generated by \( e + 1 \) elements. Thus the only minimal prime in \( S(e) \) is \( (0) \), whereas \( m_{V_i} = (h) \) is a height one prime ideal in \( R \).

**Zariski’s Main Theorem**

Let \( A \subseteq B \) be Noetherian rings. Suppose \( B \) is a finitely generated \( A \)-algebra. Denote by \( \mathbb{A} \) the integral closure of \( A \) in \( B \). Denote by \( I_{B/\mathbb{A}} \) the intersection of all elements in \( \text{Ass}_{\mathbb{A}}(B/\mathbb{A}) \). In all remaining results with the exception of Thm. 2.9 we assume that \( A \) and \( B \) are integral domains. The following result characterizes the support of \( B/\mathbb{A} \).

**Proposition 2.5.** Let \( A \subseteq B \) be integral domains. Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Then \( \mathbb{V}(I_{B/\mathbb{A}}) = \text{Supp}(B/\mathbb{A}) \).

**Proof.** If \( \text{Frac}(\mathbb{A}) \neq \text{Frac}(B) \), then \((0) \in \text{Ass}(B/\mathbb{A}) \) and trivially \( \mathbb{V}(I_{B/\mathbb{A}}) = \text{Supp}(B/\mathbb{A}) = \text{Spec}(\mathbb{A}) \). Suppose \( \text{Frac}(\mathbb{A}) = \text{Frac}(B) \). Then by \([Ran20, \text{Thm. 1.1 (i)}]\) \( I_{B/\mathbb{A}} = q_1 \cap \ldots \cap q_k \) with \( \text{ht}(q_i) = 1 \) for each \( i \). If \( q \in \mathbb{V}(I_{B/\mathbb{A}}) \), then \( q_j \subseteq q \) for some \( j \) and thus \( q_j A_q \in \text{Ass}_{\mathbb{A}}(B_q/\mathbb{A}_q) \).

Hence \( \mathbb{V}(I_{B/\mathbb{A}}) \subseteq \text{Supp}(B/\mathbb{A}) \). Suppose \( q \in \text{Supp}(B/\mathbb{A}) \). Then there is \( x/y \in B \) with \( x, y \in \mathbb{A} \), such that its image in \( B_q \) is not \( A_q \). In other words, if \( J := ((x): \mathbb{A} y) \), then \( J \subseteq q \).
But by Prp. 2.3 the minimal primes of $J$ are among the $q_i$s. Thus there exists $q_j$ such that $q_j \subset q$, i.e. $\text{Supp}_{\mathcal{A}}(B/\mathcal{A}) \subset V(I_{B/\mathcal{A}})$. □

In [EGAIII] Cor. 4.4.9 Grothendieck derives the following result as a consequence of Zariski’s main theorem (ZMT): if $g: X \to Y$ is a birational, proper morphism of noetherian integral schemes with $Y$ normal and $g^{-1}(y)$ finite for each $y \in Y$, then $g$ is an isomorphism. Below we show that in the affine case we can reach the same conclusion assuming just that $g$ is surjective. To do this we do not have to appeal to ZMT. In fact, our result proves ZMT in codimension one as shown below.

**Theorem 2.6.** Let $\mathcal{A} \subset \mathcal{B}$ be integral domains. Suppose $\mathcal{A}$ is Noetherian and $\mathcal{B}$ is a finitely generated $\mathcal{A}$-algebra. Assume $\text{Frac}(\mathcal{A}) = \text{Frac}(\mathcal{B})$ and $\text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A})$ is surjective. Then $\mathcal{B} = \mathcal{A}$.

**Proof.** Suppose there exists $q \in \text{Ass}_{\mathcal{A}}(\text{Frac}(\mathcal{B}/\mathcal{A}))$. Because $q \neq (0)$, by [Ran20] Thm. 1.1 (i)] $\mathcal{A}_q$ is a DVR. But $\mathcal{A}_q \neq \mathcal{B}_q$. Thus $\mathcal{B}_q = \text{Frac}(\mathcal{B})$. This contradicts the assumption that there exists a prime in $\mathcal{B}$ that contracts to $q$. Thus $\text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A})$ is empty, and so by Prp. 2.3 $\mathcal{B} = \mathcal{A}$. □

**Definition 2.7.** Let $Q \in \text{Spec}(\mathcal{B})$. Set $q := Q \cap \mathcal{A}$ and $p := Q \cap \mathcal{A}$. We say that $\text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A})$ is quasi-finite at $Q$ if $Q$ is isolated in its fiber, i.e. if the field extension $\kappa(p) \subset \kappa(Q)$ is finite and $\dim(B_Q/pB_Q) = 0$. We say that $\text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A})$ is quasi-finite if it is quasi-finite at each prime in $\text{Spec}(\mathcal{B})$.

The following corollary to Thm. 2.6 is a special case of ZMT.

**Corollary 2.8.** Let $\mathcal{A} \subset \mathcal{B}$ be integral domains. Suppose $\mathcal{A}$ is Noetherian and $\mathcal{B}$ is a finitely generated $\mathcal{A}$-algebra. Assume $\text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A})$ is quasi-finite at $Q$ and $\text{ht}(q) = 1$. Then there exists $f \in I_{\mathcal{B}/\mathcal{A}}$ with $f \not\in q$ such that $\mathcal{B}_f = \mathcal{A}_f$.

**Proof.** Because $\text{ht}(q) = 1$ then $\mathcal{A}_q$ is a universally catenary Noetherian ring. Applying the dimension formula for $\mathcal{A}_q$ and $\mathcal{B}_q$ we get

(5) $\text{ht}(Q) + \text{tr. deg}_{\kappa(q)}\kappa(Q) = \text{ht}(q) + \text{tr. deg}_{\mathcal{A}}\mathcal{B}$

Because $qB_Q$ is $QB_Q$-primary we have $\text{ht}(q) \geq \text{ht}(Q)$. But $\kappa(Q)$ is a finite field extension of $\kappa(q)$. So $\text{tr. deg}_{\kappa(q)}\kappa(Q) = 0$. Thus $\text{ht}(Q) = \text{ht}(q)$ and $\text{Frac}(\mathcal{A}) = \text{Frac}(\mathcal{B})$. Applying Thm. 2.6 to $\mathcal{A}_q$ and $\mathcal{B}_q$ we get $\mathcal{A}_q = \mathcal{B}_q$. Because $\text{Frac}(\mathcal{A}) = \text{Frac}(\mathcal{B})$, by Prp. 2.3 for each $b \in \mathcal{B}$ there exists a positive integer $k_b$ such that $I_{\mathcal{B}/\mathcal{A}}^{k_b}b \in \mathcal{A}$. Thus for each $f \in I_{\mathcal{B}/\mathcal{A}}$ we have $\mathcal{A}_f = \mathcal{B}_f$. By Prp. 2.5 $I_{\mathcal{B}/\mathcal{A}} \not\subset q$. So we can select $f \in I_{\mathcal{B}/\mathcal{A}}$ with $f \not\in q$. □

**Theorem 2.9** (Zariski’s Main Theorem). Let $\mathcal{A} \subset \mathcal{B}$ be Noetherian rings. Suppose $\mathcal{B}$ is a finitely generated $\mathcal{A}$-algebra. Assume $\text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A})$ is quasi-finite at $Q$. Then there exists $f \not\in p$ such that $\mathcal{B}_f = \mathcal{A}_f$.

**Proof.** We will prove that $\mathcal{A}_q = \mathcal{B}_q$. Assume $\mathcal{A}$ is local with maximal ideal $p$. Let’s show we can reduce to the case when $\mathcal{A}$ is a complete Noetherian local domain and $\mathcal{B}$ is a domain. Here we follow the proof of [Ran20] Thm. 1.1 (i)]. The main observation we use is a descent property of the integral closure under a faithfully flat base change.

Let $(\hat{\mathcal{A}}, \hat{p})$ be the completion of $\mathcal{A}$ with respect to $p$. Set $\mathcal{B} := \mathcal{B} \otimes_\mathcal{A} \hat{\mathcal{A}}$ and $\mathcal{Q} := Q\mathcal{B}$. By flatness and associativity of the tensor product $\mathcal{Q}$ is maximal in $\mathcal{B}$ (see the second paragraph in the proof of [Ran20] Thm. 1.1 (i)]). Furthermore, $\mathcal{Q}$ is isolated in its fiber over $\hat{p}$ by faithful
flatness. Indeed, suppose there is \( Q'' \subset Q' \) such that \( Q'' \cap \hat{A} = \hat{p} \). Then \( Q'' \cap B \) contracts to \( p \), and so \( Q'' \cap B = Q \), which by faithful flatness forces \( Q'' = Q' \).

If we show that \( B' \) is integral over \( A \), then that will force the integrality of \( B \) over \( A \) by [Ran20] Prp. 2.2. Let \( Q'_{\min} \) be a minimal prime of \( B' \). If we prove that \( B'/Q'_{\min} \) is integral over \( \hat{A}/(Q'_{\min} \cap \hat{A}) \) for each minimal prime \( Q'_{\min} \), then that will force \( B' \) to be integral over \( \hat{A} \) (cf. [SH06 Prp. 2.1.6]). Note that \( \hat{A}/(Q'_{\min} \cap \hat{A}) \) is a complete local Noetherian ring and \( B'/Q'_{\min} \) is a finitely generated \( \hat{A}/(Q'_{\min} \cap \hat{A}) \)-algebra. The image of \( Q' \) in \( B'/Q'_{\min} \) is isolated in its fiber over \( \hat{p}/Q'_{\min} \).

From now on assume that \((A, p)\) is a complete local Noetherian domain and \( B \) is a domain. To show that \( \hat{A}_q = B_q \) it’s enough to prove that \( B_Q \) is integral over \( A \). Consider the associated graded rings \( \text{gr}_p(A) \) and \( \text{gr}_p(B) \). From \( p^n/p^{n+1} \otimes B \to p^n B/p^{n+1} B \) one deduces that \( \text{gr}_p(B) \) is a homomorphic image of \( p^{\infty} \otimes \kappa(p) B/pB \). But \( B/pB \) is finite over \( \kappa(p) \) because \( pB \) is \( QB \)-primary and \( \kappa(Q) \) is a finite field extension of \( \kappa(p) \). Thus \( \text{gr}_p(B) \) is module-finite over \( \text{gr}_p(A) \). But \( A \) is complete and \( B_Q \) is separated in the \( p \)-adic topology. Thus by [AK18 Lem. 22.24] we conclude that \( B_Q \) is module-finite over \( A \).

Finally, \( \hat{A}_q = B_q \). Let \( b_1, \ldots, b_s \) be generators of \( B \) as an \( \hat{A}_q \)-algebra. Write each \( f_i \) in the form \( a_i/f_i \) where \( a_i \in \hat{A} \) and \( f_i \in \hat{A} / q \). Set \( f := \prod_{i=1}^s f_i \). Then \( \hat{A}_f = B_f \).

In the proof of Thm. 2.9 we showed that \( B_Q \) is integral over \( A \) assuming that the latter ring is a complete local Noetherian domain. This may seem too strong to be true for even if \( B \) is integral over \( A \) and \( q \) has at least two maximal ideals of different height, say \( Q \) is the one of smaller height, then \( B_Q \) is not integral over \( A \) (see Nagata’s \[SH06 Ex. 2.2.6\]). It’s the completeness of \( \hat{A} \) that guarantees that \( \hat{A} \) is a local ring. Indeed, let \( L \) be the integral closure of \( A \) in \( \text{Frac}(\hat{A}) \). The latter is a finite field extension of \( \text{Frac}(A) \). Thus by [SH06 Thm. 4.3.4] \( L \) is a complete Noetherian local domain. But \( L \) is the integral closure of \( \hat{A} \) in \( \text{Frac}(\hat{A}) \). Thus \( \hat{A} \) is local as well.

As an immediate corollary we get an explicit description of the quasi-finite locus in \( B \) in the domain case.

**Corollary 2.10.** Let \( A \subset B \) be integral domains. Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Set \( g : \text{Spec}(B) \to \text{Spec}(A) \) and \( U := \text{Spec}(A) - \text{V}(I_B/p) \). Then \( g^{-1}U \) is the set of primes at which \( \text{Spec}(B) \to \text{Spec}(A) \) is quasi-finite.

**Proof.** By Thm. 2.9 if \( \text{Spec}(B) \to \text{Spec}(A) \) is quasi-finite at \( Q \), then \( \hat{A}_q = B_q \) where \( q = Q \cap \hat{A} \). Applying Prp. 2.9 finishes the proof.

In general, Cor. 2.10 is usually stated as an existence result (cf. [Stks Tag 03GT] and [Ray70 Cor. 1, p.42]). The following corollary due to Grothendieck shows that we can replace \( \hat{A} \) by a subalgebra which is module-finite over \( A \). It’s what is used in algebraic geometry to show that a quasi-finite morphism factors as a composition of an open immersion and a finite morphism.

**Corollary 2.11.** [EGAIII Cor. 4.4.7] Let \( A \subset B \) be integral domains. Suppose \( A \) is a local Noetherian ring with maximal ideal \( p \) and \( B \) is a finitely generated \( A \)-algebra. Assume \( \text{Spec}(B) \to \text{Spec}(A) \) is quasi-finite at \( Q \) and \( Q \cap A = p \). Then there exists a finite \( A \)-algebra \( A' \) with a maximal ideal \( m' \) such that \( B_Q = A'_m \).

**Proof.** Let \( A' \) be the subalgebra of \( \hat{A} \) defined in the proof of Prp. 2.1. By Thm. 2.9 \( \text{Frac}(\hat{A}) = \text{Frac}(B) \). Thus there exists \( a \in A \) such that \( B_a = \hat{A}_a = \hat{A}'_a \). Moreover, \( a \in I_B/\hat{A} \). Set \( m' := Q \cap A' \). By Thm. 2.9 \( a \notin Q \). Thus \( B_Q = A'_m \).
Remark 2.12. Zariski worked under the assumption that $\mathcal{A}$ and $\mathcal{B}$ have the same field of fractions. In [Z43, pg. 522–527] he gave a lengthy proof of his main theorem using an inductive argument on the number of generators for $\mathcal{B}$ as an $\mathcal{A}$-algebra and an intricate study of conductor ideals. His approach was generalized by Peskine in [Pes66] for ring maps $\mathcal{A} \to \mathcal{B}$ of finite type (cf. [Stks, Tag 00Q9]).

In [Z49] Zariski gave a simple proof of his theorem using valuations but under the additional assumption that $\mathcal{A}$ is a local domain of finite type over a field $K$ and $\mathcal{A}$ is integrally closed in its field of fractions. In this case $\mathcal{A}$ is analytically irreducible. Using a precursor to Cohen’s structure theorem, Zariski shows that $\hat{\mathcal{A}}$ and the completion $\hat{\mathcal{B}}_Q$ of $\mathcal{B}_Q$ with respect to $Q$ are module finite over $K[[x_1, \ldots, x_d]]$ where $(x_1, \ldots, x_d)$ is a system of parameters for $\mathcal{A}$. In particular, $\mathcal{B}_Q$ is module-finite over $\hat{\mathcal{A}}$. To derive the integrality of $\mathcal{B}_Q$ over $\hat{\mathcal{A}}$, Zariski uses in an essential way that $\mathcal{A}$ is analytically irreducible: in this case, there is a bijective correspondence of divisorial valuations associated with the field of fractions of $\hat{\mathcal{A}}$ and $\mathcal{A}$, respectively, given by restriction (see [SH06, Prps. 9.3.5; 6.5.4]). Thus $\mathcal{B}_Q$ is contained in each divisorial valuation with respect to $\hat{\mathcal{A}}$, and so $\mathcal{B}_Q$ is integral over $\hat{\mathcal{A}}$.

In [EGAIII, Thm. 4.4.3] (see also [EGAIV, Thm. 8.12.6]) Grothendieck proved a global version of Zariski’s main theorem using the formal function theorem. Our use of the associated graded and completions is implicit in Grothendieck’s proof of Zariski’s connectedness theorem. The use of both techniques in this context, however, predates Grothendieck’s treatment. The method of associated graded rings ([ZS60, pg. 248; Cor. 1 pg. 259]) is used to show that the completion of a Noetherian ring is Noetherian and it appears often in valuation theory and in the theory of multiplicities. The use of passing to the completion in our setup goes back at least to the proof of the Mori–Nagata theorem [Nag55, Thm. 1].


Department of Mathematics, University of Chicago, Chicago, IL 60637, Institute of Mathematics, and Informatics, Bulgarian Academy of Sciences, Akad. G. Bonchev 8, Sofia 1113, Bulgaria