A VALUATION THEOREM FOR NOETHERIAN RINGS

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Abstract. Let \( A \subset B \) be integral domains. Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Denote by \( \overline{A} \) the integral closure of \( A \) in \( B \). We show that \( \overline{A} \) is determined by finitely many unique discrete valuation rings. Our result generalizes Rees’ classical valuation theorem for ideals. We give a simple proof of Zariski’s main theorem in the domain case and strengthen some of its versions.

1. Introduction

Let \( A \subset B \) be integral domains. Denote the integral closure of \( A \) in \( B \) by \( \overline{A} \). Suppose there exist valuation rings \( V_1, \ldots, V_r \) in \( \text{Frac}(\overline{A}) \) such that
\[
\overline{A} = \cap_{i=1}^r V_i \cap B.
\]
We say that (1) is a valuation decomposition of \( \overline{A} \). We say the decomposition is irredundant or minimal if dropping any \( V_i \) violates (1). The main result of this paper is the following valuation theorem.

Theorem 1.1. Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Then either \( \text{Ass}_{\overline{A}}(B/A) = \{0\} \), or \( \overline{A} = B \), or there exist unique discrete valuation rings \( V_1, \ldots, V_r \) in \( \text{Frac}(\overline{A}) \) such that \( \overline{A} = \cap_{i=1}^r V_i \cap B \) is minimal. Furthermore, if \( A \) is locally formally equidimensional, then each \( V_i \) is a divisorial valuation ring.

It’s well-known that \( \overline{A} \) may fail to be Noetherian [SH06, Ex. 4.10]. The proof of Thm. 1.1 rests upon three key observations. First, we show that \( \overline{A} \) is generically Noetherian. Then we use this to prove that \( \text{Ass}_{\overline{A}}(B/A) \) is finite by results of [Ran20]. We set each \( V_i \) to be the localization of \( \overline{A} \) at a prime in \( \text{Ass}_{\overline{A}}(B/A) \), which is a DVR by [Ran20] Thm. 1.1 (i)]. Finally, to get the equality in (1) we show that the minimal primes of an ideal in \( \overline{A} \) which is the annihilator of an element of \( B/A \) are in \( \text{Ass}_{\overline{A}}(B/A) \). As another application of these observations we give a simple proof of Zariski’s main theorem in the case of domains, and strengthen some of its versions.

Let \( R \) be a Noetherian domain. Suppose \( A = \oplus_{i=0}^\infty A_i \subset B = \oplus_{i=0}^\infty B_i \) is a homogeneous inclusion of finitely generated graded \( R \)-algebras with \( A_0 = B_0 = R \). For each \( n \) denote by \( \overline{A}_n \) the integral closure of \( A_n \) in \( B_n \), i.e. it’s the \( R \)-module consisting of all elements in \( B_n \) that are integral over \( A \). For the discrete valuations \( V_i \) in Thm. 1.1 set \( V_i := V_i \cap \text{Frac}(R) \). Define \( A_nV_i \cap B_n \) to be the set of elements in \( B_n \) that map to \( A_nV_i \) as a submodule of \( B_nV_i \). The following is a corollary to our main result.

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Corollary 1.2. Suppose $\mathcal{A}V = \mathcal{A}V$ for each valuation $V$ in $\text{Frac}(R)$. Then either $\mathcal{A} = \mathcal{B}$, or $\text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A}) = \{(0)\}$, or
\[ \mathcal{A}_n = \cap_{i=1}^r A_i \cap \mathcal{B}_n \]
for each $n$. Furthermore, if $V_i \neq \text{Frac}(R)$ for $i = 1, \ldots, r$, then (2) is minimal and the $V_i$s in (2) are unique.

Let $I$ be an ideal in a Noetherian domain $R$. Let $t$ be a variable. The graded algebra $R[It] := R \oplus It \oplus I^2t^2 \oplus \cdots$ is called the Rees algebra of $I$. It’s contained in the polynomial ring \[ R[t] := R \oplus Rt \oplus Rt^2 \oplus \cdots. \]
For each $n$ denote by $T_n$ the integral closure of $I^n$ in $R$. Set $A := R[It]$ and $B := R[t]$ in Cor. [1.2] Note that for each valuation ring $V$ in $\text{Frac}(R)$ we have $\mathcal{A}V = V[t]$ or $\mathcal{A}V = V[ad]$ where $IV = (a)$ for some $a \in I$. Thus $\mathcal{A}V$ is integrally closed, and so $\mathcal{A}V = \mathcal{A}V$. Cor. [1.2] recovers a classical result due to Rees [R56].

Corollary 1.3. [Rees’ valuation theorem] Let $R$ be a Noetherian domain and $I$ be a nonzero ideal in $R$. There exists unique discrete valuations $V_1, \ldots, V_r$ in the field of fractions of $R$ such that $T^n = \cap_{i=1}^r I^n V_i \cap R$ for each $n$.

As before, let $R$ be a Noetherian domain and let $A \subset B$ be a homogeneous inclusion graded finitely generated $R$-algebras. For the discrete valuations $V_i$ in Thm. [1.1] set $V_i := V_i \cap \text{Frac}(R)$. We can give a geometric interpretation of the centers of the $V_i$s using Chevalley’s constructability result assuming $R$ is locally formally equidimensional. Consider the structure map $c: \text{Proj}(A) \to \text{Spec}(R)$. For each integer $k \geq 0$
\[ S(k) := \{p \in \text{Spec}(R): \dim \text{Proj}(A \otimes_R k(p)) \geq k\}. \]
By Chevalley’s [EGAIV], Thm. 13.1.3 and Cor. 13.1.5 $S(k)$ is closed in $\text{Spec}(R)$. For each $i = 1, \ldots, r$ denote by $m_i$ the center $V_i$ in $R$. Set $e := \dim \text{Proj}(A \otimes_R \text{Frac}(R))$.

Theorem 1.4. Suppose $R$ is locally formally equidimensional. If $ht(m_i) > 1$ for some $i$, then $m_i$ is a minimal prime of $S(ht(m_i) + e - 1)$.

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2. Proofs

The proof of Thm. [1.1] is based on three key propositions.

Proposition 2.1. Suppose $A \subset B$ are integral domains. Suppose $A$ is Noetherian and $B$ is a finitely generated $A$-algebra. Then there exists $f \in \mathcal{A}$ such that $\mathcal{A}_f$ is Noetherian.

Proof. Denote by $E$ the algebraic closure of $\text{Frac}(A)$ in $\text{Frac}(B)$. By Zariski’s lemma $E$ is a finite field extension of $\text{Frac}(A)$. Because $E = \text{Frac}(\mathcal{A})$, there exist $f_1, \ldots, f_k \in \mathcal{A}$ such that $E = \text{Frac}(A)[f_1, \ldots, f_k]$. Set $A' := A[f_1, \ldots, f_k]$. Then $A'$ is Noetherian and $\text{Frac}(A') = \text{Frac}(\mathcal{A})$. By [Ran20], Prp. 2.1, $\text{Ass}_{A'}(B/A')$ is finite. But $\text{Ass}_{A'}(\mathcal{A}/A') \subset \text{Ass}_{A'}(B/A')$. So $\text{Ass}_{A'}(\mathcal{A}/A')$ is finite, too. Select $f \in A'$ from the intersection of all minimal primes in $\text{Ass}_{A'}(\mathcal{A}/A')$. Then $A'_f = \mathcal{A}_f$; hence $\mathcal{A}_f$ is Noetherian.

The next proposition strengthens [Ran20, Thm. 1.1 (ii)] in the domain case.

Proposition 2.2. Suppose $A \subset B$ are integral domains. Suppose $A$ is Noetherian and $B$ is a finitely generated $A$-algebra. Then $\text{Ass}_{A}(B/A)$ and $\text{Ass}_{A}(B/\mathcal{A})$ are finite.
Hence \( q \) is Noetherian. Clearly, \( \hat{A} \) is generated by \( q \). Suppose \( f \in q \). As before, denote by \( E \) the algebraic closure of \( \mathcal{F}(A) \) in \( \mathcal{F}(B) \). It’s a finite field extension of \( \mathcal{F}(A) \). Denote by \( L \) the integral closure of \( A \) in \( E \). By the Mori–Nagata Theorem \( L \) is a Krull domain ([Bour75] Prp. 12, pg. 209) and [SH06] Ex. 4.5). But \( L \) is also the integral closure of \( \hat{A} \) in its field of fractions. Let \( q' \) be a prime in \( L \) that contracts to \( q \). We have \( \hat{A}_q \subset L_{q'} \). By Thm. 1.1 (i) \( \hat{A}_q \) is a DVR. As \( \hat{A} \) and \( L \) have the same field of fractions, \( \hat{A}_q \) is a DVR. Thus \( \text{ht}(q') = 1 \). Because \( L \) is a Krull domain, there are finitely many height one prime ideals in \( L \) containing \( f \). Thus there are finitely many \( q \in \text{Ass}_{\hat{A}}(B/\hat{A}) \) containing \( f \). This proves the finiteness of \( \text{Ass}_{\hat{A}}(B/\hat{A}) \). Alternatively, apply directly [Ran20] Thm. 1.1 (ii) for \( \hat{A}' \) and \( B \) noting that \( \hat{A} \) and \( \hat{A}' \) have the same integral closure in \( B \).

Let \( p \in \text{Ass}_{\hat{A}}(B/\hat{A}) \). If \( f \notin p \), then \( p \) is a contraction from a prime in \( \text{Ass}_{\hat{A}}(B_f/\hat{A}) \) which is finite by [Ran20] Prp. 2.1. If \( f \in p \), then the proof of [Ran20] Thm. 1.1 (ii) shows that \( p \in \text{Ass}_{\hat{A}}(A_f/\hat{A}) \) which is finite because \( A \) is Noetherian. The proof is now complete.

**Proposition 2.3.** Suppose \( A \subset B \) are integral domains. Suppose \( A \) is Noetherian and \( B \) is a finitely generated \( A \)-algebra. Let \( b \in B \) be such that \( J := (\hat{A} : \hat{b}) \) is a nonunit ideal in \( \hat{A} \). Then the minimal primes of \( J \) are in \( \text{Ass}_{\hat{A}}(B/\hat{A}) \).

**Proof.** If \( J = (0) \), then clearly \( J \in \text{Ass}_{\hat{A}}(B/\hat{A}) \). Suppose \( J \neq (0) \). Select a nonzero \( h \in J \). Then \( J := ((h) : \hat{b} h) \). Thus the minimal primes of \( J \) are among the minimal primes of \( (h) \) each of which is of height one. Denote by \( L \) the integral closure of \( \hat{A} \) in \( \mathcal{F}(\hat{A}) \). Because \( L \) is a Krull domain, then there are finitely many minimal primes of \( hL \). But \( L \) is integral over \( \hat{A} \). So by incomparability each minimal prime of \( (h) \) is a contraction of a prime of height one in \( L \) which has to be a minimal prime of \( hL \). Therefore, \( (h) \) has finitely many minimal primes, and so does \( J \).

Denote by \( q_1, \ldots, q_l \) the minimal primes of \( J \). First, we want to show that for each \( 1 \leq i \leq l \) there exists a positive integer \( s_i \) such that \( q_i^{s_i} \subset J \hat{A}_{q_i} \). We proceed as in the proof of [Ran20] Thm. 1.1 (i)]. Set \( p_i := q_i \cap A \). We can assume that \( \hat{A} \) is local at \( p_i \). Let \( \hat{A} \) be the completion of \( A \) with respect to \( p_i \). Set \( \hat{A}' := \hat{A} \otimes_A \hat{A} \) and \( B' := B \otimes_A \hat{A} \). Replace \( \hat{A}' \) and \( B' \) by their reduced structures. Because \( \hat{A} \) is a reduced complete local ring and \( B' \) is a finitely generated \( \hat{A} \)-algebra, then by [Stks, Tag 03GH] \( A' \) is module-finite over \( \hat{A} \). In particular, \( A' \) is Noetherian. Clearly, \( JA' \) is primary to \( q_i A' \). Thus there exists \( s_i \) such that \( q_i^{s_i} A' \subset JA' \). Hence \( q_i^{s_i} b \in A' \). But \( q_i^{s_i} b \in B \). Thus by [Ran20] Prp. 2.2 \( q_i^{s_i} b \in \hat{A}_{p_i} \), and so \( q_i^{s_i} b \in \hat{A}_{q_i} \). This implies \( q_i \in J_{\hat{A}_{q_i}} \) by [ALM69] Prp. 3.14 applied for \( (A, b)/\hat{A} \).

Assume that the \( s_i \) defined above are the minimal possible. Fix \( 1 \leq j \leq l \). For each \( i \neq j \) by prime avoidance we can select \( c_i \in q_i^{s_i} \) and \( c_i \notin q_j \). Let \( c_j \in q_j^{s_j-1} \) with \( c_j \notin J_{\hat{A}_{q_j}} \). Set \( c := c_1 \cdots c_l \). Then \( q_j = (\hat{A} : \hat{c} b) \) and thus \( q_j \in \text{Ass}_{\hat{A}}(B/\hat{A}) \).

**Proof of Theorem 1.1**

We can proceed with the proof of Thm. 1.1. Suppose \( \text{Ass}_{\hat{A}}(B/\hat{A}) \neq \{(0)\} \) and \( \hat{A} \neq B \). Then by Prp. 2.2 \( \text{Ass}_{\hat{A}}(B/\hat{A}) \) contains finitely many nonzero prime ideals which we denote by \( q_1, \ldots, q_r \). By [Ran20] Thm. 1.1 (i)] \( V_i := \hat{A}_{q_i} \) is a DVR for each \( i = 1, \ldots, r \). Obviously, \( \hat{A} \subset \bigcap_{i=1}^r V_i \cap B \). Let \( b = x/y \in \bigcap_{i=1}^r V_i \cap B \). Set \( J := (y \hat{A} : x) \). We have \( JV_i = V_i \) for each \( i \). Thus \( J \notin q_i \) for each \( i \). But \( Jb \in \hat{A} \) and \( J \neq (0) \). So by Prp. 2.3 if \( J \) is a nonunit ideal, its
minimal primes are among \( q_1, \ldots, q_r \) which is impossible. Thus \( J \) has to be the unit ideal, which implies that \( \theta \in \mathcal{A} \).

To prove minimality of the valuation decomposition, suppose we can drop \( \mathcal{V}_j \) in (1) for some \( j \). Then localizing both sides of (1) at \( q_j \) we obtain that \( \mathcal{A}_{q_j} = \mathcal{B}_{q_j} \) which contradicts with \( q_j \in \text{Supp}(B/\mathcal{A}) \). Thus (1) is minimal. We are left with proving the uniqueness of the \( \mathcal{V}_i \)s. Suppose

\[
\mathcal{A} = \cap_{j=1}^s \mathcal{V}'_j \cap B
\]

is a minimal discrete valuation decomposition. Set \( B' := \text{Frac}(\mathcal{A}) \cap B \). We have \( \mathcal{A}_{q_i} \neq \mathcal{B}'_{q_i} \) for each \( i = 1, \ldots, r \). As the intersection in (3) takes place in \( B' \) we can replace in it \( B \) by \( B' \). Localizing both sides of (3) at \( q_1 \) we obtain that there exists a valuation \( \mathcal{V}'_j \) such that \( (\mathcal{V}'_{j})_{q_1} \neq \text{Frac}(\mathcal{A}) \). But \( \mathcal{V}_1 = \mathcal{A}_{q_1} \subset (\mathcal{V}'_{j})_{q_1} \). Thus \( \mathcal{V}_1 = (\mathcal{V}'_{j})_{q_1} \). Also, \( \mathcal{V}'_i \subset (\mathcal{V}'_{j})_{q_1} \), and so \( \mathcal{V}'_i = (\mathcal{V}'_{j})_{q_1} \). Continuing this process we obtain that each \( \mathcal{V}_i \) appears in (3). As (3) is minimal, we obtain that \( s = r \) and after possibly renumbering we get \( \mathcal{V}_i = \mathcal{V}'_i \) for \( i = 1, \ldots, r \).

Let \( \mathcal{A}' \) be the module-finite \( \mathcal{A} \)-algebra defined in the proof of of Prp. 2.1. Suppose \( \mathcal{A} \) is locally formally equidimensional. Then so is \( \mathcal{A}' \). Note that \( \text{Frac}(\mathcal{A}') = \text{Frac}(\mathcal{A}) \). Denote by \( m_{\mathcal{V}_i} \) the maximal ideal of \( \mathcal{V}_i \). Set \( p_i := m_{\mathcal{V}_i} \cap \mathcal{A}' \). By Cohen’s dimension inequality (see [SH06, Thm. B.2.5])

\[
\text{tr. deg}_{\kappa(p_i)}(m_{\mathcal{V}_i}) \leq \text{ht}(p_i) - 1.
\]

Because \( p_i = q_i \cap \mathcal{A}' \), then by [Ran20, Thm. 1.1 (iii)] we get \( \text{ht}(p_i) = 1 \). Therefore, \( \text{tr. deg}_{\kappa(p_i)}(m_{\mathcal{V}_i}) = \text{ht}(p_i) - 1 = 0 \). Hence each \( \mathcal{V}_i \) is a divisorial valuation ring in \( \text{Frac}(\mathcal{A}) \). □

**Remark 2.4.** As it’s well-known, an integrally closed domain equals the intersection of all valuation rings in its field of fractions that contain it. From here one derives set-theoretically that \( \mathcal{A} = \cap \mathcal{V} \cap B \) where the intersection is taken over all valuation rings in \( \text{Frac}(\mathcal{A}) \) that contain the integral closure of \( \mathcal{A} \) in \( \text{Frac}(\mathcal{A}) \). Because \( \mathcal{A}' \) and \( \mathcal{A} \) have the same integral closure and \( \mathcal{A}' \) is Noetherian (see the proof of Prp. 2.1), then in the intersection we can take only DVRs. Thus, the real contribution of Thm. 1.1 is that under the additional hypothesis that \( B \) is a finitely generated \( \mathcal{A} \)-algebra, one can take finitely many uniquely determined DVRs each of which is a localization of \( \mathcal{A} \) at a height one prime ideal.

**Proof of Cor. 1.2 and Cor. 1.3**

Suppose \( \text{Ass}_{\mathcal{A}}(B/\mathcal{A}) \neq \{(0)\} \) and \( \mathcal{A} \neq B \). Because \( \mathcal{A}V_i = \mathcal{A}V_i \cap B \). Also, by Thm. 1.1 \( \mathcal{A} = \cap_{i=1}^r V_i \cap B \). Thus, set-theoretically \( \mathcal{A}_n = \cap_{i=1}^r A_nV_i \cap B_n \). Set \( K := \text{Frac}(R) \). Suppose \( V_i \neq K \) for each \( i = 1, \ldots, r \). Showing that the decomposition is minimal and unique is done in the same way as in Thm. 1.1. Here we will show just the uniqueness of the valuations. Suppose there exist DVRs \( V'_1, \ldots, V'_s \) in \( K \) such that

\[
\mathcal{A} = \cap_{j=1}^s \mathcal{A}V'_j \cap B
\]

is minimal. As in the proof of Thm. 1.1 we can assume that \( \text{Frac}(B) = \text{Frac}(\mathcal{A}) \). If \( V'_j = K \) for some \( j \), then because (4) is minimal \( s = 1 \) and \( V'_1 = K \). So \( \mathcal{A} = AK \cap B \). Localizing at \( q_1 \) we obtain that \( \mathcal{V}_1 = (AK)_{q_1} \). But \( K \subset (AK)_{q_1} \). Thus \( \mathcal{V}_1 = K \), a contradiction. Therefore, \( V'_j \neq K \) for each \( j \). Again, by localizing (4) at \( q_1 \), we get that there is a \( j \) such that
\( \mathcal{V}_i = (\mathcal{A}V'_j)_{q_i} \). But \( V_i = \mathcal{V}_i \cap \text{Frac}(R) \) and \( V'_j \in (\mathcal{A}V'_j)_{q_i} \cap \text{Frac}(R) \). Because \( V_i \neq \text{Frac}(R) \), then \( V_i = V'_j \). Thus \( r = s \) and after possible renumbering \( V_i = V'_j \) for each \( i = 1, \ldots, r \).

Consider Cor. 1.3 Apply Cor. 1.2 with \( A := R[It] \) and \( B := R[t] \). In the introduction we proved that \( R[It]V = R[It]V \) for each valuation \( V \) in \( K \). What remains to be shown is that \( V_i \neq K \) for each \( i \). Indeed, the prime ideals in \( R[It] \) are contractions of extensions of prime ideals of \( R \) to \( R[t] \). Thus \( \text{ht}(q_i \cap R) \geq 1 \) and so \( V_i \neq K \) for each \( i \) otherwise \( q_i \bigcap R[It]_{q_i} \) is a unit ideal which is impossible. \( \square \)

A version of Cor. 1.2 for Rees algebras of modules is proved by Rees in [R87, Thm. 1.7].

Proof of Theorem 1.4

First, we show that \( m_i \in S(\text{ht}(m_i) + e - 1) \). Recall that the maximal ideal of \( \mathcal{V}_i \) contracts to \( q_i \) in \( \mathcal{A} \). Set \( p_i := q_i \cap A \). Then by [Ran20, Thm. 1.1 (iii)] \( \text{ht}(p_i) = 1 \). Consider the map \( \text{Proj}(A_{m_i}) \to \text{Spec}(R_{m_i}) \). It’s closed, surjective and of finite type. By the dimension formula \( (\text{Shks Tag 02JX}) \text{dim} A_{m_i} = \text{ht}(m_i) + e \). But \( A_{m_i} \) is a local formally equidimensional ring. Because \( \text{ht}(p_i, A_{m_i}) = 1 \), by [SH06, Lem. B.4.2] \( \text{dim} A_{m_i}/m_i A_{m_i} = \text{ht}(m_i) + e - 1 \). Thus \( m_i \in S(\text{ht}(m_i) + e - 1) \).

Next, suppose there exists a prime \( n_i \) in \( R \) with \( n_i \subset m_i \) and \( n_i \in S(\text{ht}(m_i) + e - 1) \). Then \( \text{dim} A_{n_i}/n_i A_{n_i} \geq \text{ht}(m_i) + e - 1 \). Note that \( n_i \neq (0) \) for otherwise \( n_i \in S(e) \) which forces \( \text{ht}(m_i) = 1 \), a contradiction. Because \( \text{dim} A_{n_i} = \text{ht}(n_i) + e \) then \( \text{dim} A_{n_i}/n_i A_{n_i} \leq \text{ht}(n_i) + e - 1 \). Therefore,

\[
\text{ht}(m_i) + e - 1 \leq \text{ht}(n_i) + e - 1.
\]

But \( n_i \subset m_i \). Thus \( n_i = m_i \) and \( m_i \) is a minimal prime in \( S(\text{ht}(m_i) + e - 1) \). This completes the proof of Thm. 1.4. \( \square \)

Thm. 1.4 generalizes [Ran18, Thm. 7.8], which is a result for Rees algebras of modules. To see that Thm. 1.4 is sharp, let \((R, m)\) be a Noetherian regular local ring of dimension at least 2, and let \( h \in m \) be an irreducible element. Let \( B \) be the polynomial ring \( R[y_1, \ldots, y_{e+1}] \) for some \( e \geq 0 \). Set \( A := R[y_1, \ldots, y_{e+1}] \). Thus \( A \) is a polynomial subring of \( B \). It is normal because \( R \) is regular. In the setup of Cor. 1.2 there is only one \( V_i = A_{q_i} \) where \( q_i = hA \). Note that \( S(k) = S(e) \) for all \( k \geq 0 \) because \( A \) is a polynomial ring over \( R \) generated by \( e + 1 \) elements. Thus the only minimal prime in \( S(e) \) is \((0)\), whereas \( m_{V_i} = (h) \) is a height one prime ideal in \( R \).

Zariski’s Main Theorem

Let \( A \subset B \) be integral domains with \( A \) Noetherian and \( B \) a finitely generated \( A \)-algebra. In Prp. 2.2 we showed \( \text{Ass}_A(B/A) \) is finite. Denote by \( I_{B/A} \) the intersection of all elements in \( \text{Ass}_A(B/A) \). The following result characterizes the support of \( B/A \).

Proposition 2.5. We have \( \text{Supp}_A(B/A) = \forall(I_{B/A}) \).

Proof. If \( \text{Frac}(A) \neq \text{Frac}(B) \), then \((0) \in \text{Ass}_A(B/A) \) and trivially \( \text{Supp}_A(B/A) = \forall(I_{B/A}) = \text{Spec}(A) \). Suppose \( \text{Frac}(A) = \text{Frac}(B) \). Then by [Ran20, Thm. 1.1 (i)] \( I_{B/A} = q_1 \cap \ldots \cap q_s \) with \( \text{ht}(q_i) = 1 \) for each \( i \). If \( q \in \forall(I_{B/A}) \), then \( q_j \subset q \) for some \( j \) and thus \( q_j A_q \in \text{Ass}_A(B_q/A_q) \).

Hence \( \forall(I_{B/A}) \subset \text{Supp}_A(B/A) \). Suppose \( q \subset \text{Supp}_A(B/A) \). Then there is \( x/y \in B \) with \( x, y \in A \) such that its image in \( B_q \) is not \( A_q \). In other words, if \( J := ((x): A_y) \), then \( J \subset q \).

But by Prp. 2.3 the minimal primes of \( J \) are among the \( q_j \)’s. Thus there exists \( q_j \) such that \( q_j \subset q \), i.e. \( \text{Supp}_A(B/A) \subset \forall(I_{B/A}) \). \( \square \)
Definition 2.6. Let $Q \in \text{Spec}(B)$. Set $q := Q \cap \mathcal{A}$ and $p := Q \cap A$. We say that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is quasi-finite at $Q$ if $Q$ is isolated in its fiber, i.e. if the field extension $\kappa(p) \subset \kappa(Q)$ is finite and $\dim(B_Q/pB_Q) = 0$. We say that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is quasi-finite if it is quasi-finite at each prime in $\text{Spec}(B)$.

Theorem 2.7 (Zariski’s Main Theorem). Let $A \subset B$ be integral domains. Suppose $A$ is Noetherian and $B$ is a finitely generated $A$-algebra. Assume $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is quasi-finite at $Q$. Then there exists $f \in \mathcal{I}_{B/A}$ with $f \notin q$ such that $B_f = \mathcal{A}_f$.

Proof. First we show that $\mathcal{A}_q = B_q$. Assume $\mathcal{A}$ is local at $q$. Let $b \in B$. Suppose $b \notin Q$. Using the assumption that $\kappa(p) \subset \kappa(Q)$ is a finite field extension, we will show that $b$ is a unit in $\mathcal{A}$. Set $q' := Q \cap A[b]$. Because $\kappa(p) \subset \kappa(Q)$ is a finite field extension, so is $\kappa(q) \subset \kappa(q')$. Thus $b$ satisfies the following relation

$$\beta_kb^k + \beta_{k-1}b^{k-1} + \cdots + \beta_0 = 0$$

with $\beta_i \in \mathcal{A}$ and at least one $\beta_j$ not in $q$. Thus $b \in \text{Frac}(\mathcal{A})$ because the latter is algebraically closed in $\text{Frac}(B)$. Write $b = c/d$ for $c, d \in \mathcal{A}$. Set $J := ((d) : \mathcal{A} c)$. If $J$ is nonunit ideal, then by Prp. 2.3 $\sqrt{J} = q_1 \cap \ldots \cap q_l$ with $q_i \in \text{Ass}(B/A)$ for each $i$. By Ran2 Thm. 1.1 (i) $\mathcal{A}_q := (\mathcal{V}_i, (t_i))$ is a DVR for each $i$. Denote by $v_i$ the valuation associated with $\mathcal{V}_j$.

Suppose $\beta_k \notin J$. Then there exists $1 \leq j \leq l$ such that $J = (t_j)$ and $\beta_k = t_j^{\alpha-\ell}k$ with $l_k < \alpha$ and $\beta_k$ a unit in $\mathcal{V}_j$. Multiplying both sides of (5) by $t_j^{\alpha-\ell}$, we get that the two leading terms of (5) combine to $(t_j^{\alpha}k^\alpha b + t_j^{\alpha-\ell}k^{\alpha-\ell-1})b^{k-1}$. But $t_j^{\alpha}k^\alpha b + t_j^{\alpha-\ell}k^{\alpha-\ell-1}$ is a unit in $\mathcal{V}_j$ because $v_j(t_j^{\alpha}b) = 0$ and $\alpha > l_k$. Thus (5) gives an equation of integral dependence for $b$ over $\mathcal{V}_j$. The latter is integrally closed. Thus $b \in \mathcal{V}_j$ and so $J$ is the unit ideal, i.e. $b \in \mathcal{A}$.

Suppose $\beta_k \in J$. Then there exists $d_k \in \mathcal{A}$ such that $\beta_kc = dd_k$, which yields

$$\beta_kb^k = \frac{\beta_k c^{k-1}}{d} = d_kb^{k-1}.$$ 

Because $\beta_k \in q$ and $\beta_kb = d_k$ we get $d_k \in Q$, hence $d_k \in q$. Continuing this process we get that (5) is eventually equivalent to

$$\begin{align*}
(d_{l+1} + \beta_l)b^l + \cdots + \beta_0 &= 0.
\end{align*}$$

where $d_{l+1} \in q$ and $\beta_l$ is the coefficient in (5) with largest $l$ such that $\beta_l \notin q$. Thus, $d_{l+1} + \beta_l$ is a unit in $\mathcal{A}$. Therefore, (6) gives an integral dependence relation for $b$, i.e. $b \in \mathcal{A}$.

We just proved that each $b \in B$ with $b \notin Q$ is a unit in $\mathcal{A}$; hence it is a unit in $B$. Therefore, $B$ is local with maximal ideal $Q$. By assumption $\dim(B/pB) = \dim(B/qB) = 0$. Thus $qB$ is $Q$-primary. Suppose $b \in Q$. Then $\mathcal{A}[b]$ is local with maximal ideal $q'$ such that $qA[b]$ is $q'$-primary. So there exists $u$ such that $b^u \in q[A][b]$ which gives the same relation for $b$ as (5).

Finally, we conclude that $b \in \mathcal{A}$, i.e. $\mathcal{A}_q = B_q$. In particular, $\text{Frac}(\mathcal{A}) = \text{Frac}(B)$. Therefore, for each $f \in \mathcal{I}_{B/A}$ we have $\mathcal{A}_f = B_f$. By Prp. 2.5 $\mathcal{I}_{B/A} \not\subset q$. So we can select $f \in \mathcal{I}_{B/A}$ with $f \notin q$.

As an immediate corollary we get an explicit description of the quasi-finite locus in $B$.

Corollary 2.8. Let $A \subset B$ be integral domains. Suppose $A$ is Noetherian and $B$ is a finitely generated $A$-algebra. Set $g: \text{Spec}(B) \rightarrow \text{Spec}(A)$ and $U := \text{Spec}(A) - \mathbf{V}(\mathcal{I}_{B/A})$. Then $g^{-1}U$ is the set of primes at which $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is quasi-finite.
The following corollary due to Grothendieck shows that we can replace \( \mathcal{A} \) by a subalgebra which is module-finite over \( \mathcal{A} \). It’s what is used in algebraic geometry to show that a quasi-finite morphism factors as a composition of an open immersion and a finite morphism.

**Corollary 2.9.** [EGAIII, Cor. 4.4.7] Let \( \mathcal{A} \subset \mathcal{B} \) be integral domains. Suppose \( \mathcal{A} \) is a local Noetherian ring with maximal ideal \( p \) and \( \mathcal{B} \) is a finitely generated \( \mathcal{A} \)-algebra. Assume \( \text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A}) \) is quasi-finite at \( Q \) and \( Q \cap \mathcal{A} = p \). Then there exists a finite \( \mathcal{A} \)-algebra \( \mathcal{A}' \) with a maximal ideal \( m' \) such that \( \mathcal{B}_Q = \mathcal{A}'_{m'} \).

**Proof.** Let \( \mathcal{A}' \) be the subalgebra of \( \mathcal{A} \) defined in the proof of Prp. 2.1. By Thm. 2.7 \( \text{Frac}(\mathcal{A}) = \text{Frac}(\mathcal{B}) \). Thus there exists \( a \in \mathcal{A} \) such that \( \mathcal{B}_a = \mathcal{A}_a = \mathcal{A}'_a \). Moreover, \( a \in \mathcal{I}_{\mathcal{B}/\mathcal{A}} \). Set \( m' := Q \cap \mathcal{A}' \). By Thm. 2.7 \( a \notin Q \). Thus \( \mathcal{B}_Q = \mathcal{A}'_{m'} \). \( \square \)

In [EGAIII, Cor. 4.4.9] Grothendieck applies Cor. 2.9 to derive the following result: if \( g: X \to Y \) is a birational, proper morphism of noetherian integral schemes with \( Y \) normal and \( g^{-1}(y) \) finite for each \( y \in Y \), then \( g \) is an isomorphism. Below we show that in the affine case we can reach the same conclusion assuming just that \( g \) is surjective. To do this we do not have to appeal to Thm. 2.7.

**Theorem 2.10.** Let \( \mathcal{A} \subset \mathcal{B} \) be integral domains with the same field of fractions. Suppose \( \mathcal{A} \) is Noetherian and \( \mathcal{B} \) is a finitely generated \( \mathcal{A} \)-algebra. Assume \( \text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A}) \) is surjective. Then \( \mathcal{B} = \mathcal{A} \).

**Proof.** Suppose there exists \( q \in \text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A}) \). Because \( q \neq (0) \), by [Ran20, Thm. 1.1 (i)] \( \mathcal{A}_q \) is a DVR. But \( \mathcal{A}_q \neq \mathcal{B}_q \). Thus \( \mathcal{B}_q = \text{Frac}(\mathcal{B}) \). This contradicts the assumption that there exists a prime in \( \mathcal{B} \) that contracts to \( q \). Thus \( \text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A}) \) is empty, and so by Prp. 2.5 \( \mathcal{B} = \mathcal{A} \). \( \square \)

**Remark 2.11.** In the proof of Thm. 2.7 we translate directly the quasi-finite assumption to the polynomial relation \( \mathcal{I} \), which by our results about \( \text{Ass}_{\mathcal{A}}(\mathcal{B}/\mathcal{A}) \) transforms into an integral dependence relation. In Cor. 2.8 using our characterization of the support of \( \mathcal{B}/\mathcal{A} \) we give an explicit description of \( U \) in the domain case. In general, Cor. 2.8 is usually stated as an existence result (cf. [Stks, Tag 03GT] and [Ray70, Cor. 1, p.42]).

Zariski worked under the assumption that \( \mathcal{A} \) and \( \mathcal{B} \) have the same field of fractions. In [Z43, pg. 522–527] he gave a lengthy proof of his main theorem using an inductive argument on the number of generators for \( \mathcal{B} \) as an \( \mathcal{A} \)-algebra and a study of conductor ideals. This inductive approach was used by Peskine in [Pes66] to generalize Zariski’s main theorem to ring maps \( \mathcal{A} \to \mathcal{B} \) of finite type. In [Z49] Zariski gave a simple proof of his theorem using valuations but under the additional hypothesis that \( \mathcal{A} \) is a domain of finite type over a field. In [EGAIII, Thm. 4.4.3] (see also [EGAIV, Thm. 8.12.6]) Grothendieck proved a global version of Zariski’s main theorem using the formal function theorem and cohomology.

**References**


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