Projective normality of G.I.T. quotient varieties modulo finite groups.

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Abstract

We prove that for any finite dimensional representation V of a finite group G of order n the quotient variety $G\backslash\mathbb{P}(V)$ is projectively normal with respect to descent of $\mathcal{O}(1)^{\otimes l}$ where $l = lcm\{1, 2, 3, 4, \cdots, n\}$. We also prove that for the tautological representation V of the Alternating group A_n the projective variety $A_n\backslash\mathbb{P}(V)$ is projectively normal with respect to the descent of the above line bundle.

Keywords Projective normality; Line bundle; G.I.T. Quotient.

2010 Mathematics Subject Classification. 14L30.

1 Introduction

Let V be a finite dimensional representation of a finite group G over a field K. We denote by $K[V]$ the algebra of polynomial functions on V, which we define to be the symmetric algebra on V^*, the dual of V. In other words, the space of homogeneous forms on V of degree m denoted by $K[V]_m$ is $Sym^m(V^*)$ the m^{th} symmetric power of V^* and we have $K[V] = \oplus_{m\geq 0}Sym^m(V^*)$. Then G acts on the algebra $K[V]$ by the formula $(gf)(v) := f(g^{-1}.v)$. Then the ring of G-invariant polynomials $K[V]^G := \{f \in K[V] : gf = f, \ \forall g \in G\}$ inherits a grading from $K[V]$. In [17] and [18] Emmy Noether gave two different constructive proofs for the fact that $K[V]^G$ is finitely generated K algebra. So, when K is algebraically closed, it is an interesting problem to study the GIT- quotient varieties $G\backslash V = Spec(K[V]^G)$ and $G\backslash\mathbb{P}(V)$ (see [14] and [15]). Since $G\backslash V = Spec(K[V]^G)$ is a normal variety as $K[V]^G$ is integrally clased in its field $K(V)^G$ of fractions, it is interesting to study the projective normality of the quotient veriety $G\backslash\mathbb{P}(V)$. In [10] the projective normality of the polarized variety $(G\backslash\mathbb{P}(V), \mathcal{L})$, where $\mathcal{L}$ is the descent of $\mathcal{O}(1)^{\otimes|G|}$ is considered and it is proved that when G is either a solvable group or a finite subgroup of $GL(V)$ generated by pseudo reflections the above polarized variety is projectively normal. Note that such a descent exists and is unique (see page 63, Theorem 2.3 of [4]). We also note that when G is a finite subgroup of $GL(V)$ generated by pseudo reflections, only for the representation V of G the result is known and the result is not known for other representations of G. In [3] the above results are obtained for every finite group but with the descent of the line bundle $\mathcal{O}(1)^{\otimes|G|!}$ and they are reproved in [11]. In this article we prove projective normality for any finite dimensional representation of any finite group of order n with respect to the descent of the line bundle $\mathcal{O}(1)^{\otimes l}$ where $l = lcm\{1, 2, 3, 4, \cdots, n\}$ which is much smaller than $n!$. We also show that for the tautological representation V of the alternating group A_n the polarized variety $(A_n\backslash\mathbb{P}(V), \mathcal{L})$ is projectively normal, where $\mathcal{L}$ is the descent of $\mathcal{O}(1)^{\otimes l}$.

The results in this paper arose out of an attempt to understand the quotient $W\backslash(T\backslash\backslash(G/P)^{ss}(\mathcal{L}_r))$, where G is a semisimple simply connected algebraic group, T is a fixed maximal torus in G, P is the maximal parabolic subgroup associated to a simple root α_r, W is the Weyl group of G with respect to T and $\mathcal{L}_r$ is the line bundle associated to the fundamental weight ω_r. For the details see Theorem 4.4 of [8] and Theorem 3.3 of [9]. The second author of this paper is working on understanding the projective normality of torus quotient of the Grassmannian and flag variety for the action of the Weyl group with respect to a suitable ample line bundle.

The layout of the paper is as follows. Section 2 consists of preliminary definitions and notations. In section 3 we prove the projective normality result for any finite dimensional representation of a finite group and in section 4 we prove projective normality for the tautological representation of the alternating group.

2 Preliminary notations and a lemma

Let n be a positive integer and K be an algebraically closed field of characteristic not dividing $n!$. Let $V = K^n$ be the natural representation of the symmetric group S_n, i.e., the action is by permuting the coordinates. Then the restriction of the representation to the alternating group A_n is called the tautological representation (see page 5 of [16]).

Let $\mathcal{O}(1)$ denote the ample generator of the Picard group of $\mathbb{P}(V)$. We set $\mathcal{O}(1)^{\otimes d} := \underbrace{\mathcal{O}(1) \otimes \mathcal{O}(1) \cdots \otimes \mathcal{O}(1)}_{d\ times}$.

Then we have the following lemma.

Lemma 2.1. *The line bundle $\mathcal{O}(1)^{\otimes l}$, where $l = lcm\{1, 2, 3, 4, \cdots, n\}$ on $\mathbb{P}(V)$, descends to the quotient $S_n\backslash\mathbb{P}(V)$.*

Proof. Let $G = S_n$. By a theorem of Kempf (Page 63, Theorem 2.3 of [4]), a line bundle $\mathcal{L}$ on $\mathbb{P}(V)$ descend to the quotient $G\backslash\mathbb{P}(V)$ if and only if the stabilizer G_x of each point $x \in \mathbb{P}(V)$ acts trivially on the fiber $\mathcal{L}_x$. In our case G acts on $\mathcal{O}(1)$ and G_x acts on $\mathcal{O}(1)_x$ through a character $\chi_x : G_x \to \mathbb{C}^*$ and on $\mathcal{O}(1)_x^{\otimes l}$ through the character χ_x^l. Now if $\sigma \in S_n$, we write σ as a product of disjoint cycles. Then order of σ is the lcm of lengths of cycles occur in the product and so order of σ divides l. So G_x acts trivially on $\mathcal{O}(1)_x^{\otimes l}$. Hence, $\mathcal{O}(1)^{\otimes l}$ descends to the quotient $G\backslash\mathbb{P}(V)$. □

Remark 1: The above statement holds true and the proof is exactly the same if we replace S_n by the alternating group A_n.

Remark 2: Note that if we have a finite group G acting on a projective variety X and suppose a G equivariant line bundle on X descends to the quotient $G\backslash X$, then the descendend line bundle is unique. Since G is finite we have $X^{ss} = X$ and if $\pi : X \to G\backslash X$ is the quotient morphism, then the descendend line bundle is $\pi_*(\mathcal{L})^G$.

We will denote by $\mathcal{L}$ the descent of the line bundle $\mathcal{O}(1)^{\otimes l}$ to the quotient $S_n\backslash\mathbb{P}(V)$ as well as to the quotient $A_n\backslash\mathbb{P}(V)$.

Now we recall the definition of prjective normality of a projective variety. A projective variety X is said to be projectively normal if the affine cone $\hat{X}$ over X is normal at its vertex. For a reference, see exercise 3.18, page 23 of [6]. For the practical purpose we need the following fact about projective normality of a polarized variety.

A polarized variety $(X, \mathcal{L})$ where $\mathcal{L}$ is a very ample line bundle is said to be projectively normal if its homogeneous coordinate ring $\oplus_{n\in\mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes n})$ is integrally closed and it is generated as a K-algebra by $H^0(X, \mathcal{L})$ (see Exercise 5.14, Chapter II of [6]).

The polarized variety ($S_n\backslash\mathbb{P}(V), \mathcal{L}$) is

$$Proj(\oplus_{m\in\mathbb{Z}_{\geq 0}}(H^0(\mathbb{P}(V), \mathcal{O}(1)^{\otimes m(n!)}))^{S_n}) = Proj(\oplus_{m\in\mathbb{Z}_{\geq 0}}(Sym^{m(n!)}(V^*))^{S_n}).$$

For a reference, see Theorem 3.14 and page 76 of [14].

Before ending this section we recall the definition of polarizations of a polynomial from page 5 of [21]. Let V be a finite dimensional representation of a finite group G. Let $f \in K[V]^G$ be a homogeneous polynomial of degree d. For $v_1, v_2, \cdots, v_m \in V$ and $t_1, t_2, \cdots, t_m$ are indeterminates, we consider the function $f(\sum_i t_iv_i)$. Then

$$f(\sum_i t_iv_i) = \bigoplus_{\alpha\in(\mathbb{Z}^+)^m, |\alpha|=d} f_\alpha(v_1, \cdots, v_m)t^\alpha, \tag{1}$$

where the $f_\alpha \in K[V^{\oplus m}]^G$ are multihomogeneous of the indicated degree α. Here for $\alpha = (a_1, a_2, \cdots, a_m) \in (\mathbb{Z}^+)^m$, we have $t^\alpha = t^{a_1} \ldots t^{a_m}$ and $|\alpha| = a_1 + \ldots + a_m$. We call the polynomials f_α, the *polarizations* of f.

Polarizations of a polynomial can also be defined in terms of some linear differential operators called the polarization operators (see [20]). Choosing a basis for V and writing $v_i = (x_{i1}, \cdots, x_{in})$ we define

$$D_{ij} = \sum_{k=1}^{n} x_{ik}\frac{\partial}{\partial x_{jk}}.$$

The operators D_{ij}'s are called polarization operators. They commute with the action of G on $K[V^{\oplus m}]$ and applying successively operators D_{ij} $(i > j)$ to $f \in K[V]^G$ we obtain precisely (up to a constant) the polarizations of f in any number of variables.

3 Projective normality of finite group quotients

Before proving the main results in this section we will prove a combinatorial result which is the main ingredient of the proof.

Lemma 3.1. *Given a positive integer* $r \geq 9$, *suppose* l *denotes the least common multiple of* $1, 2, \cdots, r$. *Then, we have:*

$$l > \frac{r(r^2+1)(r-1)}{4}$$

Proof. We will verify the inequality for $9 \leq r \leq 16$. Note that $lcm\{1, 2, \cdots, 10\} = lcm\{1, 2, \cdots, 9\} = 2520 > \frac{10\times9\times101}{4}$. Note that $lcm\{1, 2, \cdots, r, r+1, \cdots, r+i\} \geq lcm\{1, 2, \cdots, r\}$, for i a positive integer. Since $\frac{r(r^2+1)(r-1)}{4}$ is an increasing function of r and $27720 = lcm\{1, 2, \cdots, 11\} \geq \frac{16\times257\times15}{4} = 15420$ the inequality is verified for $9 \leq r \leq 16$.

Now we prove the inequality for $r \geq 17$. Consider the Chebyshev function defined as: $\psi(x) = \sum_{n\leq x} \Lambda(n)$, where $\Lambda(n) = \log p$ if n is a power of some prime p and is equal to 0 otherwise. Then, we'll see that our inequality follows from the fact that $l = e^{\psi(r)}$ and the inequality $\psi(x) \geq (x-2)\log 2$ for all $x \geq 4$ (see page 37 of [19]).

Then, $4l = 4e^{\psi(r)} \geq 4e^{(r-2)\log 2} = 2^r > r^4$ for $r \geq 17$, where the last inequality follows by induction. Next, we have $l > \frac{r^4}{4} > \frac{r^4-1}{4} = \frac{(r^2-1)(r^2+1)}{4} \geq \frac{(r^2-r)(r^2+1)}{4} = \frac{r(r^2+1)(r-1)}{4}$. □

Proposition 3.2. *Given* $(m_1, m_2, \cdots, m_r) \in \mathbb{Z}^r_{\geq 0}$, *such that:*

$$\sum_{i=1}^{r} i.m_i \geq q.l - \tfrac{r.(r-1)}{2}$$

where l denotes the least common multiple of $1, 2, \cdots, r$ and $q \geq 2$ is an integer. Then there exist $(m'_1, m'_2, \cdots, m'_r)$ satisfying $0 \leq m'_i \leq m_i \ \forall\ i$ such that:

$$\sum_{i=1}^{r} i.m'_i = l$$

Proof. We will give a rigorous proof for $r \geq 9$ and after proposition 3.3 we will illustrate the proof for $1 \leq r \leq 8$. For every i, we have $i|l$. Hence, if for some i, we have $i.m_i \geq l$, (i.e. $m_i \geq l/i$) we can take $m'_i = l/i$ and $m'_k = 0$ for all $k \neq i$, to get $\sum_{i=1}^{r} i.m'_i = l$. So, let us assume that $i.m_i < l$ for all i. Next, it is easy to see that for some j, we must have $j.m_j > \frac{q.l}{r} - \frac{r-1}{2}$. Fixing this j, we construct a set S as follows:

$$S = \left\{ \sum_{i \neq j} i.b_i : j|b_i, 0 \leq b_i \leq m_i \forall 1 \leq i \leq r, i \neq j \right\}.$$

Then, we will see in Proposition 3.3 which is proved below that we have an integer $s(= \sum_{i \neq j} i.s_i)$ in S that lies between $l - \frac{q.l}{r} + \frac{r-1}{2}$ and l. Note that $l - \frac{q.l}{r} + \frac{r-1}{2} > 0$. This follows from the fact that $q \leq r$ as $i.m_i < l$ for all i by assumption. Also, since $j|s$, there exists $t \geq 0$ such that $s + t.j = l$. By the bound on s, we get that $t.j = l - s \leq \frac{q.l}{r} - \frac{r-1}{2} < j.m_j$ and so $t < m_j$. So, if we define $m'_i = s_i$ for all $i \neq j$ and $m'_j = t$, we get that $\sum_{i=1}^{r} i.m'_i = l$ and this concludes the proof. ☐

Proposition 3.3. *Consider the set S defined as in Proposition 3.2. Then, there exists an element of S lying between $l - \frac{q.l}{r} + \frac{r-1}{2}$ and l.*

Proof. As we have put the condition that $j|b_i$ for all i, each member of S must be divisible by j. Let us try to get a bound on the largest number in S. For each $i \neq j$, we have $b_i \leq m_i$ and $j|b_i$. Hence, if c_i denotes the maximum possible value that b_i can take, we have $c_i \geq m_i - (j-1)$. Thus, if m is the largest number contained in S, we have:

$$m = \sum_{i \neq j} i.c_i \geq \sum_{i \neq j} i.(m_i - j + 1) = \sum_{i=1}^{r} i.m_i - j.m_j - (j-1). \sum_{i \neq j} i \geq q.l - \tfrac{r.(r-1)}{2} - j.m_j - (j-1)(\tfrac{r(r+1)}{2} - j)$$

As a function of j, $(j-1)(\frac{r(r+1)}{2} - j)$ is increasing and hence maximised when $j = r$ with the maximum value $\frac{r.(r-1)^2}{2}$. As $jm_j < l$ by assumption, the above inequality becomes:

$$m > q.l - \frac{r.(r-1)}{2} - l - \frac{r.(r-1)^2}{2} \geq l - \frac{r^2(r-1)}{2}, \text{ for } q \geq 2.$$

Using the inequality $\frac{q.l}{r} \geq \frac{2l}{r} > \frac{(r^2+1).(r-1)}{2}$ from Lemma 3.1, we have $m \geq l - \frac{q.l}{r} + \frac{r-1}{2}$. Therefore, we conclude that the set S consists of at least one number greater than $l - \frac{q.l}{r} + \frac{r-1}{2}$.

Now, let $m' = \sum_{i \neq j} i.d_i$ be the largest number in S less than $l - \frac{q.l}{r} + \frac{r-1}{2}$. By the bound obtained above, we have $m' < m$. Thus, there exists a k for which we have $d_k < c_k$. Also, as d_k and c_k are both divisible by j, we must have $d_k \leq c_k - j$. Therefore, $s := m' + j.k$ must belong to S (Note that $m' + j.k$ is obtained by replacing d_k by $d_k + j$ in the given sum). By the maximality of m', we must have $s \geq l - \frac{q.l}{r} + \frac{r-1}{2}$. Also, as $m' < l - \frac{q.l}{r} + \frac{r-1}{2}$ and $jk \leq r^2$, we get $s = m' + j.k < l - \frac{q.l}{r} + \frac{r-1}{2} + r^2 < l$ because $\frac{q.l}{r} \geq \frac{2l}{r} > \frac{(r^2+1)(r-1)}{2} > \frac{r-1}{2} + r^2$ by Lemma 3.1. Hence, we have shown that $l - \frac{q.l}{r} + \frac{r-1}{2} < s < l$.

□

Now we will verify Proposition 3.2 for $1 \le r \le 8$. We prove a lemma here which will be helpful for the verification.

Lemma 3.4. *Consider 4 non-negative integers a, b, m, n and let $s \in \mathbb{N}$ be such that $a|s$ and $b|s$. Suppose $am + bn = s + t$ for some positive integer t. Let $g = gcd(a, b)$. Suppose $t \ge g(\frac{a}{g} - 1)(\frac{b}{g} - 1)$. Then, there exist m' and n' such that $0 \le m' \le m$ and $0 \le n' \le n$ such that $am' + bn' = s$.*

Proof. Firstly, let a and b be co-prime. Then, $g = 1$. Now, if $am \ge s$, we can choose $m' = s/a$ and $n' = 0$ and we'll be done. So, let $am < s$, i.e., $bn > t$. Similarly, we can also assume that $am > t$. Now, we know that $t \ge (a-1)(b-1)$. So, by Chicken McNugget Theorem (also known as Postage Stamp Theorem or Frobenius Coin Theorem) (see Proposition 1.17 of [13]), there exist positive integers c and d such that $ac + bd = t$. So, we have $am > t \ge ac$. Thus, $m > c$. Similarly, $n > d$. Therefore, taking $m' = m - c$ and $n' = n - d$, we have $am' + bn' = am + bn - ac - bd = s$, and so, we're done.

Now, consider the general case. As before, we have $am, bn > t$. Let $a' = a/g$ and $b' = b/g$. Then $gcd(a', b') = 1$. Again, by Chicken McNugget Theorem, there exist positive integers c, d such that $a'c + b'd = t/g$. Hence, $ac + bd = ga'c + gb'd = t$. Thus, taking $m' = m - c$ and $n' = n - d$ in this case too, we are done. □

Now, consider the problem when $r = 8$. So, we have non-negtive m_i's such that $\sum_{i=1}^{8} i.m_i \ge 1652$ where we have taken $q = 2$. We transform the m_i's as follows. As long as $m_1 \ge 2$, we can reduce m_1 by 2 and increase m_2 by 1, without changing $\sum_{i=1}^{8} i.m_i$. Similarly, we can reduce m_2, m_3 and m_4 by increasing m_4, m_6 and m_8 respectively. Let us assume that these transformations change each m_i to c_i such that $c_1, c_2, c_3, c_4 \le 1$ and $\sum_{i=1}^{8} i.c_i = \sum_{i=1}^{8} i.m_i \ge 1652$. This implies that $5c_5 + 6c_6 + 7c_7 + 8c_8 \ge 1642$. We'll be done if we can find a_5, a_6, a_7, a_8 such that each $a_i \le c_i$ and $5a_5 + 6a_6 + 7a_7 + 8a_8 = 840$. Therefore without loss of generality, we may assume that $5c_5, 6c_6, 7c_7, 8c_8 < 840$.

Now, we must have $5c_5 + 7c_7 \ge 821$ or $6c_6 + 8c_8 \ge 822$. Without loss of generality, we assume that the former holds true. Thus, $5c_5 + 7c_7 \ge 821$. Now, we try to get an upper bound on $5c_5 + 7c_7$. Let $5c_5 + 7c_7 = 840 + t$ for some integer t which is not necessarily non-negative. By Lemma 3.4, if $t \ge 24$, we'll be able to get a_5 and a_7 such that $0 \le a_5 \le c_5$, $0 \le a_7 \le c_7$ and $5a_5 + 7a_7 = 840$. Hence, let us assume that $5a_5 + 7a_7 < 864$. Thus, $6a_6 + 8a_8 > 778$. The proof for the other cases are similar.

So, we have $821 \le 5c_5 + 7c_7 < 864$ and $778 < 6c_6 + 8c_8$. Now, we must have $5c_5 > 400$ or $7c_7 > 400$. Similarly, we must have $6c_6 > 300$ or $8c_8 > 300$. Without loss of generality, let $5c_5 > 400$ and $6c_6 > 300$. Next, find the maximum d_5 with $d_5 \le c_5$ such that $5d_5 + 7c_7 \le 840$. Also, by maximality of d_5, $5d_5 + 7c_7 > 800$. Next, choose the maximum d such that $d \le d_5$ and $5d + 7c_7$ is divisible by 6. Then, $d_5 - d \le 5$ and so, $5d + 7c_7 > 775$. Thus, choosing $a_5 = d$, $a_7 = c_7$, $a_8 = 0$ and $a_6 = (840 - 5d - 7c_7)/6$, we will be done. As can be seen, there was nothing special about 5 and 6, and that, each case can be resolved using similar arguments.

For $r = 7$, the argument is similar. We have $\sum_{i=1}^{7} i.m_i \ge 819$. After reducing m_1, m_2, m_3 as we did in the previous case, we get c_i's such that $4c_4 + 5c_5 + 6c_6 + 7c_7 \ge 813$. Using the same chain of arguments as done for $r = 8$, we can find a_4, a_5, a_6, a_7 such that $a_i \le c_i$ and $4a_4 + 5a_5 + 6a_6 + 7a_7 = 420$.

For $r = 6$, we have $\sum_{i=1}^{6} i.m_i \ge 105$. Applying the transformations to m_1, m_2 and m_3, we get c_i's such that $4c_4 + 5c_5 + 6c_6 \ge 99$. By Lemma 3.4, if $4c_4 + 6c_6 \ge 64$, we will be able to find the required a_i's. Thus, let $4c_4 + 6c_6 \leq 62$. Thus, $5c_5 \geq 37$. This implies that $5c_5 \geq 40$. Hence, $4c_4 + 6c_6 \leq 59$. This implies that $4c_4 + 6c_6 \leq 58$. Hence, $5c_5 \geq 41$ and being a multiple of 5, this implies $5c_5 \geq 45$. So, $4c_4 + 6c_6 \leq 54$. Also, if $5c_5 \geq 60$, we can take $a_5 = 12$ and we are done. So, let $5c_5 \leq 55$ and thus, $4c_4 + 6c_6 \geq 44$.

In conclusion, $44 \leq 4c_4 + 6c_6 \leq 54$ and $45 \leq 5c_5 \leq 55$. We must have either $c_4 \geq 5$ or $c_6 \geq 5$. Without loss of generality, let $c_4 \geq 5$. Let d be the largest integer such that $d \leq c_4$ and $4d + 6c_6$ is divisible by 5. Then, $c_4 - d \leq 4$. Thus, $4d + 6c_6 \geq 28$. As it is divisible by 5, $4d + 6c_6 \geq 30$. As $5c_5 \geq 45$, taking $a_4 = d$, $a_6 = c_6$ and $a_5 = (60 - 4d - 6c_6)/5$ we are done.

Next, let $r = 5$. We have $\sum_{i=1}^{5} i.m_i \geq 110$. Applying the transformations to m_1 and m_2, we get c_i's such that $3c_3 + 4c_4 + 5c_5 \geq 107$. By Lemma 3.4, if $3c_3 + 4c_4 \geq 66$, we will get the required a_i's. So, let $3c_3 + 4c_4 \leq 65$ and so, $5c_5 \geq 42$. This implies that $5c_5 \geq 45$. But, if $5c_5 \geq 60$, we will be done by taking $a_5 = 12$ and all the other a_i's as 0. So, let $5c_5 \leq 55$. Thus, $3c_3 + 4c_4 \geq 52$. Therefore, at least one of c_3 and c_4 is greater than or equal to 5. Then, the same argument as above gives us the required a_i's.

For $1 \leq r \leq 4$, Proposition 3.2 can be verified similarly. Therefore, Proposition 3.2 is true for $1 \leq r \leq 8$.

Corollary 3.5. *Let $l = lcm\{1, 2, \cdots, r\}$. Then the semigroup $M_l = \{(m_1, m_2, \cdots, m_r) \in \mathbb{Z}^r_{\geq 0} : \sum_{i=1}^{r} m_i i \equiv 0 \, mod \, l\}$ is generated by the set $S_l = \{(m_1, m_2, \cdots, m_r) \in \mathbb{Z}^r_{\geq 0} : \sum_{i=1}^{r} m_i i = l\}$.*

Proof. The proof follows from Proposition 3.2. ☐

Remark: The affine sub-semigroup generated by the set $S_l = \{(m_1, m_2, \cdots, m_r) \in \mathbb{Z}^r_{\geq 0} : \sum_{i=1}^{r} m_i i = l\}$ is normal. For the definition of the normality of an affine semigroup refer to page 61 of [1].

Let V be the natural representation of S_n. Then for every integer $m \geq 1$, S_n acts on the algebra $\mathbb{C}[V^m]$ of polynomial functions on the direct sum $V^m := V \oplus \cdots \oplus V$ of m copies of V via the diagonal action

$$(\sigma f)(v_1, \cdots, v_m) := f(\sigma^{-1} v_1, \cdots, \sigma^{-1} v_m), f \in \mathbb{C}[V^m], \sigma \in S_n.$$

Theorem 3.6. *Let V be the natural representation of the Symmetric group S_n. Then for $m \geq 1$, the variety $S_n \backslash \mathbb{P}(V^m)$ is projectively normal with respect to the descent of the line bundle $\mathcal{O}(1)^{\otimes l}$, where $l = lcm\{1, 2, 3, \cdots, n\}$*

Proof. The algebra $K[V]^{S_n} = K[x_1, x_2, \cdots, x_n]^{S_n}$ is the polynomial ring $K[e_1, e_2, \cdots, e_n]$, where e_i's are the elementary symmetric polynomials in x_k's. For the diagonal action of S_n on V^m by a theorem of H. Weyl (see pages 36-39 of [19]), the algebra $\mathbb{C}[V^m]^{S_n}$ is generated by polarizations of $e_1, e_2, \cdots, e_n$.

For each $i \in \{1, 2, \cdots, n\}$, let $\{e_{ij} : j = 1, 2, \cdots a_i\}$ denote the polarizations of e_i where a_i is a positive integer. Since the polarization operators $D_{ij} = \sum_{k=1}^{n} x_{ik} \frac{\partial}{\partial x_{jk}}$ do not change the total degree of the original polynomial, we have

$$\text{degree of } e_{ij} = \text{degree of } e_i = i, \ \forall \ j = 1, 2, \cdots a_i. \quad (2)$$

Let $R := \oplus_{q \geq 0} R_q$; where $R_q := (Sym^{ql}((V^m)^*))^{S_n}$. Since the K-algebra R is integrally closed, so to prove our claim, it is enough to prove that it is generated by $R_1 = (Sym^{l}((V^m)^*))^{S_n}$.

Let us take an invariant polynomial $f \in (Sym^{ql}((V^m)^*))^{S_n}$, where $q > 1$. Since e_{ij}'s generate $\mathbb{C}[V^m]^{S_n}$ with out loss of generality we can assume f is a monomial of the form $\prod_{i=1}^{n} \prod_{j=1}^{a_i} f_{ij}^{m_{ij}}$.

Since $f = \prod_{i=1}^{n} \prod_{j=1}^{a_i} e_{ij}^{m_{ij}} \in (Sym^{ql}(V^m))^{S_n}$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{a_i} m_{ij} i = ql$$

Let $m_i = \sum_{j=1}^{a_i} m_{ij}$ then we have $\sum_{i=1}^{n} m_i i = ql$, and hence $(m_1, m_2, \cdots, m_n)$ is in the semigroup $M_{\underline{l}} = \{(m_1, m_2, \cdots m_r) \in \mathbb{Z}_{\geq 0}^r : \sum_{i=1}^{r} m_i i \equiv 0 \bmod l\}$.

By corollary (3.3), the semigroup $M_{\underline{l}}$ is generated by the set $S_{\underline{d}} = \{(m_1, m_2, \cdots m_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{i=1}^{r} m_i i = l\}$. So there exists $(m'_1, m'_2, \cdots m'_n) \in \mathbb{Z}_{\geq 0}^n$ such that for each i

$$m'_i < m_i \text{ and } \sum_{i=1}^{r} m'_i i = l.$$

Again, since $m'_i < m_i = \sum_{j=1}^{a_i} m_{ij}$, for each i and j there exists $m'_{ij} \leq m_{ij}$ such that

$$m'_i = \sum_{j=1}^{a_i} m'_{ij}.$$

Then $g := \prod_{i=1}^{n} \prod_{j=1}^{a_i} e_{ij}^{m'_{ij}}$ is invariant under S_n and is in $(Sym^l((V^m)^*))^{S_n}$.

Let $f' = \frac{f}{g}$. Then $f' \in (Sym^{(q-1)l}((V^m)^*))^{S_n}$ and so by induction on q, f' is in the subalgebra generated by $(Sym^l((V^m)^*))^{S_n}$.

Hence $f = g.f'$ is in the subalgebra generated by $(Sym^l((V^m)^*))^{S_n}$. □

The next corollary gives projective normality for any finite dimensional representation of any finite group with respect to a much smaller power of $\mathcal{O}(1)$ than it is considered in [3] or [11]. The ideas of the proof is basically same as [11].

Corollary 3.7. *Let G be a finite group of order n and W be any finite dimensional representation of G over $\mathbb{C}$. Let $\mathcal{L}$ denote the descent of $\mathcal{O}(1)^{\otimes l}$, where $l = lcm\{1, 2, 3, \cdots, n\}$. Then $G \backslash \mathbb{P}(W)$ is projectively normal with respect to $\mathcal{L}$.*

Proof. Let $G = \{g_1, g_2, \cdots, g_n\}$ and let $\{w_1, w_2, \cdots, w_k\}$ be a basis of W^*. Let V be the natural representation of the permutation group S_n. Let $\{x_1, x_2, \cdots, x_n\}$ be a basis of V^*; then the set $\{x_{11}, \cdots, x_{n1}, \cdots, x_{1k}, \cdots, x_{nk}\}$ is a basis of $(V^k)^*$.

Consider the Cayley embedding $G \hookrightarrow S_n$, $g \mapsto (g_i \mapsto g_j := gg_i)$. Then

$$\eta : Sym((V^k)^*) \to Sym(W^*), \quad x_{il} \mapsto g_i(w_l)$$

is a G-equivariant and degree preserving algebra epimorphism.

Now we will use Noether's original argument (see page 2 of [17]) to show that the restriction map

$$\tilde{\eta} : (Sym((V^k)^*))^{S_n} \to (Sym(W^*))^G$$

is surjective. For any $f = f(w_1, \cdots, w_k) \in (Sym(W^*))^G$, we define

$$f' := \frac{1}{n}(f(x_{11}, x_{12}, \cdots, x_{1k}) + \ldots + f((x_{n1}, x_{n2}, \cdots, x_{nk})) \in (Sym((V^k)^*))^{S_n}.$$

Then we have

$$\tilde{\eta}(f') = \frac{1}{n}(f(g_1(w_1), g_1(w_2), \cdots, g_1(w_k)) + \ldots + f(g_n(w_1), g_n(w_2), \cdots, g_n(w_k)))$$
$$= \frac{1}{n}(g_1 f(w_1, w_2, \cdots, w_k) + \ldots g_n f(w_1, w_2, \ldots, w_k)) = f$$

Hence, $\tilde{\eta}(f') = f$ and $\tilde{\eta}$ is surjective. So the corollary follows from theorem (3.4).

□

4 Projective normality for the Alternating group quotient

Theorem 4.1. *Let V be the tautological representation of the Alternating group A_n. Then the quotient variety $A_n\backslash\mathbb{P}(V)$ is projectively normal with respect to the descent of the line bundle $\mathcal{O}(1)^{\otimes l}$, where $l = lcm\{1,2,3,\cdots,n\}$.*

Proof. The invariant ring $K[V]^{A_n}$ is generated by the elementary symmetric polynomials

$$e_r = \sum_{1\leq j_1<j_2<\ldots<j_r\leq n} x_{j_1}\cdots x_{j_r}$$

along with the discriminant $\Delta = \prod_{i<j}(x_i - x_j)$.

Let $R := \oplus_{q\geq 0}R_q$; where $R_q := (Sym^{ql}(V^*))^{A_n}$. Since the K-algebra R is integrally closed, so to prove our claim, it is enough to prove that it is generated by R_1.

Let us take an invariant polynomial $f \in (Sym^{ql}(V^*))^{A_n}$, where $q > 1$. Since e_i's and Δ generate $K[V]^{A_n}$ with out loss of generality we can assume f is a monomial of the form $\prod_{i=1}^n e_i^{m_i}\Delta^t$. Again since $\Delta^2 \in K[V]^{S_n}$, it is a polynomial in e_i's. So we may assume that $t = 0$ or 1.

Then f is of the form $\prod_{i=1}^n e_i^{m_i}\Delta$ or $\prod_{i=1}^n e_i^{m_i}$. Since $f \in (Sym^{ql}(V^*))^{A_n}$ we have either

$$\textstyle\sum_{i=1}^n im_i + \frac{n(n-1)}{2} = q.l \text{ or } \sum_{i=1}^n im_i = q.l$$

In either case by proposition (3.2) there exist $(m_1', m_2', ..., m_n')$ satisfying $m_i' \leq m_i\ \forall\ i$ such that:

$$\textstyle\sum_{i=1}^n i.m_i' = l$$

Then $g = \prod_{i=1}^n e_i^{m_i'}$ is invariant under A_n and is in $(Sym^l(V^*))^{A_n}$. Let $f' = \frac{f}{g}$. Then $f' \in (Sym^{(q-1)l}(V^*))^{A_n}$ and so by induction on q, f' is in the subalgebra generated by $(Sym^l(V^*))^{A_n}$.

Hence $f = g.f'$ is in the subalgebra generated by $(Sym^l(V^*))^{A_n}$.

□

Example 1: The alternating group A_5 has exactly two irreducible representations of degree 3 (see page 26 of [5]). In this example we will show that projective normality holds for these two representations for the descent of $\mathcal{O}(1)^{\otimes|A_5|}$. The Euclidean reflection group of type H_3 is the symmetry group of the icosahedron in $\mathbb{R}^3$ and it is abstractly isomorphic to $\mathbb{Z}_2 \times A_5$ where $\mathbb{Z}_2$ is the center $\{\pm 1\}$ and the group of orientation preserving or rotational symmetries is isomorphic to A_5 (see Section 2.13, page 46 of [7]). This shows that A_5 is isomorphic to a subgroup of $SO_3(\mathbb{R})$ as well.

The ring of invariants of the group of type H_3 is a polynomial ring with generators say, f_1, f_2 and f_3 called the basic invariants of degrees $2, 6$ and 10 (see page 59 of [7]). So the invariants for A_5 is the direct sum of the invariants for H_3 and the anti-invariants, which are functions f that transform under $w \in H_3$ to $-f$ when $w \notin A_5$. The Jacobian J of the basic invariants is an anti-invariant and every anti-invariant is a product of J and an invariant polynomial (see Proposition 3.13 of [7]). The Jacobian is of degree $1 + 5 + 9 = 15$. So we conclude that the ring of invariants of A_5 is generated by homogeneous polynomials f_1, f_2, f_3 and J of degrees $2, 6, 10$ and 15 respectively. Again since J is an anti-invariant, we have $J^2 \in \mathbb{C}[f_1, f_2, f_3]$.

Let us take a typical invariant monomial $f_1^{m_1}f_2^{m_2}f_3^{m_3}J^{m_4} \in (Sym^{q|A_5|}V^*)^{A_5}$, where $q \geq 2$ and V is the above 3 dimensional representation. Since $J^2 \in \mathbb{C}[f_1, f_2, f_3]$, we may assume that $m_4 = 0$ or 1. If $m_4 = 1$, then we have

$$2m_1 + 6m_2 + 10m_3 + 15 = 60q,$$

which is absurd. So we conclude that $m_4 = 0$. Now we have $2m_1+6m_2+10m_3 = 60q$ or $m_1+3m_2+5m_3 = 30q$. By repeatedly applying Lemma 2.1 of [10] we see that there exist $(m_1', m_2', m_3') \in \mathbb{Z}^3_{\geq 0}$, $m_i' \leq m_i$ for $i = 1, 2, 3$ such that $2m_1' + 6m_2' + 10m_3' = 60$. Then $g = f_1^{m_1'} f_2^{m_2'} f_3^{m_3'}$ is invariant under A_5 and is in $(Sym^{|A_5|}(V^*))^{A_5}$. Let $f' = \frac{f}{g}$. Then $f' \in (Sym^{(q-1)|A_5|}(V^*))^{A_5}$ and so by induction on q, f' is in the subalgebra generated by $(Sym^{|A_5|}(V^*))^{A_5}$. Hence $f = g.f'$ is in the subalgebra generated by $(Sym^{|A_5|}(V^*))^{A_5}$.

Consider the inner automorphism τ of S_5 corresponding to conjugation by the trasposition (12). It sends (12345) to (13452) and leaves invariant the other conjugacy classes of A_5. So τ is an outer automorphism of A_5. The other 3 dimensional irreducible representation of A_5 is obtained by twisting the above representation by this automorphism and hence the ring of invariants are the same. So projective normality holds for this representation as well.

Remark: As noted in the introduction the projective normality result is known only for the standard representation of the symmetric group and it is easy to see that it also holds for the trivial and sign representation. For other irreducible representations of the symmetric group finding a minimal set of generators and the relations between them seems to be a difficult problem. On the other hand it is interesting to study projective normality for these representations as well.

Example 2: Consider the irreducible representation W of S_6 corresponding to the partition $2+2+2$ which is of dimension 5. This representation can be obtained by twisting the standard 5 dimensional representation by the exceptional outer automorphism of S_6. In this way this can be seen as a reflection representation in which the products of three disjoint 2-cycles act as reflections. In other words the outer automorphism of S_6 induces an isomorphism of the ring of invariants of the standard representation and the ring of invariants of W. So $S_6 \backslash\backslash \mathbb{P}(W) \cong S_6 \backslash\backslash \mathbb{P}(V)$, where V is the standard representation of S_6. In fact this is the only irreducible representation of a non-solvable symmetric group apart from trivial, sign and standard representation for which the ring of invariants is a polynomial ring.

Acknowledgements. We would like to thank Prof. S.S. Kannan for suggesting this problem and many helpful discussions. We thank the referee for comments that lead to the improvement of the exposition of the paper. The second named author is supported by SERB-DST, Government of India through Young Scientist Scheme No-SB/FTP/MS-017/2014.

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