Inequalities

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1. (AM-GM inequality) Given positive real numbers $a_1, a_2, \ldots, a_n$, prove that
$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

2. (Young's inequality) If a, b, p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then show that
$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

3. (RMO 2012) For positive reals a and b such that $a + b = 1$, prove that
$$a^a b^b + a^b b^a \leq 1.$$

4. For any positive reals a, b, c, show that
$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

5. For any positive reals a, b, c, d, show that
$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a + b + c + d.$$

6. (Nesbitt's inequality) For any positive reals a, b, c, prove that
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

7. (BMO 1996) Let a, b, c and d be positive real numbers such that $a + b + c + d = 12$ and $abcd = 27 + ab + ac + ad + bc + bd + cd$. Find all possible values of a, b, c, d satisfying these equations.

8. (RMO 2011) Find all possible real solutions to the equation:
$$16^{x^2+y} + 16^{x+y^2} = 1.$$

9. (IMO 2012) Suppose $n \geq 2$ is a natural number. Let $a_2, a_3, \cdots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that
$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \geq n^n$$