Catalan numbers via representation theory

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A classical problem in Combinatorics requires the enumeration of the number of paths in the Cartesian plane from $(0,0)$ to (n,n) for a positive integer n, such that one only moves along lattice lines, one only goes rightwards or upwards and one always stays below or on the line $x=y$. It turns out that the answer is $C_n = \frac{1}{n+1}\binom{2n}{n}$. A number of elegant proofs of this fact can be found in the literature, along with several other combinatorial interpretations of C_n, which has become known as the n^{th} Catalan number. Here, we compute the value of the n^{th} Catalan number by making use of the representation theory of $SU(2)$.

The special linear group $SU(2) = \{A \in M_2(\mathbb{C}) : A^*A = I, \det(A) = 1\}$ is a compact Lie group whose finite dimensional irreducible complex representations can be classified by their highest weights. More precisely, for every non-negative integer n, $SU(2)$ has a unique irreducible representation V_n with highest weight n. It is known that V_n is an $(n+1)$-dimensional vector space that is spanned by unique (upto scalar multiplication) vectors corresponding to the weights $-n, -n+2, \ldots, n-2, n$. As the V_i's are, upto isomorphism, all the irreducible representations of $SU(2)$, an interesting problem is to decompose $V_i \otimes V_j$ for some $0 \leq i \leq j$ into a direct sum of irreducible representations. By some explicit computations, one sees that:

$$V_i \otimes V_j \cong V_{i+j} \oplus V_{i+j-2} \oplus \cdots \oplus V_{i-j},$$

and so, in particular, we have $V_n \otimes V_1 \cong V_{n+1} \oplus V_{n-1}$ for all positive integers n.

Now, we try to see how the Catalan numbers come in. For this, we consider the McKay graph G of the representation V_1. Then, the graph G consists of infinitely many vertices indexed by the non-negative integers, where the integer i corresponds to the representation V_i, and we have edges from i to $i+1$ and $i-1$ (except when $i=0$, when we only have the edge from 0 to 1). Now, observe that:

$$V_1 \cong V_1$$
$$V_1 \otimes V_1 \cong V_0 \oplus V_2$$
$$V_1 \otimes V_1 \otimes V_1 \cong V_1 \oplus V_1 \oplus V_3$$
$$V_1 \otimes V_1 \otimes V_1 \otimes V_1 \cong V_0 \oplus V_0 \oplus V_2 \oplus V_2 \oplus V_2 \oplus V_4$$
$$V_1 \otimes V_1 \otimes V_1 \otimes V_1 \otimes V_1 \cong V_1 \oplus V_1 \oplus V_1 \oplus V_1 \oplus V_1 \oplus V_3 \oplus V_3 \oplus V_3 \oplus V_3 \oplus V_5,$$

and so on. This motivates the fact that the number of copies of V_0 in $V_1^{\otimes 2n}$ is equal to the number of paths in the graph G from 0 to 0 that consist of exactly $2n$ steps, and a moment of thought gives us that this is in bijection with the number of paths we wanted to compute in our original combinatorial problem. Thus, we have that C_n is equal to the number of copies of V_0 in the decomposition of $V_1^{\otimes 2n}$ as a direct sum of irreducible representations.

We'll use character theory to determine the above number. If we denote by χ_i the character of $SU(2)$ associated to the representation V_i, we have by the orthonormality of characters that:

$$\langle \chi_i, \chi_j \rangle = \int_{SU(2)} \chi_i(g)\overline{\chi_j(g)}d\mu(g) = \delta_{ij},$$

where μ is the normalised Haar measure on $SU(2)$. Thus, for any representation V with character χ, the number of copies of V_i in V is given by the inner product $\langle \chi, \chi_i \rangle$. Now, as the character for $V_1^{\otimes 2n}$ is given by χ_1^{2n}, we have by the above discussion:

$$C_n = \int_{SU(2)} \chi_1^{2n}(g)\overline{\chi_0(g)}d\mu(g) = \int_{SU(2)} \chi_1^{2n}(g)d\mu(g),$$

as V_0 is the trivial one dimensional representation of $SU(2)$. To compute the above integral, we use the Weyl integration formula which gives us that:

$$C_n = \frac{1}{|W|}\int_T \det((I - Ad(t^{-1})|_{su_2/\mathfrak{t}})\chi_1^{2n}(t)d\mu'(t).$$

Here, W denotes the Weyl group of $SU(2)$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ giving that $|W| = 2$, T is the subgroup of diagonal matrices in $SU(2)$ which is a maximal torus for $SU(2)$, su_2 denotes the space of skew Hermitian 2×2 complex matrices having zero trace which is the Lie algebra of $SU(2)$, $\mathfrak{t}$ denotes the space of skew Hermitian diagonal matrices with zero trace which is the Lie agebra of T and μ' is the normalised Haar measure on T.

Now, elements of T are of the form $t_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ for $0 \leq \theta < 2\pi$. Thus, $T \cong \mathbb{S}^1$ as a Lie group and so, we get:

$$C_n = \frac{1}{4\pi}\int_0^{2\pi} \det((I - Ad(t_\theta^{-1})|_{su_2/\mathfrak{t}})\chi_1^{2n}(t_\theta)d\theta.$$

Now, the representation V_1 is given by the usual action of $SU(2)$ on $\mathbb{C}^2$ by thinking of $\mathbb{C}^2$ as column matrices. In particular, we have $\chi_1(t_\theta) = \mathrm{tr}(t_\theta) = 2\cos\theta$. Next, we want to consider $su_2/\mathfrak{t}$. Now, as a real vector space, su_2 is spanned by $t = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. As t spans $\mathfrak{t}$ as a real vector space, $\{a, b\}$ is a basis for $su_2/\mathfrak{t}$. Then, we get that:

$$t_\theta^{-1}at_\theta = \begin{bmatrix} 0 & e^{2i\theta} \\ -e^{-2i\theta} & 0 \end{bmatrix} = \cos 2\theta a + \sin 2\theta b$$

$$t_\theta^{-1}bt_\theta = \begin{bmatrix} 0 & ie^{2i\theta} \\ ie^{-2i\theta} & 0 \end{bmatrix} = -\sin 2\theta a + \cos 2\theta b.$$

Thus, in terms of the ordered basis (a, b), the matrix for the action of $I - Ad(t_{\theta^{-1}})$ is given by $\begin{bmatrix} 1 - \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & 1 - \cos 2\theta \end{bmatrix}$. In particular, as a linear map over $su_2/\mathfrak{t}$, we have $det(I - Ad(t_{\theta^{-1}})) = (1 - \cos 2\theta)^2 + \sin^2 2\theta = 4\sin^2\theta$. To sum it up,

$$C_n = \frac{1}{4\pi}\int_0^{2\pi} 2^{2n+2}\sin^2\theta\cos^{2n}\theta d\theta = \frac{2^{2n}}{\pi}(I_n - I_{n+1}),$$

where for all non-negative integers t, we define $I_t = \int_0^{2\pi} \cos^{2t}(\theta) d\theta$. Then, we have by integration by parts that $I_t = (2t-1)(I_{t-1} - I_t)$, giving the recurrence relation $I_t = \frac{2t-1}{2t} I_{t-1}$. Using the initial condition $I_0 = 2\pi$, we get

$$I_t = \frac{(2t-1) \times (2t-3) \times \cdots \times 3 \times 1}{2t \times (2t-2) \times \cdots \times 4 \times 2} \times 2\pi = \frac{\binom{2t}{t}\pi}{2^{2t-1}}.$$

Thus, we get

$$C_n = \frac{2^{2n}}{\pi}\left(\frac{\binom{2n}{n}\pi}{2^{2n-1}} - \frac{\binom{2n+2}{n+1}\pi}{2^{2n+1}}\right) = 2\binom{2n}{n} - \frac{1}{2}\binom{2n+2}{n+1} = \frac{\binom{2n}{n}}{n+1}.$$