The Index of $T_2^\text{an}$ inside $T_2$

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1 Introduction

Let $N$ be a prime number and let $S_2(\Gamma_0(N), \mathbb{Z})$ denote the modular forms of weight 2 and level $\Gamma_0(N)$ with integer coefficients, and for any other ring $R$, we denote $S_2(\Gamma_0(N), R) = S_2(\Gamma_0(N), \mathbb{Z}) \otimes R$. If $R$ is a characteristic $p$ ring, we define $S_2(\Gamma_0(N), R)^{\text{Katz}}$ to be the $R$-module of Katz forms as defined in [Kat73, Section 1.2], and define similar notation for the spaces of weight 1 forms.

For $N \nmid n$, let $T_n$ denote the $n$th Hecke operator inside $\text{End}(S_2(\Gamma_0(N), \mathbb{Z}))$, and let $U_N$ denote the $N$th Hecke operator. We let $T_{\text{an}}$ denote $\mathbb{Z}[T_3, T_5, \ldots]$, the algebra generated by $T_n$ for $(2N, n) = 1$, and we denote $T_{\text{an}}[T_2, U_N]$ by $T$. The goal of this paper is to compute the index of $T_{\text{an}}$ inside $T$.

Specifically, we prove the following theorem in sections 3 and 4:

**Theorem 1.1.** The quotient $T/T_{\text{an}}$ is purely 2-torsion, and

$$\dim_{\mathbb{F}_2} T/T_{\text{an}} = \dim_{\mathbb{F}_2} S_1(\Gamma_0(N), \mathbb{F}_2)^{\text{Katz}}.$$  

In other words, if $c = \dim_{\mathbb{F}_2} S_1(\Gamma_0(N), \mathbb{F}_2)^{\text{Katz}}$ is the dimension of the weight 1 level $\Gamma_0(N)$ Katz forms over $\mathbb{F}_2$, then the index of $T_{\text{an}}$ in $T$ is equal to $2^c$.

The setup of the paper is as follows. In section 2, we introduce some facts from the literature about modular forms and establish a duality theorem between modular forms and Hecke algebras. In section 3 we prove the first half of the theorem, that $T_{\text{an}}$ contains $2T$ as submodules of $T$, so the quotient $T/T_{\text{an}}$ is purely 2-torsion. Then in section 4 we use a theorem of Katz to relate the extra elements of $T$ to weight 1 modular forms using the duality, and finally establish the equality of Theorem 1.1 between dimensions. In section 5 we conclude with some examples, and some theorems and conjectures we propose based on the work of Cohen-Lenstra and Bhargava.

2 Preliminaries

2.1 From $\mathbb{Z}$ to $\mathbb{Z}_2$

We start by proving that $U_N \in T_{\text{an}}$, thereby reducing our work to considering $T_{\text{an}} \subseteq T_{\text{an}}[T_2]$.

**Theorem 2.1.** $U_N \in T_{\text{an}}$.

**Proof.** It is enough to check that $U_N \in T_{\text{an}} \otimes \mathbb{Z}_p$ for every $p$: if $T_{\text{an}}$ and $T_{\text{an}}[U_N]$ have different ranks as $\mathbb{Z}$-modules, then the $\mathbb{Z}_p$-ranks of $T_{\text{an}} \otimes \mathbb{Z}_p$ and $T_{\text{an}}[U_N] \otimes \mathbb{Z}_p = T_{\text{an}} \otimes \mathbb{Z}_p[U_N]$ are also

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different for every \( p \), contradiction. On the other hand, if \( \text{rank}(T^\text{an}) = \text{rank}(T^\text{an}[U_N]) \), then the quotient \( T^\text{an}[U_N]/T^\text{an} \) is finite. If it’s nontrivial, then for any prime \( p \) dividing its order, there is a surjective map \((T^\text{an}[U_N] \otimes \mathbb{Z}_p)/(T^\text{an} \otimes \mathbb{Z}_p) \rightarrow (T^\text{an}[U_N]/T^\text{an}) \otimes \mathbb{Z}_p \) with nontrivial image. So for this \( p \), \( T^\text{an}[U_N] \otimes \mathbb{Z}_p \neq T^\text{an} \otimes \mathbb{Z}_p \). Therefore, we will only check whether \( T^\text{an} \otimes \mathbb{Z}_p \) contains \( U_N \). Further, as \( T^\text{an} \otimes \mathbb{Z}_p \) is a complete semi-local ring, it splits as a direct sum of its completions at maximal ideals, so it’s further enough to check that \( U_N \) is in \( T^\text{an} \) for the completion \( T^\text{an}_m \) at each maximal ideal \( m \).

In a previous paper, we proved that \( U_N \in T^\text{an} \otimes \mathbb{Z}_2 = T^\text{an}_2 \) [ Tay20, Lemma 5.1], so the statement is true for all maximal ideals over 2. So let \( \ell \) be an odd prime, \( m \) be a maximal ideal of \( T^\text{an} \) over \( \ell \), and \( \mathfrak{a} \) be a maximal ideal of \( T \) containing \( m \).

Let \( T_{\mathfrak{a}} \) be the completion of \( T \) with respect to \( \mathfrak{a} \), and let \( A \) be the integral closure of \( T_{\mathfrak{a}} \) over \( \mathbb{Z}_\ell \), which can be written as \( A = \oplus_i \mathcal{O}_i \), where \( \mathcal{O}_i \) is a finite extension of \( \mathbb{Z}_\ell \). The maps

\[
\pi_i : T \rightarrow T_{\mathfrak{a}} \rightarrow A \rightarrow \mathcal{O}_i
\]

produce conjugacy classes of eigenforms with coefficients in \( \mathcal{O}_i \), with the coefficient \( a_{i,j} \) of \( q^j \) equal to \( \pi_i(T_j) \) if \( (j, N) = 1 \), or \( \pi_i(U_j) \) if \( N \mid j \). These are newforms as \( N \) is prime, and there are no weight 2 level 1 forms. By Eichler-Deligne-Shimura-Serre there are representations \( \rho_i : G_\mathbb{Q} \rightarrow \text{GL}_2(\mathcal{O}_i) \), unramified away from \( \ell N \), so that \( \text{Tr}(\rho_i(\text{Frob}_p)) = a_{i,p} \) for all primes \( p \not| \ell N \).

[DDT97, Theorem 3.1(e)] describes the shape of the local-at-\( N \) representation:

\[
\rho_i|_{G_{\mathbb{Q},N}} = \begin{pmatrix} \epsilon \chi & * \\ 0 & \chi \end{pmatrix}
\]

where \( \chi \) is the unramified representation taking \( \text{Frob}_N \) to \( a_{i,N} \) and \( \epsilon \) is the \( N \)-adic cyclotomic character. Additionally, \( \det \rho_i = \epsilon \), so \( \chi^2 \) is identically 1 and \( a_{i,N} \) is equal to 1 or \(-1\) for each \( i \).

We show that \( a_{i,N} \) is equal among all \( i \) over all \( \mathfrak{a} \) containing \( m \), so that the image of \( U_N \) in \( T_{\mathfrak{a}} \) is constantly 1 or \(-1\) over all \( \mathfrak{a} \), and hence, in \( T^\text{an}_m = \oplus_{m \subseteq \mathfrak{a}} T_{\mathfrak{a}} \), is inside \( T^\text{an}_m \).

By the Chebotarev density theorem, a representation is determined up to semisimplification and conjugation by its trace on the Frobenius elements of unramified primes. The \( \rho_i(\text{Frob}_p) \) have trace equal to \( a_{i,p} \), which is the image of \( T_p \) under \( \pi_i \). Because \( m \) is contained in \( \mathfrak{a} \) for all \( \mathfrak{a} \), the image of \( T_p \) under reduction of \( T^\text{an} \) mod \( m \) is the same as the reduction of \( a_{i,p} \) mod \( \mathfrak{a} \). Therefore, the semisimplifications of the reductions of \( \rho_i \) over all \( i \) and all \( \mathfrak{a} \) are all isomorphic. But we can deduce the value of \( a_{i,N} \) from the reduction of \( \rho_i \) mod \( \mathfrak{a} \), because \( \rho_i|_{G_{\mathbb{Q},N}} \) has an unramified quotient and a ramified subspace, and the same is true for the reduction mod \( \mathfrak{a} \) as \( \ell \not| 2 \). So the image of the Frobenius on the unramified quotient is either 1 or \(-1\) for one (and hence every) \( \rho_i \), and therefore \( a_{i,N} \) does not depend on \( i \) or \( \mathfrak{a} \), only on \( m \). So \( U_N \) lies in \( T^\text{an}_m \) for all \( m \), and we’re done.

We can now reduce from forms over \( \mathbb{Z} \) to forms over \( \mathbb{Z}_2 \). To do this, we first recall [Wil95, Lemma, pp. 491] which says that if \( T^1 \) is the Hecke algebra over \( \mathbb{Z} \) corresponding to level \( \Gamma_1(N) \) forms and \( T^2 \) is the subalgebra of operators relatively prime to \( 2 \), that \( T^2 \) has 2-power index in \( T^1 \). As the algebras \( T \) and \( T^\text{an} = T^\text{an}[U_N] \) are quotients of \( T^1 \) and \( T^2 \) of this lemma, the same is true for \( T \) and \( T^\text{an} \). (Alternatively, with a similar argument to the proof of Theorem 2.1, we can check that \( T_2 \) is contained in all completions at maximal ideals of \( T^\text{an} \) \( \frac{1}{2} \). This is true as 2 is unramified in, and \( T_2 \) is a trace of the modular representations over primes other than 2, so Chebotarev and completeness of \( T^\text{an}_m \) show that \( T_2 \in T^\text{an}_m \).) So we can calculate the index of \( T^\text{an} \otimes \mathbb{Z}_2 \) inside \( T \otimes \mathbb{Z}_2 \), and by abuse of notation begin to call these \( T^\text{an} \) and \( T \) instead. We know that \( T \) and \( T^\text{an} \) are semi-local rings, and as such, they can be written as a direct sum of their completions:

\[
T = \bigoplus_{\mathfrak{a} \subset T} T_{\mathfrak{a}}, \quad \text{and} \quad T^\text{an} = \bigoplus_{m \subset T^\text{an}_m} T^\text{an}_m.
\]
Additionally, because the $\mathbb{Z}_2$-ranks of $T$ and $\mathbb{T}_\text{an}$ are equal, $T_2 \in T \otimes \mathbb{Q}_2 = \mathbb{T}_\text{an} \otimes \mathbb{Q}_2 = \mathbb{T}_\text{an} \left[ \frac{1}{2} \right]$, and hence maps $\mathbb{T}_\text{an} \to K$ where $K$ is a finite extension of $\mathbb{Q}_2$ can be uniquely extended to maps $T \to K$. This means that modular forms are rigid in characteristic 0: we can determine the image of $T_2$ from the image of the remaining operators, and hence from any modular representation $\rho_f : G_{\mathbb{Q}_2} \to \text{GL}_2(K)$ we may determine the entire form $f$. We say that $\rho$ is ordinary if the restriction $\rho|_{D_2}$ of $\rho$ to the decomposition group at 2 is reducible, and we say that an eigenform is ordinary if $a_2$ is a unit mod 2. The next theorem describes the shape of $\rho_f$ at 2:

**Theorem 2.2** ([Wil88, Theorem 2]). If $f$ is an ordinary 2-adic form, then $\rho_f|_{D_2}$, the restriction of $\rho_f$ to the decomposition group at a prime above 2, is of the shape

$$\rho_f|_{D_2} \sim \begin{pmatrix} \chi \lambda^{-1} & * \\ 0 & \lambda \end{pmatrix}$$

for $\lambda$ the unramified character $G_{\mathbb{Q}_2} \to \mathbb{Z}_2^\times$ taking $\text{Frob}_2$ to the unit root of $X^2 - a_2X + 2$, and $\chi$ is the 2-adic cyclotomic character.

### 2.2 A Duality Theorem

In this section, we will compute the Pontryagin dual of one of the summands in $T$ with the following lemma. Let $\mathfrak{a}$ be any maximal ideal of $\mathbb{T}$ and let

$$S_2(\Gamma_0(N), \mathbb{Z}_2)_\mathfrak{a} = e \cdot S_2(\Gamma_0(N), \mathbb{Z}_2)$$

where $e$ is the projector $T \to T_\mathfrak{a}$.

**Lemma 2.3.** The Pontryagin dual of $T_\mathfrak{a}$ is $M = \lim_{\to} S_2(\Gamma_0(N), \mathbb{Z}_2)_\mathfrak{a}/(2^n)$ where the transition maps are multiplication by 2.

**Proof.** First, we note that $T_\mathfrak{a}$ acts on $M$ because $T_\mathfrak{a}$ acts compatibly on each level. If any element $T \in T_\mathfrak{a}$ acts trivially on $M$, then on any given modular form in $S_2(\Gamma_0(N), \mathbb{Z}_2)_\mathfrak{a}$, it acts by arbitrarily high powers of 2, and hence acts as 0. Then $T$ acts trivially on the rest of $S_2(\Gamma_0(N), \mathbb{Z}_2)$, so $T$ is the 0 endomorphism. Therefore, $M$ is a faithful $T_\mathfrak{a}$-module.

We also know that $M[\mathfrak{a}]$, the elements of $M$ killed by all of $\mathfrak{a}$, is a subspace of $S_2(\Gamma_0(N), \mathbb{Z}_2)_\mathfrak{a}/(2) = S_2(\Gamma_0(N), \mathbb{F}_2)_\mathfrak{a}$. It is a vector space over $T/\mathfrak{a}$, although through the action of $T$, not by multiplication on the coefficients. We explain why it’s a 1-dimensional $T/\mathfrak{a}$-vector space. The map

$$S_2(\Gamma_0(N), \mathbb{F}_2) \to \text{Hom}(T, \mathbb{F}_2), \quad f \mapsto \phi_f : T_n \to a_n$$

is injective by the $q$-expansion principle. The forms killed by $\mathfrak{a}$ must correspond to maps factoring through $T/\mathfrak{a}$, so the space of forms is at most the dimension of $\text{Hom}(T/\mathfrak{a}, \mathbb{F}_2) = \dim_{\mathbb{F}_2} T/\mathfrak{a}$. So the dimension as a $T/\mathfrak{a}$-vector space is at most 1.

On the other hand, there is at least 1 form in $M[\mathfrak{a}]$, because we may take the form $T_1q + T_2q^2 + T_3q^3 + \ldots \in S_2(\Gamma_0(N), T/\mathfrak{a})$ and consider its image under the trace map $T/\mathfrak{a} \to \mathbb{F}_2$. This is nonzero because the trace map is nondegenerate, and because the Hecke operators generate $T$ additively. This is in the kernel of $\mathfrak{a}$ because the trace of a form is just the sum of its conjugates, and for any expression in $\mathfrak{a}$ in terms of the Hecke operators with coefficients in $\mathbb{F}_2$, because its application to the original form is 0 by definition, its application to any of the form’s conjugates must also be 0 (because the Hecke operators act $\mathbb{F}_2$-linearly on a form’s coefficients and hence commute with Galois conjugation), and so too must its application to the sum. Because the trace form has coefficients in $\mathbb{F}_2$, we’ve found a nontrivial form in $M[\mathfrak{a}]$, and this must be dimension 1 as required.
We consider the Pontryagin dual of $M$: as $M$ is a $\mathbb{Z}_2$-module, the image of any map $M \to \mathbb{Q}/\mathbb{Z}$ must land in $\mathbb{Q}_2/\mathbb{Z}_2$. So let $M^\vee = \text{Hom}_{\mathbb{Z}_2}(M, \mathbb{Q}_2/\mathbb{Z}_2)$. We endow this with a $T_a$-module structure by letting $(T \phi)(f) = \phi(Tf)$. Because $S_2(\Gamma_0(N), \mathbb{Z}_2)_a \cong \mathbb{Z}^k_2$ for some $k$ because it is torsion free, $M \cong (\mathbb{Q}_2/\mathbb{Z}_2)^k$ as a $\mathbb{Z}_2$ module. So if $\phi(f) = 0$ for all $\phi \in M^\vee$, we know that $f = 0$. If $T \phi = 0$ for all $\phi$, then $\phi(Tf) = 0$ for all $\phi$ and $f$, and so $Tf = 0$ for all $f$, and $T = 0$. So $M^\vee$ is also a faithful $T_a$-module.

Further, $T_a$ injects into $M^\vee$: we can rewrite
\[ M = \lim_{n \to \infty} \frac{1}{2^n} S_2(\Gamma_0(N), \mathbb{Z}_2)_a/S_2(\Gamma_0(N), \mathbb{Z}_2)_a \]
where the transition maps are inclusion. Then the $T_a \times M \to \mathbb{Q}_2/\mathbb{Z}_2$ as $(T, f) \to a(Tf)$ defines the injection. By Nakayama’s lemma and the duality of $M[a]$ and $M^\vee/a$, the minimal number of generators of $M$ as a $T_a$-module is 1. So we’ve proven that $M^\vee \cong T_a$.

We may use Pontryagin Duality to find that the dual to $T_a/2 = M^\vee/2$ is $M[2]$, which is exactly $S_2(\Gamma_0(N), \mathbb{Z}_2)_a/(2) = S_2(\Gamma_0(N), \mathbb{F}_2)_a$. Thus we obtain a perfect pairing
\[ T_a/2 \times S_2(\Gamma_0(N), \mathbb{F}_2)_a \to \mathbb{F}_2, \quad (T, f) \to a(Tf). \]
We may sum these pairings over all $a$, because Hecke operators and forms with incompatible maximal ideals annihilate each other. Therefore we obtain a perfect pairing $T/2 \times S(\Gamma_0(N), \mathbb{F}_2) \to \mathbb{F}_2$.

### 3 $2T_2$ is integral

In this section we prove the following lemma:

**Lemma 3.1.** For any element $T \in T$, the element $2T \in T$ lies inside $T^{\text{an}}$.

First we prove a lemma describing the image of the representation corresponding to a non-Eisenstein ideal.

**Lemma 3.2.** Suppose $m$ does not contain the Eisenstein ideal. Then there is a representation $\rho : G_\mathbb{Q} \to \text{GL}_2(T^{\text{an}}_m)$. That is unramified outside $2N$, and which satisfies $\text{Tr}(\rho(\text{Frob}_\ell)) = T_\ell$ for $\ell \nmid 2N$.

**Proof.** Let $A = T^{\text{an}}_m$ and $A'$ is its integral closure over $\mathbb{Z}_2$, which can be written as the product $\prod_i \mathcal{O}_i$ of a collection of integer rings. We know that there exist representations $\rho'_i : G_\mathbb{Q} \to \prod_i \text{GL}_2(\mathcal{O}_i)$, by Eichler-Shimura-Deligne-Serre. The image is $\text{GL}_2(\mathcal{O}_i)$, because $G_\mathbb{Q}$ is compact, and we may choose an invariant lattice on which it acts. These $\rho'_i$ combine to give a representation
\[ \rho' = \prod_i \rho'_i : G_\mathbb{Q} \to \prod_i \text{GL}_2(\mathcal{O}_i). \]

We know that the traces of the representations at $\text{Frob}_\ell$ are the images of $T_\ell$ for all $\ell \nmid pN$, so the trace of $\rho'$ by Chebotarev Density always lands in $T^{\text{an}}_m$. We assumed $m$ did not contain the Eisenstein ideal, so we know that each $\rho'_i$, and therefore the full $\rho'$, is residually irreducible. By [Car94, Theorem 2] we find that $\rho'$ is similar to a representation
\[ \rho : G_\mathbb{Q} \to \text{GL}_2(T^{\text{an}}_m). \]
To prove Lemma 3.1, we look at the three different possible cases and deduce that the projection of $2T_2$ to $\mathbb{T}_a$ lies in $\mathbb{T}_m^{an}$ for each $m \subseteq a$. Further, we prove that $T_2^2$ lies in $\mathbb{T}_m^{an} \cdot T_2 + \mathbb{T}_m^{an}$, so that any $T \in \mathbb{T}$, being an element in $\mathbb{T}_m^{an}[T_2]$, lies in $\mathbb{T}_m^{an} \cdot T_2 + \mathbb{T}_m^{an}$, and hence is half of an element in $\mathbb{T}_m^{an}$.

3.1 $\bar{\rho}$ ordinary irreducible

We first assume that the residual representation $G_\mathbb{Q} \to \text{GL}_2(\mathbb{T}_m^{an}/m)$ is irreducible but the local residual representation at 2 is reducible. We will show that $2T_2$, as an element of $\mathbb{T}_m^{an}[T_2]$, actually lies in $\mathbb{T}_m^{an}$. This will be done by proving it is in the ring generated over $\mathbb{Z}_2$ by the traces of $\rho$. Equivalently, we will look at the traces of $\rho \otimes \mathbb{Z}_2 \mathbb{Q}_2$. This breaks the representation into a direct sum $\bigoplus_i \rho_i' \otimes \mathbb{Q}_2 : G_\mathbb{Q} \to \prod_i \text{GL}_2(E_i)$. Each of the $\rho_i'$ themselves have the same residual representation which is reducible when restricted to the decomposition group, so all these representations are ordinary.

Looking at a given $\rho_i'$, we may apply Theorem 2.2 to it to obtain a shape of $\rho_i'|_{D_2}$. In particular, the trace of an element $\rho(g)$ is equal to $\chi(g)\lambda^{-1}(g) + \lambda(g)$ with $\lambda$ the unramified character whose image of Frobenius is the unit root of $X^2 - T_2X + 2$, and $\chi$ is the cyclotomic character. If $\alpha$ denotes the unit root of $x^2 - a_{2,i}x + 2 = 0$, then letting $g$ be an element of $\text{Gal}(\mathbb{Q}_2^{ab}/\mathbb{Q}_2)$ which both is a lift of Frobenius and acts trivially on the 2-power roots of unity (so $\chi(g) = 1$), then we know $\text{Tr}(g) = \alpha + \alpha^{-1}$. If we let $h$ be a lift of Frobenius with $\chi(h) = -1$, we find that $\text{Tr}(h) = \alpha - \alpha^{-1}$. And by definition, we know $\alpha + 2\alpha^{-1} = a_{2,i}$, so $2a_{2,i} = 2\alpha + 4\alpha^{-1} = 3\text{Tr}(g) - \text{Tr}(h)$.

We now look at the product of representations. The elements $g$ and $h$ were independent of the coefficient field, so we know that the element of $\mathbb{T}_m^{an} \otimes \mathbb{Q}_2$ that is $2a_{2,i}$ in each coordinate, namely $2T_2 \otimes 1$, is equal to $3\text{Tr}(g) - \text{Tr}(h)$. So $2T_2$ is in the ring generated by the traces of elements, and thus in $\mathbb{T}_m^{an}$.

Similarly, we can prove that $T_2^2$ is in $\mathbb{T}_m^{an} + T_2 \cdot \mathbb{T}_m^{an}$; in each coordinate, we can calculate that

$$a_{2,i}^2 = \text{Tr}(g)a_{2,i} + (\text{Tr}(gh) - \text{Tr}(g^2) - 1).$$

So in $\mathbb{T}_m^{an}/T_2$, we find that $T_2^2 = \text{Tr}(g)T_2 + (\text{Tr}(gh) - \text{Tr}(g^2) - 1)$. So $T_2^2 \subseteq \mathbb{T}_m^{an} + T_2 \cdot \mathbb{T}_m^{an}$, and therefore so is every power of $T_2$. So we know that $2\mathbb{T}_m^{an}/T_2 \subseteq \mathbb{T}_m^{an}$, and the $\mathbb{T}_m^{an}$-module quotient $\mathbb{T}_m^{an}/T_2/\mathbb{T}_m^{an}$ is an $\mathbb{F}_2$ vector space. In section 4 we will calculate its dimension.

3.2 $\bar{\rho}$ reducible

We now suppose $\mathbb{T}_m^{an}$ corresponds to a reducible residual representation, so that $m$ is the Eisenstein ideal generated by 2 and $T_\ell$ for $\ell \nmid N$ (including $\ell = 2$). We claim that $T_2$ is already in $\mathbb{T}_m^{an}$. This is because by [Maz77, Proposition 17.1], the Eisenstein ideal of the full Hecke algebra is generated by $1 + \ell - T_\ell$ for any good prime. So by completeness, $T_2 - 3$ and therefore $T_2$ can be written as a power series in $T_\ell - \ell - 1$.

3.3 $\bar{\rho}$ non-ordinary

We now assume that the residual local representation at 2 is irreducible, or equivalently that in $\mathbb{T}_a$, $T_2$ is not a unit, where $a$ is some ideal of $\mathbb{T}$ above $m$ corresponding to $\rho$. We claim that $T_2$ is already in $\mathbb{T}_m^{an}$, so that $a = m$ is actually unique, and the index is 1.

**Theorem 3.3.** If $\rho$ is non-ordinary with corresponding map $\mathbb{T}_m^{an} \to \mathbb{F}$ with maximal ideal $m$, then for any $a \subseteq \mathbb{T}$ containing $m$, $T_2 \in \mathbb{T}_a$ is already contained in the image of $\mathbb{T}_m^{an}$.
4.1 Relating

In this section we prove the second half of Theorem 1.1. It is enough to look locally, so we will consider all of $\mathbb{T}^{an}/m$ to 0, and $T_2$ to 1. Recalling the perfect pairing from Lemma 2.3, we find a nonzero modular form $g \in S_2(\Gamma_0(N), \mathbb{F}_2)[m]$ with all odd coefficients equal to 0.

By part (3) of the main result of [Kat77], we know that there is some nonzero form $f \in S_1(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ with $f^2 = f$. (Here, we’re considering weight 1 Katz forms, and so the weight 2 forms we construct may be Katz forms as well. So if necessary we enlarge the spaces we’re considering, but it doesn’t affect the conclusion.) As forms with coefficients in $\mathbb{F}_2$ commute with the Frobenius endomorphism, $f(q^2)$ has the same $q$-expansion as $g$. If $T^1$ and $T^{1,an}$ are the weight 1 Hecke algebras, it is quick to check that the corresponding Hecke actions on $q$-expansions of $T^{1,an}$ are identical to those of $\mathbb{T}^{an}$. Therefore $f \in S_1(\Gamma_0(N), \mathbb{F}_2)^{Katz}[m]$. Further, we know that $f$ is alone in this space, by part (2) of [Kat77]: any other form in $S_1(\Gamma_0(N), \mathbb{F}_2)^{Katz}[m]$ has the same odd coefficients, so the difference between it and $f$ has only even-power coefficients, and hence must be 0 by Katz’s theorem. So $f$ is also an eigenform for $T_2$ in weight 1, say, with eigenvalue $b_2$.

So we’ve discovered that $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}[m]$ is at most 2 dimensional, spanned by $Vf$ and $Af$. Here, $V$ acts as $V(\sum_{n=1}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} a_n q^{2n}$ on power series, so that $Vf = g$, and can either be a weight-doubling operator, as used in [Kat77], or a level-doubling operator. Additionally, $Af$ is the multiplication of $f$ with the Hasse invariant $A$, which preserves $q$-expansions. We can hence calculate the action of $T_2$ on this space: we know that $T_2$ acts in weight 2 via $U + 2V$, where $U(\sum_{n=1}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} a_{2n} q^n$, and in weight 1 as $U + \langle 2 \rangle V$ with $\langle 2 \rangle$ the diamond operator, which is identically 1 on mod 2 forms. Further, we can compute that $UVf = Af$, as $V$ doubles each exponent and $U$ halves it.

So we find

$$T_2(Vf) = UVf = Af$$

$$T_2(Af) = U(Af) = A(Uf) = A(T_2f - \langle 2 \rangle Vf) = A(b_2f) - \langle 2 \rangle Vf$$

and the matrix for the $T_2$ action is

$$\begin{pmatrix} b_2 & -\langle 2 \rangle \\ 1 & 0 \end{pmatrix}$$

(In these computations, the distinction between the level-raising $V$ and the weight-raising $V$ has been blurred, because on $q$-expansions they are equal; we view both lines as equalities of weight 2 level $\Gamma_0(N)$ forms.) As $\langle 2 \rangle$ is trivial, the determinant of this matrix is 1, so $T_2$ is invertible. This is impossible because the form was non-ordinary. So there cannot be such a form $g$, and $\mathbb{T}^{an}[T_2]$ requires only one generator as a $\mathbb{T}^{an}_m$-module, as required.

4 Dimension of $\mathbb{T}/\mathbb{T}^{an}$

In this section we prove the second half of Theorem 1.1. It is enough to look locally, so we will localize at a maximal ideal $m$ of $\mathbb{T}^{an}$. Because completion at only ordinary non-Eisenstein ideals have $T_2$ not immediately in $\mathbb{T}^{an}_m$, we assume that $m$ is such an ideal.

4.1 Relating $\mathbb{T}/\mathbb{T}^{an}$ to $S_2$

We first recall the perfect pairing $S_2(\Gamma_0(N), \mathbb{F}_2) \times \mathbb{T}/2 \rightarrow \mathbb{F}_2$, given by $(f, T) \rightarrow a_1(Tf)$. While proving this, we proved perfect pairings $S_2(\Gamma_0(N), \mathbb{F}_2)_a \times \mathbb{T}/2 \rightarrow \mathbb{F}_2$, and we now combine all $a$ that contain $m$, to get a perfect pairing $S_2(\Gamma_0(N), \mathbb{F}_2)_m \times \mathbb{T}/m/2 \rightarrow \mathbb{F}_2$ where we denote $\mathbb{T}_m$ as the localization of $\mathbb{T}$ at the (not necessarily maximal) ideal $m \mathbb{T}$, and $S_2(\Gamma_0(N), \mathbb{F}_2)_m = e \cdot S_2(\Gamma_0(N), \mathbb{F}_2)$ for $e$ the projection from $\mathbb{T}$ to $\mathbb{T}_m$. Considering the subspace of forms killed by $A\theta$, the operator
defined in [Kat77] which acts as \( q \frac{d}{dq} \) on \( q \)-expansions and raises the weight by 3, it’s clear that the entirety of \( T_m^{an} \) annihilates it under the pairing, and we wish to prove that this is the full annihilator. For ease of notation, let us write \( V = T_m/2T_m \), \( W = S_2(\Gamma_0(N), F_2)_m \), and \( V' = T_m^{an}/2T_m \).

**Lemma 4.1.** \( S_2(\Gamma_0(N), F_2)_m \cap \text{Ker} A\theta \) and \( T_m^{an}/2T_m \) are mutual annihilators in this perfect pairing.

**Proof.** We’ve seen that they annihilate each other. Now suppose \( f = \sum_{i=1}^{\infty} a_i q^i \in W \) is annihilated by all of \( V' \). By the usual formula for the Hecke action on \( q \)-expansions, the coefficient of \( q^1 \) in \( T_n f \) is \( a_n \), so \( a_n = 0 \) for all odd \( n \). Therefore \( f \in S_2(\Gamma_0(N), F_2)_m \cap \text{Ker} A\theta \), and we can call this space \( \text{Ann}(V') \). This is enough to show they are mutual annihilators by dimension count, but we’ll prove the other direction as well.

The space \( W/\text{Ann}(V') \) is represented by sequences of odd-power coefficients that appear in forms in \( W \). We first prove that the map \( V' \rightarrow \text{Hom}(W/\text{Ann}(V'), F_2) \) induced by the pairing is surjective. Given a map \( \varphi \in \text{Hom}(W/\text{Ann}(V'), F_2) \) whose input is sequences of odd-power coefficients, we can define a map \( \varphi' \) in the double dual of \( V' \) taking maps
\[
\chi : V' \rightarrow F_2 \text{ to } \varphi(\chi(T_1), \chi(T_3), \ldots).
\]

This is the definition of \( \varphi' \) when \( (\chi(T_1), \chi(T_3), \ldots) \) appears as the odd-power coefficients of a form. And then if we’ve not defined \( \varphi' \) on all of the dual of \( V' \), we can just extend it any way we want. But because \( V' \) is finite dimensional, this \( \varphi' \) determines an element \( T_{\varphi} \in V' \) for which
\[
\chi(T_{\varphi}) = \varphi'(\chi) = \varphi(\chi(T_1), \chi(T_3), \ldots).
\]

Then because any sequence of coefficients \( (a_1, a_3, \ldots) \) is given by a character \( \chi(a_i) : T_n \rightarrow a_n \) (the restriction of such a \( \chi \) from \( T_n \), for example), the pairing truly does send \( T_{\varphi} \) to \( \varphi \).

Now given \( T \) that sends all of \( \text{Ann}(V') \) to 0, \( T f \) must only depend on the odd coefficients of \( f \). But then \( \varphi : f \rightarrow a_1(T f) \) is an element of \( \text{Hom}(W/\text{Ann}(V'), F_2) \). So by surjectivity there is some element \( T' \) of \( V' \) with \( a_1(T f) = \varphi(f) = a_1(T' f) \) for all \( f \in W/\text{Ann}(V') \). Then \( a_1((T - T') f) = 0 \) for all \( f \) either in \( \text{Ann}(V') \) or a lift of an element of \( W/\text{Ann}(V') \), and so in all of \( W \). Because the pairing is perfect, \( T = T' \in V' \) as we needed.

Now that we know these are mutual annihilators, we obtain an isomorphism
\[
V/V' \rightarrow \text{Hom}(\text{Ann}(V'), F_2),
\]
and taking dimensions and reinterpreting, we’ve proven that
\[
\dim T_m/T_m^{an} = \dim S_2(\Gamma_0(N), F_2)_m \cap \text{Ker} A\theta.
\]

So we have proven the following.

**Lemma 4.2.** The index of \( T_m^{an} \) in \( T_m \) equals 2 raised to the dimension of \( S_2(\Gamma_0(N), F_2)_m \cap \text{Ker} A\theta \).

### 4.2 Lifting from weight 1 to weight 2

Now we use the main theorem of [Kat77] to find a subspace of \( S_1(\Gamma_0(N), F_2)^{\text{Kat}} \) that maps under \( V \) to \( S_2(\Gamma_0(N), F_2)_m \cap \text{Ker} A\theta \). As in Section 3.3, we have \( T_m^{an} \)-equivariance, and so the maximal ideal \( \mathfrak{m} \) has an exact analogue in \( T_m^{an} \) and we land in the subspace \( S_1(\Gamma_0(N), F_2)^{\text{Kat}}_m \). We may not obtain the whole subspace because, while \( V f \) is in the kernel of \( A\theta \) for all \( f \in S_1(\Gamma_0(N), F_2)^{\text{Kat}}_m \), we don’t know it’s a form that is the reduction of a \( Z_2 \) form, which is what \( T_m^{an} \) parametrizes. In this section we will prove that the space of Katz forms of weight 2 actually are all standard forms.

The first case is \( N \equiv 3 \mod 4 \), which was taken care of Edixhoven:

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Theorem 4.3 ([Edi06, Theorem 5.6]). Let $N \geq 5$ be odd and divisible by a prime number $q \equiv -1 \mod 4$ (hence the stabilizers of the group $\Gamma_0(N)/\{1,-1\}$ acting on the upper half plane have odd order). Then $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ and $\mathbb{F}_2 \otimes S_2(\Gamma_0(N), \mathbb{Z})$ are equal, and the localizations at non-Eisenstein maximal ideals of the algebras of endomorphisms of $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ and $H^1_{par}(\Gamma_0(N), \mathbb{F}_2)$ generated by all $T_n$ ($n \geq 1$) coincide: both are equal to that of $S_2(\Gamma_0(N), \mathbb{Z})$ tensored with $\mathbb{F}_2$.

So for primes $N \equiv 3 \mod 4$, we’ve proven the equality in Theorem 1.1. For the remainder of this section we therefore assume $N \equiv 1 \mod 4$. Further, up until this point we’ve only worked with $\mathbb{F}_2$-forms, but we change coefficients to $\mathbb{F}_2$ so that we can find eigenforms associated to each maximal ideal. Theorem 4.3 still applies as its proof in [Edi06] can be extended to all finite extensions of $\mathbb{F}_2$.

Theorem 4.4. There are no Katz forms that are not the reduction of a form in $S_2(\Gamma_0(N), \mathbb{Z}_2)$. That is,

$$S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz} = S_2(\Gamma_0(N), \mathbb{F}_2).$$

Proof. Let $\ell$ be an arbitrary prime that is $3 \mod 4$, and we will look at $S_2(\Gamma_0(N\ell), \mathbb{F}_2)^{Katz}$. We can apply Theorem 4.3 to it and conclude that this space is exactly the characteristic 0 forms tensored with $\mathbb{F}_2$, so we may drop the Katz superscript. Further, we know that all Katz forms of level $\Gamma_0(N)$ lie in this space. So we just need to show there are no extra level $\Gamma_0(N)$ forms within this space.

As $T^{Katz}_{\ell} \otimes \mathbb{F}_2$ can be broken into a direct sum of $\mathbb{F}_2$-vector spaces on which the semi-simple action of each operator is by multiplication by a constant, $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ can be written as a direct sum of generalized eigenspaces. If we show every generalized eigenform in $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ is the reduction of a modular form from $S_2(\Gamma_0(N), \mathbb{Z}_2)$, then we’re done. So suppose $f$ is a generalized Katz eigenform for all $T_n$, including $T_2$. Let the eigenvalue corresponding to $T_\ell$ equal $a_\ell$; we will prove that if $f \not\in S_2(\Gamma_0(N), \mathbb{F}_2)$, then $a_\ell = 0$.

There are two maps from $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ to $S_2(\Gamma_0(N\ell), \mathbb{F}_2)$: the plain embedding with equality on $q$-expansions, and the map $V_\ell$ sending $f(q)$ to $f(q^\ell)$. We know $T_\ell = U_\ell + \ell V_\ell$ on $q$-expansions, so we find that

$$U_\ell(T_\ell - a_\ell) = U_\ell(U_\ell + \ell V_\ell - a_\ell) = U_\ell^2 - a_\ell U_\ell + \ell U_\ell V_\ell = U_\ell^2 - a_\ell U_\ell + \ell$$

as operators from $S_2(\Gamma_0(N), \mathbb{F}_2)^{Katz}$ to $S_2(\Gamma_0(N\ell), \mathbb{F}_2)$. Then because $f$ is a generalized eigenform, we find

$$0 = (U_\ell^k(T_\ell - a_\ell)^k) f = U_\ell^{k-1}(U_\ell^2 - a_\ell U_\ell + \ell)(T_\ell - a_\ell)^{k-1} f = \ldots = (U_\ell^2 - a_\ell U_\ell + \ell)^k f.$$

If we factor $X^2 - a_\ell X + \ell$ as $(X - \alpha)(X - \beta)$ for some lift of $a_\ell$, we’ve proven that $(U_\ell - \alpha)(U_\ell - \beta)$ acts topologically nilpotently on any lift of $f$ (which exists by Theorem 4.4). This will eventually be used to prove that one of $\alpha$ or $\beta$, and hence both, reduce to $1$ mod the maximal ideal of $\mathbb{Z}_2$.

Lemma 4.5. For any characteristic 0 newform $g$ of level $N\ell$, $U_\ell - 1$ acts topologically nilpotently.

Proof. The eigenform $g$ gives us a representation $\rho : G_3 \to \text{GL}_2(\overline{\mathbb{Q}}_2)$. The shape of this representation at the decomposition group at $\ell$ is given by [DDT97, Theorem 3.1(e)], as we recalled in the proof of Theorem 2.1, which says that

$$\rho|_{D_\ell} = \begin{pmatrix} \chi \varepsilon & * \\ 0 & \chi \end{pmatrix}$$
where $\chi$ is the unramified representation that sends $\text{Frob}_\ell$ to the $U_\ell$-eigenvalue of $g$, and $\varepsilon$ is the 2-adic cyclotomic character. Because the determinant is the 2-adic cyclotomic character as well, we know that $\chi^2 = 1$, so the $U_\ell$-eigenvalue of $g$ is $±1$. So $U_\ell - 1$ is either 0 or $−2$, which both act nilpotently.

If $\alpha - 1$ and $\beta - 1$ have valuation 0, then $(U_\ell - \alpha)(U_\ell - \beta)$ will not act nilpotently on any linear combination of eigenforms which includes at least one newform, by Lemma 4.5. As $(U_\ell - \alpha)(U_\ell - \beta)$ acts nilpotently on a lift of $f$, we know that this lift is a linear combination of only oldforms, and hence $f$ lifts to $S_2(\Gamma_0(N), \mathbb{Z}_2)$. Otherwise, one of $\alpha$ and $\beta$, and hence both, are 1 mod the maximal ideal of $\mathbb{Z}_2$, and so $\alpha + \beta \equiv 0 \equiv a_\ell$.

Therefore, we have proven that if $f$ is a generalized eigenform in $S_2(\Gamma_0(N), \mathbb{F}_2)^{\text{Katz}}$ that has no lift to characteristic 0, then $a_\ell = 0$ for any prime $\ell \equiv 3 \mod 4$, as our choice of $\ell$ was arbitrary. Letting $g$ be a true eigenform in the same eigenspace as $f$, we obtain a representation $\overline{\rho}_g : G_\mathbb{Q} \to \text{GL}_2(\mathbb{F}_2)$ with $\text{Tr}(\rho_g(\text{Frob}_p)) = a_p$. We showed that $\overline{\rho}_g$ has trace 0 on all $\text{Frob}_p$, so it must be the induction of a character from $G_{\mathbb{Q}(i)}$ to $G_\mathbb{Q}$. But such a representation is dihedral in the terminology of [KM19], and [KM19, Theorem 12(1)] proves that it’s impossible for a dihedral representation on $G_{\mathbb{Q}(i)}$ to give rise to a form of level $\Gamma_0(N)$. So there can be no Katz eigenforms of level $\Gamma_0(N)$ that don’t lift, and hence no generalized eigenforms and therefore no forms at all.

From this, we conclude that all the forms $V_2 f$, where $f$ is a weight 1 form of level $N$, are classical forms, and so the dimension of the space $S_2(\Gamma_0(N), \mathbb{F}_2)_m \cap \text{Ker} A\theta$ is exactly the dimension $S_1(\Gamma_0(N), \mathbb{F}_2)^{\text{Katz}}_m$. And so from Lemma 4.2, taking a direct sum over all $m$, we obtain Theorem 1.1.

5 Examples

In this section we use Theorem 1.1 to make nontrivial observations about the index of $T^{\text{an}}$ inside $T$.

5.1 $N \equiv 3 \mod 4$

Lemma 5.1. If $N \equiv 3 \mod 4$ is prime, the anemic Hecke algebra $T^{\text{an}}$ is equal to the full algebra $T$ if and only if the class group $\text{Cl}(\mathbb{Q}(\sqrt{-N}))$ is trivial.

Proof. If $K = \mathbb{Q}(\sqrt{-N})$ has class number greater than 1, by genus theory, since the discriminant of $K$ is $-N$ which is divisible by only a single prime, the 2-part of the class group of $K$ is trivial, so $\text{Cl}(K)$ has a nontrivial mod 2 multiplicative character which translates to an unramified mod 2 character $\chi$ of $\text{Gal}(\overline{\mathbb{Q}}/K)$. Inducing this to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we get a dihedral representation with Artin conductor equal to $N$. Wiese proves in [Wie04] that all dihedral representations give rise to Katz modular forms, and so the space $S_1(\Gamma_0(N), \mathbb{F}_2)^{\text{Katz}}$ is nontrivial, and hence $T^{\text{an}} \subseteq T$.

This shows that if $N$ is not 3, 7, 11, 19, 43, 67 or 163 (and is still a 3 mod 4 prime), $T^{\text{an}}(N) \subseteq T(N)$. On the other hand, for $N = 3$ and $N = 7$ there are no modular forms of weight 2, and for the other $N$, computer verification using the techniques of modular symbols, such as described in [Ste07], provides the following table:
These each prove that there are no Katz eigenforms of weight 1 and level \( N \) for any of these \( N \), and in turn that there are no Galois representations that could provide such forms. Of course, we knew \textit{a priori} there were no dihedral representations, as they would need to arise from the class group, but we now know that there are no larger-image representations.

5.2 \( N \equiv 1 \mod 4 \)

\textbf{Question 5.2.} Is it true that for a positive proportion of prime \( N \equiv 1 \mod 4 \), the anemic Hecke algebra \( T_{an} \) is not equal to the full algebra \( T \), and for a positive proportion of \( N \), \( T_{an} \) is equal to \( T \)?

We cannot immediately claim anything about the class group, because the Cohen-Lenstra heuristics [CL84, C11] claim that approximately 75.446\% of positive prime-discriminant quadratic extensions have trivial class group, so that there can be no dihedral modular forms.

The strong form of Serre’s conjecture due to Edixhoven [Edi97, Conjecture 1.8] is not known, where the strong form differs from the form proven by Khare and Wintenberger in [KW09] in this weight 1 case. A result of Wiese for dihedral representations [Wie04] is known, and a converse (that the corresponding representation \( \bar{\rho} \) is unramified at 2) has been proven [Wie14, Corollary 1.3]. We may also use Theorem 1.1 to construct weight 1 forms in the case that the eigenvalues of \( \text{Frob}_2 \) in the characteristic 2 representation are distinct, because there are two possible values for \( a_2 \), implying that \( T_m \neq T_{an,m} \).

We also know the subgroups of \( \text{SL}_2(\mathbb{F}_2) \), by Dickson, of four types: cyclic, upper-triangular, dihedral, and full-image (see [Suz77, Chapter 3, Theorem 6.17]). We know a modular representation must be absolutely irreducible: if not, say \( f \) is a weight 1 form for which \( \bar{\rho}_f \) is reducible. Then \( Af \) is a weight 2 form with the same representation, along with \( Vf \) in the same generalized eigenspace. But in Section 3.2 we proved that \( T_2 \) is already contained in the Hecke algebra corresponding to any eigenform with reducible representation, meaning that the dimension of \( S_2(\Gamma_0(N), \mathbb{F}_2)_m \) is dimension 1, not 2. Therefore only absolutely irreducible representations can be modular, so only dihedral and full-image representations can exist. So assuming the strong version of Serre’s conjecture, we know that for any weight 1 forms to exist at level \( N \), we need either a dihedral extension of \( \mathbb{Q} \), which must arise from inducing from the class group of \( \mathbb{Q}(\sqrt{N}) \), or we need an extension of \( \mathbb{Q} \) unramified outside \( N \) with Galois group isomorphic to \( \text{SL}_2(\mathbb{F}_{2^k}) \) for some \( k \).

Work has been done by Lipnowski [Lip16] to interpret Bhargava’s heuristics for the Galois group \( \text{GL}_2(\mathbb{F}_p) \) for \( p \) a prime, in order to count elliptic curves by their conductors through their \( p \)-adic representations. Although not done in this current note, it appears tractable to similarly analyze the groups \( \text{SL}_2(\mathbb{F}_{2^k}) \) and obtain a heuristic, explicit or not, on how many primes \( p \) have an elsewhere-unramified extension with each of these as their Galois groups. Because of the Cohen-Lenstra heuristics, it appears likely that infinitely many, even a positive proportion, of primes \( 1 \mod 4 \) have no weight 1 forms, so \( T = T_{an} \), and a positive proportion of primes have some weight 1 form so \( T_{an} \subseteq T \).
5.2.1 Explicit example: \( N = 653 \)

An instructive example is that of \( N = 653 \). Of course this is 1 mod 4, and so any dihedral representation that would give a weight 1 form would have to come from an induction of the class group of \( \mathbb{Q}(\sqrt{653}) \), but the Minkowski bound is \( \frac{1}{2}\sqrt{653} \approx 12.77 \), and 2, 3, 5 are inert and 7 = 230^2 - 653 \cdot 9^2 and \( -11 = 51^2 - 653 \cdot 2^2 \) are norms of principal ideals. So \( \mathbb{Q}(\sqrt{653}) \) has class number 1. But the Galois closure \( L \) of the field \( \mathbb{Q}[x]/(x^3 + 3x^3 - 6x^2 + 2x - 1) \) has Galois group \( A_5 = \text{SL}_2(\mathbb{F}_4) \), and is ramified only at 653 with ramification degree 2 and inertial degree 2. Therefore, Edixhoven predicts that the tautological Galois representation gives rise to a weight 1 level \( \Gamma_0(653) \) modular form. This is not a classical form, as \( \text{SL}_2(\mathbb{F}_4) \) does not embed into \( \text{GL}_2(\mathbb{C}) \), where all weight 1 characteristic 0 eigenforms must arise from.

On the other hand, \( \text{SL}_2(\mathbb{F}_4) \) does embed into \( \text{PGL}_2(\mathbb{C}) \), and by a theorem of Tate, all projective Galois representations lift. We can follow the proof given by Serre in [Ser77] to obtain a lift, unramified away from 653, and with Artin conductor 653^2. The fixed field of the kernel of this representation is a quadratic extension of \( L[x]/(x^4 - x^3 + 82x^2 - 1102x + 13537) \), which is itself the compositum of \( L \) and the quartic subfield of the 653rd roots of unity. Locally at 653 it is a faithful representation of \( \text{Gal}(\mathbb{Q}(653, \sqrt{2})/\mathbb{Q}(653)) \), a Galois group isomorphic to \( \langle x, y | x^8 = y^2 = e, yx = x^5y \rangle \).

We therefore find that, as the Artin conjecture for odd representations has been proven in [KW09], an eigenform of weight 1 and level 653^2 that reduces to the characteristic 2 form of level 653 we found above. We can additionally twist by the nontrivial character of \( \mathbb{Q}(\sqrt{653})/\mathbb{Q} \), not changing the determinant or level, to get a second Artin representation, and hence a second modular form of the same weight and nebentypus. These two eigenforms are congruent mod 2, so not changing the determinant or level, to get a second Artin representation, and hence a second level 653 we found above. We can additionally twist by the nontrivial character of \( 2 \) characteristic 0 eigenforms must arise from.

Indeed, we can find the following four (non-eigen)forms of weight 2 and level 653:

\[
\begin{align*}
\text{f}_1 &= 0q^1 + q^2 + 2q^3 - 4q^4 + 0q^5 + 2q^6 + 0q^7 + 4q^8 + 0q^9 + 4q^{10} + 0q^{11} + 1q^{12} - 6q^{13} + \ldots \\
\text{f}_2 &= 0q^1 + 0q^2 + 2q^3 - 3q^4 + 0q^5 + 2q^6 + 2q^7 + 4q^8 + 4q^9 - 3q^{10} + 4q^{11} - 6q^{12} + 0q^{13} + \ldots \\
\text{f}_3 &= 0q^1 + 0q^2 + 0q^3 + 4q^4 + 0q^5 + 4q^6 + 2q^7 + 2q^8 + 4q^9 + 5q^{10} + 2q^{11} + 0q^{12} + 4q^{13} + \ldots \\
\text{f}_4 &= 0q^1 - 2q^2 - 6q^3 + 2q^4 + 0q^5 + 2q^6 + 2q^7 - 5q^8 + 0q^9 + 0q^{10} - 2q^{11} - 6q^{12} - 2q^{13} + \ldots 
\end{align*}
\]

each of whose odd-power coefficients are all even, proving that none of \( T_2, T_4, T_6, T_8 \) or \( T_8 \) are in \( \mathbb{T}^\text{an} \) plus the other 3. But a calculation up to the Sturm bound of 109 proves that there are no other modular forms with all odd-power coefficients and coefficients of \( q^2, q^4, q^6, q^8 \) all even but some other coefficient is odd. Therefore \( \mathbb{T} = \mathbb{T}^\text{an} + 2\mathbb{T} + \langle T_2, T_4, T_6, T_8 \rangle \), so \( \mathbb{T}/(2\mathbb{T} + \mathbb{T}^\text{an}) \) is generated as an \( \mathbb{F}_2 \)-vector space by \( T_2, T_4, T_6, T_8 \). By Lemma 3.1, \( \mathbb{T}^\text{an} \) contains \( 2\mathbb{T} \), but from the above forms \( T_2, T_4, T_6, T_8 \) are independent in \( \mathbb{T}/\mathbb{T}^\text{an} \) so the index of \( \mathbb{T}^\text{an} \) in \( \mathbb{T} \) must be exactly \( 2^4 = 16 \).

References


