SOERGEL'S THEOREMS VIA WALL-CROSSING FUNCTORS.

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1. Introduction

The study of finite-dimensional representations of a (semisimple) Lie algebra $\mathfrak{g}$ over $\mathbf{C}$ is well understood and was first developed by Cartan and Weyl in the early 1900s. They classified finite-dimensional irreducible representations with an algebraic approach: consider a large module, called a Verma module, and then take the unique largest nontrivial quotient. Later, Borel and Weil used a geometric approach: consider global sections of a line bundle over a certain homogenous space, called the flag variety.

However, the study of infinite-dimensional representations of $\mathfrak{g}$ is far more complicated. In fact, it is known that $\mathfrak{sl}_2(\mathbf{C})$ is the only (semisimple) Lie algebra for which all irreducible representations are known [Maz]. Thus this leads to an interesting subcategory, the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, for which there is a classification result. In this paper, we explore two theorems (1.1, 1.2) of Soergel which describe endomorphism rings of certain projective objects in $\mathcal{O}$. Since there is an equivalence between $\mathcal{O}$ and modules over the algebra of endomorphisms of the projective generator, these theorems for example give an explicit algorithm for computing the Ext-quiver of various blocks of $\mathcal{O}$ [Str].

The importance of Soergel's fundamental theorems goes beyond category $\mathcal{O}$. For example, Soergel used these theorems to develop the study of Soergel bimodules (e.g. see [EMTW] for a comprehensive introduction), with the ultimate goal of giving a purely algebraic proof of the Kazhdan-Lusztig conjecture. This was accomplished recently in [EW]. Soergel's theorems also appear in the proof of Koszul duality for BGG category $\mathcal{O}$ [BGS].

Our approach in proving these theorems closely follows [BG]. Namely, we will use properties of translation functors deduced by theorems in $\mathcal{D}$-modules. As a result, our method is not the most direct, but the advantage is the arguments work in a general setting, and we see the main results arise naturally from the Beilinson-Bernstein correspondence $\mathfrak{g}-\text{modules} \longleftrightarrow \mathcal{D}-\text{modules}$.

All unexplained notation of the following theorems will be introduced in Section 2.

Theorem 1.1. *(Endomorphismensatz theorem) Let λ be a $\rho-$dominant integral weight. Then there is an algebra isomorphism*

$$\mathrm{End}_{U\mathfrak{g}}(P_{w_0\cdot\lambda}) \cong C^{W_\lambda}.$$

Theorem 1.2. *(Struktursatz theorem) Let λ be a ρ-dominant integral weight. Let $\mathbf{V} : \mathcal{O}_\lambda \to C^{W_\lambda}$, $M \mapsto \mathrm{Hom}_{U\mathfrak{g}}(P_{w_0\cdot\lambda}, M)$ denote the Soergel functor. Then for any projective object $Q \in \mathcal{O}_\lambda$ and any $M \in \mathcal{O}_\lambda$, the following natural morphism is an isomorphism*

$$\mathrm{Hom}_{U\mathfrak{g}}(M, Q) \to \mathrm{Hom}_{C^{W_\lambda}}(\mathbf{V}(M), \mathbf{V}(Q)).$$

2. Translation functors for extended universal enveloping algebra

We use the following notation. Fix a semisimple Lie algebra $\mathfrak{g}$ over complex numbers $\mathbf{C}$ and denote $\mathfrak{H}$ the *abstract* Cartan subalgebra. This is obtained by identifying the spaces $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ for all Borel subalgebras $\mathfrak{b}$. In particular, $\mathfrak{H}$ is not a subalgebra of $\mathfrak{g}$. This Cartan comes equipped with a root system $\Phi \subset \mathfrak{H}^*$ and with a canonical choice of simple roots $\{\alpha_1, \dots, \alpha_l\}$. See e.g [CG, Section 3.1] for details. Let $\rho \in \mathfrak{H}^*$ be the *half-sum of positive roots*. We say a weight $\lambda \in \mathfrak{H}^*$ is

ρ-dominant if $\langle \lambda+\rho, \alpha^\vee\rangle \geq 0$ for all positive coroots $\alpha^\vee$. We give a partial ordering on weights by saying $\lambda \leq \mu \Leftrightarrow \mu - \lambda \in \bigoplus_{i=1}^{l} \mathbf{Z}_{\geq 0}\alpha_i$.

Recall the (abstract) Weyl group W acts on $\mathfrak{H}^*$ in two ways. The standard action: for all simple reflections $s_\alpha \in W$ for $\alpha \in \Phi^+$ positive, define $s_\alpha.\lambda := \lambda - \langle \lambda, \alpha^\vee\rangle\alpha$. The dot action: $w\cdot\lambda := w(\lambda+\rho)-\rho$ for $w \in W, \lambda \in \mathfrak{h}^*$. We will always assume W acts on $\mathfrak{H}^*$ via the dot action unless stated otherwise. Define the stabilizer $W_\lambda := \{w \in W : w\cdot\lambda = \lambda\}$. Note, $-\rho$ is the unique weight with $W_{-\rho} = W$. We call a weight λ *regular* if $W_\lambda = \{1\}$. Given $\lambda \in \mathfrak{H}^*$, define the W-orbit $|\lambda| := W/W_\lambda$. Let $w_0 \in W$ denote the longest element of W. Let $\mathbf{C}[\mathfrak{H}^*]^W$ denote the W-dot invariant polynomials on $\mathfrak{H}^*$ and define $\mathfrak{H}^*/W := \mathrm{Specm}\mathbf{C}[\mathfrak{H}^*]^W$, the geometric quotient with respect to W-dot action. Let $C = \mathbf{C}[\mathfrak{h}^*]/\mathbf{C}[\mathfrak{h}^*]\cdot\mathbf{C}[\mathfrak{h}^*]^W_+$ denote the *coinvariant algebra*.

Let $U\mathfrak{g} := T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y])$ denote the universal enveloping algebra, and let $Z\mathfrak{g} \subset U\mathfrak{g}$ be its center. Let $G \supset B \supset U$ be the simply connected semisimple Lie group corresponding to $\mathfrak{g}$, a Borel subgroup of G, and U the unipotent radical of B. Let $T = B/U$ be the abstract maximal torus, so that $\mathrm{Lie}(T) = \mathfrak{H}$. It is well known G/B (called the "flag variety") is a projective variety and in bijection with the variety $\mathcal{B}$ consisting of all Borel subalgebras of $\mathfrak{g}$, via the map $g \mapsto g\cdot\mathfrak{b}\cdot g^{-1}$, where $\mathfrak{b}$ is some fixed Borel subalgebra of $\mathfrak{g}$.

We recall the *Harish-Chandra homomorphism* $\chi : Z\mathfrak{g} \to U\mathfrak{H}$. For any $x \in G/B$, let $\mathfrak{b}_x \supset \mathfrak{n}_x$ denote the corresponding Borel, nilpotent subalgebra, respectively. Let $M_{\mathrm{univ}} := U\mathfrak{g}/U\mathfrak{g}.\mathfrak{n}_x \cong U\mathfrak{g}\otimes_{U\mathfrak{b}_x} U\mathfrak{H}$ denote the universal Verma. Note, M_{univ} is a $(U\mathfrak{g}, U\mathfrak{H})$-bimodule. There is an inclusion of vector spaces $U\mathfrak{H} \hookrightarrow M_{\mathrm{univ}}, h \mapsto v_{\mathrm{univ}}.h$ and an algebra map $Z\mathfrak{g} \to M_{\mathrm{univ}},\ a \mapsto a.v_{\mathrm{univ}}$, where $v_{\mathrm{univ}} = 1\otimes 1$ is the generator of M_{univ}. These two maps have the same image [Gait], so there is an algebra map, called Harish-Chandra homomorphism, $\chi : Z\mathfrak{g} \to U\mathfrak{H}$, which satisfies $a.v_{\mathrm{univ}} = v_{\mathrm{univ}}.\chi(z), a \in Z\mathfrak{g}$. Moreover, [Gait] shows χ is an isomorphism onto its image, which consists of W-dot invariant polynomials on $\mathfrak{H}^*$, once we make the canonical identification $U\mathfrak{H} = \mathbf{C}[\mathfrak{H}^*]$. Thus the Harish-Chandra homomorphism $\chi : Z\mathfrak{g} \xrightarrow{\sim} \mathbf{C}[\mathfrak{H}^*]^W \hookrightarrow \mathbf{C}[\mathfrak{H}^*]$ is identified with the projection $\pi : \mathfrak{H}^* \to \mathfrak{H}^*/W$ of affine varieties. Given $\lambda \in \mathfrak{H}^*$, we define $\mathcal{J}_\lambda \in \mathrm{Specm}\mathbf{C}[\mathfrak{H}^*]$ to be the maximal ideal of polynomials vanishing on λ and define $I_{|\lambda|} := \pi^{-1}(\mathcal{J}_\lambda) \in \mathrm{Specm}Z\mathfrak{g}$. Also, we denote $\chi_\lambda : Z\mathfrak{g} \to \mathbf{C}^*,\ z \mapsto (\chi(z))(\lambda)$ the *central character* associated to λ.

In Section 5 only, we will fix a Cartan and Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ and consider the BGG category $\mathcal{O}$, which is defined as the full subcategory of $\mathrm{Mod}(U\mathfrak{g})$ consisting of finitely-generated $U\mathfrak{g}$-modules M such that $U\mathfrak{b}$ acts locally finitely and $U\mathfrak{h}$ acts semisimply. Category $\mathcal{O}$ has a direct sum decomposition $\mathcal{O} = \bigoplus_{\lambda\in\mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\lambda$, with $\mathcal{O}_\lambda$ consisting of modules for which $(z-\chi_\lambda(z))^n$ acts by 0 for large n. Let $L_\lambda \in \mathcal{O}_\lambda$ denote the simple module of highest weight λ, Δ_λ the corresponding Verma, and P_λ the corresponding indecomposable projective. It is known $\mathcal{O}_\lambda$ is a highest-weight category and is only easy to describe in the degenerate case $\lambda = -\rho$, ($\mathcal{O}_{-\rho}$ is equivalent to the category of finite-dimensional vector spaces). See [Hum] for a general discussion of properties of $\mathcal{O}$.

Definition 2.1. *We define the extended universal enveloping algebra*

$$\tilde{U} := U\mathfrak{g}\otimes_{Z\mathfrak{g}} \mathbf{C}[\mathfrak{H}^*]$$

where $Z\mathfrak{g}$ acts on $\mathbf{C}[\mathfrak{H}^*]$ via the Harish-Chandra map.

Observe, there is a natural filtration on $\tilde{U}$ so that $\mathrm{gr}(\tilde{U}) = S\mathfrak{g}\otimes_{S\mathfrak{H}^W} S\mathfrak{H} = \mathcal{O}(\mathfrak{g}\times_{\mathfrak{H}/W}\mathfrak{H})$. In particular, $\tilde{U}$ is a domain. Since $U\mathfrak{g}$ and $U\mathfrak{H}$ are free over $Z\mathfrak{g}$, $\mathbf{C}[\mathfrak{H}^*]$ and $U\mathfrak{g}$ are subalgebras of $\tilde{U}$. The W-action on $\mathfrak{H}$ induces an action on $\tilde{U}$ in a way that $\tilde{U}^W = U\mathfrak{g}$. Let $\mathrm{Mod}_{|\lambda|}(U\mathfrak{g})$, resp. $\mathrm{Mod}_\lambda(\tilde{U})$ be the category of finitely-generated $U\mathfrak{g}$, resp. $\tilde{U}$-modules M such that $I^n_{|\lambda|}\cdot M = 0$, resp. $\mathcal{J}^n_\lambda\cdot M = 0$ for sufficiently large n. We have a canonical exact functor

$$\mathrm{Res}_\lambda : \mathrm{Mod}_\lambda(\tilde{U}) \to \mathrm{Mod}_{|\lambda|}(U\mathfrak{g}).$$

Next, we introduce an intermediate algebra $\mathbf{C}[\mathfrak{H}^*]^W \subset \mathbf{C}[\mathfrak{H}^*]^{W_\lambda} \subset \mathbf{C}[\mathfrak{H}^*]$ which corresponds to the factorization $\mathfrak{H}^* \twoheadrightarrow \mathfrak{H}^*/W_\lambda \twoheadrightarrow \mathfrak{H}/W$ (since W is finite, $\mathbf{C}[\mathfrak{H}^*/W] = \mathbf{C}[\mathfrak{H}^*]^W$).

Define $\mathcal{J}_\lambda^{W_\lambda} := \mathbf{C}[\mathfrak{H}^*]^{W_\lambda} \cap \mathcal{J}_\lambda$ and let $\mathrm{Mod}_\lambda(\tilde{U}^{W_\lambda})$ denote category of finitely-generated $\tilde{U}^{W_\lambda}$-modules annihilated by large enough power of $\mathcal{J}_\lambda^{W_\lambda}$. The map $\mathfrak{H}^*/W_\lambda \to \mathfrak{H}^*/W$ is unramified over $|\lambda|$, hence the $I_{|\lambda|}$-adic completion of $Z\mathfrak{g}$ is isomorphic to $\mathcal{J}^{W_\lambda}$-adic completion of $\mathbf{C}[\mathfrak{H}^*]^{W_\lambda}$. This implies there is isomorphism of $I_{|\lambda|}$-adic completion of $U\mathfrak{g}$ with $\mathcal{J}^{W_\lambda}$-adic completion of $\tilde{U}^{W_\lambda}$. Thus

(2.1) $$\mathrm{Res}_\lambda : \mathrm{Mod}_\lambda(\tilde{U}^{W_\lambda}) \to \mathrm{Mod}_{|\lambda|} U\mathfrak{g} \quad \text{is an equivalence of categories.}$$

Now, the projection $\pi : \mathfrak{H}^* \to \mathfrak{H}^*/W_\lambda$ satisfies $\pi^{-1}\pi(\lambda) = \lambda$, i.e is totally ramified. Then we see for $M \in \mathrm{Mod}_\lambda(\tilde{U}^{W_\lambda})$, $\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} M \in \mathrm{Mod}_\lambda(\tilde{U})$. Thus $M \mapsto \tilde{U} \otimes_{\tilde{U}^{W_\lambda}} M$ is left adjoint of Res_λ, where $M \in \mathrm{Mod}_\lambda U\mathfrak{g}$ is viewed as $\tilde{U}^{W_\lambda}$-module by (2.1).

Now, we define *translation functors*. Let L be a finite-dimensional $\mathfrak{g}$-module and $M \in \mathrm{Mod}_{|\lambda|}(U\mathfrak{g})$. Then by [BeGe], $L \otimes M$ is annihilated by an ideal of $Z\mathfrak{g}$ of finite-codimension, and thus there is a direct sum decomposition

$$L \otimes M = \bigoplus_{\mu \in \mathfrak{H}^*/W} \mathrm{pr}_{|\mu|}(L \otimes M), \quad \text{where } \mathrm{pr}_{|\mu|}(L \otimes M) \in \mathrm{Mod}_{|\mu|}(U\mathfrak{g}).$$

Now suppose $\lambda, \mu \in \mathfrak{H}^*$ are ρ-dominant weights and $L_{w\cdot(\lambda-\mu)}$ is the finite-dimensional $U\mathfrak{g}$-module of highest-weight $w \cdot (\lambda - \mu)$.

Definition 2.2. *Define the translation functor*

$$\theta_\mu^\lambda : Mod_{|\mu|}(U\mathfrak{g}) \to Mod_{|\lambda|}(U\mathfrak{g}), \quad M \mapsto pr_{|\mu|}(L_{w\cdot(\lambda-\mu)} \otimes M).$$

The funtors θ_μ^λ and θ_λ^μ are easily seen to be biadjoint and exact. By (2.1), θ_μ^λ may be viewed as

$$\theta_\mu^\lambda : \mathrm{Mod}_\mu(\tilde{U}^{W_\mu}) \to \mathrm{Mod}_\lambda(\tilde{U}^{W_\lambda}).$$

Acting by $z \in \mathbf{C}[\mathfrak{H}^*]^{W_\mu}$ on $M \in \mathrm{Mod}_\mu(\tilde{U}^{W_\mu})$ is a $\tilde{U}^{W_\mu}$-module homomorphism since $Z(\tilde{U}^{W_\mu}) = \mathbf{C}[\mathfrak{H}^*]^{W_\mu}$. Hence functoriality gives an endomorphism $\theta_\mu^\lambda(a) : \theta_\mu^\lambda M \to \theta_\mu^\lambda M$ of $\tilde{U}^{W_\lambda}$-modules.

We may explicitly describe this action as follows. From now on, fix integral ρ-dominant weights $\lambda, \mu \in \mathfrak{H}^*$ such that $W_\lambda \subset W_\mu$ and set $\theta^+ := \theta_\mu^\lambda$ and $\theta^- := \theta_\lambda^\mu$. Let $T_{\lambda-\mu} : \mathbf{C}[\mathfrak{H}^*] \to \mathbf{C}[\mathfrak{H}^*]$ be the map $p(x) \to p(x + \lambda - \mu)$. Then $T_{\pm(\lambda-\mu)}$ preserves $\mathbf{C}[\mathfrak{H}^*]^{W_\lambda}$ and $T_{\pm(\lambda-\mu)}\mathbf{C}[\mathfrak{H}^*]^{W_\mu} \subset \mathbf{C}[\mathfrak{H}^*]^{W_\lambda}$. The following proposition will be proved in Section 4 using $\mathcal{D}$-modules.

Proposition 2.3. *Let $z \in \mathbf{C}[\mathfrak{H}^*]^{W_\mu}$. Then*

(1) For any $M \in Mod_\mu(\tilde{U}^{W_\mu})$ and $m \in \theta_\mu^\lambda M$, we have $z \cdot m = T_{\lambda-\mu}(z)[\theta_\mu^\lambda(z)(m)]$.

(2) For any $M \in Mod_\lambda(\tilde{U}^{W_\lambda})$ and $m \in \theta_\lambda^\mu M$, we have $z \cdot m = [\theta_\lambda^\mu(T_{\lambda-\mu}z)] \cdot m$.

So Proposition 2.3 explicitly gives: Suppose $u = g \otimes z \in U\mathfrak{g} \otimes_{Z\mathfrak{g}} \mathbf{C}[\mathfrak{H}^*]^{W_\mu} \cong \tilde{U}^{W_\mu}$ and $l \otimes m \in E_{\mu-\lambda} \otimes M$. Then, $(g \otimes z)(l \otimes m) = (1 \otimes z).(gl \otimes m + l \otimes gm) = g.l \otimes T_{\lambda-\mu}(z).m + l \otimes T_{\lambda-\mu}(z)g.m.$

Our goal is to extend these translation functors to $\tilde{U}$-modules. We have maps

$$\mathrm{Mod}_\lambda(\tilde{U}) \xrightarrow{\mathrm{Res}_\lambda} \mathrm{Mod}_{|\lambda|}(U\mathfrak{g}) \xrightarrow{\theta^-} \mathrm{Mod}_{|\mu|}(U\mathfrak{g}).$$

Also, the action of $a \in \mathbf{C}[\mathfrak{H}^*] = Z(\tilde{U})$ on $M \in \mathrm{Mod}_\lambda(\tilde{U})$ induces by functoriality an endomorphism $\theta^-(a) : \theta^-(\mathrm{Res}_\lambda M) \to \theta^-(Res_\lambda M)$. We now define a $\mathbf{C}[\mathfrak{H}^*]$ action on $\theta^-(\mathrm{Res}_\lambda(M))$ by

$$a * m := \theta^-(T_{\mu-\lambda}a) \cdot m, \;\; m \in \theta^-(\mathrm{Res}_\lambda M).$$

Proposition 2.3(2) implies this action restricted to the W_μ-invariants coincides with the $\mathbf{C}[\mathfrak{H}^*]^{W_\mu}$-action arising from restricting the $\tilde{U}^{W_\mu}$ action on $\theta^-(\mathrm{Res}_\lambda M)$ Thus, we may combine the $\mathbf{C}[\mathfrak{H}^*]$ and $\tilde{U}^{W_\mu}$ actions to get at $\tilde{U}^{W_\mu} \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{H}^*]$ action which factors to give a $\tilde{U} \cong \tilde{U}^{W_\mu} \otimes_{\mathbf{C}[\mathfrak{H}^*]^{W_\mu}} \mathbf{C}[\mathfrak{H}^*]$-action. Thus we obtain an exact functor $\tilde{\theta}^- : \mathrm{Mod}_\lambda(\tilde{U}) \to \mathrm{Mod}_\mu(\tilde{U})$.

Next, consider the composition

$$\mathrm{Mod}_\mu(\tilde{U}) \xrightarrow{\mathrm{Res}_\mu} \mathrm{Mod}_{|\mu|}(U\mathfrak{g}) \xrightarrow{\theta^+} \mathrm{Mod}_{|\lambda|}(U\mathfrak{g})$$

and define for $N \in \mathrm{Mod}_\mu(\tilde{U})$ a $\mathbf{C}[\mathfrak{H}^*]$-action on $\theta^+\mathrm{Res}_\mu N$ via

$$a * m := [T_{\lambda-\mu}\theta^+(a)] \cdot m, \quad m \in \theta^+(\mathrm{Res}_\mu N).$$

We similarly can directly check the $\tilde{U}^{W_\lambda}$- and $\mathbf{C}[\mathfrak{H}^*]$-actions are compatible. Moreover, by Proposition 2.3(1), these actions agree on $\mathbf{C}[\mathfrak{H}^*]^{W_\mu}$, but not necessarily on the full $\mathbf{C}[\mathfrak{H}^*]^{W_\lambda} \supset \mathbf{C}[\mathfrak{H}^*]^{W_\mu}$. To remedy this, define

$$\tilde{\theta}_r^+ N := \{n \in \theta^+(\mathrm{Res}_\mu N) : a * n = a \cdot n, \ \ \forall a \in \mathbf{C}[\mathfrak{H}^*]^{W_\lambda}\}, \ \text{ and}$$
$$\tilde{\theta}_l^+ N := \theta^+(\mathrm{Res}_\mu N)/\{a * n - a \cdot n, \ \ \forall a \in \mathbf{C}[\mathfrak{H}^*]^{W_\lambda}, n \in \theta^+(\mathrm{Res}_\mu N)\}.$$

Then the $\tilde{U}^{W_\lambda}$- and $\mathbf{C}[\mathfrak{H}^*]$-actions on θ^+N descend to compatible actions on $\tilde{\theta}^+N$. Hence they combine to give a $\tilde{U}$- action. We remark these functors are not in general exact. If both λ and μ are regular, then $\tilde{\theta}_r^+ = \tilde{\theta}_l^+ = \theta^+$. Also, $\tilde{\theta}_l^+, \tilde{\theta}_r^+$ are the left, right, resp. adjoint of the functor $\tilde{\theta}^-$.

3. Beilinson-Bernstein localization

Set $\tilde{\mathcal{B}} = G/U$, the "base affine space" space, and $\mathcal{B} := G/B$, the flag variety of G. There is a natural right T-action on $\tilde{\mathcal{B}}$ making $\pi : G/U \to G/B$ a principal G-equivariant T-bundle. We comment if G is adjoint, then G/U has a realization of decorated flags:

$$G/U = \{\mathfrak{b}_x, \{a_x^{\alpha_i}\} : \mathfrak{b}_x \subset \mathfrak{g} \text{ is Borel subalgebra, } a_x^{\alpha_i} \text{ generates } \mathfrak{g}/\mathfrak{b}_x, \text{ where } \alpha_i \text{ is a simple root of } \Phi^+\}$$

and the projection to G/B is given by forgetting the decorations.

The G-action by left translation translation and T-action by right translation on G/U commute, so differentiating the action map at identity yields map of Lie algebras into global algebraic vector fields on $\tilde{\mathcal{B}}$:

$$\Phi : \mathfrak{g} \times \mathfrak{H} \to \Gamma(G/U, \mathcal{T}_{G/U}), \ \ (g,h) \mapsto \partial_{g,h} : f(x) \mapsto \frac{d}{dt}|_{t=0} f(e^{-tg}xe^{th}),$$

where $f \in \mathcal{O}_{G/U}(V)$ for some $V \subset G/U$. By universal property, this can be extended to associative algebra homomorphism $\Phi : U\mathfrak{g}\otimes_{\mathbf{C}} U\mathfrak{H} \to \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T =$ algebra of right T-invariant global differential operators on $\tilde{B}$. Moreover, it is a fact that $z \otimes 1$ and $1 \otimes \chi(z)$ map to same element under Φ, where $\chi : Z\mathfrak{g} \to U\mathfrak{H}^W$ is Harish-Chandra, so Φ descends to a map $\tilde{\Phi} : \tilde{U} \to \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T$. In [BoBr, Prop. 8], it is shown by passing to the associated graded, and in a similar way to the case of $\mathrm{Ker}(U\mathfrak{g} \to \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}})) = \ker \chi_\lambda$, that $\tilde{\Phi}$ is an isomorphism:

$$\Phi : \tilde{U} \xrightarrow{\sim} \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T \cong \Gamma(\mathcal{B}, \tilde{\mathcal{D}}). \tag{3.1}$$

Let $\pi_\bullet\mathcal{D}_{\tilde{\mathcal{B}}}$ be sheaf-theoretic direct image of $\mathcal{D}_{\tilde{\mathcal{B}}}$ to $\mathcal{B}$. Define $\tilde{\mathcal{D}} \subset \pi_\bullet\mathcal{D}_{\tilde{\mathcal{B}}}$ to be the sheaf of T-invariant sections of $\pi_\bullet\mathcal{D}_{\tilde{\mathcal{B}}}$. Also, since T commutes with its own Lie algebra action, the image of $1 \otimes U\mathfrak{H} \cong \mathbf{C}[\mathfrak{H}^*]$ is contained in stalk of sheaf $\tilde{\mathcal{D}}$ at any point of $\mathcal{B}$. Thus, $\mathbf{C}[\mathfrak{H}^*] \hookrightarrow \tilde{\mathcal{D}}$ is central embedding. So for any $\lambda \in \mathfrak{H}^*$, we may define category $\mathrm{Mod}_\lambda(\tilde{\mathcal{D}})$ of coherent sheaves on $\mathcal{B}$ of $\tilde{\mathcal{D}}$-modules $\mathcal{M}$ such that $\mathcal{J}_\lambda^n\mathcal{M} = 0$ for large n. Taking global sections (via 3.1) is a functor $\Gamma_\lambda : \mathrm{Mod}_\lambda(\tilde{\mathcal{D}}) \to \mathrm{Mod}_\lambda(\tilde{U})$. Its left adjoint is the localization functor $\Delta_\lambda : M \mapsto M \otimes_{\tilde{U}} \tilde{\mathcal{D}}$.

Theorem 3.1. *(Beilinson-Bernstein Localization)*[BB, Theorem 3.3.1]

(1) If λ is ρ-dominant, then Γ_λ is exact and the unit map $\Gamma_\lambda\circ\Delta_\lambda \to Id_{Mod_\lambda(\tilde{U})}$ is an isomorphism

(2) If λ is regular and ρ-dominant, then Γ_λ is equivalence of the categories $Mod_\lambda(\mathcal{D})$ and $Mod_\lambda(\tilde{U})$, and the localization functor Δ_λ is the inverse.

Lemma 3.2. [Bez, Lemma 5.3.12] *Suppose $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories which has a left(right) adjoint $G : \mathcal{B} \to \mathcal{A}$. Let $\overline{\mathcal{A}} := \mathcal{A}/\operatorname{Ker} F$ denote the Serre quotient category. Then $\overline{F} : \overline{\mathcal{A}} \to \mathcal{B}$ is an equivalence of categories if and only the canonical morphism $\eta : Id_{\mathcal{B}} \to F \circ G$ (resp. $F \circ G \to Id_{\mathcal{A}}$) is an isomorphism.*

Thus, applying this to $F = \Gamma_\lambda$ and using the localization theorem, we deduce

Corollary 3.3. *Let λ be ρ-dominant. Then*

$$\overline{\Gamma}_\lambda : Mod_\lambda(\tilde{\mathcal{D}})/\operatorname{Ker}\Gamma_\lambda \to Mod_\lambda(\tilde{U})$$

is an equivalence of categories.

Now, suppose λ is dominant integral. This gives rise to homomorphism $\dot{\lambda} : T \to \mathbf{C}^*$. Let $\mathcal{O}(\lambda)$ denote the sheaf on $\mathcal{B}$ formed by all regular functions f on $\tilde{\mathcal{B}}$ such that $f(xt) = \dot{\lambda}(t)f(x)$ for all $x \in \tilde{\mathcal{B}},\ t \in T$. This is the sheaf of sections of the line bundle $G \times_B \mathbf{C}_\lambda$ over G/B and the Borel-Weil theorem tells us $\Gamma(G/B, \mathcal{O}(w_0 \cdot \lambda)) = E_\lambda$, the simple highest weight $U\mathfrak{g}$-module of weight λ. Let $\mathcal{D}_{\mathcal{B}}(\lambda)$ denote the sheaf of twisted differential operators on $\mathcal{B}$ (see e.g [HTT, Section 11]). Then quantum hamiltonian reduction tells us $\mathcal{D}_{\mathcal{B}}(\lambda) = \tilde{\mathcal{D}}/\tilde{\mathcal{D}} \cdot \mathcal{J}_\lambda$ and Beilinson-Bernstein (e.g [HTT, Thm 11.2.2]) implies $\Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}}(\lambda)) = U\mathfrak{g}/U\mathfrak{g} \cdot I_{|\lambda|}$.

Define the geometric translation functor $\Theta_\lambda^\mu : \mathrm{Mod}_\lambda(\tilde{\mathcal{D}}) \to \mathrm{Mod}_\mu(\tilde{\mathcal{D}})$ by $\mathcal{M} \mapsto \mathcal{O}(\mu - \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M}$. Note, we assume $\lambda - \mu$ is dominant integral. Since $\mathcal{O}(\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(-\nu) = \mathcal{O}_{\mathcal{B}}$, we have $\Theta_\lambda^\mu \Theta_\mu^\lambda = \mathrm{Id}$. Thus, geometric translation functors are always an equivalence of categories. Compare with the algebraic translation functors on $U\mathfrak{g}$, which are equivalence of category if and only if λ and μ have the same degeneracy, i.e $W_\lambda = W_\mu$.

Lemma 3.4. *Suppose λ, μ are ρ-dominant integral weights and $W_\lambda \subset W_\mu$. The following diagram commutes up to canonical equivalence of funtors:*

$$\begin{array}{ccc} Mod_\lambda(\tilde{\mathcal{D}}) & \xrightarrow{\Gamma_\lambda} & Mod_\lambda(\tilde{U}) \\ \downarrow{\scriptstyle \Theta_\lambda^\mu} & & \downarrow{\scriptstyle \tilde{\theta}_\lambda^\mu} \\ Mod_\mu(\tilde{\mathcal{D}}) & \xrightarrow{\Gamma_\mu} & Mod_\mu(\tilde{U}) \end{array}$$

Proof. Given a finite-dimensional $\mathfrak{g}$-module L, associate $L_{\mathcal{B}} := L \otimes \mathcal{O}_{\mathcal{B}}$ to the trivial sheaf of (algebraic) regular functions $f : \mathcal{B} \to L$. Given f, define $\phi_f : G \to L$ by $g \to g^{-1}f(g)$ and given ϕ such that $\phi(gb) = b^{-1}\phi(g)$, define f by $f(g.B) = g\phi(g)$. This makes the identification of sheaves on $\mathcal{B}$:

$$L_{\mathcal{B}} \cong \mathrm{Ind}_B^G L := \{\phi :\in \mathbf{C}[G] \otimes L : \phi(g.b) = b^{-1}\phi(g)\}.$$

By Lie's theorem, we can find a B-stable filtration of L by codimension 1 spaces. Since $\mathrm{Ind}_B^G(-)$ is exact on finite-dimensional B-modules, we get a filtration on $\mathrm{Ind}_B^G L$ with subquotients $\mathrm{Ind}_B^G(L_i/L_{i-1}) = \mathcal{O}(\nu_i)$ where $\nu_i : B \to \mathbf{C}$ is a character of B corresponding to L_i/L_{i-1}.

Now, suppose $\lambda-\mu$ is dominant and let $L = L_{\mu-\lambda}$. Giving a $\mathfrak{g}$-module structure on $\mathcal{M} \in \mathrm{Mod}_\lambda(\tilde{\mathcal{D}})$ and tensor product $\mathfrak{g}$-structure on $\mathcal{L}_\nu \otimes \mathcal{M}$, we get a $\mathfrak{g}$ stable filtration

$$0 \subset \mathrm{Ind}_B^G L_1 \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M} \subset \cdots \subset \mathrm{Ind}_B^G L_r \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M} = \mathrm{Ind}_B^G L \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M},$$

with successive quotients $\mathcal{O}(\nu_i) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M}$. For any $a \in \mathbf{C}[\mathfrak{H}^*]$, the action of a on $\mathcal{O}(\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M}$ is

$$a.(f \otimes m) = f \otimes (T_\nu a).m, \quad f \in \mathcal{O}(\nu),\ m \in \mathcal{M}, \tag{3.2}$$

where $T_\nu a : \mathbf{C}[\mathfrak{H}^*] \to \mathbf{C}[\mathfrak{H}^*],\quad p(x) \mapsto p(x + \nu)$ is the affine linear translation. Using this action and the fact that $Z\mathfrak{g}$ acts on $\mathcal{M} \in \mathrm{Mod}_\lambda(\tilde{\mathcal{D}})$ via central character χ_λ, we find $z \in Z\mathfrak{g}$ acts on

$\mathcal{O}(\nu)\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}$ by central character $\chi_{\lambda+\nu}$. Thus

$$\Big(\prod_{i=1}^{r}(z-\chi_{\lambda+\nu_i}(z))^{\alpha_i}\Big).\mathrm{Ind}_B^G L\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}=0,\quad \text{for some } \alpha_i\in\mathbf{Z}_{\geq 0}.$$

Hence there is a direct sum decomposition parameterized by central characters. We show the summand corresponding to μ is just

$$\mathrm{pr}_{|\mu|}\mathrm{Ind}_B^G L_{\mu-\lambda}\otimes\mathcal{M}=\mathcal{O}(\mu-\lambda)\otimes\mathcal{M}.$$

Indeed, if ν' is a weight of $L_{\mu-\lambda}$, then $\chi_{\nu'+\lambda}=\chi_{\lambda+\nu}$ ($\nu:=\mu-\lambda$) implies $\exists w\in W$ such that

$$w\cdot(\nu'+\lambda)=\lambda+\nu\Rightarrow(w(\lambda+\rho)-(\lambda+\rho))+(w(\nu')-\nu)=0.$$

But λ is ρ-dominant, so $w(\lambda+\rho)-(\lambda+\rho)\leq 0$. And ν' is weight of $L_{\mu-\lambda}$, so $w(\nu')\leq\mu-\lambda=\nu$. Thus $w\cdot\lambda=\lambda\Rightarrow w\in W_\lambda\subset W_\mu\Rightarrow w(\nu)=\nu$ and $w(\nu')=\nu\Rightarrow\nu'=\nu$, as desired. Thus the subquotient sheaf $\mathcal{O}(\mu-\lambda)\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}$ splits off from $\mathrm{Ind}_B^G E_{\mu-\lambda}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}$ as a sheaf of $Z\mathfrak{g}$-modules.

Next, observe

$$E\otimes\Gamma(\mathcal{M})=\Gamma(E_\mathcal{B}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M})=\Gamma(\mathrm{Ind}_B^G E\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}).$$

Thus we obtain

$$\begin{aligned}\tilde{\theta}_\lambda^\mu\circ\Gamma_\lambda(\mathcal{M})&=\mathrm{pr}_{|\mu|}(E_{\mu-\lambda}\otimes\Gamma(\mathcal{M}))\\&=\mathrm{pr}_{|\mu|}\Gamma(\mathrm{Ind}_B^G E_{\mu-\lambda}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M})\\&=\Gamma(\mathcal{O}(\mu-\lambda)\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M})=\Gamma_\mu\circ\Theta_\lambda^\mu\mathcal{M}.\end{aligned}$$

Finally, formula 3.2 shows the $\mathbf{C}[\mathfrak{H}^*]$-action on $\theta_\lambda^\mu\circ\Gamma_\lambda(M)$ defined in section 2 agrees with the natural $\mathbf{C}[\mathfrak{H}^*]$-action on $\Theta_\lambda^\mu M$. ☐

4. Properties of translation functors

Proof of Proposition 2.3. Since every $M\in\mathrm{Mod}_\lambda(\tilde{U}^{W_\lambda})$ is quotient of some $M'\in\mathrm{Mod}_\lambda(\tilde{U})$, it suffices to consider $M\in\mathrm{Mod}_\lambda(\tilde{U})$. By the localization theorem, we can find $\mathcal{M}\in\mathrm{Mod}_\lambda(\mathcal{D})$ such that $\Gamma_\lambda(\mathcal{M})=M$. Now part (2) follows from Lemma 3.4 and equation 3.2 for $\nu=\mu-\lambda$.

Next we prove part (i). Take $\tilde{M}\in\mathrm{Mod}_\mu(\tilde{U}),\tilde{N}\in\mathrm{Mod}_\lambda(\tilde{U})$ and let $M=\mathrm{Res}_\mu\tilde{M}\in\mathrm{Mod}_\mu(\tilde{U}^{W_\mu})$, $N=\mathrm{Res}_\lambda\tilde{N}$. Given $a\in\mathbf{C}[\mathfrak{H}^*]^{W_\mu}=Z(\tilde{U}^{W_\mu})$, act_a is a $\tilde{U}^{W_\mu}$-morphism and thus functoriality gives $\theta^+(a)\in\mathrm{End}_{\tilde{U}^{W_\lambda}}(\theta^+M)$, hence to an endomorphism $\theta^+(a)_M$ of $\mathrm{Hom}_{\tilde{U}^{W^\lambda}}(\theta^+M,N)$. We arrive at the following commutative diagram:

$$\begin{array}{ccccccccc}\mathrm{Hom}(\theta^+M,N)&\longrightarrow&\mathrm{Hom}(M,\theta^-N)&\longrightarrow&\mathrm{Hom}(M,\theta^-N)&\longrightarrow&\mathrm{Hom}(M,\theta^-N)&\longrightarrow&\mathrm{Hom}(\theta^+M,N)\\ \downarrow\theta^+(a)_M&&\downarrow a_M&&\downarrow a_{\theta^-N}&&\downarrow\theta^-(T_{\mu-\lambda}a)_{\theta^-N}&&\downarrow(T_{\mu-\lambda}a)_M\\ \mathrm{Hom}(\theta^+M,N)&\longrightarrow&\mathrm{Hom}(M,\theta^-N)&\longrightarrow&\mathrm{Hom}(M,\theta^-N)&\longrightarrow&\mathrm{Hom}(M,\theta^-N)&\longrightarrow&\mathrm{Hom}(\theta^+M,N)\end{array}$$

where Hom in columns 2, 3, and 4 are over $\tilde{U}^{W_\mu}$ and columns 1 and 5 are over $\tilde{U}^{W_\lambda}$. The first and last horizontal arrows are isomorphisms by adjunction and the two middle horizontal arrows are equality. Commutativity in squares 1 and 4 are from adjointness. Commutativity in square 2 is from a being chosen central and $\mathbf{C}[\mathfrak{H}^*]^{W_\mu}\subset\mathbf{C}[\mathfrak{H}^*]^{W_\lambda}$. Commutativity in square 3 is by part (2) proved in previous paragraph. We conclude $\theta^+(a)_M=(T_{\mu-\lambda}a)_M$ holds for all N, so Yoneda's lemma (uniqueness of representable functor) implies $\theta^+(a)=T_{\mu-\lambda}a$. ☐

Proposition 4.1. *The extended translation functors satisfy:*

1. *The functor $\tilde{\theta}^-:Mod_\lambda\tilde{U}\to Mod_\mu\tilde{U}$ induces an equivalence* $Mod_\lambda(\tilde{U})/\operatorname{Ker}\tilde{\theta}^-\to Mod_\mu\tilde{U}$.
2. *The adjunction morphisms induce isomorphisms of functors*

$$\tilde{\theta}^-\circ\tilde{\theta}_l^+\xleftarrow{\sim}Id_{Mod_\mu\tilde{U}}\xrightarrow{\sim}\tilde{\theta}^-\circ\tilde{\theta}_r^+.$$

(3) There is a natural isomorphism of functors $\tilde{\theta}_l^+ = \Gamma_\lambda \circ \Theta^+ \circ \Delta_\mu$.

Proof. We first prove (3) and then formally deduce (1) and (2) from it. Observe the left adjoint of the functors $\tilde{\theta}^-, \Gamma_\lambda, \Theta^-$ are $\tilde{\theta}_l^+, \Delta_\lambda, \Theta+$, respectively. So taking the adjoint of Lemma 3.4 yields

$$\Delta_\lambda \circ \tilde{\theta}_l^+ = \Theta_+ \circ \Delta_\mu : \mathrm{Mod}_\mu(\tilde{U}) \to \mathrm{Mod}_\lambda(\tilde{\mathcal{D}}).$$

Part (3) follows by taking Γ_λ on left of both sides and using $\Gamma_\lambda \Delta_\lambda \cong \mathrm{Id}$ by the localization theorem.

Now,

$$\tilde{\theta}^- \tilde{\theta}_l^+ = \tilde{\theta}^- \Gamma_\lambda \Theta^+ \Delta_\mu = \Gamma_\mu \Theta^- \Theta^+ \Delta_\mu = \Gamma_\mu \Delta_\mu = \mathrm{Id}_{\mathrm{Mod}_\mu(\tilde{U})},$$

where first equality uses (3), second uses Lemma 3.4, and third uses localization theorem. This proves (2) and (1) follows from "if" part of Lemma 3.2. ☐

Proposition 4.2. *There is a functorial isomorphism*

$$\theta^+ M \cong \left(\mathrm{Res}_\lambda \tilde{\theta}_l^+ (\tilde{U} \otimes_{\tilde{U}^{W_\mu}} M)\right)^{W_\lambda}, \quad M \in Mod_{|\mu|}(U\mathfrak{g}).$$

Proof. Recall $\tilde{\theta}^- = \theta^- \mathrm{Res}_\lambda$. If we forget the $\mathbf{C}[\mathfrak{H}^*]$ action, we find a functorial isomorphism

$$\theta^- \mathrm{Res}_\lambda = \mathrm{Res}_\mu \tilde{\theta}^- : \mathrm{Mod}_\lambda(\tilde{U}) \to \mathrm{Mod}_{|\mu|}(U\mathfrak{g}).$$

Then recall the left adjoints of $\mathrm{Res}_\lambda, \theta^-, \tilde{\theta}^-$ are $\mathrm{Ind}_{\tilde{U}^{W_\lambda}}^{\tilde{U}}, \theta^+, \tilde{\theta}_l^+$, respectively. So taking left adjoints on both sides (and again using Yoneda's lemma) yields an isomorphism of $\tilde{U}$-modules:

$$\widetilde{\theta^+ M} := \tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\theta^+ M) = \tilde{\theta}_l^+ (\tilde{U} \otimes_{\tilde{U}^{W_\mu}} M) = \tilde{\theta}_l^+ (\tilde{M}). \tag{4.1}$$

Since W_λ only acts on the $\tilde{U}$ component on both sides, these isomorphisms are compatible with the W_λ-actions. By the Pittie-Steinberg theorem, $\mathbf{C}[\mathfrak{H}^*] \cong \mathbf{C}[\mathfrak{H}^*]^{W_\lambda} \otimes \mathbf{C}[W_\lambda]$ as W_λ-modules. Thus

$$\tilde{U} \cong \tilde{U}^{W_\lambda} \otimes_{\mathbf{C}[\mathfrak{H}^*]^{W_\lambda}} \mathbf{C}[\mathfrak{H}^*] \cong \tilde{U}^{W_\lambda} \otimes_{\mathbf{C}[\mathfrak{H}^*]^{W_\lambda}} \mathbf{C}[\mathfrak{H}^*]^{W_\lambda} \otimes \mathbf{C}[W_\lambda] \cong \tilde{U}^{W_\lambda} \otimes_{\mathbf{C}} \mathbf{C}[W_\lambda]$$

as $(\tilde{U}^{W_\lambda}, W_\lambda)$-bimodules. Thus, $(\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\theta^+ M))^{W_\lambda} \cong \theta^+ M$ as $\tilde{U}^{W_\lambda}$-modules. Restricting to $U\mathfrak{g}$ and taking W_λ-invariants of right hand side of (4.1) yields precisely what we wanted. ☐

Proposition 4.3. *There is a functorial isomorphism*

$$\theta^- \theta^+ M \cong \mathrm{Res}_\mu (\tilde{U}^{W_\lambda} \otimes_{\tilde{U}^{W_\mu}} M).$$

Proof. As in Proposition 4.2, $\widetilde{\theta^+ M} = \tilde{\theta}_l^+ (\tilde{M})$. Now,

$$\mathrm{Res}_\mu \left(\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} (\theta^- \theta^+ M)\right) = \theta^- (\mathrm{Res}_\lambda \tilde{\theta}_l^+ (\tilde{M})) = \mathrm{Res}_\mu \tilde{\theta}^- \tilde{\theta}_l^+ (\tilde{M}) = \mathrm{Res}_\mu (\tilde{M}),$$

where the first equality uses θ^- commutes with $\mathrm{ResInd}_{\tilde{U}^{W_\mu}}^{\tilde{U}}$ and $\tilde{\theta}_l^+ (\tilde{M}) = \widetilde{\theta^+ M}$, second equality uses $\theta^- \mathrm{Res}_\lambda = \mathrm{Res}_\mu \tilde{\theta}^-$, and third equality uses Proposition 4.1(ii). Finally, W_λ commutes with every functor, so taking W_λ invariants of left hand side is by Proposition 4.2 equal to $\theta^- \theta^+ M$, and W_λ invariants of right hand side is $\mathrm{Res}_\mu (\tilde{U}^{W_\lambda} \otimes_{\tilde{U}^{W_\mu}} M)$. ☐

5. Soergel's theorems

Proof of Theorem 1.1. We use the notation $\tilde{M} = \mathrm{Ind}_{U\mathfrak{g}}^{\tilde{U}} M$. Observe we have the following isomorphism of algebras:

$$\begin{aligned}
\mathrm{End}_{U\mathfrak{g}}(P_{w_0 \cdot \lambda}) &= \mathrm{End}_{U\mathfrak{g}} \left((\mathrm{Res}_\lambda \tilde{\theta}_l^+ (\tilde{M}_{-\rho}))^{W_\lambda}\right) && \text{By } P_{w_0 \cdot \lambda} = \theta^+ M_{-\rho} \text{ and Proposition 4.2} \\
&= \mathrm{End}_{\tilde{U}^{W_\lambda}} \left((\tilde{\theta}_l^+ (\tilde{M}_{-\rho}))^{W_\lambda}\right) && \text{Equivalence of category (2.1)} \\
&= \mathrm{End}_{\tilde{U}^{W_\lambda}} \left((\tilde{M}_{-\rho})^{W_\lambda}\right) && \tilde{\theta}_l^+ \text{ is fully faithful, see Proposition 4.1} \\
&= \mathrm{End}_{\tilde{U}^{W_\lambda}} \left(\tilde{U}^{W_\lambda} \otimes_{U\mathfrak{g}} M_{-\rho}\right).
\end{aligned}$$

Now, observe the action of central subalgebra $\mathbf{C}[\mathfrak{H}^*]^{W_\lambda} \subset \tilde{U}^{W_\lambda}$ on $\tilde{U}^{W_\lambda} \otimes_{U\mathfrak{g}} M_{-\rho}$ induces an algebra isomorphism $\tau : C^{W_\lambda} \to \operatorname{End}_{\tilde{U}^{W_\lambda}}\left(\tilde{U}^{W_\lambda} \otimes_{U\mathfrak{g}} M_{-\rho}\right)$ because $M_{-\rho}$ is generated by a highest weight vector $v_{-\rho}$. Composing previous chain of algebra isomorphisms with τ^{-1} completes the proof. □

Proof of Theorem 1.2. We present a proof, only in the case of λ regular, which takes advantage of various duality properties in category $\mathcal{O}$. This proof closely follows [Bez, Prop. 6.0.1], which is motivated by the proof of the analogous Soergel Structure theorem for rational Cherednik algebras [GGOR, Thm. 5.3]. A key idea is to introduce a functor which sends Verma modules to dual Verma modules, and this can only be accomplished after passing to the derived category of $\mathcal{O}$.

Fix λ a ρ-dominant, regular weight. Let μ be a ρ-dominant weight lying on a single hyperplane s_α, and define the *wall crossing functor* $R_\alpha := \theta_\mu^\lambda \circ \theta_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$. Since translation functors are biadjoint, there exist natural (co)unit maps $\mathrm{Id} \to R_\alpha$ and $R_\alpha \to \mathrm{Id}$.

Definition 5.1. *Let μ lie on a single wall, defined by hyperplane $s_\alpha \in W$, in the closure of λ. The intertwining functors are defined as:*

$$\Theta_\alpha : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda), \quad M \mapsto \operatorname{Cone}(M \to R_\alpha M),$$

$$\Theta'_\alpha : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda), \quad M \mapsto \operatorname{Cone}(R_\alpha(M) \to M)[-1].$$

Note the intertwining functors are induced by taking the cone between exact functors on category of complexes, hence they are exact. The following lemma is a consequence of knowing how translation functors act on Vermas, see e.g [Hum, Chapter 7].

Lemma 5.2. *Let Δ_ν be the Verma of highest weight $\nu \in W\cdot\lambda$. Viewing Θ_α as functors $D^{\leq 0} \to D^{\leq 0}$, with $\Theta_\alpha(M) = (M \to R_\alpha M)$, M degree -1 and $R_\alpha M$ degree 0, and similarly for $\Theta'_\alpha : D^{\geq 0} \to D^{\geq 0}$,*

$$\Theta_\alpha : \Delta_\nu \mapsto \Delta_{s_\alpha \nu} \quad \textit{if } s_\alpha\nu > \nu, \quad \Theta'_\alpha : \Delta_\nu \mapsto \Delta_{s_\alpha\nu}, \;\; s_\alpha\nu < \nu$$

Then using this lemma, and property that a functor on a complex of Harish-Chandra bimodules is determined by the image of Verma modules, we may deduce $\{\Theta_\alpha, \Theta'_\alpha\}$ satisfy the braid group relations on $D^b(\mathcal{O}_\lambda)$ and also $\Theta_\alpha \circ \Theta'_\alpha \cong \mathrm{Id} \cong \Theta'_\alpha \circ \Theta_\alpha$ [Bez, Cor. 5.6.5].

We define a *standard object* in $\mathcal{O}$ as one which admits a filtration whose successive quotients are Verma's, and a *costandard object* as one which admits a filtration whose successive quotients are dual Verma's, denoted ∇_λ. A *tilting object* is one which is standard and costandard. For example, all projective objects in $\mathcal{O}$ are standard and the antidominant projective $P_{w_0\cdot\lambda}$ is tilting.

Lemma 5.3. *(a) If T_1, T_2 are tilting, then*

$$\operatorname{Hom}_{\tilde{\mathcal{O}}_\lambda}(T_1, T_2) \cong \operatorname{Hom}_{\tilde{\mathcal{O}}_{-\rho}}(\tilde{\theta}_\lambda^{-\rho}(T_1), \tilde{\theta}_\lambda^{-\rho}(T_2)).$$

(b) Let $w_0 = s_{\alpha_1} \cdots s_{\alpha_n}$ be a minimal expression. Then

$$\Theta_{w_0} = \Theta_{s_{\alpha_1}} \circ \cdots \circ \Theta_{s_{\alpha_n}}$$

sends projectives to tiltings.
(c) Using s_α acts on $\tilde{\mathcal{O}}_{-\rho} \cong C$-mod via $M \mapsto C \otimes_{C^{s_\alpha}} M$, there is a commutative diagram

$$\begin{array}{ccc} D^b(\mathcal{O}_\lambda) & \xrightarrow{\Theta_{s_\alpha}} & D^b(\mathcal{O}_\lambda) \\ \downarrow{\scriptstyle \tilde{\theta}_\lambda^{-\rho}} & & \downarrow{\scriptstyle \tilde{\theta}_\lambda^{-\rho}} \\ D^b(\tilde{\mathcal{O}}_{-\rho}) & \xrightarrow{s_\alpha} & D^b(\tilde{\mathcal{O}}_{-\rho}). \end{array}$$

We explain how to prove the Struktursatz Theorem 1.2 using Lemma 5.3. Let P_1, P_2 be projective and define $T_i := \Theta_{w_0} P_i$. Then

$$\begin{aligned}
\mathrm{Hom}_{U\mathfrak{g}}(P_1,P_2) &= \mathrm{Hom}_{U\mathfrak{g}}(T_1,T_2) && \Theta_{w_0} \text{ is equivalence [Bez, Cor. 5.6.5]}\\
&= \mathrm{Hom}_{\tilde{\mathcal{O}}_{-\rho}}(\tilde{\theta}_\lambda^{-\rho} T_1, \tilde{\theta}_\lambda^{-\rho} T_2) && \text{part(b) implies } T_i \text{ tilting, then use part (a)}\\
&= \mathrm{Hom}_{\tilde{\mathcal{O}}_{-\rho}}(w_0(\tilde{\theta}_\lambda^{-\rho}(P_1)), w_0(\tilde{\theta}_\lambda^{-\rho}(P_2))) && \text{part (c)}\\
&= \mathrm{Hom}_{\tilde{\mathcal{O}}_{-\rho}}(\tilde{\theta}_\lambda^{-\rho}(P_1), \tilde{\theta}_\lambda^{-\rho}(P_2)).
\end{aligned}$$

It is easy to see $\tilde{\mathcal{O}}_{-\rho}$ is equivalent to C-mod. Then we show $\mathbf{V} = \tilde{\theta}_\lambda^{-\rho}$. Indeed, $\tilde{\theta}_\lambda^{-\rho}$ is represented by a projective object $P = \oplus_{w\in W} P_{w\cdot\lambda}^{n_w}$ in $\mathcal{O}_\lambda \cong \tilde{\mathcal{O}}_\lambda$ (See 2.1, λ is regular), and θ^- kills all simples but antidominant one, so $n_w = 0$ for $w \neq w_0$. The Endomorphism theorem implies $n_{w_0} = 1$. Thus, the final term is isomorphic to $\mathrm{Hom}_C(\mathbf{V}(P_1), \mathbf{V}(P_2))$, as desired. □

Proof of Lemma 5.3. (a): Recall $\mathcal{O}_\lambda/(\ker(\theta_\lambda^{-\rho})) \to \tilde{\mathcal{O}}_{-\rho}$ is an equivalence of categories and $\ker(\theta_\lambda^{-\rho}) = \langle L_{w\cdot\lambda} : w \neq w_0\rangle$. Next we show T_1 satisfies $\mathrm{Hom}(T_1, B) = 0$ for all $B \in \ker\tilde{\theta}$ and T_2 satisfies $\mathrm{Hom}(B, T_2) = 0$ for all $B \in \ker\tilde{\theta}$. Indeed, this is done by induction on length of (co)standard filtration and using socle of $\Delta_{w\cdot\lambda}$ is $L_{w_0\cdot\lambda}$, and top is $L_{w\cdot\lambda}$. Then using a general lemma on Serre quotients [Bez, Lemma 6.1.2], we conclude $\mathrm{Hom}_{\mathcal{O}_\lambda}(T_1,T_2) = \mathrm{Hom}_{\mathcal{O}_\lambda/\ker\tilde{\theta}_\lambda^{-\rho}}(T_1,T_2)$ as desired.

(c): Using Proposition 4.3, we find a commutative diagram (see [Bez, Lemma 5.5.1] for details)

$$\begin{array}{ccc}
\mathcal{O}_\lambda & \xrightarrow{R_\alpha} & \mathcal{O}_\lambda\\
\downarrow{\scriptstyle \tilde{\theta}_\lambda^{-\rho}} & & \downarrow{\scriptstyle \tilde{\theta}_\lambda^{-\rho}}\\
\tilde{\mathcal{O}}_{-\rho} & \xrightarrow[M\mapsto C\otimes_{C^\alpha} M]{} & \tilde{\mathcal{O}}_{-\rho}
\end{array}$$

We wish to compute $\mathrm{Cone}(M \mapsto C \otimes_{C^{s_\alpha}} M)$. Let $\mathcal{O}(V)$ denote the regular functions on variety V. There is short exact sequence

$$0 \to \mathcal{O}(\{(x,x) : x \in \mathfrak{h}^*\}) \to \mathcal{O}(\mathfrak{h}^* \times_{\mathfrak{h}^*//\{1,s_\alpha\}} \mathfrak{h}^*) \to \mathcal{O}(\{(x, s_\alpha(x)) : x \in \mathfrak{h}^*\}) \to 0.$$

Tensor with $\mathbf{C}$ over $\mathrm{Sym}(\mathfrak{t})^W = Z\mathfrak{g}$, and define $C_{s_\alpha} := \mathcal{O}(\{(x, s_\alpha(x)) : x \in \mathfrak{h}^*\}) \otimes_{\mathrm{Sym}(\mathfrak{h}^W)} \mathbf{C}$ to get short exact sequence $0 \to C \to C \otimes_{C^{s_\alpha}} C \to C_{s_\alpha} \to 0$. Thus,

$$\begin{aligned}
\mathrm{Cone}(M \to C \otimes_{C_\alpha^s} M) &\cong C_{s_\alpha} \otimes_C M\\
&\cong \mathcal{O}(\{(x, s_\alpha(x)) : x \in \mathfrak{h}^*\})/(\mathrm{Sym}(\mathfrak{h})_+^W) \otimes_{\mathcal{O}(\{(x,x):x\in\mathfrak{h}^*\})/(\mathrm{Sym}(\mathfrak{h})_+^W)} M\\
&\cong s_\alpha M
\end{aligned}$$

(b): We prove a claim: Θ_{w_0} is a bijection between set of standard objects and set of costandard objects. Since Θ_{w_0} is exact, it suffices to check Verma is sent to dual Verma. By Lemma 5.2, we have $\Delta_w = \Theta'_{w_0}(\Delta_{w_0\cdot\lambda})$ and dually, $\nabla_w = \Theta_{w_0}(\nabla_{w_0\cdot\lambda})$, where λ is dominant regular. Let $w_1 := ww_0$. Then $l(w_0) = l(w^{-1}) + l(w_1)$ and we find $\Theta_{w_0} = \Theta_{w_1} \circ \Theta_{w^{-1}}$. Thus Verma gets sent to dual Verma:

$$\begin{aligned}
\Theta_{w_0}(\Delta_w) &= \Theta_{w_1} \circ \Theta_{w^{-1}}(\Theta'_w(\Delta_{w_0\cdot\lambda}))\\
&= \Theta_{w_1}(\Delta_{w_0\cdot\lambda}) = \Theta_{w_1}(\nabla_{w_0\cdot\lambda})\\
&= \nabla_{w_1}.
\end{aligned}$$

A projective object P is a standard object, hence $\Theta_{w_0}P$, which lives in degree 0, is a costandard object by the claim. By the Ext criteria [Hum, 6.13], $\Theta_{w_0}P$ is a standard object if and only if

$$\mathrm{Ext}^1(\Theta_{w_0}(P), \nabla_\mu) = 0, \ \ \forall\mu.$$

By claim, there is ν for which $\nabla_\mu = \Theta_{w_0}(\Delta_\nu)$. Since Θ_{w_0} is autoequivalence, the above Ext equals $\mathrm{Ext}^1(P, \nabla_\nu) = 0$. Thus $\Theta_{w_0}(P)$ is a standard object as well. □

References

[BG] A. Beilinson, V. Ginzburg, Wall-crossing functors and $\mathcal{D}$-modules, Represent. Theory 3 (1999), 131.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, JAMS, (1996), 473-527

[BB] A. Beilinson, J. Bernstein, Proof of Jantzen conjectures, Advances in Soviet Mathem. **16**, Gelfand Seminar, (1983), 1-50.

[Be] J. Bernstein, Trace in Categories, Proceedings of the Dixmier Colloquium, Birkhauser, Progress in Mathem. **92**, Boston, (1990), pp. 417-423.

[Bez] R. Bezrukavnikov, Lectures by Bezrukavnikov written by T. Kanstrup: Canonical bases and representation categories. http://www.math.uni-bonn.de/people/kanstrup/RomanBezCanonBases.pdf

[BeGe] J. Bernstein, S. Gelfand, Tensor products of finite and infinite dimensional representations of semisimple Lie algebras, Comp. Math. **41** (1980), 245-285.

[BoBr] J. Bernstein, J.L. Brylinski, Differential operators on flag manifolds II, MP1 (1989).

[CG] N. Chriss, V. Ginzburg, Representation theory and complex geometry, Modern Birkhauser classics, (1997)

[EMTW] B. Elias, S. Makisumi, U. Thiel, G. Williamson, Introduction to Soergel bimodules, preprint.

[EW] B. Elias, G. Williamson, The hodge theory of Soergel bimodules, Annals of Math, 180 (2014), no. 3, 1089-1136.

[Gait] D. Gaitsgory, Geometric representation theory, lecture notes, Harvard Univ, 2005. http://people.math.harvard.edu/ gaitsgde/267y/catO.pdf

[GGOR] V. Ginzburg, N. Guay, E. Opdam, R. Rouquier, On the category $\mathcal{O}$ for rational Cherednik algebras, Invent. Math, (2002)

[Hum] J. Humphreys, Representations of Semisimple Lie Algebras in BGG Category $\mathcal{O}$, AMS **94** (2008)

[Jan] J.C Jantzen, Moduln mit einem holchsten Gewicht, Lecture Notes in Math., 750, Springer-Verlag, Berlin (1979).

[HTT] R.Hotta, K. Takeuchi, T. Tanisaki, D-modules, perverse sheaves, and representation theory, Birkhauser, (2006).

[Maz] V. Mazorchuk, Lectures on $\mathfrak{sl}_2(\mathbf{C})$-modules, Imperial College Press, (2009)

[Soe] W. Soergel, Kategorie $\mathcal{O}$, perverse garben und moduln uber den koinvarianten zur weylgruppe, JAMS **3**, April (1990).

[Str] C. Stroppel, Category $\mathcal{O}$: Quivers and endomorphism rings of projectives, Representation theory, **7**, (2003).

[Sun] Y. Sun $\mathcal{D}$-modules and representation theory, part III essay. https://yisun.io/notes/dmod.pdf