SOERGEL’S THEOREMS VIA WALL-CROSSING FUNCTORS.

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1. Introduction

The study of finite-dimensional representations of a (semisimple) Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is well understood and was first developed by Cartan and Weyl in the early 1900s. They classified finite-dimensional irreducible representations with an algebraic approach: consider a large module, called a Verma module, and then take the unique largest nontrivial quotient. Later, Borel and Weil used a geometric approach: consider global sections of a line bundle over a certain homogenous space, called the flag variety.

However, the study of infinite-dimensional representations of $\mathfrak{g}$ is far more complicated. In fact, it is known that $\mathfrak{sl}_2(\mathbb{C})$ is the only (semisimple) Lie algebra for which all irreducible representations are known [Maz]. Thus this leads to an interesting subcategory, the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, for which there is a classification result. In this paper, we explore two theorems (1.1, 1.2) of Soergel which describe endomorphism rings of certain projective objects in $\mathcal{O}$. Since there is an equivalence between $\mathcal{O}$ and modules over the algebra of endomorphisms of the projective generator, these theorems for example give an explicit algorithm for computing the Ext-quiver of various blocks of $\mathcal{O}$ [Str].

The importance of Soergel’s fundamental theorems goes beyond category $\mathcal{O}$. For example, Soergel used these theorems to develop the study of Soergel bimodules (e.g. see [EMTW] for a comprehensive introduction), with the ultimate goal of giving a purely algebraic proof of the Kazhdan-Lusztig conjecture. This was accomplished recently in [EW]. Soergel’s theorems also appear in the proof of Koszul duality for BGG category $\mathcal{O}$ [BGS].

Our approach in proving these theorems closely follows [BG]. Namely, we will use properties of translation functors deduced by theorems in $D$-modules. As a result, our method is not the most direct, but the advantage is the arguments work in a general setting, and we see the main results arise naturally from the Beilinson-Bernstein correspondence $\mathfrak{g} - \text{modules} \leftrightarrow D - \text{modules}$.

All unexplained notation of the following theorems will be introduced in Section 2.

**Theorem 1.1.** (Endomorphisnensatz theorem) Let $\lambda$ be a $\rho$–dominant integral weight. Then there is an algebra isomorphism

$$\text{End}_{\mathfrak{g}}(P_{w_0, \lambda}) \cong C^{W_\lambda}.$$

**Theorem 1.2.** (Struktursatz theorem) Let $\lambda$ be a $\rho$–dominant integral weight. Let $V : \mathcal{O}_\lambda \rightarrow C^{W_\lambda}$, $M \mapsto \text{Hom}_{U_{\mathfrak{g}}}(P_{w_0, \lambda}, M)$ denote the Soergel functor. Then for any projective object $Q \in \mathcal{O}_\lambda$ and any $M \in \mathcal{O}_\lambda$, the following natural morphism is an isomorphism

$$\text{Hom}_{U_{\mathfrak{g}}}(M, Q) \rightarrow \text{Hom}_{C^{W_\lambda}}(V(M), V(Q)).$$

2. Translation functors for extended universal enveloping algebra

We use the following notation. Fix a semisimple Lie algebra $\mathfrak{g}$ over complex numbers $\mathbb{C}$ and denote $\mathfrak{h}$ the abstract Cartan subalgebra. This is obtained by identifying the spaces $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ for all Borel subalgebras $\mathfrak{b}$. In particular, $\mathfrak{h}$ is not a subalgebra of $\mathfrak{g}$. This Cartan comes equipped with a root system $\Phi \subset \mathfrak{h}^*$ and with a canonical choice of simple roots $\{\alpha_1, \ldots, \alpha_l\}$. See e.g. [CG Section 3.1] for details. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots. We say a weight $\lambda \in \mathfrak{h}^*$ is
$\rho$-dominant if $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ for all positive coroots $\alpha^\vee$. We give a partial ordering on weights by saying $\lambda \leq \mu$ if $\mu - \lambda \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0} \alpha_i$.

Recall the (abstract) Weyl group $W$ acts on $\mathfrak{h}^*$ in two ways. The standard action: for all simple reflections $s_\alpha \in W$ for $\alpha \in \Phi^+$ positive, define $s_\alpha \cdot \lambda := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. The dot action: $w \cdot \lambda := w(\lambda + \rho) - \rho$ for $w \in W$. We will always assume $W$ acts on $\mathfrak{h}^*$ via the dot action unless stated otherwise. Define the stabilizer $W_\lambda := \{ w \in W : w \cdot \lambda = \lambda \}$. Note, $-\rho$ is the unique weight with $W_- \rho = W$. We call a weight $\lambda$ regular if $W_\lambda = \{ 1 \}$. Given $\lambda \in \mathfrak{h}^*$, define the $W$-orbit $|\lambda| := W/W_\lambda$. Let $U_0 \subset W$ denote the longest element of $W$. Let $C[\mathfrak{h}^*]_W$ denote the $W$-dot invariant polynomials on $\mathfrak{h}^*$ and define $\mathfrak{h}^*/W := \text{Specm} C[\mathfrak{h}^*]_W$, the geometric quotient with respect to $W$-dot action. Let $C = C[\mathfrak{b}^*]/C[\mathfrak{b}^*] \cdot C[\mathfrak{b}^*]^+_W$ denote the coinvariant algebra.

Next, we introduce an intermediate algebra $U_{\mathfrak{b} \mathfrak{b}}$. The torus, so that $\text{Lie} (T) = \mathfrak{b}$. The extended universal enveloping algebra $\tilde{U}_{\mathfrak{b} \mathfrak{b}}$ is a domain. Since $\chi_{\mathfrak{b} \mathfrak{b}}(\tilde{U}_{\mathfrak{b} \mathfrak{b}})$ is a domain, we will fix a Cartan and Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ and consider the BGG category $O$, which is defined as the full subcategory of $\text{Mod}(U_{\mathfrak{b} \mathfrak{b}})$ consisting of finitely-generated $U_{\mathfrak{b} \mathfrak{b}}$-modules $M$ such that $U_{\mathfrak{b} \mathfrak{b}}$ acts locally finitely and $U_{\mathfrak{h} \mathfrak{h}}$ acts semisimply. Category $O$ has a direct sum decomposition $O = \bigoplus_{\lambda \in \mathfrak{h}^*/W} O_\lambda$, with $O_\lambda$ consisting of modules for which $(z - \xi(z))^n$ acts by 0 for large $n$. Let $\lambda \in \mathfrak{h}^*$ denote the simple module of highest weight $\lambda$, $\Delta_\lambda$ the corresponding Verma, and $P_\lambda$ the corresponding indecomposable projective. It is known $O_\lambda$ is a highest-weight category and is only easy to describe in the degenerate case $\lambda = -\rho$, ($O_{-\rho}$ is equivalent to the category of finite-dimensional vector spaces). See [Hum] for a general discussion of properties of $O$.

**Definition 2.1.** We define the extended universal enveloping algebra

$$\tilde{U} := U_{\mathfrak{g} \mathfrak{g}} \otimes_{\mathfrak{g} \mathfrak{g}} C[\mathfrak{h}^*]$$

where $Z_{\mathfrak{g} \mathfrak{g}}$ acts on $C[\mathfrak{h}^*]$ via the Harish-Chandra map.

Observe, there is a natural filtration on $\tilde{U}$ so that $\text{gr}(\tilde{U}) = S_{\mathfrak{g} \mathfrak{g}} \otimes_{\mathfrak{h} \mathfrak{h}} S_{\mathfrak{h} \mathfrak{h}} = O(\mathfrak{g} \times \mathfrak{h} \times \mathfrak{h})$. In particular, $\tilde{U}$ is a domain. Since $U_{\mathfrak{g} \mathfrak{g}}$ and $U_{\mathfrak{h} \mathfrak{h}}$ are free over $Z_{\mathfrak{g} \mathfrak{g}}$, $C[\mathfrak{h}^*]$ and $U_{\mathfrak{g} \mathfrak{g}}$ are subalgebras of $\tilde{U}$. The $W$-action on $\mathfrak{h} \mathfrak{h}$ induces an action on $\tilde{U}$ in a way that $\tilde{U}^W = U_{\mathfrak{g} \mathfrak{g}}$. Let $\text{Mod}_{\lambda}(U_{\mathfrak{g} \mathfrak{g}})$, resp. $\text{Mod}_{\lambda}(\tilde{U})$ be the category of finitely-generated $U_{\mathfrak{g} \mathfrak{g}}$, resp. $\tilde{U}$-modules $M$ such that $I_{\lambda}^* \cdot M = 0$, resp. $J_{\lambda}^* \cdot M = 0$ for sufficiently large $n$. We have a canonical exact functor

$$\text{Res}_\lambda : \text{Mod}_{\lambda}(\tilde{U}) \rightarrow \text{Mod}_{\lambda}(U_{\mathfrak{g} \mathfrak{g}}).$$

Next, we introduce an intermediate algebra $C[\mathfrak{h}^*]^+_W \subset C[\mathfrak{h}^*]^+_W \subset C[\mathfrak{h}^*]^+_W$ which corresponds to the factorization $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/W \rightarrow \mathfrak{h}/W$ (since $W$ is finite, $C[\mathfrak{h}^*]/W = C[\mathfrak{h}^*]^+_W$).
Define $\mathcal{J}_\lambda^{W_\lambda} := C[\mathfrak{g}]^{W_\lambda} \cap \mathcal{J}_\lambda$ and let $\text{Mod}_\lambda(\tilde{U}^{W_\lambda})$ denote the category of finitely-generated $\tilde{U}^{W_\lambda}$-modules annihilated by large enough power of $\mathcal{J}_\lambda^{W_\lambda}$. The map $\mathfrak{g}^* / W_\lambda \to \mathfrak{g}^* / W$ is unramified over $|\lambda|$, hence the $I_{|\lambda|}$-adic completion of $Z_{\mathfrak{g}}$ is isomorphic to $\mathcal{J}_\lambda^{W_\lambda}$-adic completion of $C[\mathfrak{g}^{*}]^{W_\lambda}$. This implies there is isomorphism of $I_{|\lambda|}$-adic completion of $U_{\mathfrak{g}}$ with $\mathcal{J}_\lambda^{W_\lambda}$-adic completion of $\tilde{U}^{W_\lambda}$. Thus (2.1) 
$$\text{Res}_\lambda : \text{Mod}_\lambda(\tilde{U}^{W_\lambda}) \to \text{Mod}_{|\lambda|}(U_{\mathfrak{g}})$$
is an equivalence of categories.

Now, the projection $\pi : \mathfrak{g}^* \to \mathfrak{g}^* / W$ satisfies $\pi^{-1}(\lambda) = \lambda$, i.e. is totally ramified. Then we see for $M \in \text{Mod}_\lambda(\tilde{U}^{W_\lambda})$, $\tilde{U} \otimes_{\tilde{U}^{W_\lambda}} M \in \text{Mod}_\lambda(\tilde{U})$. Thus $M \mapsto \tilde{U} \otimes_{\tilde{U}^{W_\lambda}} M$ is left adjoint of $\text{Res}_\lambda$, where $M \in \text{Mod}_\lambda(U_{\mathfrak{g}})$ is viewed as $\tilde{U}^{W_\lambda}$-module by (2.1).

Now, we define translation functors. Let $L$ be a finite-dimensional $\mathfrak{g}$-module and $M \in \text{Mod}_{|\lambda|}(U_{\mathfrak{g}})$. Then by [BeGe], $L \otimes M$ is annihilated by an ideal of $Z_{\mathfrak{g}}$ of finite-codimension, and thus there is a direct sum decomposition
$$L \otimes M = \bigoplus_{\mu \in \mathfrak{g}^*/W} \text{pr}_{\mu}(L \otimes M), \text{ where } \text{pr}_{\mu}(L \otimes M) \in \text{Mod}_{|\mu|}(U_{\mathfrak{g}}).$$

Now suppose $\lambda, \mu \in \mathfrak{g}^*$ are $p$-dominant weights and $L_{w(\lambda - \mu)}$ is the finite-dimensional $U_{\mathfrak{g}}$-module of highest-weight $w \cdot (\lambda - \mu)$.

**Definition 2.2.** Define the translation functor
$$\theta_\mu^\lambda : \text{Mod}_\mu(U_{\mathfrak{g}}) \to \text{Mod}_\lambda(U_{\mathfrak{g}}), \quad M \mapsto \text{pr}_{|\mu|}(L_{w(\lambda - \mu)} \otimes M).$$

The functors $\theta_\mu^\lambda$ and $\theta_\mu^\lambda$ are easily seen to be biadjoint and exact. By (2.1), $\theta_\mu^\lambda$ may be viewed as
$$\theta_\mu^\lambda : \text{Mod}_\mu(\tilde{U}^{W_\nu}) \to \text{Mod}_\lambda(\tilde{U}^{W_\nu}).$$

Acting by $z \in C[\mathfrak{g}^*]^{W_\nu}$ on $M \in \text{Mod}_\mu(\tilde{U}^{W_\nu})$ is a $\tilde{U}^{W_\nu}$-module homomorphism since $Z(\tilde{U}^{W_\nu}) = C[\mathfrak{g}^*]^{W_\nu}$. Hence functoriality gives an endomorphism $\theta_\mu^\lambda(a) : \theta_\mu^\lambda M \to \theta_\mu^\lambda M$ of $\tilde{U}^{W_\nu}$-modules.

We may explicitly describe this action as follows. From now on, fix integral $p$-dominant weights $\lambda, \mu \in \mathfrak{g}^*$ such that $W_\lambda \subset W_\mu$ and set $\pm := \theta_\mu^\lambda$ and $\pm := \theta_\mu^\lambda$. Let $T_{\lambda-\mu} : C[\mathfrak{g}^*] \to C[\mathfrak{g}^*]$ be the map $p(x) \to p(x + \lambda - \mu)$. Then $T_{\pm(\lambda, \mu)}$ preserves $C[\mathfrak{g}^*]^{W_\lambda}$ and $T_{\pm(\lambda, \mu)}C[\mathfrak{g}^*]^{W_\nu} \subset C[\mathfrak{g}^*]^{W_\nu}$. The following proposition will be proved in Section 4 using $D$-modules.

**Proposition 2.3.** Let $z \in C[\mathfrak{g}^*]^{W_\nu}$. Then

1. For any $M \in \text{Mod}_\mu(\tilde{U}^{W_\nu})$ and $m \in \theta_\mu^\lambda M$, we have $z \cdot m = T_{\lambda-\mu}(z)(\theta_\mu^\lambda(z)(m))$.
2. For any $M \in \text{Mod}_\mu(\tilde{U}^{W_\nu})$ and $m \in \theta_\mu^\lambda M$, we have $z \cdot m = [\theta_\mu^\lambda(T_{\lambda-\mu}z)] \cdot m$.

So Proposition 2.3 explicitly gives: Suppose $u = g \otimes z \in U_{\mathfrak{g}} \otimes Z_{\mathfrak{g}} C[\mathfrak{g}^*]^{W_\nu} \cong \tilde{U}^{W_\nu}$ and $l \otimes m \in E_{\mu-\lambda} \otimes M$. Then, $(g \otimes z)(l \otimes m) = (l \otimes z)(g \otimes m) = g.l \otimes T_{\lambda-\mu}(z).m + l \otimes T_{\lambda-\mu}(z)g.m$.

Our goal is to extend these translation functors to $\tilde{U}$-modules. We have maps
$$\text{Mod}_\lambda(\tilde{U}) \xrightarrow{\text{Res}_\lambda} \text{Mod}_{|\lambda|}(U_{\mathfrak{g}}) \xrightarrow{\theta_\mu} \text{Mod}_{|\mu|}(U_{\mathfrak{g}}).$$

Also, the action of $a \in C[\mathfrak{g}^*] = Z(\tilde{U})$ on $M \in \text{Mod}_\lambda(\tilde{U})$ induces by functoriality an endomorphism $\theta_\mu^\lambda(a) : \theta_\mu^\lambda(\text{Res}_\lambda M) \to \theta_\mu^\lambda(\text{Res}_\lambda M)$. We now define a $C[\mathfrak{g}^*]$ action on $\theta_\mu^\lambda(\text{Res}_\lambda M)$ by
$$a \cdot m := \theta_\mu^\lambda(T_{\lambda-\mu}a) \cdot m, \quad m \in \theta_\mu^\lambda(\text{Res}_\lambda M).$$

Proposition 2.3(2) implies this action restricted to the $W_\mu$-invariants coincides with the $C[\mathfrak{g}^*]^{W_\nu}$-action arising from restricting the $\tilde{U}^{W_\nu}$ action on $\theta_\mu^\lambda(\text{Res}_\lambda M)$. Thus, we may combine the $C[\mathfrak{g}^*]$ and $\tilde{U}^{W_\nu}$ actions to get at $\tilde{U}^{W_\nu} \otimes C[\mathfrak{g}^*]^{W_\nu} C[\mathfrak{g}^*]$-action which factors to give a $\tilde{U} \cong \tilde{U}^{W_\nu} \otimes C[\mathfrak{g}^*]^{W_\nu} C[\mathfrak{g}^*]$-action. Thus we obtain an exact functor $\theta_\mu^\lambda : \text{Mod}_\lambda(\tilde{U}) \to \text{Mod}_{|\mu|}(U_{\mathfrak{g}})$. 
Next, consider the composition
\[ \text{Mod}_\mu(\tilde{U}) \xrightarrow{\text{Res}_\mu} \text{Mod}_\mu(U \mathfrak{g}) \xrightarrow{\theta^+} \text{Mod}_\lambda(U \mathfrak{g}) \]
and define for \( N \in \text{Mod}_\mu(\tilde{U}) \) a \( C[\mathfrak{g}^*] \)-action on \( \theta^+ \text{Res}_\mu N \) via
\[ a \star m := [T_{\lambda-\mu} \theta^+(a)] \cdot m, \quad m \in \theta^+(\text{Res}_\mu N). \]
We similarly can directly check the \( \tilde{U}^W \lambda^- \) and \( C[\mathfrak{g}^*] \)-actions are compatible. Moreover, by Proposition 2.3(1), these actions agree on \( C[\mathfrak{g}^*]^{W_\mu} \), but not necessarily on the full \( C[\mathfrak{g}^*]^{W_\lambda} \supset C[\mathfrak{g}^*]^{W_\mu} \).
To remedy this, define
\[ \tilde{\theta}^+_i N := \{ n \in \theta^+(\text{Res}_\mu N) : a \star n = a \cdot n, \forall a \in C[\mathfrak{g}^*]^{W_\lambda} \}, \quad \text{and} \]
\[ \tilde{\theta}^+_i N := \theta^+(\text{Res}_\mu N)/\{ a \star n - a \cdot n, \forall a \in C[\mathfrak{g}^*]^{W_\lambda}, n \in \theta^+(\text{Res}_\mu N) \}. \]
Then the \( \tilde{U}^W \lambda^- \) and \( C[\mathfrak{g}^*] \)-actions on \( \theta^+ N \) descend to compatible actions on \( \tilde{\theta}^+_i N \). Hence they combine to give a \( \tilde{U}^- \) action. We remark these functors are not in general exact. If both \( \lambda \) and \( \mu \) are regular, then \( \tilde{\theta}^+_i = \tilde{\theta}^+_i = \theta^+ \). Also, \( \tilde{\theta}^+_i, \tilde{\theta}^+_i \) are the left, right, resp. adjoint of the functor \( \tilde{\theta}^- \).

3. Beilinson-Bernstein localization

Set \( \tilde{\mathcal{B}} = G/U \), the “base affine space” space, and \( \mathcal{B} := G/B \), the flag variety of \( G \). There is a natural right \( T \)-action on \( \tilde{\mathcal{B}} \) making \( \pi : G/U \to G/B \) a principal \( G \)-equivariant \( T \)-bundle. We comment if \( G \) is adjoint, then \( G/U \) has a realization of decorated flags:
\[ G/U = \{ b_x, \{ a_{x_i}^{\alpha_i} \} : b_x \subset \mathfrak{g} \text{ is Borel subalgebra, } a_{x_i}^{\alpha_i} \text{ generates } \mathfrak{g}/b_x, \text{ where } \alpha_i \text{ is a simple root of } \Phi^+ \} \]
and the projection to \( G/B \) is given by forgetting the decorations.

The \( G \)-action by left translation translation and \( T \)-action by right translation on \( G/U \) commute, so differentiating the action map at identity yields map of Lie algebras into global algebraic vector fields on \( \tilde{\mathcal{B}} \):
\[ \Phi : \mathfrak{g} \times \mathfrak{g} \to \Gamma(G/U, \mathcal{T}_{G/U}), \quad (g, h) \mapsto \partial_{g,h} : f(x) \mapsto \frac{d}{dt}{|}_{t=0} f(e^{-tgxe^{th}}), \]
where \( f \in \mathcal{O}_{G/U}(V) \) for some \( V \subset G/U \). By universal property, this can be extended to associative algebra homomorphism \( \Phi : U \mathfrak{g} \otimes_G U \mathfrak{g} \to \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T = \text{algebra of right } T \text{-invariant global differential operators on } \tilde{\mathcal{B}}. \) Moreover, it is a fact that \( z \otimes 1 \) and \( 1 \otimes \chi(z) \) map to same element under \( \Phi \), where \( \chi : Z_{\mathfrak{g}} \to U \mathfrak{g} \) is Harish-Chandra, so \( \Phi \) descends to a map \( \tilde{\Phi} : \tilde{U} \to \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T. \) In [BoBr, Prop. 8], it is shown by passing to the associated graded, and in a similar way to the case of \( \text{Ker}(U \mathfrak{g} \to \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}})) = \ker \chi_\lambda \), that \( \Phi \) is an isomorphism:
\[ \tilde{\Phi} : \tilde{U} \xrightarrow{\sim} \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^T \cong \Gamma(\tilde{B}, \tilde{\mathcal{B}}). \]

Let \( \pi_* \mathcal{D}_{\tilde{\mathcal{B}}} \) be sheaf-theoretic direct image of \( \mathcal{D}_{\mathcal{B}} \) to \( \mathcal{B} \). Define \( \tilde{\mathcal{D}} \subset \pi_* \mathcal{D}_{\tilde{\mathcal{B}}} \) to be the sheaf of \( T \)-invariant sections of \( \pi_* \mathcal{D}_{\tilde{\mathcal{B}}} \). Also, since \( T \) commutes with its own Lie algebra action, the image of \( 1 \otimes U \mathfrak{g} \) is contained in stalk of sheaf \( \tilde{\mathcal{D}} \) at any point of \( \mathcal{B} \). Thus, \( C[\mathfrak{g}^*] \to \tilde{\mathcal{D}} \) is central embedding. So for any \( \lambda \in \mathfrak{g}^* \), we may define category \( \text{Mod}_\lambda(\tilde{\mathcal{D}}) \) of coherent sheaves on \( \mathcal{B} \) of \( \tilde{\mathcal{D}} \)-modules \( M \) such that \( J_\lambda^* M = 0 \) for large \( n \). Taking global sections (via 3.1) is a functor \( \Gamma : \text{Mod}_\lambda(\tilde{\mathcal{D}}) \to \text{Mod}_\lambda(\tilde{U}) \). Its left adjoint is the localization functor \( \Delta : M \to \mathcal{M} \otimes_{\tilde{U}} \tilde{\mathcal{D}}. \)

**Theorem 3.1.** (Beilinson-Bernstein Localization)[BB Theorem 3.3.1]

1. If \( \lambda \) is \( \rho \)-dominant, then \( \Gamma \) is exact and the unit map \( \Gamma \circ \Delta : \text{Id}_{\text{Mod}_\lambda(\tilde{U})} \) is an isomorphism.
2. If \( \lambda \) is regular and \( \rho \)-dominant, then \( \Gamma \) is equivalence of the categories \( \text{Mod}_\lambda(\mathcal{D}) \) and \( \text{Mod}_\lambda(\tilde{U}) \), and the localization functor \( \Delta \) is the inverse.
Lemma 3.2. [Bez Lemma 5.3.12] Suppose $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories which has a left (right) adjoint $G : \mathcal{B} \to \mathcal{A}$. Let $\overline{\mathcal{A}} := \mathcal{A}/\ker F$ denote the Serre quotient category. Then $\overline{F} : \overline{\mathcal{A}} \to \mathcal{B}$ is an equivalence of categories if and only if only the canonical morphism $\eta : \text{Id}_B \to F \circ G$ (resp. $F \circ G \to \text{Id}_A$) is an isomorphism.

Thus, applying this to $F = \Gamma_\lambda$ and using the localization theorem, we deduce

**Corollary 3.3.** Let $\lambda$ be $\rho$-dominant. Then

$$ \overline{\Gamma}_\lambda : \text{Mod}_\lambda(\overline{D})/\ker \Gamma_\lambda \to \text{Mod}_\lambda(\overline{U}) $$

is an equivalence of categories.

Now, suppose $\lambda$ is dominant integral. This gives rise to homomorphism $\lambda : T \to \mathbb{C}^*$. Let $\mathcal{O}(\lambda)$ denote the sheaf on $\mathcal{B}$ formed by all regular functions $f$ on $\mathcal{B}$ such that $f(xt) = \lambda(t)f(x)$ for all $x \in \mathcal{B}$, $t \in T$. This is the sheaf of sections of the line bundle $G \times_B \mathbb{C}_\lambda$ over $G/B$ and the Borel-Weil theorem tells us $\Gamma(G/B, \mathcal{O}(w_0 \cdot \lambda)) = E_\lambda$, the simple highest weight $U_g$-module of weight $\lambda$. Let $\mathcal{D}_B(\lambda)$ denote the sheaf of twisted differential operators on $\mathcal{B}$ (see e.g. [HTT Section 11]). Then quantum hamiltonian reduction tells us $\mathcal{D}_B(\lambda) = \overline{\mathcal{D}}/\overline{\mathcal{J}}_\lambda$ and Beilinson-Bernstein (e.g. [HTT Thm 11.2.2]) implies $\Gamma(B, \mathcal{D}_B(\lambda)) = U_g/U_g : I_{\lambda}$.

Define the geometric translation functor $\Theta^\mu_\lambda : \text{Mod}_\lambda(\overline{D}) \to \text{Mod}_\mu(\overline{D})$ by $M \mapsto \mathcal{O}(\mu - \lambda) \otimes \mathcal{O}_B M$. Note, we assume $\lambda - \mu$ is dominant integral. Since $\mathcal{O}(\nu) \otimes \mathcal{O}_B \mathcal{O}(-\nu) = \mathcal{O}_B$, we have $\Theta^\mu_\lambda \Theta^\nu_\lambda = \text{Id}$. Thus, geometric translation functors are always an equivalence of categories. Compare with the algebraic translation functors on $U_g$, which are equivalence of category if and only if $\lambda$ and $\mu$ have the same degeneracy, i.e. $W_\lambda = W_\mu$.

**Lemma 3.4.** Suppose $\lambda, \mu$ are $\rho$-dominant integral weights and $W_\lambda \subset W_\mu$. The following diagram commutes up to canonical equivalence of functors:

$$
\begin{array}{ccc}
\text{Mod}_\lambda(\overline{D}) & \xrightarrow{\Gamma_\lambda} & \text{Mod}_\lambda(\overline{U}) \\
\parallel & & \parallel \\
\text{Mod}_\mu(\overline{D}) & \xrightarrow{\Gamma_\mu} & \text{Mod}_\mu(\overline{U})
\end{array}
$$

**Proof.** Given a finite-dimensional $\mathfrak{g}$-module $L$, associate $L_B := L \otimes \mathcal{O}_B$ to the trivial sheaf of (algebraic) regular functions $f : B \to L$. Given $f$, define $\phi_f : G \to B$ by $g \to g^{-1}f(g)$ and given $\phi$ such that $\phi(gb) = b^{-1}\phi(g)$, define $f$ by $f(g, B) = g\phi(g)$. This makes the identification of sheaves on $B$:

$$L_B \cong \text{Ind}_B^G L := \{ \phi : \mathbb{C}[G] \otimes L : \phi(g, b) = b^{-1}\phi(g) \}.$$

By Lie’s theorem, we can find a $B$-stable filtration of $L$ by codimension 1 spaces. Since $\text{Ind}_B^G(-)$ is exact on finite-dimensional $B$-modules, we get a filtration on $\text{Ind}_B^G L$ with subquotients $\text{Ind}_B^G(L_i/L_{i-1}) = \mathcal{O}(\nu_i)$ where $\nu_i : B \to \mathbb{C}$ is a character of $B$ corresponding to $L_i/L_{i-1}$.

Now, suppose $\lambda - \mu$ is dominant and let $L = L_{\mu - \lambda}$. Giving a $\mathfrak{g}$-module structure on $M \in \text{Mod}_\lambda(\overline{D})$ and tensor product $\mathfrak{g}$-structure on $L_\nu \otimes \mathcal{M}$, we get a $\mathfrak{g}$ stable filtration

$$0 \subset \text{Ind}_B^G L_1 \otimes \mathcal{O}_B \mathcal{M} \subset \cdots \subset \text{Ind}_B^G L_r \otimes \mathcal{O}_B \mathcal{M} = \text{Ind}_B^G L \otimes \mathcal{O}_B \mathcal{M},$$

with successive quotients $\mathcal{O}(\nu_i) \otimes \mathcal{O}_B \mathcal{M}$. For any $a \in \mathbb{C}[\mathfrak{g}^*]$, the action of $a$ on $\mathcal{O}(\nu) \otimes \mathcal{O}_B \mathcal{M}$ is

$$a(f \otimes m) = f \otimes (T_\nu a).m, \quad f \in \mathcal{O}(\nu), \quad m \in \mathcal{M},$$

where $T_\nu a : \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathfrak{g}^*], \quad p(x) \mapsto p(x + \nu)$ is the affine linear translation. Using this action and the fact that $Z_{\mathfrak{g}}$ acts on $\mathcal{M} \in \text{Mod}_\lambda(\overline{D})$ via central character $\chi_\lambda$, we find $z \in Z_{\mathfrak{g}}$ acts on
$O(\nu) \otimes_{O_S} \mathcal{M}$ by central character $\chi_{\lambda+\nu}$. Thus
\[
\left( \prod_{i=1}^{r} (z - \chi_{\lambda+\nu_i}(z))^{\alpha_i} \right) \cdot \text{Ind}^{G}_{B} L \otimes_{O_S} \mathcal{M} = 0, \quad \text{for some } \alpha_i \in \mathbb{Z}_{\geq 0}.
\]
Hence there is a direct sum decomposition parameterized by central characters. We show the summand corresponding to $\mu$ is just
\[
\text{pr}_{\mu} \cdot \text{Ind}^{G}_{B} L_{\mu-\lambda} \otimes \mathcal{M} = O(\mu - \lambda) \otimes \mathcal{M}.
\]
Indeed, if $\nu'$ is a weight of $L_{\mu-\lambda}$, then $\chi_{\nu'+\lambda} = \chi_{\lambda+\nu} (\nu := \mu - \lambda)$ implies $\exists w \in W$ such that
\[
w \cdot (\nu' + \lambda) = \lambda + \nu \Rightarrow (w(\lambda + \rho) - (\lambda + \rho)) + (w(\nu') - \nu) = 0.
\]
But $\lambda$ is $\rho$-dominant, so $w(\lambda + \rho) - (\lambda + \rho) \leq 0$. And $\nu'$ is weight of $L_{\mu-\lambda}$, so $w(\nu') \leq \mu - \lambda = \nu$. Thus $w \cdot \lambda = \lambda \Rightarrow w \in W_{\lambda} \subset \mathcal{W}_{\mu} \Rightarrow w(\nu) = \nu$ and $w(\nu') = \nu' = \nu$, as desired. Thus the subquotient sheaf $O(\mu - \lambda) \otimes_{O_S} \mathcal{M}$ splits off from $\text{Ind}^{G}_{B} E_{\mu-\lambda} \otimes_{O_S} \mathcal{M}$ as a sheaf of $Z_g$-modules.

Next, observe
\[
E \otimes \Gamma(\mathcal{M}) = \Gamma(E_S \otimes_{O_S} \mathcal{M}) = \Gamma(\text{Ind}^{G}_{B} E \otimes_{O_S} \mathcal{M}).
\]
Thus we obtain
\[
\tilde{\theta}^{\mu}_{\lambda} \circ \Gamma(M) = \text{pr}_{\mu} \cdot (E_{\mu-\lambda} \otimes \Gamma(M))
\]
\[
= \text{pr}_{\mu} \cdot \Gamma(\text{Ind}^{G}_{B} E_{\mu-\lambda} \otimes_{O_S} \mathcal{M})
\]
\[
= \Gamma(O(\mu - \lambda) \otimes_{O_S} \mathcal{M}) = \Gamma \circ \Theta^{\mu}_{\lambda} \mathcal{M}.
\]
Finally, formula [3.2] shows the $C[\delta^{*}]$-action on $\theta^{\mu}_{\lambda} \circ \Gamma(M)$ defined in section 2 agrees with the natural $C[\delta^{*}]$-action on $\Theta^{\mu}_{\lambda} \mathcal{M}$.

\section{Properties of Translation Functors}

\textbf{Proof of Proposition 2.3} Since every $M \in \text{Mod}_{\lambda}(\tilde{U}^{W_{\lambda}})$ is quotient of some $M' \in \text{Mod}_{\lambda}(\tilde{U})$, it suffices to consider $M \in \text{Mod}_{\lambda}(\tilde{U})$. By the localization theorem, we can find $\mathcal{M} \in \text{Mod}_{\lambda}(D)$ such that $\Gamma_{\lambda}(\mathcal{M}) = M$. Now part (2) follows from Lemma 3.4 and equation [3.2] for $\nu = \mu - \lambda$.

Next we prove part (i). Take $M \in \text{Mod}_{\lambda}(\tilde{U})$, $\tilde{N} \in \text{Mod}_{\lambda}(\tilde{U})$ and let $\tilde{M} = \text{Res}_{\lambda} M \in \text{Mod}_{\mu}(\tilde{U}^{W_{\mu}})$, $\tilde{N} = \text{Res}_{\lambda} \tilde{N}$. Given $a \in C[\delta^{*}]^{W_{\mu}} = Z(\tilde{U}^{W_{\mu}})$, act$_a$ is a $\tilde{U}^{W_{\mu}}$-morphism and thus functoriality gives $\theta^{+}(a) \in \text{End}_{\mathcal{G}_{W_{\lambda}}}(\theta^{+} M)$, hence to an endomorphism $\theta^{+}(a)_{M}$ of $\text{Hom}_{\mathcal{G}_{W_{\lambda}}}(\theta^{+} M, N)$. We arrive at the following commutative diagram:
\[
\begin{array}{ccccccccc}
\text{Hom}(\theta^{+} M, N) & \rightarrow & \text{Hom}(M, \theta^{-} N) & \rightarrow & \text{Hom}(M, \theta^{-} N) & \rightarrow & \text{Hom}(M, \theta^{-} N) & \rightarrow & \text{Hom}(\theta^{+} M, N) \\
\downarrow^{\theta^{+}(a)_{M}} & & \downarrow^{\theta^{-}(a)_{M}} & & \downarrow^{\theta^{-}(a)_{M}} & & \downarrow^{\theta^{-}(a)_{M}} & & \downarrow^{\theta^{-}(a)_{M}} \\
\text{Hom}(\theta^{+} M, N) & \rightarrow & \text{Hom}(M, \theta^{-} N) & \rightarrow & \text{Hom}(M, \theta^{-} N) & \rightarrow & \text{Hom}(M, \theta^{-} N) & \rightarrow & \text{Hom}(\theta^{+} M, N)
\end{array}
\]
where $\text{Hom}$ in columns 2, 3, and 4 are over $\tilde{U}^{W_{\mu}}$ and columns 1 and 5 are over $\tilde{U}^{W_{\lambda}}$. The first and last horizontal arrows are isomorphisms by adjunction and the two middle horizontal arrows are equality. Commutativity in squares 1 and 4 are from adjointness. Commutativity in square 2 is from $a$ being chosen central and $C[\delta^{*}]^{W_{\mu}} \subset C[\delta^{*}]^{W_{\lambda}}$. Commutativity in square 3 is by part (2) proved in previous paragraph. We conclude $\theta^{+}(a)_{M} = (T_{\mu-\lambda} a)_{M}$ holds for all $N$, so Yoneda’s lemma (uniqueness of representable functor) implies $\theta^{+}(a) = T_{\mu-\lambda} a$. 

\textbf{Proposition 4.1.} The extended translation functors satisfy:

1. The functor $\tilde{\theta}^{-} : \text{Mod}_{\lambda}(\tilde{U}) \rightarrow \text{Mod}_{\lambda}(\tilde{U})$ induces an equivalence $\text{Mod}_{\lambda}(\tilde{U})/\text{Ker} \tilde{\theta}^{-} \rightarrow \text{Mod}_{\lambda}(\tilde{U})$.

2. The adjunction morphisms induce isomorphisms of functors
\[
\tilde{\theta}^{-} \circ \tilde{\theta}^{+} \cong \text{Id}_{\text{Mod}_{\lambda} \tilde{U}} \cong \tilde{\theta}^{-} \circ \tilde{\theta}^{+}.
\]
(3) There is a natural isomorphism of functors \( \tilde{\theta}_i^+ = \Gamma_\lambda \circ \Theta^+ \circ \Delta_\mu \).

Proof. We first prove (3) and then formally deduce (1) and (2) from it. Observe the left adjoint of the functors \( \theta^- \), \( \Gamma_\lambda \), \( \Theta^- \) are \( \tilde{\theta}_i^+ \), \( \Delta_\lambda \), \( \Theta^+ \), respectively. So taking the adjoint of Lemma 3.4 yields
\[
\Delta_\lambda \circ \tilde{\theta}_i^+ = \Theta^+ \circ \Delta_\mu : \text{Mod}_\mu(\bar{U}) \to \text{Mod}_\lambda(\bar{D}).
\]
Part (3) follows by taking \( \Gamma_\lambda \) on both sides and using \( \Gamma_\lambda \Delta_\lambda \cong \text{Id} \) by the localization theorem. Now,
\[
\tilde{\theta}^- \tilde{\theta}_i^+ = \tilde{\theta}^- \Gamma_\lambda \Theta^+ \Delta_\mu = \Gamma_\mu \Theta^- \Theta^+ \Delta_\mu = \Gamma_\mu \Delta_\mu = \text{Id}_{\text{Mod}_\mu(\bar{U})},
\]
where first equality uses (3), second uses Lemma 3.4 and third uses localization theorem. This proves (2) and (1) follows from “if” part of Lemma 3.2.

**Proposition 4.2.** There is a functorial isomorphism
\[
\theta^+ M \cong (\text{Res}_\lambda \tilde{\theta}_i^+ (\bar{U} \otimes_{\bar{G}} w_\mu M))^W_\lambda, \quad M \in \text{Mod}_\mu(U_{\mathfrak{g}}).
\]
Proof. Recall \( \tilde{\theta}^- = \theta^- \text{Res}_\lambda \). If we forget the \( C[\mathfrak{g}^*] \) action, we find a functorial isomorphism
\[
\theta^- \text{Res}_\lambda = \text{Res}_\mu \tilde{\theta}^- : \text{Mod}_\lambda(\bar{U}) \to \text{Mod}_\mu(U_{\mathfrak{g}}).
\]
Then recall the left adjoints of \( \text{Res}_\lambda, \theta^-, \tilde{\theta}^- \) are \( \text{Ind}_{\bar{G}W_\lambda}, \theta^+, \tilde{\theta}_i^+ \), respectively. So taking left adjoints on both sides (and again using Yoneda’s lemma) yields an isomorphism of \( \bar{U} \)-modules:
\[
(4.1) \quad \tilde{\theta}^+ M := \bar{U} \otimes_{\bar{G}} w_\lambda (\theta^+ M) = \tilde{\theta}_i^+ (\bar{U} \otimes_{\bar{G}} w_\mu M) = \tilde{\theta}_i^+ (\bar{M}).
\]

Since \( W_\lambda \) only acts on the \( \bar{U} \) component on both sides, these isomorphisms are compatible with the \( W_\lambda \)-actions. By the Pittie-Steinberg theorem, \( C[\mathfrak{g}^*] \cong C[\mathfrak{g}^*]^W_\lambda \otimes C[W_\lambda] \) as \( W_\lambda \)-modules. Thus
\[
\bar{U} \cong \bar{U}^{W_\lambda} \otimes_{C[\mathfrak{g}^*]^W_\lambda} C[\mathfrak{g}^*] \cong \bar{U}^{W_\lambda} \otimes_{C[\mathfrak{g}^*]^W_\lambda} C[\mathfrak{g}^*] \otimes C[W_\lambda] \cong \bar{U}^{W_\lambda} \otimes C C[W_\lambda]
\]
as \( (\bar{U}^{W_\lambda}, W_\lambda) \)-bimodules. Thus, \( (\bar{U} \otimes_{\bar{G}} w_\lambda (\theta^+ M))^W_\lambda \cong \theta^+ M \) as \( \bar{U}^{W_\lambda} \)-modules. Restricting to \( U_{\mathfrak{g}} \) and taking \( W_\lambda \)-invariants of right hand side of (4.1) yields precisely what we wanted.

**Proposition 4.3.** There is a functorial isomorphism
\[
\theta^- \theta^+ M \cong \text{Res}_\mu(\bar{U}^{W_\lambda} \otimes_{\bar{G}} w_\mu M).
\]

Proof. As in Proposition 4.2 \( \tilde{\theta}^+ M = \tilde{\theta}^{\tilde{\theta}_i^+} (\bar{M}) \). Now,
\[
\text{Res}_\mu(\bar{U} \otimes_{\bar{G}} w_\lambda (\theta^- \theta^+ M)) = \theta^+ (\text{Res}_\lambda \tilde{\theta}_i^+ (\bar{M})) = \text{Res}_\mu \theta^- \tilde{\theta}_i^+ (\bar{M}) = \text{Res}_\mu (\bar{M}),
\]
where the first equality uses \( \theta^- \) commutes with \( \text{Res} \text{Ind}_{\bar{G}}^{\bar{U}} \) and \( \tilde{\theta}_i^+ (\bar{M}) = \theta^+ M \), second equality uses \( \theta^- \text{Res}_\lambda = \text{Res}_\mu \tilde{\theta}^- \), and third equality uses Proposition 4.1(ii). Finally, \( W_\lambda \) commutes with every functor, so taking \( W_\lambda \) invariants of left hand side is by Proposition 4.2 equal to \( \theta^- \theta^+ M \), and \( W_\lambda \) invariants of right hand side is \( \text{Res}_\mu(\bar{U}^{W_\lambda} \otimes_{\bar{G}} w_\mu M) \).

5. Soergel’s Theorems

Proof of Theorem 1.1. We use the notation \( \bar{M} = \text{Ind}_{\bar{G}^\mathfrak{g}}^{\bar{U}} M \). Observe we have the following isomorphism of algebras:
\[
\text{End}_{U_{\mathfrak{g}}}(P_{\mathfrak{w}_0 \lambda}) = \text{End}_{U_{\mathfrak{g}}}(\left(\text{Res}_\lambda \tilde{\theta}_i^+ (\bar{M}_{-\rho})\right)^W_\lambda) \quad \text{By } P_{\mathfrak{w}_0 \lambda} = \theta^+ M_{-\rho} \text{ and Proposition 4.2}
\]
\[
= \text{End}_{\bar{U}^{W_\lambda}} (\left(\tilde{\theta}_i^+ (\bar{M}_{-\rho})\right)^W_\lambda) \quad \text{Equivalence of category (2.1)}
\]
\[
= \text{End}_{\bar{U}^{W_\lambda}} (\left(\bar{M}_{-\rho}\right)^W_\lambda) \quad \tilde{\theta}_i^+ \text{ is fully faithful, see Proposition 4.1}
\]
\[
= \text{End}_{\bar{U}^{W_\lambda}} (\bar{U}^{W_\lambda} \otimes_{U_{\mathfrak{g}}} M_{-\rho}).
\]
Now, observe the action of central subalgebra $\mathbf{C}[\mathfrak{g}^*]^{W_\lambda} \subset \tilde{U}^{W_\lambda}$ on $\tilde{U}^{W_\lambda} \otimes_{\mathfrak{g}} M_{-\rho}$ induces an algebra isomorphism $\tau : C^{W_\lambda} \to \text{End}_{\tilde{U}^{W_\lambda}} (\tilde{U}^{W_\lambda} \otimes_{\mathfrak{g}} M_{-\rho})$ because $M_{-\rho}$ is generated by a highest weight vector $v_{-\rho}$. Composing previous chain of algebra isomorphisms with $\tau^{-1}$ completes the proof. □

**Proof of Theorem 5.2.** We present a proof, only in the case of $\lambda$ regular, which takes advantage of various duality properties in category $\mathcal{O}$. This proof closely follows [Bez, Prop. 6.0.1], which is motivated by the proof of the analogous Soergel Structure theorem for rational Cherednik algebras [CGOR, Thm. 5.3]. A key idea is to introduce a functor which sends Verma modules to dual Verma modules, and this can only be accomplished after passing to the derived category of $\mathcal{O}$. Fix $\lambda$ a $\rho$-dominant, regular weight. Let $\mu$ be a $\rho$-dominant weight lying on a single hyperplane $s_\alpha$, and define the wall crossing functor $R_\alpha := \theta_\mu^\alpha \circ \theta_\lambda^\alpha : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$. Since translation functors are biadjoint, there exist natural (co)unit maps $\text{Id} \to \text{Id}$.

**Definition 5.1.** Let $\mu$ lie on a single wall, defined by hyperplane $s_\alpha \in W$, in the closure of $\lambda$. The intertwining functors are defined as:

$$\Theta_\alpha : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda), \quad M \mapsto \text{Cone}(M \to R_{s_\alpha}M),$$

$$\Theta'_\alpha : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda), \quad M \mapsto \text{Cone}(R_{s_\alpha}(M) \to M)[-1].$$

Note the intertwining functors are induced by taking the cone between exact functors on category of complexes, hence they are exact. The following lemma is a consequence of knowing how translation functors act on Verma’s, see e.g [Hum, Chapter 7].

**Lemma 5.2.** Let $\Delta_\nu$ be the Verma of highest weight $\nu \in W \cdot \lambda$. Viewing $\Theta_\alpha$ as functors $D^{\leq 0} \to D^{\leq 0}$, with $\Theta_\alpha(M) = (M \to R_{\alpha}M)$, $M$ degree $-1$ and $R_{\alpha}M$ degree 0, and similarly for $\Theta'_\alpha : D^{\geq 0} \to D^{\geq 0}$,

$$\Theta_\alpha : \Delta_\nu \mapsto \Delta_{\sigma_\alpha \nu} \quad \text{if} \quad s_\alpha \nu > \nu, \quad \Theta'_\alpha : \Delta_\nu \mapsto \Delta_{\sigma_\alpha \nu}, \quad s_\alpha \nu < \nu.$$  

Then using this lemma, and property that a functor on a complex of Harish-Chandra bimodules is determined by the image of Verma modules, we may deduce $\{\Theta_\alpha, \Theta'_\alpha\}$ satisfy the braid group relations on $D^b(\mathcal{O}_\lambda)$ and also $\Theta_\alpha \circ \Theta'_\alpha \cong \text{Id} \cong \Theta'_\alpha \circ \Theta_\alpha$ [Bez, Cor. 5.6.5].

We define a standard object in $\mathcal{O}$ as one which admits a filtration whose successive quotients are Verma’s, and a costandard object as one which admits a filtration whose successive quotients are dual Verma’s, denoted $\nabla_\lambda$. A tilting object is one which is standard and costandard. For example, all projective objects in $\mathcal{O}$ are standard and the antidominant projective $P_{w_0 \cdot \lambda}$ is tilting.

**Lemma 5.3.** (a) If $T_1, T_2$ are tilting, then

$$\text{Hom}_{\mathcal{O}_\lambda}(T_1, T_2) \cong \text{Hom}_{\mathcal{O}_{\sigma_\rho}}(\tilde{\theta}^\rho_{\lambda}(T_1), \tilde{\theta}^\rho_{\lambda}(T_2)).$$

(b) Let $w_0 = s_{\alpha_1} \cdots s_{\alpha_n}$ be a minimal expression. Then

$$\Theta_{w_0} = \Theta_{s_{\alpha_1}} \circ \cdots \circ \Theta_{s_{\alpha_n}}$$

sends projectives to tiltings.

(c) Using $s_\alpha$ acts on $\tilde{\mathcal{O}}_{-\rho} \cong C - \text{mod}$ via $M \mapsto C \otimes_{C_{s_\alpha}} M$, there is a commutative diagram

$$\begin{array}{ccc}
D^b(\mathcal{O}_\lambda) & \xrightarrow{\Theta_{s_\alpha}} & D^b(\mathcal{O}_\lambda) \\
\downarrow_{\tilde{\theta}^\rho_{\lambda}} & & \downarrow_{\tilde{\theta}^\rho_{\lambda}} \\
D^b(\tilde{\mathcal{O}}_{-\rho}) & \xrightarrow{s_{\alpha}} & D^b(\tilde{\mathcal{O}}_{-\rho}).
\end{array}$$
We explain how to prove the Struktursatz Theorem \[\text{1.2}\] using Lemma \[\text{5.3}\]. Let \( P_1, P_2 \) be projective and define \( T_i := \Theta_{w_0} P_i \). Then
\[
\text{Hom}_{\mathcal{U}g}(P_1, P_2) = \text{Hom}_{\mathcal{U}g}(T_1, T_2)
\]
\[
= \text{Hom}_{\mathcal{O}_{-\rho}}(\tilde{\theta}_\lambda^\rho T_1, \tilde{\theta}_\lambda^\rho T_2)
\]
part (b) implies \( T_i \) tilting, then use part (a)
\[
= \text{Hom}_{\mathcal{O}_{-\rho}}(w_0(\tilde{\theta}_\lambda^{-\rho}(P_1)), w_0(\tilde{\theta}_\lambda^{-\rho}(P_2)))
\]
part (c)
\[
= \text{Hom}_{\mathcal{O}_{-\rho}}(\tilde{\theta}_\lambda^{-\rho}(P_1), \tilde{\theta}_\lambda^{-\rho}(P_2)).
\]

It is easy to see \( \tilde{\mathcal{O}}_{-\rho} \) is equivalent to \( C \)-mod. Then we show \( \mathcal{V} = \tilde{\theta}_\lambda^\rho \). Indeed, \( \tilde{\theta}_\lambda^{-\rho} \) is represented by a projective object \( P = \oplus_{w \in W} P_{w\lambda}^{w_0} \) in \( \mathcal{O}_\lambda \cong \tilde{\mathcal{O}}_\lambda \) (See \[\text{2.1}\] \( \lambda \) is regular), and \( \theta^{-} \) kills all simples but antidominant one, so \( n_w = 0 \) for \( w \neq w_0 \). The Endomorphism theorem implies \( n_{w_0} = 1 \). Thus, the final term is isomorphic to \( \text{Hom}_{C}(\mathcal{V}(P_1), \mathcal{V}(P_2)) \), as desired.

\[\text{Proof of Lemma \[\text{5.3}\]}\] (a): Recall \( \mathcal{O}_\lambda/(\ker(\theta_\lambda^\rho)) \to \tilde{\mathcal{O}}_{-\rho} \) is an equivalence of categories and \( \ker(\theta_\lambda^{-\rho}) = \langle L_{w,\lambda} : w \neq w_0 \rangle \). Next we show \( T_1 \) satisfies \( \text{Hom}(T_1, B) = 0 \) for all \( B \in \ker \tilde{\theta} \) and \( T_2 \) satisfies \( \text{Hom}(B, T_2) = 0 \) for all \( B \in \ker \theta \). Indeed, this is done by induction on length of (co)standard filtration and using socle of \( \Delta_{w,\lambda} \) is \( L_{w,\lambda} \), and top is \( L_{w,\lambda} \). Then using a general lemma on Serre quotients \[\text{[Bez, Lemma 6.1.2]}\], we conclude \( \text{Hom}_{\mathcal{O}_\lambda}(T_1, T_2) = \text{Hom}_{\mathcal{O}_{\lambda/\ker(\theta_{-\rho})}}(T_1, T_2) \) as desired.

(c): Using Proposition \[\text{[4.3]}\] we find a commutative diagram (see \[\text{[Bez, Lemma 5.5.1]}\] for details)
\[
\begin{array}{ccc}
\mathcal{O}_\lambda & \xrightarrow{\text{R}_a} & \mathcal{O}_\lambda \\
\downarrow{\tilde{\theta}_\lambda^{-\rho}} & & \downarrow{\tilde{\theta}_\lambda^{-\rho}} \\
\tilde{\mathcal{O}}_{-\rho} & \xrightarrow{M \otimes_{\mathcal{O}_\lambda} M} & \tilde{\mathcal{O}}_{-\rho} \\
\end{array}
\]
We wish to compute \( \text{Cone}(M \mapsto C \otimes_{C_{s_\alpha}} M) \). Let \( \mathcal{O}(V) \) denote the regular functions on variety \( V \).

There is short exact sequence
\[
0 \to \mathcal{O}\{(x, x) : x \in \mathfrak{h}^*\} \to \mathcal{O}(\mathfrak{h}^* \times \mathfrak{h}^*/\{1, s_\alpha\} \mathfrak{h}^*) \to \mathcal{O}\{(x, s_\alpha(x)) : x \in \mathfrak{h}^*\} \to 0.
\]

Tensor with \( C \) over \( \text{Sym}(t)^W = Z\mathfrak{g} \), and define \( C_{s_\alpha} := \mathcal{O}\{(x, s_\alpha(x)) : x \in \mathfrak{h}^*\} \otimes_{\text{Sym}(\mathfrak{h}^W)} C \) to get short exact sequence \( 0 \to C \to C \otimes_{C_{s_\alpha}} C \to C_{s_\alpha} \to 0 \).

Thus,
\[
\text{Cone}(M \to C \otimes_{C_{s_\alpha}} M) \cong C_{s_\alpha} \otimes_C M
\]
\[
\cong \mathcal{O}\{(x, s_\alpha(x)) : x \in \mathfrak{h}^*\} / (\text{Sym}(\mathfrak{h}^W) \otimes \mathcal{O}\{(x, x, x) : x \in \mathfrak{h}^*\} / (\text{Sym}(\mathfrak{h}^W) \mathfrak{h}^*\}) M
\]
\[
\cong s_{\alpha} M
\]

(b): We prove a claim: \( \Theta_{w_0} \) is a bijection between set of standard objects and set of costandard objects. Since \( \Theta_{w_0} \) is exact, it suffices to check Verma is sent to dual Verma. By Lemma \[\text{5.2}\] we have \( \Delta_w = \Theta_{w_0}(\Delta_{w_0,\lambda}) \) and dually, \( \nabla_w = \Theta_{w_0}(\nabla_{w_0,\lambda}) \), where \( \lambda \) is dominant regular. Let \( w_1 := w_{w_0} \). Then \( l(w_0) = l(w^{-1}) + l(w_1) \) and we find \( \Theta_{w_0} = \Theta_{w_0} \circ \Theta_{w^{-1}} \). Thus Verma gets sent to dual Verma:
\[
\Theta_{w_0}(\Delta_w) = \Theta_{w_1} \circ \Theta_{w^{-1}}(\Theta_{w_0}(\Delta_{w_0,\lambda}))
\]
\[
= \Theta_{w_1}(\Delta_{w_0,\lambda}) = \Theta_{w_1}(\nabla_{w_0,\lambda})
\]
\[
= \nabla_{w_1}.
\]

A projective object \( P \) is a standard object, hence \( \Theta_{w_0} P \), which lives in degree 0, is a costandard object by the claim. By the Ext criteria \[\text{[Hum, 6.13]}\], \( \Theta_{w_0} P \) is a standard object if and only if
\[
\text{Ext}^1(\Theta_{w_0}(P), \nabla_\mu) = 0, \forall \mu.
\]
By claim, there is \( \nu \) for which \( \nabla_\nu = \Theta_{w_0}(\Delta_\rho) \). Since \( \Theta_{w_0} \) is autoequivalence, the above Ext equals
\[
\text{Ext}^1(P, \nabla_\nu) = 0.
\]
Thus \( \Theta_{w_0}(P) \) is a standard object as well. \( \square \)
REFERENCES