Let $k = \text{alg. closed field}$ (of char. 0 or $p$)

$X = \text{smooth projective curve} /k$.

$G = \text{semi-simple group} /k$, $\mathfrak{g} = \mathfrak{le}(G)$

$\text{Bun}_G X = \text{Moduli stack of principal } G\text{-bundles} / X$.

$\mathcal{M} = \{ P \in \text{Bun}_G(X)(k) \mid H^0(X, \mathcal{J}_P) = \mathcal{O}_X^p \}$, $\mathcal{J}_P := \mathcal{g}_X \mathcal{P} / G$

= Moduli space of principal $G$-bundles over $X$, with

$\text{finite group of } \mathcal{G}\text{-automorphisms}$ (so is the smooth locus of

the moduli stack $\text{Bun}_G X$)


Goal: Give several concrete realizations of the infinitesimal

get (= higher order cotangent space) spaces of $\mathcal{M}$.

This is accomplished by giving two canonical pairings with

contour blocks. For Affine Kac-Moody Lie algebras:

§1 1st Order Jets

Let $x \in X$ (Later, we'll explain the independence of choosing

a base point!)

Let $\mathcal{O}_x := \mathcal{O}_X, \mathcal{K}_x \cong k[[t_x]]$ $\Rightarrow D_x = \text{Spec } \mathcal{O}_x =$ "Formal disk

$\varnothing \subset X$"

$K_x := \text{Frac } \mathcal{O}_x \cong k((t_x))$ $\Rightarrow D^\circ_x = \text{Spec } K_x =$ "Formal

punctured disk $\varnothing \subset X$"

$\mathcal{O}_{out} := \mathcal{O}_X \mid_{X \setminus x} \Rightarrow$ punctured curve.

Thus [Breuil-Laszlo] There exists a 1-point uniformizer

$\pi : G(\mathcal{O}_{out}) \backslash G(\mathcal{K}_x) / G(\mathcal{O}_x) \to \text{Bun}_G(X)$

isomorphism of stacks.
As a consequence, we find:

\[ T_p \text{Bun}_G(X) = \mathcal{O}_{\text{kr}} / (\mathcal{O}_{\text{kr}} e^t + \mathcal{O}_{\text{out}}) \]

where \( \mathcal{O}_{\text{kr}} = \mathcal{O} \otimes \mathcal{O}(\mathbb{T}^s) \), \( \mathcal{O}_{\text{kr}} = \mathcal{O} \otimes \mathcal{O}(\mathbb{T}) \), \( \mathcal{O}_{\text{out}} = \mathcal{O} \otimes \mathcal{O}(\mathbb{X}(x)) \), \( GCK \xrightarrow{\pi} \text{Bun}_G \)

Using a Cech cover \( \{ \mathbb{X}, D_x \} \), can also show

\[ T_p \text{Bun}_G(X) = \mathcal{O}_{\text{kr}} / (\mathcal{O}_{\text{kr}} e^t + \mathcal{O}_{\text{out}} \sim H^1(X, \mathcal{O}_p) \sim H^0(X, \mathcal{O}_p \otimes \mathcal{O}(x)) \]

So dually, this provides 1st order approximation of functions on \( \text{Bun}_G(X) \).

5.2 Higher order jets.

Since we treat char 0 & p simultaneously, we must introduce a notion of "divided powers."

Suppose first \( V = k^n \) (fin dim vector space)

Define \( \text{Sym}^{(p)}(V) := \bigoplus_{i=0} \mathcal{O}_i^{(p)} / \langle \mathcal{O}_i = \mathcal{O}(\mathbb{W}^i) \rangle \]

So, if \( n \), \( \text{Sym}^{(p)}(k) = k[t^{(p)}] / (t^{(p)} t^{(q)}) \]

There's a canonical perfect pairing \( \langle \cdot, \cdot \rangle : \text{Sym}^{(p)}(V) \times \text{Sym}(V^*) \rightarrow k \)

Upsshot: in char 0, perfect pairings come when one side attaches denominators & the other does not.
Now, sheafify: let $\mathcal{M} = \text{smooth scheme}/k$.

Let $\Theta^e := \Theta_{\mathcal{M}} \otimes \Theta_{\mathcal{M}}$, $I = \ker [\Theta^e \xrightarrow{\psi} \Theta_{\mathcal{M}}] = \langle \psi_1 - \psi_2 \rangle$.

**Def**

1. $J^n(M) := \Theta^e/I \otimes_{\mathcal{O}} \Theta_{\mathcal{M}}$ is the sheaf of $n$-jets of functions on $\mathcal{M}$.

2. $J_{n, PD}^n(M) := \Gamma_I(\Theta^e)/\Gamma_{n+1}$ is the $n$th PD-neighbourhood.

Affine locally, $\Gamma_I(\Theta^e)$ is a free PD-polynomial algebra over $\Theta_{\mathcal{M}}$.

$\Gamma_{n+1}$ is the ideal generated by symbols of degree $n+1$.

(Again, (1) $\cong$ (2) for other $k = 0$).

The fiber of $J_{n, PD}^n(M)$ over $P \in \mathcal{M}$ is called the vector space of $n$th order $\text{affiliated}$ divided-power jet spaces of $P \in \mathcal{M}$.

**Def**

$D_{\text{cons}}^n(M) := \text{Hom}_\mathcal{O}(J_{n, PD}^n(M), \mathcal{O})$ is the sheaf of crystalline differential operators.

$\implies \langle , \rangle : D_{\text{cons}}^n(M) \times J_{n, PD}^n(M) \to k$ is perfect pairing by definition.

By comparing basis, $D_{\text{cons}}^n(M) \to U(T_P \mathcal{M})$ is a sum of associative objects.

Recall, we computed $T_P \mathcal{M}$, and as a corollary,

$U(T_P \mathcal{M}) \cong U(o_{\mathcal{M}})/((Ad_{g_{\mathcal{M}}}) o_{\mathcal{M}} + o_{\mathcal{M}} g_{\mathcal{M}} \cdot \text{out})$

Denote by $M_{\text{out}}^\mathcal{M}$ the space of covariants.

Have induced PBW filtration, & as a corollary.
There is a natural perfect pairing
\[
\psi : \mathcal{M}^{\text{et}}_{\mathcal{O}_k} \times \mathcal{J}^{\text{PD}}_P(M) \to k.
\]

Remark: Let \( \psi \in \mathcal{G}(K) \). Then, a vacuum module w/ central charge 0 is
\[
\mathcal{M}_c := \mathcal{U}_{y_k}/(\text{Ad}_y \cdot y_0 \mathcal{U}_{y_k}) \sim \text{Ind}^G_{y_0} \mathcal{U}_{y_0}
\]
& a corollary may be interpreted as saying: infinitesimal jet of \( \mathcal{D} \) are precisely the conformal blocks associated to affine tc-MdL L.: sq w/ central charge 0.

\section{Log Differential Forms}

We provide another description of \( \mathcal{J}^{\text{PD}}_P(M) \) which illustrates the role of configuration spaces.

Recall the Fulton-MacPherson construction:

\[
P : \hat{X}^n \to X^n \text{ w/ properties:}
\]

(i) \( P^*(D) = U_{[n+1]} \hat{I} \) is a normal crossing divisor, where \( D = \bigcup \mathcal{X}_i = x_i \).

(ii) \( \hat{I} / [n] = [n] / I = [n] / I = [n] / I \cup [n] / I \).

(iii) \( \hat{X}^n = \bigcup_{T \in \text{Gr}} S_T \) is the stratification,
\[
\overrightarrow{S_T} \sim \hat{P}_T \times \hat{X}^T, \quad T = \# \text{ connected components}
\]

Remark: \( \hat{X}^n \) is a module over the topological operad \( \mathcal{X}^n \).

Define: \( \mathcal{S}^n_{\hat{X}, \hat{X}} \) denote the sheaf of differential forms on \( \hat{X}^n \).
regular on $\hat{X}$, but with simple logarithmic poles along $\hat{\Theta} = \hat{X}^n \setminus \hat{x}^n$.

\[ \Omega(\hat{X}^n, \hat{x}^n) = \Gamma(\hat{X}^n, \hat{\Theta} \otimes \hat{\Theta}) \]

\[ \xi = \Omega(\hat{\Theta}^{c,n}, \hat{\Theta}^{c,n}) = \left\{ \lambda_{12} \text{dlog}(\xi_{12}) + \lambda_{13} \text{dlog}(\xi_{13}) + \lambda_{23} \text{dlog}(\xi_{23}) : \right\} \]

\[ \lambda_{i,j} \in k^* \quad \lambda_{12} + \lambda_{13} + \lambda_{23} = 0, \quad \xi_{i,j} = \xi - \xi_{i,j} \]

Lemma: $\text{Res} : L^* \otimes \Omega(\hat{\Theta}^{c,n}, \hat{\Theta}^{c,n}) \to k$ is a perfect pairing.

Let $\psi^*_E : g^* \to g^*_E \boxtimes \mathcal{L}(k)$, $\mathcal{L} = \mathcal{L}(I)$, $I = [0, I]$

\[ A \to (x, \otimes) \to A(\sum_{\sigma \in S_n} x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \]

We may upgrade $\psi^*_E$ to an $\hat{S}_n$-module hom.

Let $\psi^*_E : \mathfrak{p}_E^* \to \hat{g}^*_E \boxtimes \hat{\Theta} \otimes \hat{\Theta} \otimes \hat{\Theta}$

Remark: Fiberwise, $\psi^*_E = \text{Id} \times \psi^*_E$.

$\psi^*_E$ embedding

Remark: $\psi^*_E$ injective because of semi-simplicity, &

$\text{Im}(\psi^*_E)$ is a locally free sheaf on $\hat{S}_n$ w/ fiber of $g^* \otimes \xi_{c,n}$

Main Def: Define the "BG sheaf" on $\hat{X}$ by

\[ \mathcal{G} := \left\{ w : (\hat{g}^*_E \boxtimes \mathcal{L}(\xi_{c,n}, \xi_{c,n}^*) : \right\} \]

\[ \text{Res} : (w) \in \text{Im} \psi^*_E \forall \xi^{c,n} \]

Remark: the constraint says the coefficients of log forms are built by
\[ \text{We define analogues} \quad \mathfrak{S}(\mathcal{G}_n), \quad \mathfrak{S}(\mathcal{G}_n) \Rightarrow X_0 \quad \text{such that} \quad \mathfrak{S}(\mathcal{G}_n) \quad \Rightarrow \quad \mathfrak{S}(\mathcal{G}_n) \quad \Rightarrow \quad X_0 \quad \text{as above.} \]
\[ \bigoplus \Gamma(\hat{X}, \hat{\mathcal{O}}_{\hat{X}} \otimes \omega_{\hat{X}})^{S_n} \cong \bigoplus J^{\text{red}}(\mu) \]

comes from the Frobenius structure too.

Example:

\[ \text{Res}_{\tilde{E}} : \hat{\mathcal{O}}_{\tilde{E}} \rightarrow \left( \hat{\mathcal{O}}_{\tilde{E}} \otimes \hat{\mathcal{O}}_{\tilde{E}}^{\mu-1} \right) \otimes \tilde{\mathcal{O}}_{\tilde{E}} \]