Symmetric Polynomials and Representation Theory

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Symmetric Polynomials

Consider the ring $\mathbf{Z}[x_1, \ldots, x_n]$ of polynomials in n variables with integer coefficients. The symmetric group S_n acts on this ring by permuting the variables. An element $f \in \mathbf{Z}[x_1, \ldots, x_n]$ is called a **symmetric polynomial** if $\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in S_n$.

Ring of Symmetric Polynomials

The set of all symmetric polynomials of n variables forms a subring

$$\Lambda_n := \mathbf{Z}[x_1, \ldots, x_n]^{S_n},$$

which is graded by the degree:

$$\Lambda_n = \bigoplus_{d \geq 0} \Lambda_n^d$$

where Λ_n^d consists of symmetric polynomials of n variables and degree d.

Symmetric Functions

The number of variables is often irrelevant provided that it is large enough.

Ring of Symmetric Functions

The ring of **symmetric functions** is

$$\Lambda := \bigoplus_{d \geq 0} \Lambda^d,$$

where $\Lambda^d = \varprojlim \Lambda_n^d$ denotes the ring of symmetric polynomials of degree d of arbitrary number of variables.

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1. Monomial Symmetric Functions

Definition of monomial symmetric function m_λ

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ and denote x^α the monomial

$$x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Let λ be any partition of length $\leq n$, i.e $\lambda = (\lambda_1, \dots \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Then define

$$m_\lambda(x_1, \dots, x_n) := \sum x^\alpha$$

where the sums runs through distinct permutations α of $(\lambda_1, \dots, \lambda_n)$.

Examples: $n = 3$,

$$m_{(3)} = x_1^3 + x_2^3 + x_3^3$$

$$m_{(2,1)} = x_1^2 x_2 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + x_1^2 x_3 + x_3^2 x_1$$

$$m_{(1,1,1)} = x_1 x_2 x_3$$

2. Elementary Symmetric Functions

Definition of elementary symmetric function e_λ

For $r \geq 0$, define

$$e_r := \sum_{i_1 < \cdots < i_r} x_{i_1} \ldots x_{i_r} = m_{(1^r)}$$

and its generating function $E(t) := \sum_{r \geq 0} e_r t^r = \prod(1 + x_i t)$. For a partition $\lambda = (\lambda_1, \ldots)$, define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \ldots$$

Examples: $n = 3$:

$$e_{(3)} = e_3 = x_1 x_2 x_3$$

$$\begin{aligned} e_{(2,1)} = e_2 e_1 &= (x_1 x_2 + x_2 x_3 + x_1 x_3)(x_1 + x_2 + x_3) \\ &= x_1^2 x_2 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + x_1^2 x_3 + x_3^2 x_1 + 3 x_1 x_2 x_3 \end{aligned}$$

$$e_{(1,1,1)} = e_1^3 = (x_1 + x_2 + x_3)^3$$

3. Complete Symmetric Functions

Definition of complete symmetric functions h_λ

For $r \geq 0$, define

$$h_r := \sum_{|\lambda|=r} m_\lambda$$

and its generating function
$H(t) := \sum_{r\geq 0} h_r t^r = \prod(1 + x_i t + x_i^2 t^2 + \dots) = \prod(1 - x_i t)^{-1}$.
Observe

$$H(t)E(-t) = 1.$$

Since e_r are algebraically independent, may define homomorphism of graded rings $w : \Lambda \to \Lambda$ sending e_r to h_r. This satisfies $w^2 = 1$, hence is isomorphism, hence h_r are independent and $\Lambda = \mathbf{Z}[h_1, h_2, \dots]$.
Example: n=3,

$$\begin{aligned} h_3 &= m_{(3)} + m_{(2,1)} + m_{(1,1,1)} \\ &= x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + x_1^2 x_3 + x_3^2 x_1 + x_1 x_2 x_3 \end{aligned}$$

4. Power Sum Symmetric Functions

Definition of power sum symmetric functions p_λ

For $r \geq 1$, define the rth power sum as

$$p_r := \sum x_i^r = m_{(r)}$$

and generating function

$$\begin{aligned} P(t) &= \sum_{r\geq 1} p_r t^{r-1} = \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} \\ &= \sum_{i\geq 1} \frac{x_i}{1-x_i t} = \sum \frac{d}{dt}\log\frac{1}{1-x_i t} \\ &= \frac{d}{dt}\log H(t) \end{aligned}$$

Example: $n = 3$,

$$p_{(2,1)} = p_2 p_1 = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$$

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Schur Function

Let $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and consider the polynomial a_α obtained by antisymmetrizing x^α:

$$a_\alpha := \sum_{w \in S_n} \epsilon(w).w(x^\alpha)$$

where $\epsilon(w)$ is the sign (± 1) of the permutation w. Observe,

1. a_α is anti-symmetric: $w(a_\alpha) = \epsilon(w) a_\alpha$
2. $a_\alpha = 0$ unless $\alpha = (\alpha_1, \dots, \alpha_n)$ are all distinct.
3. Thus, we may assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$ and rewrite $\alpha = \lambda + \delta$, where $\delta = (n-1, n-2, \dots, 1, 0)$.
4. $a_{\lambda+\delta} = \sum \epsilon(w).w(x^{\lambda+\delta}) = \det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}$.
5. This determinant is divisible in $\mathbf{Z}[x_1, \dots, x_n]$ by each of $x_i - x_j$, hence by $\prod_{i \neq j}(x_i - x_j) = \det(x_i^{n-j}) = a_\delta$. "Vandermonde determinant."

Schur Function

Thus the following is well-defined (i.e is a polynomial)

Definition of Schur Function

For $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition of length $\leq n$, define the **Schur function**

$$s_\lambda := \frac{a_{\lambda+\delta}}{a_\delta}$$

s_λ is symmetric because $a_{\lambda+\delta}, a_\delta$ are anti-symmetric.
Examples ($n = 3$):

$$s_{(2,1,0)}(x_1, x_2, x_3) = \frac{1}{\Delta} \det \begin{bmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^0 & x_2^0 & x_3^0 \end{bmatrix} = (x_1+x_2)(x_1+x_3)(x_2+x_3)$$

$$s_{(2,2,0)}(x_1, x_2, x_3) = \frac{1}{\Delta} \det \begin{bmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^3 & x_2^3 & x_3^3 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

Schur Function

There is a useful theorem for combinatorially computing the Schur polynomials, namely

Formula for Schur Functions

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T$$

where "SSYT" means a Young tableau of shape λ with the boxes weakly increasing along each row and strictly increasing up each column

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A Few Identities

Cauchy Identities

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda h_\lambda(x)m_\lambda(y) = \sum_\lambda s_\lambda(x)s_\lambda(y)$$

Scalar Product

For $n \geq 0$, let $(u_\lambda), (v_\mu)$ be **Q**-bases of $\Lambda^n_{\mathbf{Q}}$, indexed by partitions of n. Then TFAE:

1. $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$ for all λ, μ
2. $\sum_\lambda u_\lambda(x)v_\lambda(y) = \prod_{i,j}(1-x_iy_j)^{-1}$.

Orthogonality

We define a **Z**-valued bilinear product (i.e scalar product) on Λ by requiring the complete symmetric functions to be dual to the monomial symmetric functions:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$$

By the Cauchy identity, we have

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \qquad (1)$$

so that s_λ, for $|\lambda| = n$ form orthonormal basis of Λ^n. Any other orthonormal basis must be obtained by orthogonal matrix with integer coefficients. The only such matrices are signed permutation matrices, thus (1) characterizes s_λ up to sign.

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Finite Group Representation Theory

Basic Definitions

A **representation** of a finite group G is (V, ρ) where V is finite-dimensional (complex) vector space and $\rho : G \to GL(V)$ is a homomorphism. An **irreducible representation** is a representation such that there is no nonzero proper G-invariant subspace $W \subset V$.

Complete Reducibility Theorem for Finite Group Representations

Any representation of a finite group is a direct sum of irreducible representations.

Example: $G = S_3$.

- $V = \mathbf{C}$, $g.z = z \forall g \in G, z \in \mathbf{C}$
- $V = \mathbf{C}$, $g.z = \text{sgn}(g)z, \forall g \in G, z \in \mathbf{C}$
- $V = \mathbf{C}^3$, $g.(z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)})$. This has 2-dimensional (irreducible) invariant subspace $V = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0\}$.

Character Theory

Definition of Character

Let (V, ρ) be a G-representation. To V, we associate the **Character of V**, χ_V, as a complex-valued function on the group defined by

$$\chi_V(g) := \mathrm{Tr}(\rho(g))$$

Note, χ_V is not necessarily a representation, but in the case V is 1-dimensional, it is.

Properties of Characters

Let V, W be representations of G. Then

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \cdot \chi_W \qquad \chi_{V^*} = \bar{\chi}_V$$

Inner Product on Characters

Inner product

The space of **C**-valued class functions $(f : f(g) = f(h^{-1}gh))$ on G have a natural inner-product

$$\langle \alpha, \beta \rangle_G := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}$$

A representation is determined by its character

The **irreducible characters** χ_V form an **orthonormal basis** for the space of class-functions.

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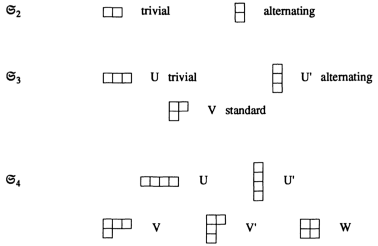
Classification Theorem for S_n-representations

Theorem

The irreducible representations of S_n are parameterized by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \ldots \lambda_k = n$



Proof of Classification Theorem of S_n-reps

Proof: Let R^n denote the $\mathbf{Z}$-module generated by irreducible characters of S_n and let

$$R = \bigoplus_{n \geq 0} R^n.$$

This carries a ring structure:

$$f \in R^m, g \in R^n, f.g := \mathrm{ind}_{S_m \times S_n}^{S_{n+m}} f \times g;$$

and a scalar product: for $f = \sum f_n, g = \sum g_n \in R$,

$$\langle f, g \rangle := \sum_n \langle f_n, g_n \rangle_{S_n}.$$

Next, define $\psi : S_n \to \Lambda^n$, $\psi(w) := p_{\rho(w)}$ where $\rho(w) = (\rho_1, \rho_2, \dots)$ is the cycle type of w and (recall) p_λ is the power-sum symmetric polynomial.

Proof continued

Next, we define a **Z**-linear mapping, called the *characteristic map*

$$\mathrm{ch} : R^n \to \Lambda^n_{\mathbf{C}}$$

$$f \to \langle f, \psi \rangle_{S_n} = \frac{1}{n!} \sum_{w \in S_n} f(w)\psi(w)$$

We may extend ch to a map $R \to \Lambda_{\mathbf{C}}$ which satisfies:

1. $\langle ch(f), ch(g) \rangle = \langle f, g \rangle_{S_n}$ for $f, g \in R^n$
2. $ch(f.g) = ch(f).ch(g)$, $f \in R^m, g \in R^n$ (Frobenius reciprocity)

Furthermore, ch is an isometric isomorphism of $R^n \to \Lambda^n_{\mathbf{C}}$, so

$$\chi^\lambda := ch^{-1}(s_\lambda) \in R^n$$

satisfies

$$\langle \chi^\lambda, \chi^\mu \rangle_{S_n} = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

Furthermore the number of conjugacy classes in S_n equals number of partitions of n, so χ^λ exhausts all irreducible characters of S_n.

Summary

1. Conjugacy classes of S_n correspond to power-sum symmetric functions p_λ, where λ is a partition of n.
2. Irreducible representations of S_n correspond to Schur functions s_λ, where λ is a partition of n.
3. The character table of S_n is the matrix for expressing Schur functions as linear combinations of power-sum functions.
4. By Schur-Weyl duality, we thus also have complete description of representations of general linear group $GL(n)$.

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Littlewood-Richardson Coefficients

Littlewood-Richardson Rule

Let λ, μ, ν be partitions. Since the Schur functions $\{s_\nu\}$ form a basis, there exists integral coefficients $c^{\nu}_{\lambda,\mu}$ such that

$$s_\lambda s_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} s_\nu.$$

The Littlewood-Richardson rule states that $c^{\nu}_{\lambda,\mu}$ is equal to the number of Littlewood-Richardson tableaux of skew shape ν/λ and of weight μ.

Littlewood-Richardson Coefficients in Representation Theory

By our theorem, $c^{\nu}_{\lambda,\mu}$ is precisely the multiplicity of an irreducible representation V_ν of $S_{|\nu|}$ occurring in $\text{ind}^{S_{|\nu|}}_{S_{|\lambda|}\times S_{|\mu|}} V_\lambda \otimes V_\mu$, where V_λ, V_μ are irreducible representations of $S_{|\lambda|}, S_{|\mu|}$, respectively. Namely,

$$\text{ind}^{S_{|\nu|}}_{S_{|\lambda|}\times S_{|\mu|}} V_\lambda \otimes V_\mu = \bigoplus_{\nu} c^{\nu}_{\lambda,\mu} V_\nu.$$

Major Open Problem (Kronecker Coefficients)

Let λ, μ, ν be partitions of n. Find a combinatorial formula for the multiplicity of V_ν in $V_\mu \otimes V_\lambda$, i.e find $g_{\lambda,\mu,\nu}$ where

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} V_\nu^{g_{\lambda,\mu,\nu}}.$$

Here, V_λ, V_μ, V_ν are all irreducible S_n-representations.

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Lie Theory (Algebra)

We use the following notation:

1. $\mathfrak{g}$: simple Lie algebra over $\mathbf{C}$
2. $\mathfrak{b}$: Borel subalgebra = maximal solvable subalgebra
3. $\mathfrak{h}$: Cartan subalgebra = maximal abelian subalgebra such that $ad\mathfrak{h}$ consists of diagonalizable operators
4. W = Weyl group = (finite) complex reflection group
5. $ad : \mathfrak{g} \to GL(\mathfrak{g})$ where $X \to (ad_X : Y \to [X, Y])$ gives root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

6. Λ = weight space = $\{\lambda \in \text{span}_{\mathbf{R}}(\Phi) : (\lambda, \check{\alpha}) \in \mathbf{Z} \forall \alpha \in \Phi\}$
7. Λ^+ = dominant weights = $\{\lambda \in \Lambda : (\lambda, \check{\alpha}) \geq 0 \forall \alpha \in \Phi^+\}$

Lie Theory (Algebra)

Theorem of Highest Weights

Let V be irreducible $\mathfrak{g}$-module. Then there exists unique dominant weight λ such that

- $\dim V_\lambda = 1$
- $\forall \mu$ weight of V, $\mu = \lambda - \sum n_i \alpha_i$, $n_i \geq 0$.
- $\mathfrak{g}_\alpha . V_\lambda = 0$ for $\alpha \in \Phi^+$

For irreducible $\mathfrak{g}$-representation, $V(\lambda)$, define its character as

$$\operatorname{ch} V(\lambda) := \sum_{\mu \in \Lambda} (\dim V(\lambda)_\mu) e^\mu \in \mathbf{Z}[\Lambda]$$

Weyl's Formula

Weyl Character Formula

Let $\lambda \in \Lambda^+$. Then

$$(\sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma\rho}) * \text{ch}(V(\lambda)) = \sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma.(\lambda+\rho)}$$

where $\rho = 1/2 \sum_{\mu \in \Phi^+} \mu$ and $*$ is the convolution product for $\mathbf{Z}[\Lambda]$.

Recall the Schur function was defined as

$$s_\lambda = \frac{\sum \epsilon(w).w(x^{\lambda+\delta})}{\sum \epsilon(w).w(x^\delta)}$$

Writing $x_i = e^{\lambda_i}$ and letting $\mathfrak{g} = \mathfrak{sl}(n+1)$, we see $W = S_n$, $\delta = \rho$ and thus Weyl's character formula is literally the Schur function. Same thing with Weyl charactor formula for Lie group $GL(n)$.

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