Symmetric Polynomials and Representation Theory

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Outline

1. Symmetric Functions
   - Ring of Symmetric Functions
   - Four Bases of \( \Lambda \)
   - One More Basis of \( \Lambda \): Schur Functions
   - Orthogonality

2. Characters of \( S_n \)
   - Finite Group Representation Theory
   - \( S_n \)-reps

3. Further Applications
   - Littlewood-Richardson Rule
   - Lie Theory
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Symmetric Polynomials

Consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in $n$ variables with integer coefficients. The symmetric group $S_n$ acts on this ring by permuting the variables. An element $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is called a symmetric polynomial if $\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in S_n$.

Ring of Symmetric Polynomials

The set of all symmetric polynomials of $n$ variables forms a subring

$$\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{S_n},$$

which is graded by the degree:

$$\Lambda_n = \bigoplus_{d \geq 0} \Lambda_n^d,$$

where $\Lambda_n^d$ consists of symmetric polynomials of $n$ variables and degree $d$. 
The number of variables is often irrelevant provided that it is large enough.

Ring of Symmetric Functions

The ring of **symmetric functions** is

$$\Lambda := \bigoplus_{d \geq 0} \Lambda^d,$$

where $\Lambda^d = \lim_{n \to \infty} \Lambda_n^d$ denotes the ring of symmetric polynomials of degree $d$ of arbitrary number of variables.
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1. Monomial Symmetric Functions

Definition of monomial symmetric function $m_\lambda$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and denote $x^\alpha$ the monomial

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$ 

Let $\lambda$ be any partition of length $\leq n$, i.e. $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Then define

$$m_\lambda(x_1, \ldots, x_n) := \sum x^\alpha$$

where the sums runs through distinct permutations $\alpha$ of $(\lambda_1, \ldots, \lambda_n)$.

Examples: $n = 3$,

$$m_{(3)} = x_1^3 + x_2^3 + x_3^3$$

$$m_{(2,1)} = x_1^2 x_2 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + x_1^2 x_3 + x_3^2 x_1$$

$$m_{(1,1,1)} = x_1 x_2 x_3$$
2. Elementary Symmetric Functions

Definition of elementary symmetric function $e_\lambda$

For $r \geq 0$, define

$$e_r := \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} = m_{(1^r)}$$

and its generating function $E(t) := \sum_{r \geq 0} e_r t^r = \prod (1 + x_i t)$. For a partition $\lambda = (\lambda_1, \ldots)$, define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$$

Examples: $n = 3$:

$$e_{(3)} = e_3 = x_1 x_2 x_3$$

$$e_{(2,1)} = e_2 e_1 = (x_1 x_2 + x_2 x_3 + x_1 x_3)(x_1 + x_2 + x_3)$$

$$= x_1^2 x_2 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + x_1^2 x_3 + x_3^2 x_1 + 3 x_1 x_2 x_3$$

$$e_{(1,1,1)} = e_1^3 = (x_1 + x_2 + x_3)^3$$
3. Complete Symmetric Functions

Definition of complete symmetric functions $h_\lambda$

For $r \geq 0$, define

$$h_r := \sum_{|\lambda|=r} m_\lambda$$

and its generating function

$$H(t) := \sum_{r\geq0} h_r t^r = \prod(1 + x_i t + x_i^2 t^2 + \ldots) = \prod(1 - x_i t)^{-1}.$$  

Observe

$$H(t) E(-t) = 1.$$  

Since $e_r$ are algebraically independent, may define homomorphism of graded rings $w : \Lambda \to \Lambda$ sending $e_r$ to $h_r$. This satisfies $w^2 = 1$, hence is isomorphism, hence $h_r$ are independent and $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$.

Example: $n=3$,

$$h_3 = m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$$

$$= x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + x_1^2 x_3 + x_3^2 x_1 + x_1 x_2 x_3$$
4. Power Sum Symmetric Functions

Definition of power sum symmetric functions $p_\lambda$

For $r \geq 1$, define the $r$th power sum as

$$p_r := \sum x_i^r = m(r)$$

and generating function

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i \geq 1} \sum_{r \geq 1} x_i^r t^{r-1}$$

$$= \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \sum \frac{d}{dt} \log \frac{1}{1 - x_i t}$$

$$= \frac{d}{dt} \log H(t)$$

Example: $n = 3$,

$$p_{(2,1)} = p_2 p_1 = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$$
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Schur Function

Let $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and consider the polynomial $a_\alpha$ obtained by antisymmetrizing $x^\alpha$:

$$a_\alpha := \sum_{w \in S_n} \epsilon(w).w(x^\alpha)$$

where $\epsilon(w)$ is the sign ($\pm 1$) of the permutation $w$. Observe,

1. $a_\alpha$ is anti-symmetric: $w(a_\alpha) = \epsilon(w)a_\alpha$
2. $a_\alpha = 0$ unless $\alpha = (\alpha_1, \ldots, \alpha_n)$ are all distinct.
3. Thus, we may assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$ and rewrite $\alpha = \lambda + \delta$, where $\delta = (n-1, n-2, \ldots, 1, 0)$.
4. $a_{\lambda+\delta} = \sum \epsilon(w).w(x^{\lambda+\delta}) = \det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n}$.
5. This determinant is divisible in $\mathbb{Z}[x_1, \ldots, x_n]$ by each of $x_i - x_j$, hence by $\prod_{i \neq j}(x_i - x_j) = \det(x_i^{n-j}) = a_\delta$. “Vandermonde determinant.”
Thus the following is well-defined (i.e is a polynomial)

Definition of Schur Function

For \(\lambda = (\lambda_1, \ldots, \lambda_n)\) a partition of length \(\leq n\), define the Schur function

\[
s_\lambda := \frac{a_{\lambda+\delta}}{a_{\delta}}
\]

\(s_\lambda\) is symmetric because \(a_{\lambda+\delta}, a_{\delta}\) are anti-symmetric.

Examples \((n = 3)\):

\[
s_{(2,1,0)}(x_1, x_2, x_3) = \frac{1}{\Delta} \det \begin{bmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^0 & x_2^0 & x_3^0 \end{bmatrix} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)
\]

\[
s_{(2,2,0)}(x_1, x_2, x_3) = \frac{1}{\Delta} \det \begin{bmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^3 & x_2^3 & x_3^3 \\ 1 & 1 & 1 \end{bmatrix} = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2
\]
Schur Function

There is a useful theorem for combinatorially computing the Schur polynomials, namely

Formula for Schur Functions

\[ s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \]

where "SSYT" means a Young tableau of shape \( \lambda \) with the boxes weakly increasing along each row and strictly increasing up each column.
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A Few Identities

Cauchy Identities

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} h_\lambda(x) m_\lambda(y) = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

Scalar Product

For $n \geq 0$, let $(u_\lambda), (v_\mu)$ be $\mathbb{Q}$-bases of $\Lambda_n^\mathbb{Q}$, indexed by partitions of $n$. Then TFAE:

1. $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$ for all $\lambda, \mu$

2. $\sum_{\lambda} u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$. 
Orthogonality

We define a \(\mathbb{Z}\)-valued bilinear product (i.e. scalar product) on \(\Lambda\) by requiring the complete symmetric functions to be dual to the monomial symmetric functions:

\[
\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}
\]

By the Cauchy identity, we have

\[
\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}
\]  

so that \(s_\lambda\), for \(|\lambda| = n\) form orthonormal basis of \(\Lambda^n\). Any other orthonormal basis must be obtained by orthogonal matrix with integer coefficients. The only such matrices are signed permutation matrices, thus (1) characterizes \(s_\lambda\) up to sign.
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Finite Group Representation Theory

Basic Definitions

A representation of a finite group $G$ is $(V, \rho)$ where $V$ is finite-dimensional (complex) vector space and $\rho : G \to GL(V)$ is a homomorphism. An irreducible representation is a representation such that there is no nonzero proper $G$-invariant subspace $W \subset V$.

Complete Reducibility Theorem for Finite Group Representations

Any representation of a finite group is a direct sum of irreducible representations.

Example: $G = S_3$.

- $V = \mathbb{C}$, $g.z = z \forall g \in G, z \in \mathbb{C}$
- $V = \mathbb{C}$, $g.z = \text{sgn}(g)z, \forall g \in G, z \in \mathbb{C}$
- $V = \mathbb{C}^3$, $g.(z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)})$. This has 2-dimensional (irreducible) invariant subspace $V = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0\}$. 
Character Theory

Definition of Character

Let \((V, \rho)\) be a \(G\)-representation. To \(V\), we associate the Character of \(V\), \(\chi_V\), as a complex-valued function on the group defined by

\[
\chi_V(g) := \text{Tr}(\rho(g))
\]

Note, \(\chi_V\) is not necessarily a representation, but in the case \(V\) is 1-dimensional, it is.

Properties of Characters

Let \(V, W\) be representations of \(G\). Then

\[
\chi_{V \oplus W} = \chi_V + \chi_W, \quad \chi_{V \otimes W} = \chi_V \cdot \chi_W \quad \chi_{V^*} = \bar{\chi}_V
\]
Inner Product on Characters

Inner product

The space of $\mathbb{C}$-valued class functions $(f : f(g) = f(h^{-1}gh))$ on $G$ have a natural inner-product

$$\langle \alpha, \beta \rangle_G := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}$$

A representation is determined by its character

The irreducible characters $\chi_V$ form an orthonormal basis for the space of class-functions.
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The irreducible representations of $S_n$ are parameterized by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \ldots \lambda_k = n$. 

**Diagram**

- $S_2$: trivial, alternating
- $S_3$: trivial, alternating, standard
- $S_4$: U, U', V, V', W
Proof of Classification Theorem of $S_n$-reps

**Proof:** Let $R^n$ denote the $\mathbb{Z}$-module generated by irreducible characters of $S_n$ and let

$$R = \bigoplus_{n \geq 0} R^n.$$  

This carries a ring structure:

$$f \in R^m, g \in R^n, f \cdot g := \text{ind}_{S_m \times S_n}^{S_{n+m}} f \times g;$$

and a scalar product: for $f = \sum f_n, g = \sum g_n \in R,$

$$\langle f, g \rangle := \sum_n \langle f_n, g_n \rangle_{S_n}.$$  

Next, define $\psi : S_n \rightarrow \Lambda^n$, $\psi(w) := p_{\rho(w)}$ where $\rho(w) = (\rho_1, \rho_2, \ldots)$ is the cycle type of $w$ and (recall) $p_\lambda$ is the power-sum symmetric polynomial.
Next, we define a \( \mathbb{Z} \)-linear mapping, called the *characteristic map*

\[
ch : R^n \to \Lambda^n_C
\]

\[
f \to \langle f, \psi \rangle_{S_n} = \frac{1}{n!} \sum_{w \in S_n} f(w) \psi(w)
\]

We may extend \( ch \) to a map \( R \to \Lambda_C \) which satisfies:

1. \( \langle ch(f), ch(g) \rangle = \langle f, g \rangle_{S_n} \) for \( f, g \in R^n \)
2. \( ch(f.g) = ch(f).ch(g), f \in R^m, g \in R^n \) (Frobenius reciprocity)

Furthermore, \( ch \) is an isometric isomorphism of \( R^n \to \Lambda^n_C \), so

\[
\chi^\lambda := ch^{-1}(s_\lambda) \in R^n
\]

satisfies

\[
\langle \chi^\lambda, \chi^\mu \rangle_{S_n} = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.
\]

Furthermore the number of conjugacy classes in \( S_n \) equals number of partitions of \( n \), so \( \chi^\lambda \) exhausts all irreducible characters of \( S_n \).
Summary

1. Conjugacy classes of $S_n$ correspond to power-sum symmetric functions $p_\lambda$, where $\lambda$ is a partition of $n$.

2. Irreducible representations of $S_n$ correspond to Schur functions $s_\lambda$, where $\lambda$ is a partition of $n$.

3. The character table of $S_n$ is the matrix for expressing Schur functions as linear combinations of power-sum functions.

4. By Schur-Weyl duality, we thus also have complete description of representations of general linear group $GL(n)$. 
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Littlewood-Richardson Coefficients

**Littlewood-Richardson Rule**

Let $\lambda, \mu, \nu$ be partitions. Since the Schur functions $\{s_\nu\}$ form a basis, there exists integral coefficients $c^\nu_{\lambda,\mu}$ such that

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

The Littlewood-Richardson rule states that $c^\nu_{\lambda,\mu}$ is equal to the number of Littlewood-Richardson tableaux of skew shape $\nu/\lambda$ and of weight $\mu$. 
By our theorem, $c_{\lambda, \mu}^{\nu}$ is precisely the multiplicity of an irreducible representation $V_{\nu}$ of $S_{|\nu|}$ occurring in $\text{ind}_{S_{|\lambda|} \times S_{|\mu|}}^{S_{|\nu|}} V_\lambda \otimes V_\mu$, where $V_\lambda$, $V_\mu$ are irreducible representations of $S_{|\lambda|}$, $S_{|\mu|}$, respectively. Namely,

$$\text{ind}_{S_{|\lambda|} \times S_{|\mu|}}^{S_{|\nu|}} V_\lambda \otimes V_\mu = \bigoplus_{\nu} c_{\lambda, \mu}^{\nu} V_\nu.$$ 

**Major Open Problem (Kronecker Coefficients)**

Let $\lambda, \mu, \nu$ be partitions of $n$. Find a combinatorial formula for the multiplicity of $V_\nu$ in $V_\mu \otimes V_\lambda$, i.e find $g_{\lambda, \mu, \nu}$ where

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} V_\nu^{g_{\lambda, \mu, \nu}}.$$ 

Here, $V_\lambda$, $V_\mu$, $V_\nu$ are all irreducible $S_n$-representations.
Lie Theory (Algebra)

We use the following notation:

1. $\mathfrak{g}$: simple Lie algebra over $\mathbb{C}$
2. $\mathfrak{b}$: Borel subalgebra = maximal solvable subalgebra
3. $\mathfrak{h}$: Cartan subalgebra = maximal abelian subalgebra such that $ad\mathfrak{h}$ consists of diagonalizable operators
4. $W = \text{Weyl group} = \text{(finite) complex reflection group}$
5. $ad : \mathfrak{g} \to GL(\mathfrak{g})$ where $X \to (ad_X : Y \to [X, Y])$ gives root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

6. $\Lambda = \text{weight space} = \{\lambda \in \text{span}_R(\Phi) : (\lambda, \check{\alpha}) \in \mathbb{Z} \forall \alpha \in \Phi\}$
7. $\Lambda^+ = \text{dominant weights} = \{\lambda \in \Lambda : (\lambda, \check{\alpha}) \geq 0 \forall \alpha \in \Phi^+\}$
Lie Theory (Algebra)

Theorem of Highest Weights

Let $V$ be irreducible $\mathfrak{g}$-module. Then there exists unique dominant weight $\lambda$ such that

- $\dim V_\lambda = 1$
- $\forall \mu$ weight of $V$, $\mu = \lambda - \sum n_i \alpha_i$, $n_i \geq 0$.
- $\mathfrak{g}_\alpha \cdot V_\lambda = 0$ for $\alpha \in \Phi^+$

For irreducible $\mathfrak{g}$-representation, $V(\lambda)$, define its character as

$$
\text{ch} V(\lambda) := \sum_{\mu \in \Lambda} (\dim V(\lambda)_\mu) e^\mu \in \mathbb{Z}[\Lambda]
$$
Weyl’s Formula

Weyl Character Formula

Let $\lambda \in \Lambda^+$. Then

$$
\left( \sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma \rho} \right) \ast \text{ch}(V(\lambda)) = \sum_{\sigma \in W} \text{sgn}(\sigma) e^{\sigma \cdot (\lambda + \rho)}
$$

where $\rho = 1/2 \sum_{\mu \in \Phi^+} \mu$ and $\ast$ is the convolution product for $\mathbb{Z}[\Lambda]$.

Recall the Schur function was defined as

$$
S_{\lambda} = \frac{\sum \epsilon(w).w(x^{\lambda+\delta})}{\sum \epsilon(w).w(x^{\delta})}
$$

Writing $x_i = e^{\lambda_i}$ and letting $g = \mathfrak{sl}(n+1)$, we see $W = S_n$, $\delta = \rho$ and thus Weyl’s character formula is literally the Schur function. Same thing with Weyl character formula for Lie group $GL(n)$. 
References I

