$\mathbf{Bun}_G$ AND HIGGS BUNDLES.

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1. Sheaf of Categories

We have now seen several times that a stack is a "sheaf of groupoids." Groupoids are, of course, categories, and we have mentioned how stacks can be thought of as a special case of the more general structure of "fibered categories."

We have also seen how many natural moduli stacks live in "bigger" and "more linear" categories. For example, the moduli stack of vector bundles over a curve Σ gives rise, over a test scheme S, to the category of vector bundles on $S \times \Sigma$. The stack only remembers automorphisms of such bundles. But the full category of vector bundles has many desirable properties; in particular the collection $\mathrm{Hom}_S(V_1, V_2)$ is a $k[S]$-module.

1.1. **Moduli of Categories.** Throughout, we work over an algebraically closed field k. Given a category, we would like construct something which parameterizes isomorphism classes of objects of that category. This will be called the moduli space of such a category. But first, we need to make sense of families of objects in a category, and this leads to the following three definitions.

Definition 1.1. *A sheaf of categories is a functor $\mathcal{C}$ which to each affine scheme S associates a $k[S]$-linear category $\mathcal{C}(S)$, and to each morphism $f : S' \to S$, associates a $k[S]$-linear functor $f^* : \mathcal{C}(S') \to \mathcal{C}(S)$ which induces an equivalence of categories*

$$k[S'] \otimes_{k[S]} \mathcal{C}(S) \xrightarrow{\sim} \mathcal{C}(S').$$

Definition 1.2. *An object $M \in \mathcal{C}(S)$ is flat over S if $End_{\mathcal{C}(S)}(M)$ is flat over $k[S]$.*

Definition 1.3. *Given a sheaf of categories $\mathcal{C}$, the moduli stack $\mathcal{M}(\mathcal{C})$ is a functor which associates to an affine scheme S the groupoid $\mathcal{M}(\mathcal{C})(S)$ whose objects are those $M \in \mathcal{C}(S)$ which are flat over S, and whose morphisms are the isomorphisms in $\mathcal{C}(S)$.*

There are two basic examples.

Example 1.4. *Let $\mathcal{C} = A - mod$, for A a k-algebra. Then $\mathcal{C}(S)$ is the category of $k[S] \otimes A$-modules. The corresponding moduli stack is the groupoid, whose objects are S-flat $k[S] \otimes A$-modules, and whose morphisms are S-isomoprhisms.*

Example 1.5. *Fix a scheme X and let $\mathcal{C}(S)$ be the category of coherent sheaves on $S \times X$. Then a sheaf $\mathcal{F}$ on $X \times S$ is flat over S if it is flat in the usual sense. We denote the corresponding moduli stack $\mathcal{M}(Coh(X))$.*

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[1]We take full responsibility for all typos or mistakes in these notes! Please email us if you spot any.

Now, consider the full subcategory $\mathrm{Vect}_X \subset \mathrm{Coh}(X)$ of locally free sheaves. We see that the moduli stack

$$\mathcal{M}(\mathrm{Vect}_X) = \bigsqcup_n \mathrm{Bun}_n(X)$$

is an open substack of $\mathcal{M}(\mathrm{Coh}(X))$. Loosely speaking, it is open because a small deformation of a locally free sheaf cannot pick up torsion. We caution that Vect_X is not an abelian category because, e.g., cokernels do not exist (indeed, the cokernel sheaf of a vector bundle map may well have torsion). So, in general, we will start with cohomology on $\mathrm{Coh}(X)$ and then restrict to Vect_X.

1.2. **Homological Dimension.** From now on, we assume all stacks $\mathfrak{X}$ are algebraic. This means there exists an atlas $X \to \mathfrak{X}$ which is a smooth (surjective) morphism. (X is a scheme, not necessarily smooth.)

Definition 1.6. *A stack $\mathfrak{X}$ is smooth if there exists a smooth atlas $X \to \mathfrak{X}$ such that X itself is smooth.*

We remark $\mathfrak{X}$ is smooth if and only if all its fibers are smooth. This means that if S is a scheme and $S \to \mathfrak{X}$ is a smooth morphism, then S itself is smooth.

Definition 1.7. *Let $\mathcal{C}$ be an abelian category. The homological dimension of $\mathcal{C}$ is*

$$hdim(\mathcal{C}) = max\{i \geq 0 : Ext^i(M, N) \neq 0 \text{ for all } M, N \in \mathcal{C}\}$$

Here are some fascinating facts/heuristics: first, $\mathrm{hdim}(\mathcal{C}) = 0$ if and only if $\mathcal{C}$ is semisimple.

Next, a moduli stack $\mathcal{M}(\mathcal{C}))$ is smooth if and only $\mathrm{hdim}(\mathcal{C}) \leq 1$. Roughly speaking, this follows from deformation theory. Namely, the tangent space measures infinitesimal deformations. Such deformations are usually paramterized by an H^1, while obstructions to deformations usually live in some H^2. Thus we usually need H^2 to vanish.

A sufficient, and perhaps necessary, condition for $\mathrm{hdim}(\mathcal{C}) \leq 1$, in the case $\mathcal{C} =$ A-mod, is for $\mathrm{Ker}(A \otimes A \to A)$ to be a projective $A \otimes A$-subbimodule. In this case bimodule resolution of A consists of 2 terms, which implies $0 = \mathrm{Ext}^2_{A \otimes A}(k, M) = \mathrm{Ext}^2_A(A, M)$. This condition is satisfied, for example, when A is free associative k-algebra (in the *non-commutative* sense!), or the path algebra of a quiver with no relations. Note that the commutative ring $k[x, y]$ is not free associative (indeed, $xy = yx$) and it certainly has nonzero Ext^2.

As another example, consider X a smooth affine scheme. Then $\mathrm{QCoh}(X) \simeq k[X]{-}\mathrm{mod}$ and $\mathrm{hdim}(\mathrm{Qcoh}(X)) = \dim(X)$ by the syzygy theorem. Now, $\mathcal{M}(\mathrm{Vect}_X)$ is open substack of $\mathrm{Coh}(X)$, hence it has the same hdim. Thus, in order for the moduli stack to be smooth, we need X to be curve (not necessarily affine!). This is why we always consider Bun_G(curve). When X is not a curve, one would have to consider derived stacks to account for the non-smoothness.

2. The Tangent and Cotangent Stack

2.1. **Tangent Spaces to Schemes.** In the context of schemes, we have a standard method for describing tangent vectors via the dual numbers. For intuition we will review it in some detail.

Definition 2.1. *Let $D = \mathrm{Spec}\, k_\epsilon := \mathrm{Spec}\, k[\epsilon]/\epsilon^2$. The ring k_ϵ is usually called the ring of dual numbers.*

We can define a tangent vector as follows. Let $S = \operatorname{Spec} A$ be a test scheme, assumed to be affine. Let $f : S \to X$ be an S-point of X.

Definition 2.2. *A tangent vector to the S-point f is a dotted map making the following diagram commutative:*

$$\begin{tikzcd} & S \times D \arrow[dr, dotted] & \\ S \arrow[ur] \arrow[rr, "f"] & & X \end{tikzcd}$$

(Here the map $S \to S \times D$ is given by sending $\epsilon \mapsto 0$ under $A \otimes k_\epsilon \to A$.)

In the case where $S = \operatorname{Spec} k$ (k our base field), note that the map f is the inclusion of a point; i.e., a map of rings $\pi : A \to k$ giving by quotienting by a maximal ideal. Then a lift of f to $D \times \operatorname{Spec} k$ is precisely a homomorphism $A \to k[\epsilon]/\epsilon^2$, such that $a \mapsto \pi(a) + \theta(a)\epsilon$, where $\theta : A \to k$ is a function (*not* a ring homomorphism!). The fact that this sum is a multiplicative homomorphism implies that $\theta(aa') = \theta(a)\pi(b) + \pi(a)\theta(b)$; i.e., that θ satisfies the Leibniz rule (or, equivalently, that θ is a k-valued derivation). And in the case where $S = X$, and f is the identity morphism (viewed as an X-point), we see that the tangent space to f is, by the same logic, equivalent to the space of "global" derivations of the algebra $k[X]$: $\operatorname{Der}(k[X], k[X])$, which is the usual algebraic defintion for a vector field.

We expect that the collection of tangent vectors to an S-point of X will have the structure of a vector space. Now, recall that the scheme D *corepresents* a tangent vector (i.e., $\operatorname{Maps}_x(D, X) = T_x X$, or tangent vectors are given by maps *out* of D). Thus addition of tangent vectors should be defined by a *cogroup* structure on D:

$$D \xrightarrow{+} D \coprod_* D.$$

We see that $D \coprod_* D = \operatorname{Spec}(k[\varepsilon]/\varepsilon^2 \times_k k[\delta]/\delta^2)$ where $\times$ means the *fiber product* (not tensor product!) in the category of rings; i.e., it is $\operatorname{Spec}(k[\varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta))$. The cogroup structure on D is given by the ring map $(a + b\varepsilon + c\delta) \mapsto a + (b + c)\epsilon$. One can check that this satisfies the appropriately-defined associativity, commutativity, inverse, and 0 properties.

Now, we must levarage this to add tangent vectors "in families." The first thing to notice is the following algebraic identity relating products and coproducts in the category of schemes:

$$(S \times D) \coprod_S (S \times D) \simeq S \times \left(D \coprod_* D \right).$$

This can be seen fairly easily at the level of rings:

$$A[\varepsilon]/\varepsilon^2 \times_A A[\delta]/\delta^2 \simeq A[\varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta) \simeq A \otimes_k k[\varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta).$$

Now: say that we have two "tangent vector" $\tilde{f}_1, \tilde{f}_2 : S \times D \to X$ lifting the S-point $f : S \to X$. We define their sum via the composite:

$$\begin{tikzcd}
S \times D \arrow[ddr] \arrow[dddr, bend right, "\tilde{f}_1"'] & S \times D \arrow[d, dashed, "(\mathrm{Id},+)"] & S \times D \arrow[ddl] \arrow[dddl, bend left, "\tilde{f}_2"] \\
& S \times (D \coprod_* D) \arrow[d, dashed, "\simeq"] & \\
& (S \times D) \coprod_S (S \times D) \arrow[d] & \\
& X &
\end{tikzcd}$$

where the maps "+" and "$\simeq$" are as defined above. Note that we implicitly assume that the $\tilde{f}_i$ are lifts of the single S-point f when we write the universal maps from $S \times D$ maps to $(S \times D) \coprod_S (S \times D)$.

As for scalar multiplication, we note that for each $\lambda \in k$, we have the map $D \xleftarrow{\lambda} D$ given by $\epsilon \mapsto \lambda\epsilon$. These are simply all the k-maps of k_ϵ. Thus any such map gives a "co-scaling" on D which in turn gives us scaling on tangent vectors when we consider maps from D to some scheme X. To (co-)scale "in families" is to give the data of an S-map $S \times D \leftarrow S \times D$; i.e., a map that $S \times D \leftarrow S \times D$ that commutes with projection onto S. And of course we still have "constant scaling" by $\lambda \in k$ in families: these are given by the maps $S \times D \xleftarrow{\lambda} S \times D$; $A[\epsilon]/\epsilon^2 \to A[\epsilon]/\epsilon^2 : a + a'\epsilon \mapsto a + \lambda a'\epsilon$.

We can check that scaling has the correct associative, commutative, 0 and 1 properties, and, most importantly, that it has the correct distributive property with respect to the above-defined addition.

As a final comment, we note that we may define the entire tangent bundle to a scheme X via the mapping stack construction from Lecture I. Indeed, $\mathrm{Maps}(D, X) \simeq TX$. Since this is valued in sets, it is an algebraic space; in fact it is a scheme, too, which can be seen by the usual "relative Spec" tangent bundle construction.

2.2. **Tangent Stack.** Thus to define a tangent vector to an S-point of a stack, we just repeat the above verbatim, replacing X with $\mathfrak{X}$. Of course, stacks are functors valued in groupoids, rather than just sets. So tangent spaces to a point of a stack will be groupoids, not just sets. And, as above, the co-group and co-scaling operations on D give us extra structure on such groupoids. We will analyze this extra structure in detail.

Definition 2.3. *Let S be an affine test scheme and $\mathfrak{X}$ a stack; let $f : S \to \mathfrak{X}$ be an S-point of $\mathfrak{X}$ (which is just a subgroupoid of $\mathfrak{X}(S)$ containing one isomorphism class of object). A tangent vector to the S-point f is a dotted map making the following diagram commutative:*

$$\begin{tikzcd}
& S \times D \arrow[dr, dotted] & \\
S \arrow[ur] \arrow[rr, "f"] & & \mathfrak{X}
\end{tikzcd}$$

The collection of such dotted maps forms a subgroupoid of $\mathfrak{X}(S \times D)$. The zero object of this groupoid consists of the diagram

$$
\text{zero} = \begin{tikzcd} & S \times D \arrow[dr, "f \circ pr_2"] & \\ S \arrow[ur] \arrow[rr, "f"] & & \mathfrak{X}. \end{tikzcd}
$$

Notice that all the extra structures of tangent spaces come from properties of the scheme D; that is to say, they all can be transferred to this setting without any extra work. Now, let $\mathfrak{T}_f$ be the subgroupoid of $\mathfrak{X}(S \times D)$ tangent to f. We see that the co-group structure of D gives us a mapping (i.e., a functor) $\mathfrak{T}_f \times \mathfrak{T}_f \to \mathfrak{T}_f$. Co-scaling gives us functors $\mathfrak{T}_f \to \mathfrak{T}_f$. We see that $\mathfrak{T}_f$ has a zero object, and an action of $k^\times$ given by "constant" scaling. We also see that each object A in $\mathfrak{T}_f$ has an additive inverse $-A$, such that $(A, -A) \mapsto 0 \in \mathfrak{T}_f$ under the addition functor.

Assembling this data shows that our groupoids have an additional "symmetric monoidal functor", namely "+", with a k-action. Moreover, every object is invertible with respect to this symmetric monoidal structure.

Definition 2.4. *A Picard category $\mathfrak{T}$ is a symmetric monoidal category in which every object is invertible with respect to the symmetric monoidal structure. If $\mathfrak{T}$ has an additional k-action by a field k, such that distributive identities hold between the k-action and the symmetric monoidal structure, $\mathfrak{T}$ is sometimes called a "category in vector spaces." For such a category, the isomorphism classes of objects form a vector space. A Picard groupoid is a Picard caetgory that is also a groupoid; i.e., every morphism and every object is invertible.*

In particular, we see that $\mathfrak{T}_f$ is a Picard Groupoid in vector spaces. Thus we see, by application of the monoidal structure to "translate" any object to 0, that the group of automorphisms of any object is isomorphic to the group of automorphisms of the 0 object. Note that the automorphisms of the zero object are given by the fiber product:

$$
\begin{tikzcd} \mathrm{Aut}(0) \arrow[r] \arrow[d] & \mathfrak{X}(S) \arrow[d, "(pr_2 \circ f)^*"] \\ \mathfrak{X}(S) \arrow[r, "(pr_2 \circ f)^*"'] & \mathfrak{X}(D \times S). \end{tikzcd}
$$

Now, since $\mathfrak{X}$ is algebraic, Aut (0) is a set; in fact, it will be represented by an algebraic space. (In reality, the condition required to ensure that Aut (0) is representable by an algebraic space is the representability of the diagonal map $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. This does not depend upon the existence of an atlas.) Moreover, we see, using the same kind of pushout argument as above, that Aut (0) has the structure of a k-vector space, just like the isomorphism classes of objects of $T_f(\mathfrak{X})$.

Finally, as in the context of schemes, we note that the collection of all tangent spaces can be packaged into a single stack via the mapping-stack construction: $\mathrm{Maps}(D, \mathfrak{X})$.

2.3. **Two-Term Complexes.** The standard references for this subsection are [BF, pg. 15] and [R09]. Thus far we have not made use of the smooth Atlas $\pi : X \to \mathfrak{X}$, which lets us say a good deal more. Indeed, T_X is a vector bundle over X, and for any test scheme S and map $f : S \to \mathfrak{X}$, we can consider the diagram

$$\begin{array}{ccc} f^*T_X & \longrightarrow & T_X \\ \downarrow & & \downarrow \\ S \times_{\mathfrak{X}} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow[f]{} & \mathfrak{X}. \end{array}$$

We will attempt to describe the structure of the tangent bundle to $\mathfrak{X}$ using such a diagram. In particular, to understand any property of a stack, we must "pull it back" to test schemes, so we would *really* like to be able to describe $f^*(T\mathfrak{X})$ as some kind of sheaf over S.

In the usual context of schemes, if $\pi : X \to Y$ is a smooth surjective map, then we have the following "Euler" exact sequence of sheaves over X:

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \pi^*(T_Y) \longrightarrow 0,$$

where $T_{X/Y}$ is the "relative" tangent bundle (which is a vector bundle since π is smooth). We note that the exactness of this sequence is equivalent to saying that there is a quasi-isomorphism:

$$\left(T_{X/Y} \longrightarrow T_X\right) \simeq \pi^* T_Y,$$

where we imagine $\pi^*(T_Y)$ living in purely degree 0, while the lefthand sequence lives in degrees -1 and 0. Of course, π^*TY is simply the cokernel of this map (since the map $T_{X/Y} \longrightarrow T_X$ is injective, a consequence of flatness of smooth maps). This may seem like a pedantic point, but, as we will see, the "right" definition of the tangent stack will be in terms of *complexes* of sheaves, rather than cokernels – these complexes will contain the right "automorphism" data.

So let us upgrade everything to stacks. We are going, essentially, to reverse the above reasoning: we will *define* the tangent complex in terms of complexes on smooth covers. So let $Y = \mathfrak{X}$. Now consider the smooth site over $\mathfrak{X}$; i.e., the collection of schemes U with a smooth map $\varphi : U \to \mathfrak{X}$ (where covers are given by jointly surjective families of such maps).

To define $T_{\mathfrak{X}}$, we must "pull back" to open sets in our cover, i.e., define $\varphi^*T_{\mathfrak{X}}$ for each such $\varphi : U \to \mathfrak{X}$. We can accomplish this if we can define $T_{U/\mathfrak{X}} \longrightarrow T_U$ for each U. So now we will try to define $T_{U/\mathfrak{X}}$.

We expect that $T_{U/\mathfrak{X}}$ will be a sheaf (in fact a vector bundle!) over U. This lives in the world of schemes; so we should be able to get a handle on it. Unfortunately, we must construct it rather indirectly. For any test scheme S and map $f : S \to \mathfrak{X}$, we may pull back everything to S as in the diagram above:

$$\begin{array}{ccc} f^*T_U & \longrightarrow & T_U \\ \downarrow & & \downarrow \\ S \times_{\mathfrak{X}} U & \longrightarrow & U \\ \downarrow & & \downarrow \\ S & \xrightarrow[f]{} & \mathfrak{X}. \end{array}$$

Then we note that the relative tangent bundle $T_{(S \times_{\mathfrak{X}} X)/S}$ is a well-defined scheme; and, morally, it ought to equal $f^*(T_{U/\mathfrak{X}})$. In fact, if we restrict our S's to some collection $S_i \to \mathfrak{X}$ forming an fpqc (or fppf, or smooth) cover, then we find that the collection of vector bundles $T_{(S_i \times_{\mathfrak{X}} U)/S_i}$ over each $S_i \times_{\mathfrak{X}} U$ satisfy the conditions for (quasicoherent) descent; thus, there is indeed a vector bundle $T_{U/\mathfrak{X}}$ whose pullback to each S_i is $T_{(S_i \times_{\mathfrak{X}} U)/S_i}$. We then readily see that this pulls back to $T_{(S \times_{\mathfrak{X}} U)/S}$ for all S (up to canonical isomorphism). (Moreover, note that, fixing $S \to \mathfrak{X}$ and varying the U, the descent argument sketched above extends to showing that $T_{S/\mathfrak{X}}$ is well-defined as a vector bundle over S for any test-scheme S.)

So now we have $T_{U/\mathfrak{X}}$ for each $U \to \mathfrak{X}$ in the smooth site over $\mathfrak{X}$.

Definition 2.5. *The tangent stack of $\mathfrak{X}$ as the functor associating to each U the two-term complex*

$$T_{U/\mathfrak{X}} \to T_U,$$

of quasicoherent sheaves over U concentrated in degrees -1 and 0.

One checks that, for compositions of pullbacks $W \xrightarrow{\varphi} V \xrightarrow{\psi} U \to \mathfrak{X}$, the canonical map from the two-term complex $\varphi^*\psi^*(T_{U/\mathfrak{X}} \to T_U)$ to $(\psi \circ \varphi)^*(T_{U/\mathfrak{X}} \to T_U)$ defines a quasi-isomorphism. These amount to compatibility requirements; they show that our definition, provided for each U in the site, is indeed defining a structure "over $\mathfrak{X}$."

Notice that we can extend this two-term complex to all test-schemes $S \to \mathfrak{X}$, via the two-term complex $T_{S/\mathfrak{X}} \to T_S$.

Definition 2.6. *Let $S \to \mathfrak{X}$ be any smooth S-point of $\mathfrak{X}$. Then define dim$(\mathfrak{X}) = $ rk$(T_S) -$ rk$(T_{S/\mathfrak{X}})$.*

This number is independent of $S \to \mathfrak{X}$ (since $\mathfrak{X}$ is smooth).

Now, it would appear that we have offered two competing notions of tangent space: one defined in terms of functors valued in (Picard) groupoids, and one defined as functors valued in two-term complexes. Indeed, it is not even clear that the two-term complex definition gives us a stack: it is, very technically, a functor valued in sections of the fibered category QCoh/Sch, not in Gpd.

The relationship between the two presentations is given by considering a two term complex as a kind of quotient groupoid h^1/h^0 (to adopt the terminology of [BF]).

To any two-term complex $\cdots \to C^{-1} \to C^0 \to 0 \to \cdots$, we may consider the action groupoid $[C^0] \,/\, [C^{-1}]$ where C^{-1} acts on C^0 via the boundary map d. To be very explicit, the objects are elements of C^0, while the morphisms $x \xrightarrow{\sim} y$ in $[C^0] \,/\, [C^{-1}]$ are given by elements of $v \in C^{-1}$ such that $dv = y - x$.

In our case, the stack we have defined assigns, to every S-point $f : S \to \mathfrak{X}$, the groupoid $[T_S] / [T_{S/\mathfrak{X}}]$, viewed as relative (abelian) group schemes over S, with the action given by the S-morphism $T_{S/\mathfrak{X}} \to T_S$. One can check that this is equivalent to the definition of $\mathfrak{T}_f$ provided above.

Manifestly the cohomology of the double complex will have geometric significance.

Definition 2.7. *The (naive) sheaf of tangent vectors to $f : S \to \mathfrak{X}$ is given by the quasicoherent sheaf $H^0(T_{S/\mathfrak{X}} \to T_S)$ on S. The naive tangent stack on $\mathfrak{X}$ is given by the association $S \mapsto H^0(T_{S/\mathfrak{X}})$.*

This, of course, is simply the isomorphism classes of objects of $\mathfrak{T}_f$. Notice that it is in general *not* a vector bundle over S, even in the case where the stack $\mathfrak{X}$ is smooth. We will see the phenomenon in many examples. This sheaf is what we would get if we took only the cokernel of the two-term complex in analogy with the Euler exact sequence – forgetting the extra data given by the full double complex.

Definition 2.8. *The inertia sheaf of $f : S \to \mathfrak{X}$ is given by the quasicoherent sheaf $H^{-1}(T_{S/\mathfrak{X}} \to T_S)$ on S. The association of sheaves $S \mapsto H^{-1}(T_{S/\mathfrak{X}} \to T_S)$ gives us the inertia stack of $\mathfrak{X}$.*

This inertia sheaf of f is, of course, equivalent to automorphisms of the zero object of $\mathfrak{T}_f$. They tell us about "infinitesimal automorphisms" of the S-point f.

Example 2.9. *Let us consider the basic example of an affine algebraic group G acting on a smooth scheme X. Consider the quotient stack $\mathfrak{X} = X/G$. What is $T\mathfrak{X}$? We have that $X \to X/G$ is a smooth atlas for $\mathfrak{X}$.*

The two-term complex is given by

$$\mathfrak{g} \times \mathcal{O}_X \xrightarrow{act} T_X,$$

where act is the infinitesimal action, i.e differentiating the G action on X. Consider a point $x \in X$. Then we see that $T_x(X/G)$, as a Picard groupoid, has objects given by the elements of the vector space T_xX, and morphisms between two objects $v, w \in T_x(X/G)$ given by an element $\phi \in \mathfrak{g} \times X$ such that $act(\phi) = v - w$. Forgetting the groupoid structure, we see the naive tangent space at $x \in X$ is given by

$$ob(T_x(X/G))/Isom = coker(act) = T_xX/\mathfrak{g}_x = \text{Normal space to } G.x$$

Also,

$$Z := Ker(act) = \{(x, a) \in X \times \mathfrak{g} : a \in Lie(Stab_G(x))\},$$

which we see is the infinitesimal perturbations of x by G that leave x "inert." Hence the name "inertia stack."

When G acts freely on X, the infinitesimal action map is injective. In this case, the naive sheaf of tangent vectors is a vector bundle, the inertia stack is trivial, and the Picard groupoid is equivalent to a set. We see that $\dim T_x(X/G)^{naive} = \dim T_xX - \dim \mathfrak{g}$.

On the other extreme, when the action of G on X is trivial, the action map is 0, and each vector in T_xX is in its own isomorphism class. Here too the naive tangent sheaf is a vector bundle, isomorphic to T_X, while the inertia stack is the bundle $\mathfrak{g} \times X$.

The inertia sheaf and naive tangent sheaf fail to be vector bundles when the rank of the action map changes; i.e., when the stabilizer of a point under the G-action changes dimension.

2.4. **Cotangent Stack.**

Definition 2.10. *Given an S-point of stack $\mathfrak{X}$, we define the cotangent stack*

$$T^*_{S\to\mathfrak{X}}\mathfrak{X} = Spec_S((Sym_S(T_S))/T_{S/\mathfrak{X}}).$$

Here, $T_{S/\mathfrak{X}}$ refers to the ideal generated by the subspace $T_{S/\mathfrak{X}}$ in $\mathrm{Sym}_S(T_S)$.

Example 2.11. *Let us return to the example of $\mathfrak{X} = X/G$. The cotangent stack $T^*(X/G)$ comes from the Hamiltonian reduction of T^*_X along the moment map $\mu : T^*_X \to \mathfrak{g}^*$. The moment map is induced by dualizing $act^t : T^*_X \to \mathfrak{g}^* \times X$ and then projecting to $\mathfrak{g}^*$. So*

$$T^*(X/G) = \mu^{-1}(0)/G,$$

which is understood as the stacky quotient. The objects are parameterized by $Ker(act^*)$.

We make two general remarks, due to Beilinson-Drinfeld.

(1) The cotangent stack is almost never smooth. This is related to the concept that "doubling the quiver" (which corresponds to taknig the cotangent bundle) should "double" the homological dimension; as we have discussed, homological dimension greater than 1 means we will lose smoothness.
(2) We have, in general $\dim(T^*(\mathfrak{X})) \geq 2\dim(\mathfrak{X})$ for $\mathfrak{X} = X/G$ and even $\mathrm{Bun}_G(\Sigma)$ for some Σ and G. When there is equality, we call the stack $\mathfrak{X}$ good. We will sketch an argument (due to Beilinson-Drinfeld) for why $\mathrm{Bun}_G(\Sigma)$ is good when G a semisimple group and Σ a smooth projective curve of genus $g > 1$.

There is another equivalent characterization for a stack $\mathfrak{X}$ to be good. Let $f : S \to \mathfrak{X}$ be a smooth S-point and consider the tangent complex on $T^\bullet_{\mathfrak{X}} := (T_{S/\mathfrak{X}} \to T_S)$. Then $\mathfrak{X}$ is *good* if the complex $\mathrm{Sym}(T^\bullet_{\mathfrak{X}})$ has cohomology only in degree 0. Notice that the degree 0 cohomology is precisely $T^*_{S\to\mathfrak{X}}\mathfrak{X} = \mathrm{Spec}((\mathrm{Sym}_S(T_S))/T_{S/\mathfrak{X}})$.

Example 2.12. *Let us consider a semisimple group G acting on its Lie algebra $X = \mathfrak{g}$ via the adjoint action. Then $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \cong \mathfrak{g} \times \mathfrak{g}$ (via the Killing form) and the moment map is just*

$$\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (\xi_1, \xi_2) \mapsto [\xi_1, \xi_2].$$

Thus $\pi^{-1}(0) = \mathcal{N}$ is the variety of commuting matrices. But if we take a generic, regular semisimple element, $\xi \in \mathfrak{g}$, then its centralizer is the torus. So $\dim \mu^{-1}(\xi) = rank(\mathfrak{g})$ and we conclude

$$\dim \mu^{-1}(0) = \dim(\mathfrak{g}) + rk(\mathfrak{g}).$$

This shows $\dim T^*(\mathfrak{g}/G) = \dim \mathfrak{g} + rk(\mathfrak{g}) - \dim(G) = rk(\mathfrak{g})$, which is clearly not equal to $2\dim(\mathfrak{g}/G) = 0$. Hence $\mathfrak{g}/G$ is not good.

Example 2.13. *$BG = */G$ is not good. This is because Hamiltonian reduction in this case implies* $\dim(T^*(*/G)) = \dim(*/G) = -\dim(G) \neq 2\dim(*/G)$.

Example 2.14. *$Bun_{GL_n}(\Sigma)$ is not good. Identifying Bun_{GL_n} with vector bundles, then we can see that $\mathbb{G}_m$ lies in the automorphism group of any vector bundle – even generically. This shows, for example, semi-simplicity of G is needed, if we want $Bun_G(\Sigma)$ to be good. We could "correct" this issue with GL_n by killing automorphisms and considering $Bun_{PGL_n}(\Sigma)$, which is good (for genus > 1). This is essentially saying that $Bun_{GL_n}(\Sigma)$ is a so-called "gerbe" – more specifically, a $\mathbb{G}_m$-gerbe – over $Bun_{PGL_n}(\Sigma)$.*

2.5. **Tangent space to $\mathbf{Bun}_G(\Sigma)$.** Let us now compute the tangent stack for $\mathrm{Bun}_G(\Sigma)$. The main idea is we first compute the tangent space for the moduli stack of vector bundles, and then use Tannakian formalism to reduce to this case. This offers an example of an application of the "sheaves of categories" method described in the beginning of the lecture.

Let $\mathfrak{X} = \mathcal{M}(\mathrm{QCoh}(\Sigma))$ with Σ a smooth projective curve. Let $\mathcal{E}$ be a vector bundle over Σ. Then, as described above, we have:

$$T_{\mathcal{E}}\mathcal{M} = \{k_\varepsilon \otimes A - \text{modules } \tilde{\mathcal{E}} \text{ flat over } k_\varepsilon \text{ such that } \tilde{M}/\varepsilon\tilde{\mathcal{E}} = M\}.$$

An $\mathcal{O}_\Sigma \otimes k_\varepsilon$ module $\tilde{\mathcal{E}}$ corresponds to an extension

$$0 \to \varepsilon\tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \to \mathcal{E} \to 0$$

of A modules. And $\tilde{\mathcal{E}}$ being flat as k_ε-module just means $\mathcal{E} \xrightarrow{\varepsilon} \varepsilon\tilde{\mathcal{E}}$ is an isomorphism. Thus

$$\mathrm{Ob}(T_{\mathcal{E}}\mathfrak{X})/\mathrm{Isom} = \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}).$$

The addition operation in Ext^1 is just the Baer sum of short exact sequences, and the scaling is given as follows: for $\lambda \in k$, scaling $0 \to \mathcal{E} \to \tilde{\mathcal{E}} \xrightarrow{f} \mathcal{E} \to 0$ by λ means replacing f by λf.

To describe the morphisms in the groupoid structure of $T_{\mathcal{E}}(\mathfrak{X})$, it suffices to just give Aut(zero section) by our discussion with the Picard groupoid. In this case,

$$\mathrm{Aut}(\text{zero object}) = \mathrm{Isom}(\mathcal{E}, \mathcal{E}) \subset \mathrm{Ext}^0(\mathcal{E}, \mathcal{E}).$$

Now, observe that:

$$T_{\mathcal{E}}(\mathrm{Bun}_n) = \mathrm{Ext}^1_{\mathcal{O}_\Sigma}(\mathcal{E}, \mathcal{E}) = \mathrm{Ext}^1(\mathcal{O}_\Sigma, \mathcal{E}^* \otimes \mathcal{E}) = H^1(\Sigma, \mathrm{End}(\mathcal{E})) = H^1(\Sigma, \mathfrak{gl}_n).$$

Thus

$$T_{\mathcal{E}}(\mathrm{Bun}\,\mathrm{GL}_n) = H^1(\Sigma, \mathfrak{gl}_n).$$

We will find that this generalizes to $\mathrm{Bun}\,G$ for all affine algebraic G.

Let Σ be a smooth projective curve, G an affine algebraic group, and P a principal G-bundle over Σ. Given V a G-representation, we may associate the vector bundle $V_P := P \overset{G}{\times} V$. In particular, for $V = \mathfrak{g}$ the adjoint representation, we call $\mathfrak{g}_P$ the associated adjoint bundle.

Theorem 2.1.

$$T_P(\mathrm{Bun}_G(\Sigma)) \cong H^1(\Sigma, \mathfrak{g}_P).$$

Proof. The first thing to do is to write Bun_G as a more "linear" sheaf of categories. For $G = \mathrm{GL}_n$ this comes easily from the correspondence between GL_n-torsors and vector bundles. For general G This is accomplished by the"Tannakian formalism":

Proposition 2.15. *Let G be an affine algebraic group. A G-bundle on Σ is the same as a monoidal exact functor Rep(G) $\to$ Vect(Σ), where Rep(G) is the tensor category of finite-dimensional representations of G, and Vect(Σ) is the tensor category of vector bundles on Σ.*

Sketch of proof of Prop. 2.15: The construction in one direction is $P \mapsto (V \mapsto V_P)$. It is easy to check this is monoidal and exact. On the other hand, given a functor F, we consider the (infinite-dimensional) regular representation of G, $\mathrm{Reg}(G)$ consisting of regular functions on G. Since F is monoidal, we may apply F to the multiplication map $\mathrm{Reg}(G) \otimes \mathrm{Reg}(G) \to \mathrm{Reg}(G)$ to get a (commutative) algebra structure on $F(\mathrm{Reg}(G))$. Then we take $P := \mathrm{Spec}_\Sigma(F(\mathrm{Reg}G))$.

Let use this formalism to describe sections of $\mathfrak{g}_P$. The point is that such sections "act" as a kind of derivation on all the vector bundles associated to P. More precisely, giving a section a of $\mathfrak{g}_P$ is equivalent to giving a collection of $a_V \in \mathrm{End}(V_P)$, for each $V \in \mathrm{Rep}(G)$, such that

$$a_{V\otimes W} = a_V \otimes \mathrm{Id}_{W_P} + \mathrm{Id}_{V_p} \otimes a_W. \tag{1}$$

To see this, note that a section a is equivalent to a G-equivariant map $\alpha : P \to \mathfrak{g}$ (with G acting on $\mathfrak{g}$ via the adjoint). Let $\rho : G \to GL(V)$ be the representation V. We will denote by $[p, v] \in V \overset{G}{\times} P$ the point lying below $(p, v) \in P \times V$ under the quotient map. Then we define:

$$a \cdot [p, v] = [p, d\rho\,(\alpha(p)) \cdot v],$$

where $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is the derivative of the representation map $\rho : G \to GL(V)$ at the identity of G. This gives us an element of $\mathrm{End}(V_P)$ for all G-representations V; the identity (1) comes from the Leibniz rule for the action of $\mathfrak{g}$ on the tensor product of representations. This gives us a Tannakian description of $H^0(\Sigma, \mathfrak{g}_P)$.

In fact, notice that the above argument gives us more than just a map of global sections: it gives us a bona fide map of sheaves over Σ, $\phi_V : \mathfrak{g}_P \to \mathrm{End}(V_P)$, for very G-representation V. Moreover, the maps ϕ_V satisfy the differential monoidal property $\phi_{V\otimes W} = \phi_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \phi_W$.

Thus we can now give a Tannakian description for $H^1(\Sigma, \mathfrak{g}_P)$, too: if $e \in H^1(\Sigma, \mathfrak{g}_p)$, we consider the collection of corresponding elements of $e_V \in H^1(\Sigma, \mathrm{End}(V_P))$ for each representation V induced by this map $\phi_V : \mathfrak{g}_P \to \mathrm{End}(V_P)$ of sheaves over Σ. Now, as discussed above, $H^1(\Sigma, \mathrm{End}(V_P))$ corresponds to equivalence classes of extensions of V_P by itself. Thus we find that:

$$\begin{aligned}H^1(\Sigma, \mathfrak{g}_P) \Leftrightarrow \{&\text{Functor from RepG to equivalence classes of SES} \\ &V \mapsto e_V : (0 \to V_P \to \tilde{V}_P \to V_P \to 0) \text{ such that} \\ &e_{V\otimes W} = e_V \otimes W_P + V_P \otimes e_W\},\end{aligned}$$

where "+" means Baer Sum.

Now we can prove Theorem 2.1, by checking that $T_P(\mathrm{Bun}_G)$ gives us the same data. We have the following equivalences, coming from definition of tangent space and Tannakian formalism:

$$\begin{aligned}\mathcal{P} \in T_P(\mathrm{Bun}_G)(S) &\Leftrightarrow \{\text{map } S \xrightarrow{\mathcal{P}} \mathrm{Bun}_G \text{ which lift to } D \to \mathrm{Bun}_G\} \\ &\Leftrightarrow \{\tilde{\mathcal{P}} \text{ a G-torsor over } \Sigma \times D \times S \text{ such that } \tilde{\mathcal{P}} \times_D \mathrm{Spec}(k) = \mathcal{P}\} \\ &\Leftrightarrow \text{Exact functor from RepG to } \mathrm{Vect}(\Sigma \times D \times S) : V \mapsto V_{\tilde{\mathcal{P}}} \\ &\quad\ \text{such that } \tilde{\mathcal{P}} \times_D \mathrm{Spec}(k) = \mathcal{P} \text{ and } V_{\tilde{\mathcal{P}}} \otimes W_{\tilde{\mathcal{P}}} = (V \otimes W)_{\tilde{\mathcal{P}}}\end{aligned}$$

For simplicity, we let $S = \mathrm{Spec}(k) = *$ so $\mathcal{P} = P$. Now, write $\pi : \Sigma \times D \to \Sigma$ for the projection. Associated to $V_{\tilde{\mathcal{P}}}$ we have the exact sequence

$$0 \to \mathrm{Ker} \to \pi_*(V_{\tilde{\mathcal{P}}}) \xrightarrow{\otimes_{k_\epsilon} k} V_P \to 0.$$

Let $\tilde{V}_P := \pi_*(V_{\tilde{\mathcal{P}}})$ We can see $\mathrm{Ker} = V_P$. The exactness of the functor means that given V_P, W_P, (which are locally free sheaves on Σ), we have short exact sequences (of sheaves on Σ)

$$\begin{aligned}0 \to V_P \otimes W_P \to \tilde{V}_P \otimes W_P \to V_P \otimes W_P \to 0 \\ 0 \to V_P \otimes W_P \to \tilde{W}_P \otimes V_P \to V_P \otimes W_P \to 0\end{aligned}$$

The Baer sum is

$$(\tilde{V}_P \otimes W_P \oplus V_P \otimes \tilde{W}_P)/(\epsilon v_p \otimes w_p, -v_p \otimes \epsilon w_p)$$

And this is naturally identified with $\tilde{V}_P \otimes \tilde{W}_P$. So we conclude

$$\begin{aligned}P \in T_P(\mathrm{Bun}_G)(*) \Leftrightarrow \{&\text{Functor from RepG to SES } V \mapsto e_V : (0 \to V_P \to \tilde{V}_P \to V_P \to 0) \\ &\text{such that } e_{V\otimes W} = e_V \otimes W_P + V_P \otimes e_W\} \\ \Leftrightarrow\, & e \in H^1(\Sigma, \mathfrak{g}_P),\end{aligned}$$

where + means Baer sum. This is precisely $H^1(\Sigma, \mathfrak{g}_P)$ as interpreted above. □

Notice that we have really only established a bijection of sets here; between $H^1(\Sigma, \mathfrak{g}_P)$ and the isomorphism classes of objects in $T_P(\mathrm{Bun}_G)(*)$. We can upgrade this bijection of sets to an equivalence of groupoids: we simply keep track of the automorphisms of each extension class in $H^1(\Sigma, \mathfrak{g}_P)$ and remember the automorphisms of the tangent space $T_P(\mathrm{Bun}_G)$. In fact, following the arguments given above, we see that the automorphisms of $T_P(\mathrm{Bun}G)$ at P are precisely $H^0(\Sigma, \mathfrak{g}_P)$. This readily upgrades the proof given above to an isomorphism of stacks.

3. Higgs Bundles

By dualizing, we can describe the cotangent stack of $\mathrm{Bun}_G := \mathrm{Bun}_G(\Sigma)$ via

$$T^*_P(\mathrm{Bun}_G) = (T_P(\mathrm{Bun}_G))^* = H^1(\Sigma, \mathfrak{g}_P)^* = H^0(\Sigma, K_\Sigma \otimes \mathfrak{g}^*_P).$$

We used Serre duality in last equality; $K_\Sigma = \Omega^1_\Sigma$ means the dualizing sheaf. Since $\mathfrak{g}$ is semisimple, we will freely identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form; thus we map identify $\mathfrak{g}_P$ with $\mathfrak{g}^*_P$. An section $\phi : \Sigma \to K_\Sigma \otimes \mathfrak{g}^*_P$ is called a Higgs field. Denote the stack Higgs to consist of pairs (P, ϕ) where P is a G-bundle and ϕ is a corresponding Higgs field. Theorem 2.1 proves, at least point-wise (and everything works over families S as well), that

$$T^*\mathrm{Bun}_G = \mathrm{Higgs}.$$

Let us now sketch an argument that Bun_G is good, i.e $\dim T^*\mathrm{Bun}_G = 2\dim \mathrm{Bun}_G$.

Consider the categorical (or "GIT") quotient $\mathfrak{g} /\!\!/ G := \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g})^G)$, which, by Chevalley's Restriction Theorem is isomorphic to the affine space $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{h})^W)$. There is a canonical map from stacky quotient of adjoint G action on $\mathfrak{g}$ to the categorical quotient:

$$\tilde{\pi} : \mathfrak{g}/G \to \mathfrak{g} /\!\!/ G.$$

We would like to consider the induced map $\mathrm{Maps}(\Sigma, \mathfrak{g}/G) \to \mathrm{Maps}(\Sigma, \mathfrak{g} /\!\!/ G)$. But $\mathfrak{g} /\!\!/ G$ is affine and Σ is a projective curve, so $\mathrm{Maps}_\Sigma(\Sigma, \mathfrak{g} /\!\!/ G) = 0$. Thus, we must twist $\tilde{\pi}$ by the canonical sheaf K_Σ in order to get an interesting map, as we shall now explain.

The $\mathbf{C}^*$-action on $\mathfrak{g}$ (viewed as the scalar action on the vector space) induces a $\mathbf{C}^*$ action on $\mathfrak{g} /\!\!/ G$. Now, we will view the canonical sheaf K_Σ (which is a line bundle on Σ) as the corresponding $\mathbf{C}^*$-torsor over Σ. Thus we can construct the associated bundle $K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g} /\!\!/ G$, and we define:

$$\mathrm{Hitch} := \Gamma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g} /\!\!/ G).$$

Later we will see this affine space has dimension $(g-1)\dim G$.

We would like to consider the "stack of sections" of the various structures living over Σ. So, say we have a stack $\mathfrak{X}$ with a fixed morphism $\phi : \mathfrak{X} \to \Sigma$. Given a test scheme S, we define

$$\mathrm{Map}_\Sigma(\Sigma, \mathfrak{X})(S) := \{F : \Sigma \times S \to \mathfrak{X} \text{ such that } \phi \circ F = \mathrm{pr}_1 : \Sigma \times S \to \Sigma\}$$

and call it the "section stack of $\mathfrak{X} \to \Sigma$". This is a closed substack of the mapping stack $\mathrm{Maps}(\Sigma, \mathfrak{X})$.

Now we can consider the map of section stacks:

$$\pi : \mathrm{Maps}_\Sigma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g}/G) \to \mathrm{Maps}_\Sigma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g} /\!\!/ G).$$

Observe that $\mathrm{Maps}_\Sigma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g}/G)$ is $T^*(\mathrm{Bun}_G) = \mathrm{Higgs}$. Indeed, if we let P be a G-torsor over Bun_G, then, by Serre Duality we have $T^*_P(\mathrm{Bun}_G) = (H^1(\Sigma, \mathfrak{g}_P))^* = H^0(\Sigma, K_\Sigma \times_\Sigma \mathfrak{g}_P)$. On the other hand, the fiber of $\mathrm{Maps}_\Sigma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g}/G)$ over $P \in \mathrm{Maps}_\Sigma(\Sigma, */G)$ is $H^0(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g})$. Meanwhile, we note that $\mathrm{Maps}_\Sigma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g} /\!\!/ G) = \mathrm{Hitch}$. So π is really a map of stacks:

$$\pi : \mathrm{Higgs} \to \mathrm{Hitch}.$$

This is called the Hitchin map.

Alternatively, we could have constructed the Hitchin map (pointwise) as follows. The quotient map $\mathfrak{g}^* \to \mathfrak{g}^* /\!\!/ G$ induces a map

$$\mathfrak{g}^*_P = \mathfrak{g}^* \overset{G}{\times} P \to (\mathfrak{g}^* /\!\!/ G) \times \Sigma.$$

Then, twisting this with K_Σ and taking global sections induces

$$\pi_P : \Gamma(\Sigma, \mathfrak{g}_P \times K_\Sigma) = T^*_P(\mathrm{Bun}_G) \to \Gamma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g}^* /\!\!/ G) = \mathrm{Hitch}.$$

Observe that the 0 section of $\mathrm{Hitch} = \Gamma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g}^* /\!\!/ G)$ is a distinguished point. The fiber of $\mathrm{Maps}_\Sigma(\Sigma, K_\Sigma \times_{\mathbf{C}^*} \mathfrak{g}/G)$ lying over it is a closed substack of Higgs.

Definition 3.1. *We call the zero fiber $\mathcal{N} := \pi^{-1}(0)$ the global nilpotent cone.*

Example 3.2. *Let us compute the Hitchin map for the basic example of $G = GL_n$. The standard reference is [H87; N10]. Let $\mathfrak{g} = \mathfrak{gl}_n = End(V)$. It is a classical fact $(Sym\,\mathfrak{g})^G = \mathbf{C}[e_1, \ldots, e_n]$, where $e_i : G \mapsto \mathbf{C}$ is $e_i(g) = Trace(\wedge^i g : \wedge(\mathbf{C}^n) \to \wedge(\mathbf{C}^n))$. Thus $\mathfrak{g} /\!\!/ G = \mathbf{A}^n$ has basis e_i, and the $\lambda \in \mathbf{C}^*$-action is just $\lambda.e_i(g) = e_i(\lambda g) = \lambda^i e_i(g)$. This allows us to decompose*

$$Hitch = \Gamma(\Sigma, K \oplus K^{\otimes 2} \oplus \cdots \oplus K^{\otimes n}) = \bigoplus_{i=1}^n \Gamma(\Sigma, \mathcal{K}^{\otimes i})$$

(where, as before, $K = \Omega^1_\Sigma$). If P is a GL_n-torsor, a Higgs field ϕ associated to P is equivalent to a linear map $\phi : V_P \to K \otimes V_P$. So considering the coefficients of $det(x - \phi)$ as a polynomial in x, we see the i^{th} degree coefficient, denoted $e_i(\phi)$, is in $\Gamma(\Sigma, K^{\otimes i})$. This produces a map

$$\pi(\phi) = (e_1(\phi), e_2(\phi), \ldots, e_n(\phi)) \in \bigoplus_{i=1}^n \Gamma(\Sigma, K^{\otimes i}) = Hitch.$$

With this setup, it is clear π is $\mathbf{C}^*$-equivariant, and that the global nilpotent cone, $\mathcal{N} = \pi^{-1}(0)$, consists of "nilpotent" matrices ϕ.

In general, there is a similar decomposition

$$\mathrm{Hitch} = \bigoplus_{i=1}^{n} \Gamma(\Sigma, K^{\otimes d_i})$$

where d_i are the exponents of a semisimple Lie group G. Recall, $\mathrm{Sym}(\mathfrak{g})^G$ is a free polynomial algebra, so we may take homogeneous generators. Then the exponents of $\mathfrak{g}$ are the degrees of these generators.

Now, we compute $\dim \mathrm{Hitch}$. Suppose Σ is genus $g > 1$, which ensures that the canonical sheaf K_Σ is ample. Thus we may use Riemann-Roch to compute

$$\dim H^0(\Sigma, K^{\otimes d_i}) = (2d_i - 1)(g-1).$$

Now, there is a classical formula $\sum_{i=1}^{\mathrm{rk}(\mathfrak{g})}(2d_i - 1) = \mathrm{rk}(\mathfrak{g})$ (e.g [CG97, Cor 6.7.22]). This computes

$$\dim \mathrm{Hitch} = (g-1)\dim(G).$$

Next we compute $\dim \mathrm{Bun}_G$ (following [R09]). The tangent complex of Bun_G may be identified (in particular, using Theorem 2.1) with $(H^0(\mathfrak{g}_P) \to H^1(\mathfrak{g}_P))$ (in degree -1 and 0 with the 0 map between them), and the dimension of Bun_G is (defined as) the Euler characteristic of this, $-\chi(\mathfrak{g}_P)$. Now, G is reductive so $\mathfrak{g}_P \simeq \mathfrak{g}_P^*$, hence $\deg(\mathfrak{g}_P) = \deg(\mathfrak{g}_P^*) = -\deg(\mathfrak{g}_P)$. Thus $\deg(\mathfrak{g}_P) = 0$. Now, Riemann-Roch tells us

$$\chi(\mathfrak{g}_P) = \deg(\mathfrak{g}_P) + \mathrm{rk}(\mathfrak{g}_P)(1-g) = \dim(G)(1-g).$$

Thus $\dim \mathrm{Hitch} = \dim \mathrm{Bun}_G$. We comment that this formula also suggests why take Σ to have genus $g > 1$.

Lets return to the Hitchin map $\pi : \mathrm{Higgs} \to \mathrm{Hitch}$. Observe $\mathbf{G}_m$ acts on Higgs via $\lambda.(P, \phi) \mapsto (P, \lambda\phi)$, and $\mathbf{G}_m$ acts on Hitch by scaling. Using the explicit construction in Example 3.1 of π, for general semisimple G, we conclude π is $\mathbf{G}_m$-equivariant.

Now, Higgs is the contangent bundle of $\mathrm{Bun}(G)$; thus it possesses a canonical symplectic structure. Now, it turns out that $\mathcal{N}$ is isotropic. This result was first proven for GL_n in [L88] and in general in [F93]. There is an elementary proof in [G00]. On the other hand, we already know $\dim T^*\mathrm{Bun}_G \geq 2\dim \mathrm{Bun}_G$ holds in general. Hence $\mathcal{N}$ is Lagrangian. Furthermore, we have $\dim \pi^{-1}(h) \leq \dim \pi^{-1}(0)$ for all $h \in \mathrm{Hitch}$, since $\pi^{-1}(0)$ is the special fiber, and $0 \in \mathrm{Hitch}$ is in the closure of any $\mathbf{C}^*$-orbit. This implies that all components of all fibers of π are equidimensional. Hence $\dim T^*\mathrm{Bun}_G = 2\dim \mathrm{Higgs} = 2\dim \mathrm{Bun}_G$. Thus Bun$G$ is good. (This argument also implies $T^*\mathrm{Bun}_G$ is a complete intersection.)

We end with a **conjecture** by Hitchin ([H10] Question 2, page 2): it is known that functions in $\mathcal{O}_{T^*\mathrm{Bun}_G}$ Poisson commute. The conjecture is that the Schouten bracket on $\wedge^\bullet T_{\mathrm{Bun}_G}$ is also zero.

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