Manifolds, Higher Categories and Topological Field Theories

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1. "Topological (Quantum) Field Theories", i.e. tensor functors

$$Z : Bord_n \to \mathcal{C}$$

2. Sheaves on Man_n, the category of n-dimensional manifolds (with morphisms open embeddings)

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For a symmetric monoidal (∞, n)-category $\mathcal{C}$, an n-dimensional $\mathcal{C}$-valued TQFT is a symmetric monoidal functor

$$Z : Bord_n \to \mathcal{C}.$$

Theorem ("Cobordism Hypotheis" (Lurie))

Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then there is an equivalence

$$\{\mathcal{C}\text{-valued TQFTs}\} \simeq \{\text{fully dualizable objects in } \mathcal{C}\}$$

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1. *This theorem indicates a intimate connection between the theory of manifolds and higher categories. It was inspirational in our work, but we don't make use of it.*
2. *In practice, it is difficult to produce examples of TQFT's using this theorem.*

Let Man_n be the (quasi-)category of framed n-dimensional manifolds and framed embeddings. It has a natural Grothendieck topology. Thus, we can consider local invariants of n-manifolds as sheaves on Man_n.

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Idea: we can improve matters by considering additional structure on manifolds; namely, the notion of transversality.

Definition

Let $nMan^{\pitchfork}$ *denote the* $(\infty, 1)$*-category whose objects are pairs*

$$(M, S \subset M)$$

where M *is a framed n-dimensional manifold and* S *is a compact subcomplex of* M *stratified by submanifolds, such that* M *admits a cover by basic opens, each of which has a non-empty intersection with* S*.*

A morphism $(M, S) \to (N, T)$ *is a framed embedding* $f : M \hookrightarrow N$ *together with a path* γ *from* $f(S)$ *to* T *in the space of stratified submanifolds of* N*.*

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Let $nMan_0^{\pitchfork} \subset nMan^{\pitchfork}$ *be the subcategory consisting of morphisms such that the corresponding path of stratified submanifolds does not decrease the number of components of strata.*

Definition

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$$\Psi : nMan^{\pitchfork} \to Spaces$$

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Gromov's h-principle gives a geometric way of producing sheaves with transversality. Namely, a sheaf valued in the (ordinary) category of topological spaces

$$\Phi : nMan \to Top$$

together with a notion of transversality which satisfies the h-principle produces a sheaf with transversality.

The requirement for objects on $nMan^{\pitchfork}$ *to have covers by basics intersecting the subcomplex is related to the failure of a sheaf satisfying the h-principle to be a homotopy sheaf.*

There is a natural functor

$$oblv : \underline{n}Man^{\pitchfork} \to nMan^{\pitchfork}$$

Given a sheaf with transversality on *nMan*

$$\Psi : nMan^{\pitchfork} \to Spaces,$$

we obtain a sheaf with transversality on $\underline{n}Man$, by pulling back along *oblv*.

By the theorem David described, this gives a functor

$$\rho : \{\pitchfork\text{-sheaves on } nMan\} \to \{(\infty, n)\text{-categories}\}.$$

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Question

How close is ρ to being an equivalence?

Let $\mathcal{C}$ be a 2-category. A morphism

$$f : x \to y$$

has a left ajoint $f^L : y \to x$ if there are 2-morphisms

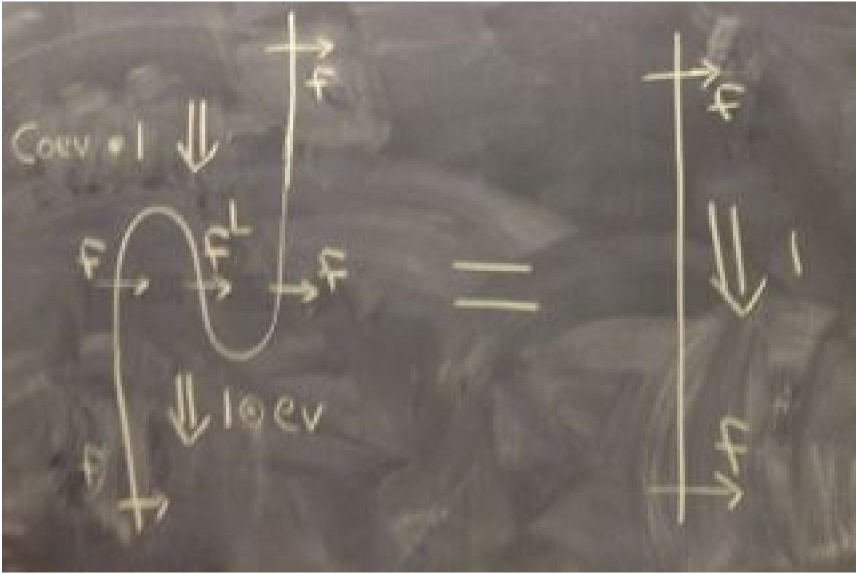
$$ev : f^L \circ f \to id_x \quad \text{and} \quad coev : id_y \to f \circ f^L$$

satisfying Zorro's lemma; i.e., the composite

$$f \xrightarrow{coev \otimes id} f \circ f^L \circ f \xrightarrow{id \otimes ev} f$$

is equal to the identity.

Similarly for right adjoints. In the same way, we have the notion of left and right adjoints for k-morphisms in an n-category (where $0 < k < n$).


coev ⊙ 1
1 ⊙ ev
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An (∞, n)-category with adjoints is an (∞, n)-category $\mathcal{C}$ such that every k-morphism has left and right adjoints for $0 < k < n$.

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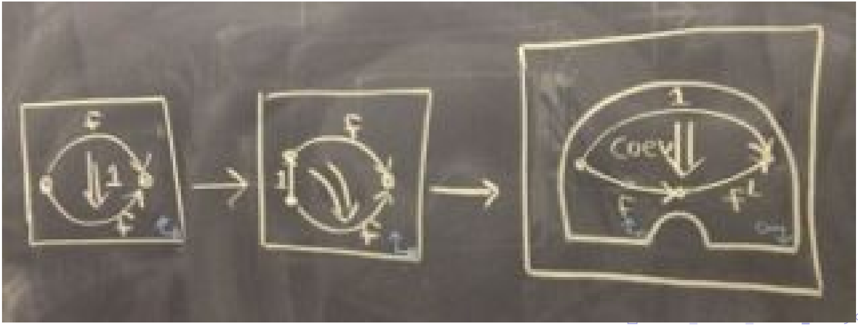

Theorem 80 (Ayala, R.)

The functor

$$\rho : \{\pitchfork\text{ -sheaves on nMan}\} \to \{(\textit{infty}, n)\text{-categories with adjoints}\}$$

is an equivalence.

Manifolds without Subcomplexes

There is a natural forgetful functor $nMan^{\pitchfork} \to nMan$
Given a sheaf with transversality

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we obtain the functor

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by left Kan extension along the forgetful functor. In this way, we obtain from Ψ invariants of manifolds.

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- When Ψ is an E_n-algebra, $\int_{\Psi}$ is given by topological chiral homology defined by Lurie.
- Should be related to blob homology defined by Morrison and Walker.

For a k-manifold M^k, let

$$\int_M^{cat} \Psi$$

be the sheaf with transversality on $(n-k)Man$ given by

$$\int_M^{cat} \Psi(N, T) = \Psi(N \times M, T \times M).$$

Proposition

There is an isomorphism

$$\int_M \Psi \simeq B(\rho(\int_M^{cat} \Psi)).$$

Theorem (Ayala, R.)

The functors $\int^{cat} \Psi$ *give an n-dimensional TQFT*

$$Z_\Psi : Bord_n \to Cat^{corr}_{\infty,n}.$$

$Cat^{corr}_{\infty,n}$ is the $(\infty, n+1)$ category of correspondences of (∞, n)-categories.

For $n = 1$, $Cat^{corr}_{\infty,1}$ has objects $(\infty, 1)$-categories and

$$Cat^{corr}_{\infty,1}(\mathcal{C}, \mathcal{D}) := Func(\mathcal{C} \times \mathcal{D}^{op}, Spaces).$$

For higher n, $Cat^{corr}_{\infty,n}$ is defined inductively.

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Remark

The n-dualizable objects of $Cat^{corr}_{\infty,n}$ are exactly categories with adjoints.

Picture of 1-Morphisms