Harmonic analysis and representation theory of p-adic reductive groups

updated on 19th May 2016

These are the notes of my spring 2016 class at the University of Chicago on the representation theory of p-adic groups. These notes will keep updating as the lectures progress. All comments are very welcome.

The theory of representations of p-adic groups has been initiated by F.I. Mautner in the pioneer work [9] dated in the late fifties. First general results have been obtained by F. Bruhat [3] who adopted Schwartz's theory of distributions as the proper language for studying harmonic analysis on p-adic groups. Next significant progress was due I. Satake who determined the spherical functions on reductive p-adic groups [12]. Later developments in the theory benefited a lot from the monumental work of F. Bruhat and J. Tits [4] on the internal structure of reductive p-adic groups by, also know as the theory of buildings.

At the same time, a general theory of harmonic analysis on p-adic groups has been built up by Harish-Chandra on the model of Lie groups. Harish-Chandra has enunciated the cusp form philosophy and proved the Plancherel formula that are both tremendously influential.

A part from studying representations of p-adic groups for its own sake, a great source of motivation stems from the realization of automorphic forms as representations of adelic groups whose representations of p-adic groups are local components. This approach has been worked out first in the case of GL(2) by greatly influential works of Gelfand's school and Jacquet-Langlands. The Langlands conjecture formulated in the late sixties have been a driving force in the development of both theories of automorphic representations and representation of reductive p-adic groups.

The theory of representation of p-adic reductive groups has nowadays attained a mature stage of developments. A large class of cuspidal representations have been constructed. The local Langlands correspondence is now established in many cases. On the other hand, many deep questions remain open.

The purpose of these notes is twofold. It aims to on the one hand lay out the foundation of the theory in a way that is accessible to graduate students. On the other hand, it should map out more recent developments in a way that is helpful for young researchers.

Earlier basic references include [7], [2] [5]. There have been later some excellent lecture notes that remain a semi-official status. The most influential ones are probably the lecture notes of Casselman and Bernstein [1]. More recent notes of Debacker [6], Murnaghan [10] and Savin [13] can also be very helpful. A very detailed account of the basic materials has been given in a recently published book of Renard [11]. My own understanding of the subject benefitted a lot from a class taught by Waldspurger in Jussieu in 1994. The writing up of these notes draws directly from the reading of the above references, I restrict myself to organize the materials following my state and fill in details here and there when they deem necessary.

It would be a tremendous task to properly attribute credits in a theory that has been building up in the last fifty years. Any help or suggestion on this matter is welcome.

Contents

1 On *td*-spaces, their smooth functions and distributions

A Haussdorf locally compact topological space X is said to be totally disconnected, *td*-**space** for short, if every point $x \in X$ admits a base of neighborhoods consisting of compact open subsets. As a Haussdorf space, any two distinct points $x, y \in X$ belong to disjoint open neighborhoods. As *td*-space, the disjoint neighborhoods of x and y can be made both compact and open. In this sense, the space X is totally disconnected for the only subsets of X that can't be divided in disjoint union of closed and open nonempty subsets are singletons. Nevertheless, in contrast with discrete sets, in general, singletons aren't open subsets in a *td*-space.

Typical example of *td*-space is the field $\mathbf{Q}_p$ of p-adic numbers. The field $\mathbf{Q}_p$ is constructed as the completion of the field of rational numbers with respect to the p-adic absolute value

$$\left|\pm\frac{m}{n}\right|_p = p^{-\mathrm{ord}_p(m)+\mathrm{ord}_p(n)} \tag{1.1}$$

where $\mathrm{ord}_p(m)$, respectively $\mathrm{ord}_p(n)$, is the exponent of p in the factorization of natural integers m, respectively n, as product of prime numbers. The formula (1.1) defines a homomorphism of abelian groups $\mathbf{Q}^\times \to \mathbf{R}_+^\times$ that we extend as a function $|\ |_p : \mathbf{Q} \to \mathbf{R}_+$ by assigning to 0 the absolute value 0. The p-adic absolute value can be extended by continuity to $\mathbf{Q}_p$ in a unique manner.

The ring of p-adic integers $\mathbf{Z}_p$ consists of $x \in \mathbf{Q}_p$ such that $|x|_p \leq 1$ is a compact subset of $\mathbf{Q}_p$. In some respects, it is similar to the unit interval $[0,1]$ in $\mathbf{R}$. The subdivision of the unit interval $[0,1]$ into smaller closed intervals:

$$[0,1] = [0,1/n] \cup [1/n, 2/n] \cup \cdots \cup [(n-1)/n, 1] \tag{1.2}$$

is similar to the subdivision of $\mathbf{Z}_p$

$$\mathbf{Z}_p = p\mathbf{Z}_p \sqcup (1 + p\mathbf{Z}_p) \sqcup \cdots \sqcup (p - 1 + p\mathbf{Z}_p). \tag{1.3}$$

In the first case, we can't avoid the overlaps of smaller intervals as $[0,1]$ is connected whereas in the second case, $\mathbf{Z}_p$, being totally disconnected, can be partitioned as a disjoint union of closed as open subsets, as small as desired. In general, p-adic analysis can be carried out in very much the same manner as real analysis only with less technical difficulty, and in particular without the recourse to the sempiternal epsilons and deltas.

The purpose of this section is to carry out elementary analysis on *td*-spaces, in particular the notion of smooth functions and distributions. In contrast with the case of real manifolds, smooth functions on *td*-spaces do not refer to any notion of derivatives: they are simply locally constant functions. There are enough locally constant functions on a *td*-space to reconstruct the space by means of an avatar of the Gelfand duality. There are also enough, in fact a lot, of distributions to make the study of smooth functions and distributions on *td*-spaces meaningful, at the same time elementary.

Locally profinite spaces

A *td*-space is the union of its compact open subsets, which are themselves td spaces. As compact *td*-spaces can be subdivided a finite disjoint union of compact open subsets which can be as small as desired, they are in fact profinite sets. In this sense, locally, *td*-spaces are profinite sets.

Proposition 1.1. *A compact td-space is profinite set i.e limit of a projective system of finite set, equipped with the projective limit topology.*

Proof. If X is the limit of a projective system $\mathscr{X} = \{X_\alpha | \alpha \in \alpha_{\mathscr{X}}\}$ consisting of finite sets X_α, then X is compact by the Tychonov theorem. If $p_\alpha : X \to X_\alpha$ denotes the canonical projection, then for every $x \in X$, the sets $p_\alpha^{-1}(p_\alpha(x))$, for α varying in the index sets $\alpha_{\mathscr{X}}$, form a base of neighborhoods of x. Those sets are themselves profinite, and hence compact. Every $x \in X$ has therefore a base of neighborhoods consisting of compact open subsets.

Conversely, let X be a compact *td*-space. We consider all partitions $\mathscr{U}$ of X as disjoint union of open subsets $X = \bigsqcup_{\alpha \in \mathscr{U}} U_\alpha$. For X is compact, such a partition $\alpha_{\mathscr{U}}$ is necessarily finite. We consider the order on the set of such partitions: $\mathscr{U} \geq \mathscr{U}'$ if $\mathscr{U}$ is a refinement of $\mathscr{U}'$. We claim that the natural map from X to $\varprojlim \mathscr{U}$ is a homeomorphism. □

Every *td*-space X is the union of its compact open subsets $X = \cup_{\alpha \in \alpha_X} U_\alpha$. We will say that X is **countable at** ∞ if $X = \cup_{\alpha \in \alpha_X} U_\alpha$ for a countable family of compact open subsets U_α. In this case, there exists a sequence of compact open subsets $U_1 \subset U_2 \subset \cdots$ of X such that $X = \bigcup_{n \in \mathbf{N}} U_n$.

Sets of points valued in a p-adic field

The *td*-spaces that we are interested in are all related to algebraic groups over nonarchimedean local field. A local nonarchimedean field is a field F that is complete with respect to a discrete valuation $\mathrm{ord}_F : F^\times \to \mathbf{Z}$ and whose residue field is finite. If R_F is the ring i.e. consisting $x \in F$ such that $\mathrm{ord}_F(x) \geq 0$ and $\mathfrak{m}$ its ideal consisting of $x \in F$ such that $\mathrm{ord}_F(x) > 0$, then R_F is the projective limit of $R_F/\mathfrak{m}^n$ as $n \to \infty$. In particular, R_F is compact, and F is the union of compact open subsets of the form $x + R_F$ with $x \in F$. As a topological space, F is thus a *td*-space.

Let $\mathbf{X}$ an affine algebraic variety of finite type over F, then $X = \mathbf{X}(F)$ can be realized as a closed subset of $F^n = \mathbb{A}^n(F)$. By restriction, X will be equipped with a topology of closed subset of F^n, and it is a *td*-space with respect to that topology. We will leave to the reader the unpleasing task to check that the topology on X constructed in this way does not depend on the choice of the embedding of $\mathbf{X}$ into an affine space. With this independence granted, the construction can be generalized to all algebraic varieties of finite type over F for because they can be covered by affine algebraic varieties of finite type.

Other *td*-spaces of interest are open subsets of $X = \mathbf{X}(F)$ for a certain algebraic variety $\mathbf{X}$ of finite type over F.

Proposition 1.2. *If X is a projective variety over a nonarchimedean local field F, then $X(F)$ is a compact td-space.*

Proof. If $X \to \mathbb{P}^d$ is a projective embedding of X, $X(F)$ is a closed subset of $\mathbb{P}^d(F)$. We only need to prove that $\mathbb{P}^d(F)$ is compact. By the valuative criterion, we have $\mathbb{P}^d(F) = \mathbb{P}^d(R_F)$ where R_F is the ring of integers of F. On the other hand, $\mathbb{P}^d(R_F)$ is a profinite set, being the projective limit of $\mathbb{P}^d(R_F/\mathfrak{m}^m)$ as $n \to \infty$. □

It should be noticed that the compactness of $\mathbb{P}^d(F)$ is proved is a completely similar way as the real projective space $\mathbb{P}^d(\mathbf{R})$. The real projective space $\mathbb{P}^d(\mathbf{R})$ is the quotient of $\mathbf{R}^{d+1}-\{0\}$ by the scalar multiplication by $\mathbf{R}^\times$. We can consider the quotient $\mathbf{R}^{d+1}-\{0\}$ by the positive scalar multiplication by $\mathbf{R}_+^\times$ which is a double covering of $\mathbb{P}^d(\mathbf{R})$. Now, up to positive scalar multiplication every vector $(x_0,\dots,x_d)\in\mathbf{R}^{d+1}-\{0\}$ is equivalent to a unique vector $(u_0,\dots,u_d)\in\mathbf{R}^{d+1}-\{0\}$ such that

$$\max_{i\in\{1,\dots,d\}}\{|u_i|\}=1 \tag{1.4}$$

and the set of such vectors is known to be compact by the Borel-Heine theorem.

The Borel-Heine theorem states that the unit interval $[0,1]$ in the set of real numbers is compact. Its counterpart in the nonarchimedean context can be stated as the compactness of $\mathbf{Z}_p$. One may also observe that the proof of compactness of $\mathbf{Z}_p$ by realizing $\mathbf{Z}_p$ as a profinite set is also completely similar to the proof of the compactness of $[0,1]$ by subdividing $[0,1]$ in to small intervals.

Smooth functions

Let $\mathbf{C}$ be an algebraically closed field of characteristic zero without topology, or more accurately, equipped with the discrete topology. A function $f:X\to\mathbf{C}$ is said to be smooth if it is locally constant. Since $\mathbf{C}$ is equipped with the discrete topology, a function $f:X\to\mathbf{C}$ is smooth if and only if it is continuous. A function $f:X\to\mathbf{C}$ is said to have compact support if there exists an open compact subset $K\subset X$ such that f vanishes on the complement of K.

We will denote by $\mathscr{C}^\infty(X)$ the space of continuous $\mathbf{C}$-valued functions, and $\mathscr{C}_c^\infty(X)$ the subspace of continuous functions with compact support. If $K_1\subset K_2$ are compact open subsets of X, we have the restriction map $\mathscr{C}^\infty(K_2)\to\mathscr{C}^\infty(K_1)$ and also the map $\mathscr{C}^\infty(K_1)\to\mathscr{C}^\infty(K_2)$ defined by the extension by zero. The space of smooth functions $\mathscr{C}^\infty(X)$ can be realized as the projective limit:

$$\mathscr{C}^\infty(X)=\varprojlim_K \mathscr{C}^\infty(K) \tag{1.5}$$

whereas the space $\mathscr{C}_c^\infty(X)$ can be realized as an inductive limit

$$\mathscr{C}_c^\infty(X)=\varinjlim_K \mathscr{C}^\infty(K). \tag{1.6}$$

It follows that, in the case where X is not compact, $\mathscr{C}_c^\infty(X)$ is a nonunital algebra, and $\mathscr{C}^\infty(X)$ is the unital algebra obtained from $\mathscr{C}_c^\infty(X)$ by with completion with respect to the topology defined by the system of ideals $I(K)$, ranging over the set of compact open subsets $K\subset X$, where $I(K)$ is the ideal of functions vanishing on K.

The space of $\mathscr{C}^\infty(X)$ is naturally equipped with the projective limit topology. A sequence ϕ_n in $\mathscr{C}^\infty(X)$ converges to $\phi \in \mathscr{C}^\infty(X)$ if and only if for every compact open set C of X, there exists N such that for all $n \geq N$, $\phi_n|_C = \phi_n|_C$. We will call this topology of $\mathscr{C}^\infty(X)$ the **compact convergence topology**.

Proposition 1.3. *1. If X and Y are td-space, $\mathscr{C}_c^\infty(X \times Y)$ can be naturally identified with the space of locally constant functions $X \to \mathscr{C}_c^\infty(Y)$. There is a canonical isomorphism:*

$$\mathscr{C}_c^\infty(X \times Y) = \mathscr{C}_c^\infty(X) \otimes \mathscr{C}_c^\infty(Y). \tag{1.7}$$

2. *For all td-spaces X and Y, the space of smooth functions with compact support $\mathscr{C}_c^\infty(X \times Y)$ can be identified with the space of locally constant functions $X \to \mathscr{C}_c^\infty(Y)$.*

3. *For all td-spaces X and Y, the space of smooth functions $\mathscr{C}^\infty(X \times Y)$ can be identified with the space of continuous functions $X \to \mathscr{C}^\infty(Y)$, $\mathscr{C}^\infty(Y)$ being equipped with the compact convergence topology.*

Proof. 1. Let $f : X \times Y \to \mathbf{C}$ be a smooth function with compact support. For each point $(x, y) \in X \times Y$, there exists a compact neighborhood $K_{(x,y)}$ of $(x, y) \in X \times Y$ such that ϕ is constant on $K_{(x,y)}$. There exists a compact neighborhood K_x of $x \in X$, and K_y of $y \in Y$ such that $K_x \times K_y \subset K_{(x,y)}$. Suppose that f is supported by a compact open $K \subset X \times Y$. By compactness, there exists finitely many points (x_i, y_i) such that K is covered by the union of $K_{x_i} \times K_{y_i}$. By further subdivision, we can construct finitely many compact open and disjoint subset $K_{X_1}, \dots, K_{X_n}$ of X and $K_{Y_1}, \dots, K_{Y_m}$ of Y such that K is covered by the union of tiles $\bigsqcup_{i=1,j=1}^{n,m} K_{X_i} \times K_{Y_j}$ and f is constant on each "tile" $K_{X_i} \times K_{Y_j}$.

2.

3. Let $\phi : X \times Y \to \mathbf{C}$ be a smooth function. For every x, the function $y \mapsto \phi_x(y) = \phi(x, y)$ is a smooth function on Y and therefore ϕ defines a map $X \to \mathscr{C}^\infty(Y)$. We claim that the map $x \mapsto \phi_x$ in continuous with respect to the compact convergence topology of $\mathscr{C}^\infty(Y)$. Let C_Y be a compact open subset of Y and consider the neighborhood $U(C_Y)$ of $0 \in \mathscr{C}^\infty(Y)$ consisting of functions ψ vanishing on C_Y. Let $x \in X$ and for every $y \in Y$, there exists a neighborhood of the form $U_x(y) \times U_y$ of (x, y) over which ϕ is constant. Since C_Y is compact, there exists $y_1, \dots, y_h$ such that $C_Y \subset \bigcup_{i=1}^n U_{y_i}$. If $U_x = \bigcap_{i=1}^n U_x(y_i)$ then for all $x' \in U_x$, we have $\phi_{x'}|_{C_Y} = \phi_X|_{C_Y}$. It follows that the preimage of $\phi_x + U(C_X)$ in X contains the neighborhood U_x of x. In other words, ϕ is continuous at x.

Conversely, let $x \mapsto \phi_x$ be a continuous function $X \to \mathscr{C}^\infty(Y)$. We have to prove that the function $\phi(x, y) = \phi_x(y)$ is locally constant. Let $(x, y) \in X \times Y$ and C_y a compact open neighborhood of $y \in Y$ such that ϕ_x is constant on C_y. Since $x \mapsto \phi_x$ is continuous, there exists an open neighborhood U_x of x such that for all $x' \in U_x$, $\phi_{x'} \in \phi_x + U(C_y)$. This implies that ϕ is constant on $U_x \times C_y$.

□

Proposition 1.4. *If $\phi : X \to Y$ is a continuous map of td-spaces, then*

$$\phi^* : \mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(X)$$

is a continuous with respect to the compact convergence topology of $\mathscr{C}^\infty(X)$ and $\mathscr{C}^\infty(Y)$.

Proof. It is enough to prove that if K_X is a compact open subset of X, $I(K_X)$ is the ideal of smooth functions vanishing on K_X, then $(\phi^*)^{-1}(I(K_X))$ is an open subspace of $C(Y)$. By definition, $(\phi^*)^{-1}(I(K_X))$ is the ideal of $C(Y)$ consisting of functions vanishing on the image of K_X in Y. That image, denoted $\phi(K_X)$, is a compact subset of Y. For $\phi(K_X)$ is compact, there exists a compact open subset K_Y of Y such that $\phi(K_X) \subset K_Y$. It follows that $I(K_Y) \subset (\phi^*)^{-1}(I(K_X))$. Since $I(K_Y)$ is an open subspace of $C(Y)$, so is $(\phi^*)^{-1}(I(K_X))$. □

It is possible to characterize the algebras arising as the space of smooth functions with compact support in a *td*-space, and recover the *td*-space from this algebra, equipped with an appropriate structure.

For every compact open subset $K \subset X$, the characteristic function e_K defines an idempotent element of $A = \mathscr{C}_c^\infty(X)$. For e_K is an idempotent, there is a decomposition in direct sum $A = Ae_K \oplus A(1-e_K)$ where Ae_K can be identified with $\mathscr{C}^\infty(K)$, $A(1-e_K)$ with $I(K)$, the projection map $A \to Ae_K$ is the restriction to K, and the inclusion map $Ae_K \to A$ is the extension by zero outside K. A possibly nonunital commutative algebra A is said to be idempotented if for every $a \in A$ there exists an idempotent $e \in A$ such that $ae = a$.

For every commutative algebra A, possibly nonunital, we denote $E(A)$ the set of its idempotents. For every $e \in A$, Ae is a unital algebra with unit e. This set is equipped with the partial order: for all $e, e' \in E(A)$, we stipulate $e \le e'$ if $ee' = e$. If $e \le e'$ and $e' \le e''$, we have $ee'' = ee'e'' = ee' = e$ and thus $e \le e''$. If A is unital, its unit is the maximal element. The algebra A is said to be **idempotented** if A is the union of its unital subalgebras Ae for e ranging over $E(A)$.

If $A = \mathscr{C}_c^\infty(X)$, for every compact open subset U of X, the characteristic function $e_U = \mathbb{I}_U$ is an idempotent of A. Moreover, all idempotents elements of A are of this form. The partial order on the set of idempotents $E(A)$ correspond to the inclusion relation: $e_U \le e_{U'}$ if and only if $U \subset U'$. The unital algebra Ae_U corresponds to the algebra of smooth functions on X with support contained in U, in other words

$$Ae_U = \mathscr{C}^\infty(X; U) = \mathscr{C}^\infty(U).$$

Since $\mathscr{C}_c^\infty(X)$ is the union of $\mathscr{C}^\infty(X; U)$ as U ranges over the set of compact open subsets of X, A is an idempotented algebra.

Proposition 1.5. *The algebra $A = \mathscr{C}_c^\infty(X)$ of compactly supported smooth functions on a td-space is an idempotented commutative algebra. Moreover, for every idempotent $e \in A$, Ae is an inductive limit of finite dimensional algebras. Inversely if A is an idempotented algebra satisfying the above property, the space X of all non zero homomorphisms of algebras $x : A \to \mathbf{C}$ is td-and A can be canonically identified with $\mathscr{C}_c^\infty(X)$.*

Proof. □

Let A be a commutative idempotented algebra. An A-module M is said to be **nondegenerate** if M is the union of eM as e ranges over $E(A)$.

Distributions

A distribution on X is a linear form $\xi : \mathscr{C}_c^\infty(X) \to \mathbf{C}$ on the space of smooth functions with compact support. We will denote $\mathscr{D}(X)$ the space of distributions on X.

A distribution with compact support is a continuous linear form $\xi : \mathscr{C}^\infty(X) \to \mathbf{C}$ with respect to the compact convergence topology of $\mathscr{C}^\infty(X)$ and the discrete topology of $\mathbf{C}$. We will denote $\mathscr{D}_c(X)$ the space of distributions with compact support on X.

A comment on the continuity condition satisfied by elements of $\mathscr{D}_c(X)$ is in order. For we refer to the discrete topology of $\mathbf{C}$, a linear functional $\xi : \mathscr{C}^\infty(X) \to \mathbf{C}$ is continuous if and only if its kernel is a open. Now in the compact convergence topology of $\mathscr{C}(X)$, the ideals $I(K)$ of smooth functions vanishing on a given compact open subset K form a system of neighborhoods of 0 as K ranges over all compact open subsets of X. It follows that a functional $\xi : \mathscr{C}^\infty(X) \to \mathbf{C}$ is continuous if and only if there exists a compact open subset K such that $\xi|_{I(K)} = 0$. The condition $\xi|_{I(K)} = 0$ means that the support of the distribution ξ is contained in K.

Both $\mathscr{D}(X)$ and $\mathscr{D}_c(X)$ are equipped with a variation of weak star topology. If V is a topological $\mathbf{C}$-vector space, V^* is space of all linear form $\xi : V \to \mathbf{C}$, continuous with respect to the discrete topology of $\mathbf{C}$. The discrete weak* topology on V^* is the weakest topology on V^* such that for all $v \in V$, the linear form $v : V^* \to \mathbf{C}$ is continuous with respect to the discrete topology of $\mathbf{C}$. The only difference with the usual weak star topology is that in the usual weak star topology, we require $v : V^* \to \mathbf{C}$ is continuous with respect to the usual topology of $\mathbf{C}$. For they are defined as spaces of linear forms of certain vector spaces, $\mathscr{D}(X)$ and $\mathscr{D}_c(X)$ are both equipped with the discrete weak star topology.

For every $x \in X$, we define the delta distribution $\delta_x \in \mathscr{D}_c(X)$ to be the linear form $\delta_x(\phi) = \phi(x)$ for all $\phi \in \mathscr{C}(X)$. We observe that the map $X \to \mathscr{D}_c(X)$ defined by $x \mapsto \delta_x$ is continuous. Indeed, for every $x \in X$, a base of neighborhood of δ_x in $\mathscr{D}_c(X)$ is given by $V_{\phi_\bullet}$ where $\phi_\bullet = \{\phi_1, \ldots, \phi_n\} \subset \mathscr{C}^\infty(X)$ is a finite subset of smooth functions and $\xi \in V_{\phi_\bullet}$ if and only if $\langle \xi, \phi_i \rangle = \phi_i(x)$ for all $i \in \{1, \ldots, n\}$. Now since $\phi_1, \ldots, \phi_n$ are locally constant, there exists a neighborhood V of x such that $\phi_i(x) = \phi_i(x')$ for all $x' \in V$, or in other words $\delta_{x'} \in V_{\phi_\bullet}$. This proves that the map $x \mapsto \delta_x$ is continuous at x.

One may think of $\mathscr{D}_c(X)$ loosely as a sort of linear span X. If $\phi : X \to Y$ is a continuous map between *td*-spaces, we have a continuous map $\phi^* : \mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(X)$. By duality, we have a linear map $\phi_* : \mathscr{D}_c(X) \to \mathscr{D}_c(Y)$. The linear application $\phi_* : \mathscr{D}_c(X) \to \mathscr{D}_c(Y)$ can thus be seen as the linear extension of $\phi : X \to Y$.

Let X and Y be *td*-spaces and $\xi_X \in \mathscr{D}(X), \xi_Y \in \mathscr{D}(Y)$ are distributions on X and Y respectively. As ξ_X, ξ_Y are linear forms on $\mathscr{C}_c^\infty(X)$ and $\mathscr{C}_c^\infty(Y)$ respectively, their tensor product defines a linear form on $\mathscr{C}_c^\infty(X \times Y) = \mathscr{C}_c^\infty(X) \times \mathscr{C}_c^\infty(Y)$:

$$\xi_X \boxtimes \xi_Y : \mathscr{C}_c^\infty(X \times Y) \to \mathbf{C}. \tag{1.8}$$

We have thus defined a linear map

$$\mathscr{D}(X) \otimes \mathscr{D}(Y) \to \mathscr{D}(X \times Y). \tag{1.9}$$

If $\xi_X \in \mathscr{D}(X, K_X)$ and $\xi_Y \in \mathscr{D}(Y, K_Y)$ are distributions supported in compacts sets $K_X \subset X$ and $K_Y \subset Y$ respectively, then $\xi_X \boxtimes \xi_Y$ is supported in $K_X \times K_Y$. In particular, if ξ_X and ξ_Y are distributions with compact support, then so is $\xi_X \boxtimes \xi_Y$. We have thus defined a linear map

$$\mathscr{D}_c(X) \otimes \mathscr{D}_c(Y) \to \mathscr{D}_c(X \times Y). \tag{1.10}$$

Bibliographical comments

Most of the materials exposed in this section can be traced back to Bruhat's thesis [3]. I follow the exposition given by Bernstein in [1]. Notations have been evolving with time. Bruhat adopted Schwartz's notation and use the letter $\mathscr{D}(X)$ denote $\mathscr{C}_c^\infty(X)$, and the letter $\mathscr{D}'(X)$ for the space of distributions. Bernstein used letter $\mathscr{S}(X)$ for $\mathscr{C}_c^\infty(X)$, $\mathscr{S}(X)$ signifying the space of Schwartz-Bruhat functions,and $\mathscr{D}(X)$ for the space of distributions. We follow notational conventions of later references as [8].

2 On *td*-groups and their representations

The main purpose of this document is to study continuous representations of *td*-groups, and in particular of reductive p-adic groups. Among all continuous representations of *td*-groups stand out the class of smooth representations where no topology is needed on the representation space. There are enough of smooth representations so that one can usually construct general representations from smooth representation by some kind of completion. For instant, given a *td*-space acted on by a *td*-group G, the space $\mathscr{C}^\infty(X)$ of all smooth functions on X afforded an action of G. The representation of G on $\mathscr{C}^\infty(X)$ is not smooth in general, but the action on the subspace $\mathscr{C}_c^\infty(X)$ of smooth functions with compact support is smooth.

The smoothness of $\mathscr{C}_c^\infty(X)$ as a representation of G boils down to a local description of *td*-groups on *td*-spaces. In a sense that will be made precise, locally the action of of a *td*-group on a *td*-space is a projective limit of actions of finite groups on finite sets. This idea can be traced back to work of van Dantzig in the 30'.

In the study of representations of finite groups, it is convenient, and even necessary to trade representation of finite groups G for modules over its group algebra $\mathbf{C}[G]$. In the context of *td*-groups, the role of the group algebra is played by the algebra $\mathscr{D}_c(G)$ of distributions with compact support on G. Based on the elementary analysis on *td*-spaces developed in the previous section, we will construct the algebra structure on $\mathscr{D}_c(G)$, its action on $\mathscr{C}^\infty(G)$ and $\mathscr{D}(G)$ by convolution product as well the canonical action of $\mathscr{D}_c(G)$ on every smooth representation of G.

On van Dantzig's lemmas

The concept *td*-spaces may be defined either as a locally compact Haussdorf topological space in which every point has a base of neighborhood consisting of compact open subsets, or a locally profinite topological space. Proposition 1.1 shows that these two definitions are equivalent. As to *td*-group, there may be at least three different ways to define it as

(1) a *td*-space equipped with a group structure that is continuous with respect to the underlying topology;

(2) a topological group of which the identity element has a base of neighborhoods consisting of compact open subgroups;

(3) a topological group of which the identity element has a base of neighborhoods consisting of profinite groups i.e projective limit of finite groups.

It is obvious that the three possible definitions have been ordered of increasing strength i.e. (3) implies (2) that implies (1). It is less obvious to prove that these three definitions are equivalent, or in other words, (1) implies (3). This is a result due to van Dantzig.

Proposition 2.1. *Let G be a topological group whose underlying topological space is a td-space. Then the unit element of G has a basis of neighborhoods consisting of compact open subgroups.*

Proof. We will prove that for every compact open neighborhood X of the identity element e_G of G, there exists a compact open subgroup K that is contained in X. For every $x \in X$, by the continuity of the group action, there exists a neighborhood V_x of e_G such that $xV_x^2 \subset X$. Since $e_G \in V_x$, we have $xV_x \subset xV_x^2 \subset X$. From the open subsets xV_x that form a covering of the compact set X, we will extract a finite covering $X = \bigcup_{i=1}^n x_i V_{x_i}$. We consider the symmetric neighborhood of e_G defined by

$$V = \bigcap_{i=1}^n V_{x_i} \cap \bigcap_{i=1}^n V_{x_i}^{-1}.$$

We have

$$XV = \left(\bigcup_{i=1}^{n} x_i V_{x_i}\right)\left(\bigcap_{i=1}^{n} V_{x_i}\right) \subset \bigcup_{i=1}^{n} x_i V_{x_i}^2 \subset V. \tag{2.1}$$

If K denotes the subgroup of G generated by V, then we have $XK \subset X$, and in particular $K \subset X$. Since K contains a neighborhood of e_G, K is an open subgroup. It is thus also a closed subgroup. Since K is contained in a compact set X, K is a compact open subgroup. We have proved that every compact open neighborhood X of e_G contains a compact open subgroup. □

Proposition 2.2. *Compact td-groups are profinite groups.*

Proof. We will prove that if G is a compact *td*-group, the identity element of G has a base of neighborhoods consisting of normal compact open subgroups. In other words, we have to prove that every compact open subgroup K of G contains a normal compact open subgroup K_1 of G.

Since G is compact, the quotient G/K is finite. For an arbitrary set of representatives of $x_1, \ldots, x_n$ of right K-cosets, the intersection $K_1 = \bigcap_{i=1}^n x_i K x_i^{-1}$ is a compact open subgroup. Moreover, it is clearly independent of the choice of the representatives $x_1, \ldots, x_n$. For every $g \in G$, we have $gK_1g^{-1} = \bigcap_{i=1}^n x_i' K {x'}_i^{-1}$ where $x_i' = gx_i$ form another system of representatives of right K-cosets. It follows that $gK_1g^{-1} = K_1$ for all $g \in G$ and hence K_1 is a normal compact open subgroup of G. □

Arguments similar to van Dantzig's will help us to analyze the local structure of actions of *td*-groups on *td*-spaces. Let X be a *td*-space and G a *td*-group. An **action** of G on X is a continuous map $G \times X \to X$ satisfying all the familiar axioms for the action of an abstract group on an abstract set. In this case, we will also say that X is a *td*-G-**set**.

Proposition 2.3. *Let G be a td-group acting continuously on a td-space X. For every compact open subset U of X there exists an open compact subgroup K of G such that $KU = U$.*

Proof. Let U be a compact open subset of X. For every $x \in U$ there exists open neighborhood V_x of e_G in G, and open neighborhood U_x of x such that $V_x^2 x \subset U$. The compact set U being covered by the open subsets $V_x U_x$, there exists finitely many points $x_1, \ldots, x_n$ such that $U = \bigcup_{i=1}^n V_{x_i} U_{x_i}$. As in the proof of the van Dantzig theorem, if V is the symmetric neighborhood of e_G defined by

$$V = \bigcap_{i=1}^{n} V_{x_i} \cap \bigcap_{i=1}^{n} V_{x_i}^{-1}$$

then we have $VU = U$. If K is the subgroup generated by V then V is a compact open subgroup such that $KU = U$. □

As compact *td*-sets are profinite sets, compact *td*-groups are profinite groups, we can expect that any action of a compact *td*-group on a compact *td*-space is the limit of a projective system consisting of finite groups acting on finite sets. This is equivalent to say that if G is a compact *td*-group, then every compact *td*-G-set is a projective limit of finite G-sets, which is the content of the following:

Proposition 2.4. *Every compact td-space X acted on by a compact td-group G can be realized a the limit of a projective system of finite G-sets. In other words, every continuous map from X to a finite set can be dominated by a G-equivariant map from X to a finite G-set.*

Proof. Let $p_a : X \to a$ be a continuous map from a tdc space X to a finite set a. We will prove that p_a factors through a G-equivariant map $X \to X_a$ where X_a is a finite G-set. For every $\alpha \in a$, the fiber U_α over an element $\alpha \in a$ is a compact open subset of X. There exists then a compact open subgroup K_a of G such that $KU_\alpha = U_\alpha$ for all $\alpha \in a$. Since G is compact, we may also assume that K_a is a normal subgroup. We set $G_a = G/K_a$. The function $G \times X \to A$ defined by $(g,x) \mapsto p_a(gx)$ factors through $G/K_a \times X \to A$ and thus defines a continuous G-equivariant map $X \to X_a$ with $X_a = \mathrm{Map}(G/K_a, A)$ being a finite G-set. The map $p_a : X \to a$ is the composition of the G-equivariant map $X \to X_a$ and the map of finite sets $X_a \to a$ that assigns $\lambda \in \mathrm{Map}(G/K_a, A)$ the element $\alpha = \lambda(e_{G/K_a}) \in a$ where e_{G/K_a} is the unit element of G/K_a. □

Torsors over *td*-spaces

Let X and Y be *td*-spaces and G a *td*-group acting on X and acting trivially on Y. A G-equivariant map $f : X \to Y$ is a said to be a G-torsor if for every $y \in Y$, there exists an open neighborhood U of y, such that there exists a G-equivariant isomorphism $f^{-1}(U) = G \times U$.

If G is a *td*-group and H is a closed subgroup of G, we will consider the quotient $X = H\backslash G$ of left cosets of H. X will be equipped with the finest topology so that the quotient map $G \to H\backslash G$ is continuous: open subsets of X are of the form $H\backslash HU$ where U is open subset of G. With respect to this topology, X is a *td*-space: its compact open subsets are of the form $H\backslash HC$ where C is a compact open subset of G.

Proposition 2.5. *If G is a td-group and H is a closed subgroup of G then $G \to H\backslash G$ is a H-torsor.*

Proof. If K is a compact open subgroup of G, then $H\backslash HK = (H \cap K)\backslash K$ is a compact open subset of $H\backslash G$. Using the right translation in G, we are reduced to prove that $K \to (H \cap K)\backslash K$ is a $(H \cap K)$-torsor.

We will prove in fact a stronger statement in the case of compact *td*-group: if G is a compact *td*-group, H is a closed subgroup of G, then $G \to H\backslash G$ has a section. The assertion is obvious for finite groups: a section of $G \to H\backslash G$ is just a choice of system of representatives of H-left cosets in G. Now every compact *td*-group G is a projective limit of finite groups G_i, every closed subgroup H of G is a projective limit of subgroups H_i of G_i. For every $i < j$ we have a homomorphism of groups $p_i^j : G_j \to G_i$ such that $p_i^j(H_j) \subset H_i$. For every i, we can choose a system of representatives N_i for H_i-left cosets in G_i. By the axiom of choices, one can choose N_i consistently i.e such that for every $i < j$, we have $p_i^j(N_j) \subset N_i$. It follows that the closed subset $N = \varprojlim_i N_i$ of G defines a section of $G \to H\backslash G$. □

Representations and smooth representations

A (continuous) **representation** of a topological group G on a topological vector space V is a homomorphism of groups $\pi : G \to \mathrm{GL}(V)$ from G to the group $\mathrm{GL}(V)$ of all linear transformations of V such that for each $v \in V$, the map $g \to \pi(g)v$ is continuous. If G is a *td*-group, the representation $\pi : G \to \mathrm{GL}(V)$ is a **smooth representation** if for each $v \in V$, the map $g \to \pi(g)v$ is smooth i.e locally constant.

For every representation (π, V) of a *td*-group G, a vector $v \in V$ is a **smooth vector** if the induced map $g \to \pi(g)v$ is smooth. We will denote V^{sm} the subspace of V consisting of smooth vectors. By definition V^{sm} is a smooth representation of G.

Let $\mathrm{Rep}(G)$ denote the category of all continuous representations of G: its objects are (π, V_π) where V_π is a topological $\mathbf{C}$-vector space, $\pi : G \to \mathrm{GL}(V_\pi)$ is a homomorphism of groups such that for every $v \in V_\pi$, the map $g \mapsto \pi(g)v$ is continuous. If $\pi = (\pi, V_\pi)$ and $\sigma = (\sigma, V_\sigma)$ are object of $\mathrm{Rep}(G)$ then the space of morphisms is $\mathrm{Hom}_G(\pi, \sigma)$ consisting of all G-linear continuous maps $V_\pi \to V_\sigma$.

Let $\mathrm{Rep}^{\mathrm{sm}}(G)$ denote the category of all smooth representations of G: its objects are (π, V_π) where V_π is a $\mathbf{C}$-vector space, $\pi : G \to \mathrm{GL}(V_\pi)$ is a homomorphism of groups such that for every $v \in V_\pi$, the map $g \mapsto \pi(g)v$ is locally constant. If $\pi = (\pi, V_\pi)$ and $\sigma = (\sigma, V_\sigma)$ are object of $\mathrm{Rep}(G)$ then the space of morphisms is $\mathrm{Hom}_G(\pi, \sigma)$ consisting of all G-linear maps $V_\pi \to V_\sigma$.

If $(\pi, V_\pi) \in \mathrm{Rep}^{\mathrm{sm}}(G)$ is a smooth representation then by assigning to V_π the discrete topology we obtain an object in $\mathrm{Rep}(G)$. We obtain in this way a fully faithful functor $\beta : \mathrm{Rep}^{\mathrm{sm}}(G) \to \mathrm{Rep}(G)$. The functor $\mathrm{sm} : \mathrm{Rep}(G) \to \mathrm{Rep}^{\mathrm{sm}}(G)$ given by $(\sigma, V_\sigma) \mapsto (\sigma^{\mathrm{sm}}, V_\sigma^{\mathrm{sm}})$ is a right adjoint to β. Indeed we have an isomorphism of functors

$$\mathrm{Hom}_G(\pi, \sigma) = \mathrm{Hom}_G(\pi, \sigma^{\mathrm{sm}}) \tag{2.2}$$

for every smooth representation $\pi \in \mathrm{Rep}^{\mathrm{sm}}(G)$ and continuous representation $\sigma \in \mathrm{Rep}(G)$.

Group actions and representations

An action of a *td*-group G on a *td*-space X is a continuous map $G \times X \to X$ satisfying usual axioms of action of abstract group on abstract set. If we write the action as $(g, x) \mapsto gx$, then we have $e_G x = x$ and $(gg')x = g(g'x)$.

We will denote by $\alpha : G \times X \to X$ the map $(g,x) \mapsto g^{-1}x$. For every $\phi \in \mathscr{C}^\infty(X)$, $\alpha^*(\phi)$ is a smooth function on $G \times X$

$$\alpha^*(\phi) \in \mathscr{C}^\infty(G \times X). \tag{2.3}$$

By Proposition 1.3, $\alpha^*(\phi)$ consists in a continuous function $g \mapsto \phi_g$ from G to $\mathscr{C}^\infty(X)$ with $\phi_g(x) = \phi(g^{-1}x)$. In other words, the action of G on $\mathscr{C}^\infty(X)$ gives rise to a continuous representation. Similarly, for every $\phi \in \mathscr{C}_c^\infty(G)$, $\alpha^*(\phi)$ is a smooth function on $G \times X$ whose support is proper over G. It follows that the function $g \mapsto \phi_g$ from G to $\mathscr{C}_c^\infty(X)$ with $\phi_g(x) = \phi(g^{-1}x)$ is locally constant.

In other words, the representation of G on $\mathscr{C}_c^\infty(X)$ gives rise to a smooth representation. We will summarize these facts in the following statement for which we will give another proof based on the Van Dantzig lemma.

Proposition 2.6. *Let X be a td-space acted on by a td-group G. Then the action of G by translation on $\mathscr{C}_c^\infty(X)$ is smooth. The action of G on $\mathscr{C}^\infty(X)$ is continuous with respect to the compact convergence topology.*

Proof. By compactness, every locally constant function $\phi : X \to \mathbf{C}$ with compact support is a linear combination of characteristic functions of compact open subspaces. In order to prove that the representation of G on $\mathscr{C}_c^\infty(X)$ is smooth, it is enough to show that for every compact open subset U of X, the characteristic function $\mathbb{I}_U$ is a smooth vector. This is equivalent to saying that there exists a compact open subgroup K of G such that $KU = U$, which is the content of Proposition 2.3. The second statement also follows from 2.3. The third statement follows from the density of the subspace $\mathscr{C}_c^\infty(X)$ of $\mathscr{C}^\infty(X)$. □

A particularly important case is the case of a *td*-group G acting on itself by left and right translation. On the space of smooth functions $\mathscr{C}^\infty(G)$, the **left and right translations** of G are given by the following formulas

$$l_x\phi(y) = \phi(x^{-1}y) \text{ and } r_x\phi(y) = \phi(yx). \tag{2.4}$$

They also define action of G by left and right translation on $\mathscr{C}_c^\infty(G)$.

The action $G \times G$ on $\mathscr{C}_c^\infty(G)$ by left and right translation is a smooth representation. Indeed, if $\phi : G \to \mathbf{C}$ is a smooth function with support contained in a compact subset C, for every $x \in C$ there exists a compact open subgroup K_x such that ϕ is constant on $K_x x K_x$. For C is compact, there exists finitely many elements $x_1, \ldots, x_n$ such that C is contained in $\bigcup_{i=1}^n K_{x_i} x_i K_{x_i}$. It follows that ϕ is left and right translation invariant under $K = \bigcup_{i=1}^n K_i$.

On the other hand, the action of $G \times G$ on $\mathscr{C}^\infty(G)$ is not smooth unless G is compact. Indeed there are smooth functions on G which are not left or right invariant under any given compact open subgroup. The action of $G \times G$ on $\mathscr{C}^\infty(G)$ is nevertheless continuous with respect to the compact convergence topology of $\mathscr{C}^\infty(G)$ because a sequence ϕ_n in $\mathscr{C}^\infty(G)$ converges to $\phi \in \mathscr{C}^\infty(G)$ if for every compact open set C of G, there exists N such that for all

$n \geq N$, $\phi_n|_C = \phi_n|_C$, and the restriction of ϕ to C is left and right invariant under a certain compact open subgroup.

The subspace $\mathscr{C}^\infty(G)^{\mathrm{sm}(l_G)}$ of smooth vector with respect to the left translation is the inductive limit $\mathscr{C}^\infty(G)^{\mathrm{sm}(l_G)} = \varinjlim_K \mathscr{C}(K\backslash G)$ where $\mathscr{C}(K\backslash G)$ is the space of left K-invariant functions on G. Similarly, the subspace $\mathscr{C}^\infty(G)^{\mathrm{sm}(r_G)}$ of smooth vector with respect to the right translation is the inductive limit $\mathscr{C}^\infty(G)^{\mathrm{sm}(r_G)} = \varinjlim_K \mathscr{C}(G/K)$ where $\mathscr{C}(G/K)$ is the space of right K-invariant functions on G. Their intersection

$$\mathscr{C}^\infty(G)^{\mathrm{sm}(l_G \times r_G)} = \mathscr{C}^\infty(G)^{\mathrm{sm}-l_G} \cap \mathscr{C}^\infty(G)^{\mathrm{sm}(r_G)}$$

is the subspace of smooth vector under the action of $G \times G$:

$$\mathscr{C}^\infty(G)^{\mathrm{sm}(l_G \times r_G)} = \varinjlim_K \mathscr{C}(K\backslash G/K). \tag{2.5}$$

where $\mathscr{C}(K\backslash G/K)$ is the space of functions on G which are left and right invariant under K. We have the inclusions:

$$\mathscr{C}_c^\infty(G) \subset \mathscr{C}^\infty(G)^{\mathrm{sm}()} \subset \mathscr{C}^\infty(G)$$

where $\mathscr{C}^\infty(G)^{\mathrm{sm}()}$ can be the space of smooth vectors of $\mathscr{C}^\infty(G)$ with respect the left or/and right translation of G. In particular, those spaces of smooth vectors are dense in $\mathscr{C}^\infty(G)$ with respect to the compact convergence topology.

Convolution of distributions

Let G be a *td*-group. The multiplication $\mu : G \times G \to G$ induces a linear maps between the spaces of distribution with compact support

$$\mu_* : \mathscr{C}_c^\infty(G \times G) \to \mathscr{C}_c^\infty(G).$$

By composition with (1.10), we obtain a linear map

$$\mathscr{D}_c(G) \otimes \mathscr{D}_c(G) \to \mathscr{D}_c(G). \tag{2.6}$$

and thus a structure of algebra on the space distribution with compact support on G. For $\xi, \xi' \in \mathscr{D}_c(G)$, we write $\xi \star \xi'$ for their convolution product. The convolution product extends the multiplication in G in the sense that $\delta_x \star \delta_y = \delta_{xy}$ for all $x, y \in G$. One can prove that $\mathscr{D}_c(G)$ is an associative unital algebra with unit δ_{e_G}; it is commutative if and only if G itself is commutative.

If G acts on a *td*-space X, then we can also define an action of $\mathscr{D}_c(G)$ on $\mathscr{C}^\infty(X)$, $\mathscr{C}_c^\infty(X)$ and dually on $\mathscr{D}(X)$ and $\mathscr{D}_c(X)$. Let $\xi \in \mathscr{D}_c(G)$ a distribution with compact support in G, and let K_ξ be a compact open subset of G such that ξ is supported in K_ξ. Let C_X be an compact open subset of X. For every smooth function $\phi \in \mathscr{C}^\infty(X)$, we consider the restriction of $\alpha^*(\phi)$ to $K_\xi \times C_X$. By Proposition 1.3, we know that $\alpha^*(\phi)|_{K_\xi \times C_X}$ can be written in the form

$$\alpha^*(\phi)|_{K_\xi \times C_X} = \sum_{i=1}^{n} \psi_i \boxtimes \phi_i \tag{2.7}$$

where $\psi_i \in \mathscr{C}^\infty(G; K_\xi)$ and $\phi_i \in \mathscr{C}^\infty(X; C_X)$. The function $\sum_{i=1}^n \xi(\psi_i)\phi_i$ on C_X does not depend on the choice of the decomposition in tensors (2.7), and we define $\xi \star \phi$ to be the unique function $\xi \star \phi \in \mathscr{C}^\infty(X)$ such that

$$\xi \star \phi|_{C_X} = \sum_{i=1}^{n} \xi(\psi_i)\phi_i. \tag{2.8}$$

for every compact open subset C_X provided (2.7).

In particular, the algebra $\mathscr{D}_c(G)$ acts on $\mathscr{C}^\infty(G)$, $\mathscr{C}_c^\infty(G)$ and also on $\mathscr{D}(G)$. It is quite convenient to express the left and right translation of G in terms of convolution:

$$l_g\phi = \delta_g \star \phi \text{ and } r_g\phi = \phi \star \delta_{g^{-1}}. \tag{2.9}$$

Smooth representations as $\mathscr{D}_c(G)$-modules

We have seen that if X is a *td*-space acted on by a *td*-group G, the induced action of G on $\mathscr{C}^\infty(X)$ and $\mathscr{C}_c^\infty(X)$ can be extended as an action of $\mathscr{D}_c(G)$. This possibility is not shared by all continuous representations but the smooth ones.

For every smooth representation (π, V) of G, one can equip V_π with a structure of $\mathscr{D}_c(G)$-module. Let $\xi \in \mathscr{D}_c(G)$ be a distribution with compact support and $v \in V$ we will define $\pi(\xi)v \in V$ as follows. Let K be a compact open subset of G containing the support of ξ. The map $g \mapsto \pi(g)v$ being locally constant, restricted to K will be of the form $\sum_{i=1}^n \mathbb{I}_{K_i} v_i$ where $K_1, \ldots, K_n$ are compact open subsets of K and $v_1, \ldots, v_n \in V$. In other words, the formula

$$\pi(g)v = \sum_{i=1}^{n} \mathbb{I}_{K_i}(g) v_i \tag{2.10}$$

holds for all $g \in K$. We set

$$\pi(\xi)v = \sum_{i=1}^{n} \xi(\mathbb{I}_{K_i}) v_i. \tag{2.11}$$

This formula endows V with a structure of $\mathscr{D}_c(G)$-module. From the structure of module over the algebra $\mathscr{D}_c(G)$, we can recover the action of G by setting $\pi(g)v = \pi(\delta_g)v$ with δ_g being the delta distribution associated to the element $g \in G$.

Proposition 2.7. *The formula* (2.11) *gives rise to structure of $\mathscr{D}_c(G)$-module on V*

$$\alpha_\pi : \mathscr{D}_c(G) \otimes V \to V. \tag{2.12}$$

Moreover α_π is G-equivariant with respect to the action of G on $\mathscr{D}_c(G) \otimes V$ given by $g(\xi \otimes v) = (\delta_g \star \xi) \otimes v$ and the action of G on V given by $v \mapsto \pi(g)v$.

Bibliographical comments

3 Haar measures and the Hecke algebra

We have seen in Prop. 2.11 that smooth representations of a *td*-group G are equipped with a structure of $\mathscr{D}_c(G)$-modules. This structure is very useful technical device to work with smooth representations for distributions with compact support on G provide essentially all operations one can perform on representations. On the other hand, we don't have yet a good understanding of $\mathscr{D}_c(G)$, which is a huge algebra, to deepen our understanding of smooth representations. In particular, the representation of $G \times G$ on $\mathscr{D}_c(G)$ is not smooth. Smooth vectors of $\mathscr{D}_c(G)$ form a nonunital subalgebra, the Hecke algebra $\mathscr{H}(G)$, is a far smaller and more accessible. The purpose of this section is to define the Hecke algebras and to initiate the study of smooth representations of G as nondegenerate module of over $\mathscr{H}(G)$. The construction of $\mathscr{H}(G)$ begins with the existence of uniqueness of Haar distributions, an avatar of classical Haar measures in the world of *td*-groups.

Haar distribution on *td*-groups

The classical Haar theorem postulates the existence and uniqueness of invariant linear form on the space complex valued continuous functions with compact support on a locally compact group where the notion of continuous functions refers to the usual topology of the field of complex numbers. Here, we will prove the same statement but starting from the discrete topology of **C**. The proof follows essentially the same pattern as the classical proof but sparring the sempiternal epsilons and deltas.

We first recall the definition of left and right translations on a group, its smooth functions and distributions. This may be a source of confusion if we don't follow the rule of thumb: for nonabelian group G there is only one consistent definition of the action of $G \times G$ on G by left and right translation by G. The action of $G \times G$ on spaces of functions and distributions should be derived accordingly.

Let G be a *td*-group, $G \times G$ acts on G by left and right translation by the formulas

$$l_x y = xy \text{ and } r_x y = yx^{-1}. \tag{3.1}$$

The induced actions on $\mathscr{C}_c^\infty(G)$ are given by the formulas

$$l_x \phi(y) = \phi(x^{-1}y) \text{ and } r_x \phi(y) = \phi(yx) \tag{3.2}$$

These formulas induce by duality actions of G on $\mathscr{D}(G)$: for every $x \in G$, $\xi \in \mathscr{D}(G)$, we define $l_x\xi$ and $r_x\xi$ by the formulas

$$\langle l_x\xi, \phi\rangle = \langle \xi, l_{x^{-1}}\phi\rangle \text{ and } \langle r_x\xi, \phi\rangle = \langle \xi, r_{x^{-1}}\phi\rangle. \tag{3.3}$$

If $\xi = \delta_y$, we have $l_x(\delta_y) = \delta_{xy}$ and $r_x(\delta_y) = \delta_{yx^{-1}}$.

Proposition 3.1. *The space $\mathscr{D}(G)^{l(G)}$ of left invariant distributions on a td-group G is one dimensional as* **C***-vector space.*

Proof. Let $K_i, i \in I$ denote the system of neighborhoods of the identity element e_G consisting of compact open subgroups. Let $\mathscr{C}_c^\infty(G/K_i)$ denote the space of compactly supported functions on G that are right invariant under K_i. If $K_j \subset K_i$, we have a natural inclusion

$$\mathscr{C}_c^\infty(G/K_i) \to \mathscr{C}_c^\infty(G/K_j) \tag{3.4}$$

so that the spaces $\mathscr{C}_c^\infty(G/K_i)$ form an inductive system. The inductive limit of this system is

$$\mathscr{C}_c^\infty(G) = \varinjlim_{K_i} \mathscr{C}_c^\infty(G/K_i). \tag{3.5}$$

as every smooth function with compact support in G is right invariant under a certain compact open subgroup K_i, for K_i small enough. It follows that

$$\mathscr{D}(G) = \varprojlim_{K_i} \mathscr{D}(G/K_i)$$

and

$$\mathscr{D}(G)^{l(G)} = \varprojlim_{K_i} \mathscr{D}(G/K_i)^{l(G)} \tag{3.6}$$

where G acts on the discrete set G/K_i by left translation.

For each K_i, the space $\mathscr{C}_c^\infty(G/K_i)$ has a basis $\mathbb{I}_{xK_i}$ consisting of characteristic functions of right cosets xK_i. A distribution $\xi \in \mathscr{D}(G/K_i)$ is G-invariant if and only if $\xi(\mathbb{I}_{xK_i}) = \xi(\mathbb{I}_{K_i})$ for all $x \in G$. In other words, the map $\mathscr{D}(G/K_i)^{l(G)} \to \mathbf{C}$ given by $\xi \mapsto \xi(\mathbb{I}_{K_i})$ is an isomorphism. In particular $\mathscr{D}(G/K_i)^{l(G)}$ is one dimensional.

If $K_i \supset K_j$ are two compact open subgroups contained one in another, we have the inclusion $\mathscr{C}_c^\infty(G/K_i) \subset \mathscr{C}_c^\infty(G/K_j)$ and a surjection $\mathscr{D}(G/K_j) \to \mathscr{D}(G/K_i)$ and a map between one-dimensional spaces

$$\mathscr{D}(G/K_j)^{l(G)} \to \mathscr{D}(G/K_i)^{l(G)}.$$

Let ξ_j be an element of $\mathscr{D}(G/K_j)^{l(G)}$ and $\xi_i \in \mathscr{D}(G/K_i)^{l(G)}$ its image. We know that ξ_j and ξ_i are completely determined by the numbers $\xi_j(\mathbb{I}_{K_j})$ and $\xi_i(\mathbb{I}_{K_i})$. Since K_i in a disjoint union of $\#(K_i/K_j)$ right K_j-cosets, those numbers satisfy the relation

$$\xi_i(\mathbb{I}_{K_i}) = \#(K_i/K_j)\xi_j(\mathbb{I}_{K_j}), \tag{3.7}$$

where the constant $\#(K_i/K_j)$ are invertible elements of $\mathbf{C}$. It follows that elements of (3.6) consists in a system of elements $\alpha_i \in \mathbf{C}$ satisfying the relation $\alpha_i = \#(K_i/K_j)\alpha_j$, which forms a one dimensional $\mathbf{C}$-vector space. □

Proposition 3.2. *Let K be a compact td-group. Then a left invariant distribution on K is also right invariant.*

Proof. By Proposition 2.2, the identity of K has a base of neighborhoods K_i consisting of normal compact open subgroups of K. As K_i are normal subgroups, a right K_i-coset is also a left K_i-cosets. In other words, there is a canonical bijection between the discrete sets K/K_i and $K_i\backslash K$. It follows a canonical isomorphism $\mathscr{C}_c^\infty(K/K_i) = \mathscr{C}_c^\infty(K_i\backslash K)$ such that the one-dimensional spaces of K-invariant linear form on $\mathscr{C}_c^\infty(K/K_i)$ and $\mathscr{C}_c^\infty(K_i\backslash K)$ correspond:

$$\mathscr{D}(K/K_i)^{l(K)} = \mathscr{D}(K_i\backslash K)^{r(K)}.$$

By passing to the limit as K_i ranging over all normal compact open subgroups of K, as in the proof of Proposition 3.1, we get

$$\mathscr{D}(K)^{l(K)} = \mathscr{D}(K)^{r(K)},$$

in other words, a left invariant distribution on K is also right invariant. □

Modulus character and unimodular groups

Since the actions of G on $\mathscr{C}_c^\infty(G)$ by left and right translation commute one with each other, the space $\mathscr{D}(G)^{l(G)}$ of distributions invariant under the left translation is stable under the right translation of G. Since $\mathscr{D}(G)^{l(G)}$ is one-dimensional, there exists a unique homomorphism of groups

$$\Delta_G : G \to \mathbf{C}^\times \tag{3.8}$$

such that for every $\mu \in \mathscr{D}(G)^{l(G)}$, we have $r_g\mu = \Delta_G(g)\mu$ for all $g \in G$. We call it the **modulus character.** [1] By Proposition 3.2, the restriction of Δ_G to every compact open subgroup K of G is trivial. In particular, the modulus character Δ_G is a smooth character of G. A *td*-group G is said to be **unimodular** if its modulus character Δ_G is trivial.

Proposition 3.3. *1. For every smooth character $\chi : G \to \mathbf{C}^\times$, the space $\mathscr{D}(G)^{l(G,\chi)}$ of all distributions μ such that $l_g\mu = \chi(g)\mu$ for all $g \in G$, is one-dimensional. Moreover the map $\mu \mapsto \chi^{-1}\mu$ defines an isomorphism of $\mathbf{C}$-vector spaces $\mathscr{D}(G)^{l(G)} \to \mathscr{D}(G)^{l(G,\chi)}$.*

[1] I followed the sign convention of Bernstein-Zelenvinski's paper [2, p.11] which seems to be opposite to Weil's [15, p.39].

2. *For every smooth character $\chi : G \to \mathbf{C}^\times$, the space $\mathscr{D}(G)^{r(G,\chi)}$ of all distributions μ such that $r_g\mu = \chi(g)\mu$ for all $g \in G$, is one-dimensional. Moreover the map $\mu \mapsto \chi\mu$ defines an isomorphism of $\mathbf{C}$-vector spaces $\mathscr{D}(G)^{r(G)} \to \mathscr{D}(G)^{r(G,\chi)}$.*

3. *We have*
$$\mathscr{D}(G)^{l(G,\chi)} = \mathscr{D}(G)^{r(G,\Delta_G\chi^{-1})}.$$
In particular, $\mathscr{D}(G)^{l(G)} = \mathscr{D}(G)^{r(G,\Delta_G)}$ and $\mathscr{D}(G)^{l(G,\Delta_G)} = \mathscr{D}(G)^{r(G)}$.

4. *If $\mu \in \mathscr{D}(G)^{l(G)}$ is a left invariant distribution, then $\Delta_G\mu$ is a right invariant distribution.*

Proof. 1. It is enough to check that for all smooth characters χ, χ' of G, the multiplication operator $\mu \mapsto \chi\mu$ defines a map $\mathscr{D}(G)^{l(G,\chi')} \to \mathscr{D}(G)^{(G,\chi^{-1}\chi')}$. Indeed, this assertion being granted, $\mu' \to \chi^{-1}\mu'$ would define its inverse map and therefore $\mathscr{D}(G)^{l(G,\chi')} \to \mathscr{D}(G)^{(G,\chi^{-1}\chi')}$ is an isomorphism. What we need to check is that if $\mu \in \mathscr{D}(G)^{l(G,\chi')}$ then $\chi\mu \in \mathscr{D}(G)^{(G,\chi^{-1}\chi')}$. This statement follows from the commutation relation between the action of the left translation l_g and the multiplication by χ on the space of distributions: the relation
$$l_g\chi\mu = \chi(g)^{-1}\chi l_g\mu \tag{3.9}$$
holds for all $\mu \in \mathscr{D}(G)$. This relation follows from a similar commutation relation of $l_{g^{-1}}$ and χ on $\mathscr{C}_c^\infty(G)$ which can be checked directly upon definitions.

2. The second statement is completely similar to the first and follows from the commutation relation
$$r_g\chi\mu = \chi(g)\chi r_g\mu \tag{3.10}$$
for all $\mu \in \mathscr{D}(G)$.

3. It follows from the commutation relation $r_g\chi\mu = \chi(g)\chi r_g\mu$ that the multiplication by χ defines an isomorphism $\mathscr{D}(G)^{r(G,\chi')} \to \mathscr{D}(G)^{r(G,\chi\chi')}$. Since $\mathscr{D}(G)^{l(G)} = \mathscr{D}(G)^{r(G,\Delta_G)}$, by the very definition of Δ_G, for every χ we have $\mathscr{D}(G)^{l(G,\chi)} = \mathscr{D}(G)^{r(G,\Delta_G\chi^{-1})}$.

4. The last statement follows immediately from the three first.

□

Here is a typical example of a *td*-group with nontrivial modulus character. Let F be a nonarchimedean field, R its ring of integers, and q the cardinal of the field of residues. Let $F^\times$ be its multiplicative group. We have $F^\times$ acting on F by multiplication $(t,x) \mapsto tx$ and form the semidirect product
$$G = F \rtimes F^\times \tag{3.11}$$
from this action. The multiplication rule is G is to be given by the formula
$$(x_1,t_1)(x_2,t_2) = (x_1 + t_1x_2, t_1t_2). \tag{3.12}$$

Let us calculate explicitly the modulus character of G.

For the purpose of the calculation, we will identify F and $F^\times$ with subgroups of G by mapping $x \in F$ on $(x,1) \in G$ and $t \in F^\times$ on $(0,t) \in G$. We have the following formulas for the left an right translations on G by an elements of $x \in F$

$$l_x(x_1,t_1) = (x+x_1,t_1) \text{ and } r_{-x}(x_1,t_1) = (x_1+x,t_1), \tag{3.13}$$

and $t \in F^\times$:

$$l_t(x_1,t_1) = (tx_1,tt_1) \text{ and } r_{t^{-1}}(x_1+t_1) = (x_1,t_1 t) \tag{3.14}$$

according to (3.1). If $\phi = \mathbb{I}_R \times \mathbb{I}_{R^\times}$ is the characteristic function of the compact open subset $R \times R^\times$ of G then we have

$$l_t\phi = \mathbb{I}_{tR} \times \mathbb{I}_{tR^\times} \text{ and } r_{t^{-1}}\phi = \mathbb{I}_R \times \mathbb{I}_{tR^\times}. \tag{3.15}$$

If μ is a left invariant distribution on G then on the one hand we have

$$\langle \mu, l_t\phi \rangle = \langle \mu, \phi \rangle \tag{3.16}$$

and on the other hand

$$\langle \mu, r_{t^{-1}}\phi \rangle = \langle r_t\mu, \phi \rangle = \Delta_G(t)\langle \mu, \phi \rangle, \tag{3.17}$$

and therefore

$$\Delta_G(t) = \frac{\langle \mu, r_{t^{-1}}\phi \rangle}{\langle \mu, l_t\phi \rangle} \tag{3.18}$$

provided that $\langle \mu, \phi \rangle \neq 0$ for ϕ is the characteristic function of a compact open subset.

Assume that $t \in R$, then we deduce from (3.15) that

$$r_{t^{-1}}\phi = \sum_{x \in R/tR} l_x(l_t\phi) \tag{3.19}$$

where x ranges over a set of representatives of tR-cosets in R. Since μ is a left invariant distribution we have

$$\frac{\langle \mu, r_{t^{-1}}\phi \rangle}{\langle \mu, \phi \rangle} = \frac{\langle \mu, r_{t^{-1}}\phi \rangle}{\langle \mu, l_t\phi \rangle} = \#(R/tR) = q^{\mathrm{ord}(t)}. \tag{3.20}$$

It follows that for every $(x,t) \in G = F \rtimes F^\times$ we have

$$\Delta_G(x,t) = |t|_F^{-1}. \tag{3.21}$$

The inverse operator

The inverse map $g \mapsto g^{-1}$ of $G \to G$ induces an involution on the spaces of functions $\mathscr{C}^\infty(G)$ and $\mathscr{C}_c^\infty(G)$ as well as the spaces of distribution $\mathscr{D}_c(G)$ and $\mathscr{D}(G)$. We will denote this involution by $\xi \mapsto \check{\xi}$ that we will call the inverse involution. For every $g \in G$, we have $\check{\delta}_g = \delta_{g^{-1}}$ where δ_g is the delta distribution at g. We have the formula:

$$(\xi_1 \star \xi_2)^\vee = \check{\xi}_2 \star \check{\xi}_1. \tag{3.22}$$

The inverse involution plays the role of the adjoint operator in the following sense: for every $\xi \in \mathscr{D}_c(G)$, $\xi_1 \in \mathscr{D}_c(G)$ and $\phi_1 \in \mathscr{C}^\infty(G)$ we have

$$\langle \xi \star \xi_1, \phi_1 \rangle = \langle \xi_1, \check{\xi} \star \phi_1 \rangle. \tag{3.23}$$

The same formula holds if we assume $\xi \in \mathscr{D}_c(G)$, $\xi_1 \in \mathscr{D}(G)$ and $\phi_1 \in \mathscr{C}_c^\infty(G)$ or if $\xi \in \mathscr{D}(G)$, $\xi_1 \in \mathscr{D}_c(G)$ and $\phi_1 \in \mathscr{C}_c^\infty(G)$.

Proposition 3.4. *The inverse involution $\xi \mapsto \check{\xi}$ defines an isomorphism $\mathscr{D}(G)^{l(G)} \to \mathscr{D}(G)^{r(G)}$ of one-dimensional* **C***-vector spaces. More precisely, for every $\mu \in \mathscr{D}(G)^{l(G)}$, we have*

$$\check{\mu} = \Delta_G^{-1}\mu. \tag{3.24}$$

Proof. Both distributions $\tau(\mu)$ and $\Delta^{-1}\mu$ are nonzero vectors in the one-dimensional space $\mathscr{D}(G)^{r(G)} = \mathscr{D}(G)^{l(G,\Delta_G)}$. Assume that $\mu \neq 0$. To prove Equality (3.24), it is enough to find a test function $\phi \in \mathscr{C}_c^\infty(G)$ such that $\langle \tau(\mu), \phi \rangle = \langle \Delta_G^{-1}\mu, \phi \rangle$ is a nonzero element of $\mathbf{C}$. We can just pick $\phi = \mathbb{I}_K$ the characteristic function of any compact open subgroup K. □

The Hecke algebra

A distribution with compact support $\xi \in \mathscr{D}_c(G)$ is said to be smooth with respect to the left translation if and only if the map $g \mapsto l_g\xi$ is smooth, in other words, ξ is a smooth vector with respect to the left translation of G. We will see that this is equivalent to be smooth with respect to the right translation. We denote $\mathscr{H}(G)$ the space of smooth distributions with compact support.

For every compact open subgroup K of G, we will denote

$$e_K = \mathbb{I}_K \mu(K)^{-1}\mu \tag{3.25}$$

where $\mathbb{I}_K$ is the characteristic function of K, $\mu \in \mathscr{D}(K)^{l(H)}$ is a left invariant distribution, and $\mu(K)$ is the μ-measure of K. Note that μ is only well defined up to a scalar, which is offset by the factor $\mu(K)^{-1}$, thus e_K is independent of all choices. For K is compact, in particular $\mathscr{D}(K)^{l(K)} = \mathscr{D}(K)^{r(K)}$, in the formula (3.25) we can take μ to be a right invariant distribution as well. We have $\delta_g \star e_K = e_K \star \delta_g = e_K$ for all $g \in K$. It is also easy to see that e_K is an idempotent element of $\mathscr{D}_c(G)$ i.e. $e_K \star e_K = e_K$. We also observe that e_K is stable under the inverse operator $\check{e}_K = e_K$.

Proposition 3.5. *A vector $\xi \in \mathscr{D}_c(G)$ is smooth under the left action of G if and only if there exists a compact open subgroup K of G such that $e_K \star \xi = \xi$.*

Proof. If $l_g\xi = \xi$ for all $g \in K$, then $e_K \star \xi = \xi$ by the very definition of the convolution product. If $e_K \star \xi = \xi$ then for every $g \in K$, we have $l_g \xi = \delta_g \star \xi = \delta_g \star e_K \star \xi = e_K \star \xi = \xi$. □

Proposition 3.6. 1. *An element $\xi \in \mathscr{D}_c(G)$ is smooth with respect to the left translation if and only if it is of the form $\xi = \phi\mu$ where $\mu \in \mathscr{D}(G)^{r(G)}$ and $\phi \in \mathscr{C}_c^\infty(G)$ is a smooth function with compact support.*

2. *If ξ is smooth with respect to the left translation if and only if it is also smooth with respect to the right translation. In other words, $\xi \in \mathscr{H}(G)$ if and only if there exists a compact open subgroup K of G such that $e_K \star \xi = \xi \star e_K = \xi$.*

3. *If $\xi_1 \in \mathscr{H}(G)$ and $\xi_2 \in \mathscr{D}_c(G)$ then $\xi_1 \star \xi_2$ and $\xi_2 \star \xi_1$ belong to $\mathscr{H}(G)$. In other words, $\mathscr{H}(G)$ is a two sided ideal of $\mathscr{D}_c(G)$. In particular, $\mathscr{H}(G)$ is a subalgebra of $\mathscr{D}_c(G)$, which is nonunital unless G is discrete.*

Proof. 1. If $\xi \in \mathscr{D}_c(G)$ is a smooth vector with respect to the left translation of G, then there exists a compact open subgroup K of G such that for all $g \in K$ we have $l_g\xi = \xi$. This is equivalent to say that $\xi = e_K \star \xi$. If C denotes the support of ξ, then C is a compact open subset of G which is invariant under the left translation of K.

For every $\psi \in \mathscr{C}(G)$, we have

$$\langle \xi, \psi \rangle = \langle e_K \star \xi, \psi \rangle = \langle \xi, \check{e}_K \star \psi \rangle = \langle \xi, e_K \star \psi \rangle$$

and therefore the linear form $\xi : \mathscr{C}(G) \to \mathbf{C}$ factorizes through the endomorphism of $\mathscr{C}(G)$ given by $\psi \mapsto e_K \star \psi$. The image of $\psi \mapsto e_K \star \psi$ is the subspace of $\mathscr{C}(G)$ consisting of left K-invariant functions on G. This subspace can be identified with the space of functions on the discrete set $K\backslash G$, equipped with the compact convergence topology. A continuous linear form on $\mathscr{C}^\infty(K\backslash G)$ is given by a function $\phi : K\backslash G \to \mathbf{C}$ with finite support. If we identify ϕ with a function with compact support in G and left invariant under K, then $\xi = \phi\mu$ where μ is the right invariant distribution on G such that $\mu(\mathbb{I}_K) = 1$.

2. If μ is a right invariant distribution on G then by Proposition 3.3, $\Delta_G\mu$ is a left invariant distribution. It follows that $\xi \in \mathscr{D}_c(G)$ is a smooth vector with respect to the left translation if and only if it is a smooth vector with respect to the right translation.

3. If $\xi_1 \in \mathscr{H}(G)$, there exists a compact open subgroup K such that $e_K \star \xi_1 = \xi_1 \star e_K = \xi_1$. Then we have $\xi_1 \star \xi_2 = e_K \star \xi_1 \star \xi_2$ and $\xi_2 \star \xi_1 = \xi_2 \star \xi_1 \star e_K$. It follows that both distributions $\xi_1 \star \xi_2$ and $\xi_2 \star \xi_1$ are smooth.

□

Proposition 3.7. *Let G be a unimodular td-group and μ a nonzero Haar distribution on G. Then the map $\phi \mapsto \phi\mu$ induces a $G \times G$-equivariant isomorphism*

$$\mu : \mathscr{C}_c^\infty(G) \to \mathscr{H}(G). \tag{3.26}$$

Proof. This is essentially a short reformulation of the previous proposition. As to the $G \times G$-equivariant property, we only need to use the obvious formula:

$$l_{g_1} r_{g_2}(\phi\mu) = (l_{g_1} r_{g_2}\phi)(l_{g_1} r_{g_2}\mu) \tag{3.27}$$

for all $\phi \in \mathscr{C}_c^\infty(G)$ and $\mu \in \mathscr{D}(G)$. □

We will call element $\mathscr{H}(G)$ a smooth measure with compact support for it is the Haar measure multiplied by a smooth function with compact support. We observe that $\mathscr{H}(G)$ is in general nonunital since the unit δ_{e_G} of $\mathscr{D}_c(G)$ is not a smooth measure unless G is discrete. Although $\mathscr{H}(G)$ doesn't have an unit, it is endowed with a lot of idempotents including the elements e_K defined in (3.25). The system of idempotents of $\mathscr{H}(G)$ replaces in some sense its unit. For every compact open subgroup K, we will consider the subalgebra of $\mathscr{H}(G)$

$$\mathscr{H}_K(G) = e_K \star \mathscr{H}(G) \star e_K \tag{3.28}$$

of distributions with compact support on G left and right invariant un der K. The idempotent e_K is the unit of $\mathscr{H}_K(G)$. We have

$$\mathscr{H}(G) = \bigcup_K \mathscr{H}_K(G). \tag{3.29}$$

We consider more generally an arbitrary associative algebra $\mathscr{A}$. We will denote $E(\mathscr{A})$ the set of idempotents of $\mathscr{A}$. This set is equipped with a partial order: if e and f are idempotents in $\mathscr{A}$ we say that $e \le f$ if and only if $ef = fe = e$. If $e \in E(\mathscr{A})$, $e\mathscr{A}e$ is an unital subalgebra of $\mathscr{A}$ of unit e. If $e \le f$ then $e\mathscr{A}e \subset f\mathscr{A}f$. We will say an associative $\mathscr{A}$ is **idempotented** if

$$\mathscr{A} = \bigcup_{e \in E(\mathscr{A})} e\mathscr{A}e. \tag{3.30}$$

In the Hecke algebra $\mathscr{H}(G)$, we have $e_K \subset e_{K'}$ if and only if the compact open subgroup K contains the compact open subgroup K'. Moreover, every idempotent $e \in e_K$ is dominated by an idempotent of the form e_K where K is a compact open subgroup of G. By (3.29), $\mathscr{H}(G)$ is an idempotented algebra.

Nondegenerate modules over the Hecke algebra

Let G be a *td*-group and (π, V) a smooth representation of G. The action of G on V can be extended to the action of the algebra of distributions with compact support $\mathscr{D}_c(G)$ by (2.11). By restricting to $\mathscr{H}(G)$, we see that V is equipped with a structure of module over the Hecke algebra. The smoothness of V as a representation of G can be translated very simply in a property of the corresponding module over $\mathscr{H}(G)$.

Proposition 3.8. *Let (π, V) be a representation of a td-group G such that the action of G can be extended to an action of $\mathscr{D}_c(G)$. For every compact open subgroup K of G, $v \in V^K$ is a fixed vector of K if and only if $v = \pi(e_K)v$. We have $V^K = \pi(e_K)V$.*

Proof. If $\pi(g)v = v$ for all $g \in K$ then we have $\pi(e_K)v = v$ by the very definition (2.11) of the action of $\mathscr{D}_c(G)$ on V. Conversely, if if $\pi(e_K)v = v$ then for every $g \in K$, we have

$$\pi(g)v = \pi(g)\pi(e_K)v = \pi(e_K)v = v$$

Moreover, if $v \in \pi(e_K)V$ then $\pi(e_K)v = e_K$ for e_K is idempotent. □

Let $\mathscr{A}$ be an idempotented algebra. A $\mathscr{A}$-module M is said to be **nondegenerate** if

$$M = \bigcup_{e \in E(\mathscr{A})} eM. \tag{3.31}$$

We note that for every $e \in E(\mathscr{A})$, eM is a module over the unital subalgebra $e\mathscr{A}e$ of $\mathscr{A}$. If $\mathscr{A} = \mathscr{H}(G)$ and M is a nondegenerate $\mathscr{H}(G)$-module then

$$M = \bigcup_K e_K M. \tag{3.32}$$

the union ranging over all compact open subgroups K of G, for every idempotent $e \in E(\mathscr{H}(G))$ is dominated by some e_K.

Proposition 3.9. *Let G be a td-group. There is an equivalence of category between the category smooth representations of G and the category of nondegenerate modules over the Hecke algebra $\mathscr{H}(G)$.*

Proof. If V is a smooth representation of G, for every $v \in V$, the map $g \mapsto \pi(g)v$ is locally constant. In particular, there exists a compact open subgroup K of G such that $\pi(g)v = v$ for all $g \in K$. It follows that V is union of $\pi(e_K)V$ while K ranges over the set of compact open subgroups of G, and therefore as $\mathscr{H}$-module, V is nondegenerate.

Inversely, if V is a nondegenerate $\mathscr{H}$-module i.e.

$$V = \bigcup_{e \in E(\mathscr{H})} \pi(e)V,$$

we can extend the action of $\mathscr{H}$ to an action of $\mathscr{D}_c(G)$. For every $\xi \in \mathscr{D}_c(G)$ and $v \in V$, we choose a compact open subgroup K such that $\pi(e_K)v = v$. Then we set $\pi(\xi)v = \pi(\xi \star e_K)v$ where $\xi \star e_K \in \mathscr{H}(G)$. This definition is independent of the choice of e_K. Indeed if $e \le e'$ are two idempotents of $\mathscr{H}$ then $e' \star e = e$ then we have $\pi(\xi \star e')v = \pi(\xi \star e')\pi(e)v = \pi(\xi \star e)v$ for every $v \in e_K V \subset e_{K'} V$.

For every $g \in G$, we set $\pi(g)v = \pi(\delta_g)v$. Since $\delta_{gg'} = \delta_g \delta_{g'}$ this gives rise to a homomorphism of groups $G \to \mathrm{GL}(V)$. We claim that for every $v \in V$, the induced map $g \mapsto \pi(g)v$ is smooth. Let K be a compact open subgroup such that $\pi(e_K)v = v$. The formula shows $\pi(g)v = \pi(\delta_g \star e_K)v$ the function $g \mapsto \pi(g)v$ is right K-invariant, and therefore smooth. $\square$

Let (π, V) be an representation of G, not necessarily smooth. A vector $v \in V$ is said to be smooth if the function $g \mapsto \pi(g)v$ is smooth or in other words, if v is fixed by a certain compact open subgroup K of G. The space of smooth vectors in V is:

$$V^{\mathrm{sm}} = \bigcup_K V^K \tag{3.33}$$

Assume that the action of G on V can be extended as a structure of $\mathscr{D}_c(G)$-module on V. Then for every compact open subgroup K of G, we have $\pi(e_K)V = V^K$, and

$$V^{\mathrm{sm}} = \bigcup_K \pi(e_K)V. \tag{3.34}$$

Contragredient and admissible representations

Let (π, V) be a smooth representation of a *td*-group G. As in (2.11), the action of G on V can be extended canonically to an action of $\mathscr{D}_c(G)$. Let V^* denote the space of all linear forms $v^* : V \to \mathbf{C}$. This is a $\mathbf{C}$-vector space with an action of G given by $v^* \mapsto \pi^*(g)v^*$ satisfying

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle. \tag{3.35}$$

V^* is also equipped with a structure of $\mathscr{D}_c(G)$-module defined by the formula

$$\langle \pi^*(\xi)v^*, v\rangle = \langle v^*, \pi(\check{\xi})v\rangle. \tag{3.36}$$

where $\xi \mapsto \check{\xi}$ is the inverse operator on $\mathscr{D}_c(G)$ defined in (3.22). In general, the representation π^* of G on V^* is not smooth. We define the **contragredient** of V as the subspace of V^* consisting of smooth vectors of V^* that are $v' \in V^*$ such that there exists a compact open subgroup K of G such that $\pi^*(e_K)v' = v'$.

Let V be $\mathbf{C}$-vector space possibly of infinite dimension and V^* the space of all linear forms on V. There is a natural associative algebra structure on $V \otimes V^*$ given by

$$(v_1 \otimes v_1^*)(v_2 \otimes v_2^*) = \langle v_1^*, v_2\rangle (v_1 \otimes v_2^*).$$

Let $\mathrm{End}(V)$ the algebra of all linear transformations of V and $\mathrm{End}_{\mathrm{fin}}(V)$ the subalgebra $\mathrm{End}(V)$ of all linear transformations of V with finite dimensional image.

Proposition 3.10. *Let V be $\mathbf{C}$-vector space possibly of infinite dimension. By assigning to each vector $w = \sum_{i=1}^n v_i \otimes v_i^* \in V \otimes V^*$ the linear transformation $f_w(v) = \sum_{i=1}^n v_i \langle v_i^*, v\rangle$ we define an isomorphism of algebras*

$$V \otimes V^* \to \mathrm{End}_f(V). \tag{3.37}$$

Proof. For every $w = \sum_{i=1}^n v_i \otimes v_i^* \in V \otimes V^*$, the image of this linear transformation $f_w(v) = \sum_{i=1}^n v_i \langle v_i^*, v\rangle$ is contained in the finite dimensional subspace generated by $v_1, \ldots, v_n$, and thus finite dimensional. It can be checked directly upon the formulas that $w \mapsto f_w$ is a homomorphism of nonunital associative algebras $V \otimes V^* \to \mathrm{End}_f(V)$. To prove that it is an isomorphism, it is enough to construct an inverse. Let $f \in \mathrm{End}_{\mathrm{fin}}(V)$ be a linear transformation of V with finite dimensional image. If $v_1, \ldots, v_n$ is a basis of $\mathrm{im}(f)$ then there exists unique vectors $v_1^*, \ldots, v_n^*$ such that $f(v) = \sum_{i=1}^n v_i \langle v_i^*, v\rangle$. Moreover the vector $\sum_{i=1}^n v_i \otimes v_i^* \in V \otimes V^*$ is then independent of the choice of the basis of $\mathrm{im}(f)$ therefore gives rise to the inverse map of (3.37). □

If (π, V) is a smooth representation of G then $G \times G$ acts on $\mathrm{End}(V)$ by the formula

$$(g_1, g_2)f = \pi(g_1) \circ f \circ \pi(g_2^{-1}). \tag{3.38}$$

This formula induces an action on the subalgebra $\mathrm{End}_{\mathrm{fin}}(V)$ of $\mathrm{End}(V)$. For $\mathrm{im}(f)$ is finite dimensional for $f \in \mathrm{End}_{\mathrm{fin}}(V)$, the left action of G on $\mathrm{End}_{\mathrm{fin}}(V)$. This can also be derived from the fact that the isomorphism (3.37) is $G \times G$-equivariant, and the action of G on V is smooth whereas its action on V^* isn't. We have an $G \times G$-equivariant isomorphism of algebras

$$V \otimes V' \to \mathrm{End}_{\mathrm{fin}}(V)^{\mathrm{sm}} \tag{3.39}$$

where V' is the contragredient representation of V.

A smooth representation (π, V) of a *td*-group G is said to be **admissible** if for every compact open subgroup K of G, the subspace $V^K = \pi(e_K)V$ is finite dimensional.

Proposition 3.11. *If (π, V) is an admissible representation then for every $\phi \in \mathscr{H}(G)$, the operator $\pi(\phi)$ has finite dimensional image.*

Proof. For every $\phi \in \mathscr{H}(G)$, there exists a compact open subgroup K of G such that $e_K \star \phi = \phi$. It follows that for all $v \in V$, $\pi(\phi)v \in V^K$. Since V^K is finite dimensional, the operator $\pi(\phi)$ has finite dimensional image. □

For admissible representation (π, V), the homomorphism of algebras

$$\pi : \mathscr{H}(G) \to \mathrm{End}(V)$$

factorizes through $\mathrm{End}_{\mathrm{fin}}(V)$. Combined with the isomorphism (3.39), we obtain a homomorphism

$$\pi : \mathscr{H}(G) \to V \otimes V'. \tag{3.40}$$

Proposition 3.12. *If V is a smooth admissible representation of a td-G, then its contragredient V' is also admissible. Let V'' denote the contragredient of V'. The double dual map $V \to V''$, assigning to each vector $v \in V$ the linear form $v' \mapsto \langle v, v' \rangle$ it defines on V', is an isomorphism $V \to V''$ of G-modules.*

Proof. For e_K is idempotent, we have a decomposition in direct sum $V = \pi(e_K)V \oplus (1 - \pi(e_K))V$. For the contragredient representation, we have the dual decomposition $V' = \pi'(e_K^*)V' \oplus (1-\pi'(e_K^*))V'$, and for the double contragredient $V'' = \pi''(e_K)V'' \oplus (1-\pi''(e_K))V''$. In these decompositions, $\pi'(e_K^*)V'$ is the dual vector space of $\pi(e_K)V$, and $\pi(e_K)V''$ is the dual vector space of $\pi'(e_K^*)V'$. For $\pi(e_K)V$ is finite dimensional, the double dual map $\pi(e_K)V \to \pi''(e_K)V''$ is an isomorphism. The double contragredient map $V \to V''$ induces an isomorphism $V^K \to (V'')^K$ on subspaces of K-points for each compact open subgroup. As V and V'' are smooth, the map $V \to V''$ is an isomorphism. □

Let (π, V) be an admissible representation of a *td*-group G. For every $\xi \in \mathscr{H}(G)$, the endomorphism $v \mapsto \pi(\xi)v$ has finite dimensional image. Indeed, if K is a compact open subgroup such that $\xi = e_K \star \xi$ then $\pi(\xi)v \subset V^K$. We can define the trace of $\pi(\xi)$ as the trace of the restriction of $\pi(\xi)$ to its finite dimensional image

$$\mathrm{tr}_\pi(\xi) = \mathrm{tr}(\pi(\xi)|_{\mathrm{im}\pi(\xi)}). \tag{3.41}$$

Each admissible representation π of G thus gives rise to a linear form on the Hecke algebra $\mathscr{H}(G)$, called the character of π. The space of all linear forms on $\mathscr{H}(G)$ is called the space of generalized functions as it contains $\mathscr{C}^\infty(G)$ as a subspace. The character of an admissible representation of G is a generalized function.

If G is a reductive p-adic group, all irreducible smooth representations of G are admissible. This highly nontrivial fact can only be proven after making a deep inroad into the structure of smooth representations of reductive p-adic groups. This fact is also obviously wrong for a general *td*-group. For instant, if G is a discrete group, and V is an infinite dimensional irreducible representation V then V is not admissible. Such a example exists for infinite nonabelian discrete group.

Reduction to a finite level

The reduction to a finite level consists in considering the subspace V^K of K-invariant vectors in a smooth representation V of a *td-G*, K being a compact open subgroup of G. The terminology of finite level refers to the level in the theory of modular forms.

Let G be a *td*-group, (π, V) a smooth representation of G. Recall that after Prop. 3.9, V is equipped with a structure of non-degenerate module over the Hecke algebra $\mathscr{H}(G)$. For every compact open subgroup K, the subspace of K-fixed vector inherits a structure of $\mathscr{H}_K(G)$-module. Indeed, for every $\phi \in \mathscr{H}_K(G)$ and $v \in V$ we have

$$\phi v = (e_K \star \phi)v = e_K(\phi v) \in V^K. \tag{3.42}$$

We obtain in this way a functor $V \mapsto V^K$ from the category of nondegenerate $\mathscr{H}(G)$-modules to the category of $\mathscr{H}_K(G)$-modules.

If $K \subset K'$ are compact open subgroups of G, M is a $\mathscr{H}_K(G)$-module then $e_{K'}M$ is a $\mathscr{H}_{K'}(G)$-module. Thus we obtain a functor from the category of $\mathscr{H}_K(G)$-modules to the category of $\mathscr{H}_{K'}$-modules. We understand better this functor in putting ourselves in the following general context:

Proposition 3.13. *1. Let A be an associative algebra with unit e. Let e' be an idempotent, but not necessarily central, element $e' \in A$, and $A' = e'Ae'$. Then $M \mapsto e'M$ is an exact functor from the category Mod_A of A-modules to the category $\mathrm{Mod}_{A'}$ of A'-modules.*

2. *The subcategory $\mathrm{Nil}_{e'}\mathrm{Mod}_A$ of Mod_A consisting of all A-modules M annihilated by e' is a Serre subcategory of Mod_A i.e. if N is an object of $\mathrm{Nil}_{e'}\mathrm{Mod}_A$ all subquotients of N are also objects of $\mathrm{Nil}_{e'}\mathrm{Mod}_A$, and all extensions of objects of $\mathrm{Nil}_{e'}\mathrm{Mod}_A$ remain in $\mathrm{Nil}_{e'}\mathrm{Mod}_A$.*

3. *The quotient of the abelian category Mod_A by its Serre subcategory $\mathrm{Nil}_{e'}\mathrm{Mod}_A$ can be identified with $\mathrm{Mod}_{A'}$. More precisely for every A-linear map $\phi : M_1 \to M_2$, the induced map $e'\phi : e'M_1 \to e'M_2$ is an isomorphism of A'-modules if and only if both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ belong to $\mathrm{Nil}_{e'}\mathrm{Mod}_A$.*

4. *For every A'-module M', the natural map $\alpha : e'(A \otimes_{A'} M') \to M'$ is an isomorphism. Its inverse gives rise to a pair of adjoint functors: the functor $M' \mapsto A \otimes_{A'} M'$ from $\mathrm{Mod}_{A'}$ to Mod_A is a left adjoint to the functor $M \mapsto e'M$ from Mod_A to $\mathrm{Mod}_{A'}$.*

5. *If M' is a (nonzero) simple A'-module, then the quotient of $A \otimes_{A'} M'$ by its largest A-submodule annihilated by e' is a simple A-module M. Moreover, we have then $e'M = M'$.*

6. *If M is a simple A-module such that $e'M \neq 0$, then $M' = e'M$ is a simple A'-module. Moreover, the adjunction map $A \otimes_{A'} M' \to M$ is a surjective map whose kernel is the largest A-submodule of $A \otimes_{A'} M'$ annihilated by e'.*

Proof. 1. Since e' is an idempotent element, we have a decomposition $M = e'M \oplus (e - e')M$ as abelian groups (we note that since e' is not necessarily central, neither $e'M$ nor $(e - e')M$ are necessarily A-modules). If follows that the functor $M \mapsto e'M$ is an exact functor form Mod_A to the category of abelian groups. Since $e'M$ is automatically endowed with a structure of A'-modules, the functor $M \mapsto e'M$ is an exact functor from the category of A-modules to the category of A'-modules.

2. If N is an A-module annihilated by e', it is obvious that all subobjects and quotients of N are also annihilated by e'. The category $\mathrm{Nil}_{e'}\mathrm{Mod}_A$ is therefore stable under subquotients. Let us check that it is also stable under extension. We consider an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

where M_1 and M_3 are annihilated by e'. Let x_2 be an arbitrary element of M_2, x_3 its image in M_3. Since $e'x_3 = 0$, we have $e'x_2 \in M_1$. It follows that $e'(e'x_2) = e'x_2 = 0$. This proves that M_2 is annihilated by e'.

3. Let $\phi : M_1 \to M_2$ be an A-linear map such that the induced map $e'M_1 \to e'M_2$ is an isomorphism. Since the functor $M \to e'M$ is exact, this implies that both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are annihilated by e'.

4. That α is an isomorphism is obvious to see once we write down its formula:

$$e' \sum_{i=1}^{n} a_i \otimes_{A'} m'_i \mapsto \sum_{i=1}^{n} e' a_i m'_i$$

with $a_i \in A$ and $m'_i \in M'$. Since every $m' \in M'$ can be written in the form $\sum_{i=1}^{n} e' a_i m'_i$ with just $n = 1$, $m'_1 = m'$ and $a_i = 1$, the map α is surjective. On the other hand, for we can rewrite

$$e' \sum_{i=1}^{n} a_i \otimes_{A'} m'_i = \sum_{i=1}^{n} 1 \otimes_{A'} e' a_i e' m'_i$$

if $\sum_{i=1}^{n} e' a_i m'_i = 0$ then $e' \sum_{i=1}^{n} a_i \otimes m'_i = 0$. Therefore α is injective.

Since $\alpha : e'(A \otimes_{A'} M') \to M'$ is an isomorphism of A'-modules, we have an inverse $M' \to e'(A \otimes_{A'} M')$. If $A \otimes_{A'} M' \to M$ is an A-linear map, the composition $M' \to e'(A \otimes_{A'} M') \to e'M$ gives rise to an A-linear map $M' \to e'M$. We thus obtain a morphism of functors

$$\mathrm{Hom}_A(A \otimes_{A'} M', M) \to \mathrm{Hom}_{A'}(M', e'M). \tag{3.43}$$

Inversely, given a A'-linear map $M' \to e'M$, we obtain by composition $A \otimes_{A'} M' \to A \otimes_{A'} e'M \to M$ an A-linear map $A \otimes_{A'} M' \to M$. We thus obtain

$$\mathrm{Hom}_{A'}(M', e'M) \to \mathrm{Hom}_A(A \otimes_{A'} M', M) \tag{3.44}$$

One can check easily that (3.43) and (3.44) are inverse one of each other.

5. Let M' be a simple A'-module and m' a generator of M'. Let M denote the quotient of $A \otimes_{A'} M'$ by its largest A-submodule N killed by e'. Then we have $M' = e(A \otimes_{A'} M') = e'M$. In particular $M \neq 0$.

By construction the image of $1 \otimes_{A'} m'$ in M is a generator of M. To prove that M is a simple A-module, we prove that for any nonzero element $m \in M$ is also a generator of M. This is equivalent to saying that there exists $a \in A$ such that $am = 1 \otimes_{A'} m'$ in M. Let m_1 be an element of $A \otimes_{A'} M'$ whose image in M is m. There exists $a_1 \in A$ such that $m = a_1 \otimes_{A'} m'$. Since $m_1 \notin N$, there exists $a_2 \in A$ such that $e' a_2 m_1 \neq 0$. Now we have

$$e' a_2 m_1 = e' a_2 a_1 \otimes_{A'} m' = e' a_2 a_1 e' \otimes_{A'} m' = 1 \otimes_{A'} (e' a_2 a_1 e') m'.$$

Since $e'a_2m_1 \neq 0$, we have $(e'a_2a_1e')m' \neq 0$. Since M' is a simple A'-module, there exists $a_3 \in A$ such that $(e'a_3e')(e'a_2a_1e')m' = m'$. It follows that

$$(e'a_3e')(e'e_2)m_1 = (e'a_3e')(e'e_2e') \otimes_{A'} m' = 1 \otimes_{A'} m'.$$

In other words $am_1 = 1 \otimes_{A'} m'$ for $a = (e'a_3e')(e'e_2) \in A$. This proves that M is a simple A-module.

The quotient map $A \otimes_{A'} M' \to M$ correspond by adjunction to a map

$$M' = e'(A \otimes_{A'} M') \to e'M.$$

The map $A \otimes_{A'} M' \to M$ being nonzero, the map $M' \to e'M$ is also nonzero. On the other hand, it is surjective for the functor $M \mapsto e'M$ is exact. Since M' is a simple A-module, the map $M' \to e'M$ is an isomorphism.

6. Let M be a simple A-module such that $e'M \neq 0$. Since $e'(A \otimes_{A'} e'M) = e'M$, both kernel and cokernel the map $A \otimes_{A'} e'M \to M$ are annihilated by e'. For M is a simple A-module such that $e'M \neq 0$, it has neither nontrivial submodule or quotient annihilated by e'. It follows that the map $A \otimes_{A'} e'M \to M$ is surjective and its kernel is the largest submodule of $A \otimes_{A'} e'M$ killed by A.

   It only remains to prove that $M' = e'M$ is a simple A'-module. Let M'_1 be a nonzero A'-submodule of M'. We will prove that $M'_1 = M'$. We consider the A-linear map

   $$\phi_1 : A \otimes_{A'} M'_1 \to A \otimes_{A'} M'.$$

   For every A-module P, we denote $N_{e'}(P)$ the largest submodule of P annihilated by e' and $Q(P) = M/N_e(P)$. We now have a map

   $$Q(\phi_1) : Q(A \otimes_{A'} M'_1) \to Q(A \otimes_{A'} M') = M$$

   Since $e'Q(A \otimes_{A'} M'_1) = M'_1$ and $e'Q(A \otimes_{A'} M') = M'$, the induced map

   $$e'Q(\phi_1) : e'Q(A \otimes_{A'} M'_1) \to e'Q(A \otimes_{A'} M')$$

   is just the inclusion map $M'_1 \to M'$, and in particular, it is nonzero. It follows that $Q(\phi_1)$ is nonzero. Since M si a simple A-module, $Q(\phi_1)$ has to be surjective. It follows that the inclusion map $M'_1 \to M_1$ is also surjective. In other words, $M'_1 = M'$ and M' is a simple A'-module.

□

Bibliographical comments

We followed [**Hewitt-Ross**] in the proof of the van Dantzig theorem.

4 The Schur lemma and matrix coefficients

The Schur lemma and the orthogonality of matrix coefficients of irreducible representations are the cornerstones of the theory of representations of finite groups. The purpose of this section is to incorporate these materials into the theory of representations of *td*-groups.

An immediate consequence of the Schur lemma is that all smooth irreducible representations of abelian *td*-groups are one dimensional i.e. characters. We will develop an avatar of the Pontryagin duality for abelian *td*-groups. In contrast with the theory of Pontryagin duality for abelian locally compact groups, we won't restrict ourselves to unitary characters for at this point, we attempt to stay clear from any norm or topology on the coefficient field of representations. In particular, instead of the Fourier transform in the Pontryagin theory, we will study an avatar of the classical Fourier-Laplace transform for the case of abelian *td*-groups.

The Schur lemma is also an essential ingredient for the proof of the orthogonality relation among matrix coefficients and the Peter-Weyl theorem. We will discuss the Peter-Weyl theorem in the case of compact *td*-groups. For noncompact *td*-groups, the the orthogonality relation among matrix coefficients can be generalized to the class of compact representations.

The Schur lemma

A smooth representation V of a *td*-group G is said to be **irreducible** if it has no G-invariant proper subspace other than 0.

Proposition 4.1. *If (π, V_π) and $(\pi', V_{\pi'})$ are two irreducible representations of a td-group V, then all non zero G-equivariant maps $V_\pi \to V_{\pi'}$ are invertible. Then if π and π' are not isomorphic then* $\mathrm{Hom}_G(\pi, \pi') = 0$.

Proof. Let $\phi : V_\pi \to V_{\pi'}$ be a nonzero G-equivariant map. Then $\mathrm{im}(\phi)$ is a nonzero G-invariant subspace of $V_{\pi'}$. Since $V_{\pi'}$ is irreducible, we must have $\mathrm{im}(\phi) = V_{\pi'}$ or in other words, ϕ is surjective. On the other hand, $\ker(\phi)$ is a proper G-invariant subspace of V_π. Since π is irreducible, we must have $\ker(\phi) = 0$ i.e. ϕ is injective. It follows that ϕ is invertible, and in particular π and π' are isomorphic. If π and π' aren't isomorphic, we must have $\mathrm{Hom}_G(\pi, \pi') = 0$. □

Proposition 4.2 (Schur lemma)**.** *Let G be a td-group countable at ∞ i.e it is a countable union of compact open subsets. Assume* $\mathbf{C}$ *is algebraically closed and uncountable. Then for every smooth irreducible representation (π, V) of G, we have* $\mathrm{End}_G(V) = \mathbf{C}$.

Proof. If V is an irreducible representation of G and $\phi \in \mathrm{End}_G(V)$ is a nonzero G-linear endomorphism of V, then γ is invertible by Prop. 4.1. In other words, $\mathrm{End}_G(V)$ is a **skew field** i.e. an unital associative ring in which all nonzero elements are invertible.

Assume that the inclusion $\mathbf{C} \subset \mathrm{End}_G(V)$ is strict and $T \in \mathrm{End}_G(V) - \mathbf{C}$. The action of t on V gives rise to a homomorphism of algebras $\phi : \mathbf{C}[t] \to \mathrm{End}_G(V)$ with $\phi(t) = T$. We claim that ϕ is injective. Indeed, if it is not, it is of the form $A = \mathbf{C}[t]/I$ where I is a nonzero ideal of $\mathbf{C}[t]$, and hence A is a finite $\mathbf{C}$-algebra. As a subalgebra of $\mathrm{End}_G(V)$, A needs to be an integral domain, thus A is a finite extension of $\mathbf{C}$. Since $\mathbf{C}$ is assumed to be algebraically closed, the only possibility is $A = \phi(\mathbf{C})$ that would contradict $T = \phi(t) \notin \phi(\mathbf{C})$.

Now as ϕ is injective, and all nonzero elements of $\mathrm{End}_G(V)$ are invertible, the homomorphism $\phi : \mathbf{C}[t] \to \mathrm{End}_G(V)$ can be extended to $\mathbf{C}(t)$ where $\mathbf{C}(t)$ is the field of rational functions of variable t. We observe that rational functions of the form $(t-\alpha)^{-1}$ with $\alpha \in \mathbf{C}$ are linearly independent and therefore $\mathbf{C}(t)$ can't have countable basis. Since for every nonzero vector $v \in V$, the map $\mathbf{C}[t] \to V$ given by $f \mapsto \phi(f)v$ is injective, the space V can't have countable basis.

On the other hand, the assumption G being countable at ∞ implies that V has a countable basis. Indeed, let v be a nonzero vector of V. There exists a compact open subgroup K such that $\pi(K)v = v$. Since G is countable at ∞, G/K is a countable set. In particular the set of vectors of the form $\pi(g)v$ is thus countable. Since V is irreducible, the subspace of V generated by the vectors $\pi(g)v$ is V itself. It follows that V has a countable basis. Starting from the assumption $\mathrm{End}_G(V) \neq \mathbf{C}$, we have just reached a contradiction. □

If Z is the center of a *td*-group G, and (π, V_π) is a smooth irreducible representation of G, then there exists a smooth character $\chi : Z \to \mathbf{C}^\times$ such that for every $v \in V_\pi$, we have $\pi(z)v = \chi(z)v$ for all $z \in Z$. We say that χ is the **central character** of π.

In particular, every irreducible smooth representation of an abelian *td*-group G is one dimensional. Smooth irreducible representations of G are **smooth characters** $\chi : G \to \mathbf{C}^\times$ i.e. those homomorphism of groups $\chi : G \to \mathbf{C}^\times$ that are trivial on a compact open subgroup of G.

Amitsur's separation lemma

The separation lemma for *td*-groups is based on the following fact of noncommutative algebra due to Amitsur.

Proposition 4.3. *Let R be a countably infinite dimensional algebra over $\mathbf{C}$ and $\phi \in R$ a non nilpotent element. Then there exists an irreducible R-module M on which ϕ is nonzero.*

Proof. First, we prove that there exists $\alpha \in \mathbf{C}^\times$ such that $\phi - \alpha \notin R^\times$. Assume the opposite is true, we consider the vectors $\phi - \alpha$ as $\alpha \in \mathbf{C}$. Since $\mathbf{C}$ is uncountable and R has countable dimension, these vectors ought to be linearly independent. Thus there exists non zero numbers $a_1, \dots, a_n \in \mathbf{C}$ and distinct numbers $\alpha_1, \dots, \alpha_n \in \mathbf{C}$ such that $\sum_{i=1}^n a_i(\phi - \alpha_i)^{-1} = 0$. By multiplying with $\prod_{i=1}^n (\phi - \alpha_i)$ we obtain a an expression $P(\phi) = 0$ with $P \in \mathbf{C}[t]$ given by

$$P = \prod_{i=1}^n (t - \alpha_i) \sum_{i=1}^n c_i (t - \alpha_i)^{-1}.$$

We have $P \neq 0$ as $P(\alpha_i) \neq 0$ for every $i = 1, \ldots, n$. The set of all polynomials $Q \in \mathbf{C}[t]$ such that $Q(\phi) = 0$ is a nonzero ideal of $\mathbf{C}[t]$. We will assume that P is a generator of this ideal i.e. the polynomial of minimal degree annihilating ϕ. Since ϕ is not nilpotent, P has a nonzero root α. Then $\phi - \alpha$ is a divisor of zero in R and hence $\phi - \alpha \notin R^\times$.

Next, we consider the left proper ideal $R(\phi - \alpha)$ of R. By the Zorn lemma, there exists a maximal left proper ideal I containing $R(\phi - \alpha)$. The quotient R/I is then a simple R-module. Since R may not be commutative, in general, it is not true that ϕ acts on R/I as the scalar $\alpha \in \mathbf{C}^\times$. We claim nevertheless that ϕ doesn't acts trivially on R/I. Indeed, if it does, then $a.1_R \in I$. Since $\phi - \alpha$ also lies in I, it follows that $\alpha \in I$ where $\alpha \in \mathbf{C}^\times$. It would follow a contradiction for I is assumed to be a proper ideal. Therefore ϕ acts non trivially on the simple module M. □

Proposition 4.4. *Let G be a unimodular td-group, non necessarily compact. For every nonzero element $\phi \in \mathscr{H}(G)$, there exists a smooth irreducible representation (π, V_π) of G such that $\pi(\phi) \neq 0$.*

Proof. According to the Amitsur lemma Prop. 4.3, it is enough to find ϕ' so that $\phi \star \phi'$ is not nilpotent. For the construction of ϕ' seems to require the complex conjugation on $\mathbf{C}$ and the fact that the field $\mathbf{R}$ of real numbers is totally ordered.

We will assume that G is unimodular. We will choose a left invariant distribution μ on G so that for every compact open subgroup K of G, $\mu(\mathbb{I}_K)$ is a positive rational number. Using μ we will identify $\mathscr{C}_c^\infty(G) \simeq \mathscr{H}(G)$. The induced convolution product on $\mathscr{C}_c^\infty(G)$ is given by the formula

$$\phi \star \phi'(g) = \int_{G,\mu} \phi(h)\phi'(h^{-1}g). \tag{4.1}$$

We will choose ϕ' to be the complex conjugate of $\check{\phi}$ where $\check{\phi}(g) = \phi(g^{-1})$. Thus

$$\phi \star \phi'(e_G) = \int_{G,\mu} \phi(h)\overline{\phi(h)} \tag{4.2}$$

which is a strictly positive number as long as $\phi \neq 0$. In particular $\phi \star \phi' \neq 0$.

The same argument but applied to $\phi \star \phi'$ instead of ϕ shows that $\phi \star \phi' \star \phi \star \phi' \neq 0$. By induction, we have $(\phi \star \phi')^{*2^n} \neq 0$ for all n. We infer that $\phi \star \phi'$ is not nilpotent. □

Abelian *td*-groups

For every abelian *td*-group G, we will denote $\Omega(G)$ the space of all smooth characters of G. While it is obvious that $\Omega(G)$ is equipped with a structure of abelian group, it is less clear what is the natural geometric structure of $\Omega(G)$. The equivalent question is what are the "natural functions" on $\Omega(G)$.

The algebra of "natural functions" on $\Omega(G)$ should certainly include the algebra $\mathscr{D}_c(G)$ of distributions with compact support on G. We know that $\mathscr{D}_c(G)$ acts on every smooth representation of G by (2.11). If $\chi : G \to \mathbf{C}^\times$ is a smooth character, the action of $\mathscr{D}_c(G)$ on $V_\chi = \mathbf{C}$ is given by a homomorphism of algebras $\xi \mapsto \langle \xi, \chi \rangle$. Thus $\xi \in \mathscr{D}_c(G)$ gives rise to a function

$$\hat{\xi} : \Omega(G) \to \mathbf{C} \tag{4.3}$$

defined by $\chi \mapsto \langle \xi, \chi \rangle$. The map $\xi \mapsto \hat{\xi}$ is a generalization of the Fourier-Laplace transform. We will equip $\Omega(G)$ the coarsest topology such that for every $\xi \in \mathscr{D}_c(G)$ the subset of $\Omega(G)$ consisting of $\chi \in \Omega(G)$ such that $\hat{\xi}(\chi) = 0$ is a closed subset.

Proposition 4.5. *For every compact open subgroup K of G, let $\Omega(G;K)$ denote the subgroup of $\Omega(G)$ consisting of characters $\chi : G \to \mathbf{C}^\times$ trivial on K. Then $\Omega(G;K)$ is an open subset of $\Omega(G)$.*

Proof. For every compact open subgroup K of G, we denote V_K the closed subset of $\Omega(G)$ determined by the vanishing of $\hat{\xi}$ for all $\xi \in \mathscr{D}_c(G) \star e_K$. Then $\Omega(G;K)$ is the complement of of V_K in $\Omega(G)$, and therefore is an open subset. □

The case of multiplicative group

Let F nonarchimedean local field, $R^\times$ its ring of integers whose residue field is the finite field $\mathbf{F}_q$ with q elements. The multiplicative group $F^\times$ is an abelian *td*-group whose maximal compact subgroup is $R^\times$. We have an exact sequence

$$0 \to R^\times \to F^\times \xrightarrow{\text{ord}} \mathbf{Z} \to 0 \tag{4.4}$$

where for every $x \in F^\times$, $\text{ord}(x)$ is the unique integer such that $x = \mathrm{u}_F^{\text{ord}(x)}\alpha$ where u_F is a generator of the maximal ideal of R and $\alpha \in R^\times$. We every every choice of u_F we have an isomorphism $F^\times = R^\times \times \mathbf{Z}$ with $x \mapsto (\arg(x), \text{ord}(x))$ where $\arg(x) \in R^\times$ and $\text{ord}(x) \in \mathbf{Z}$ such that $x = \arg(x)\mathrm{u}_F^{\text{ord}(x)}$.

Every smooth character $\chi : F^\times \to \mathbf{C}^\times$ is of the form

$$\chi(x) = \omega(\arg(x))t^{\text{ord}(x)} \tag{4.5}$$

where $t \in \mathbf{C}^\times$, and $\omega : R^\times \to \mathbf{C}^\times$ is a smooth character of the compact group $R^\times$. As a smooth character of $R^\times$, ω has to be trivial on a compact open subgroup of the form $1 + \mathrm{u}_F^n R$, and in particular, ω has to be of finite order. The group of all smooth characters of $R^\times$ is the inductive limit of the finite groups of characters of $R^\times/(1 + \mathrm{u}_F^n R)$

$$\Omega(R^\times) = \varinjlim_n \Omega(R^\times/(1 + \mathrm{u}_F^n R)), \tag{4.6}$$

which is a infinite discrete torsion group. The space of all smooth characters of $F^\times$

$$\Omega(F^\times) = \Omega(R^\times) \times \mathbf{C}^\times \tag{4.7}$$

is a disjoint union of countably many copies of $\mathbf{C}^\times$. In particular, it is equipped with a structure of algebraic variety of $\mathbf{C}$. Components of $\Omega(F^\times)$ are to be indexed by characters ω of $R^\times$, and for every ω, the corresponding component $\Omega(F^\times, \omega)$ is isomorphic to $\mathbf{C}^\times$. The component $\Omega(F^\times, \omega)$ consists in all character $\chi : F^\times \to \mathbf{C}^\times$ whose restriction is ω, and in particular, it doesn't depend on the choice of u_F. However, the isomorphism $\Omega(F^\times, \omega) \simeq \mathbf{C}^\times$ does depend on the choice of u_F.

Proposition 4.6. *There is an homomorphism of algebras $\mathscr{D}_c(F^\times) \to \Gamma(\Omega(F^\times), \mathscr{O})$, from the algebra of distributions with compact support on G to the algebra of all algebraic regular functions on $\Omega(F^\times)$. By restriction, it induces an isomorphism of $\mathscr{H}(F^\times)$ and the subalgebra $\Gamma_0(\Omega(F^\times), \mathscr{O})$ of algebraic regular functions on $\Omega(F^\times)$ vanishing on all but finitely many components of $F^\times$.*

Proof. Over the component $\Omega(F^\times, \omega)$, we have the smooth representation $(\pi_\omega, \mathbf{C}[t, t^{-1}])$ where

$$\pi_\omega(x)\phi = \omega(\arg(x)) t^{\mathrm{ord}(x)} \phi \tag{4.8}$$

for every $\phi \in \mathbf{C}[t, t^{-1}]$. We obtain irreducible representations of $F^\times$ in this component by specializing t to elements of $\mathbf{C}^\times$.

For every $\xi \in \mathscr{D}_c(G)$, and $\omega \in \Omega(R^\times)$, the action $\pi_\omega(\xi)$ on $\mathbf{C}[t, t^{-1}]$ is a endomorphism of $\mathbf{C}[t, t^{-1}]$-modules. It follows that there exists a unique element $\hat{\xi}_\omega \in \mathbf{C}[t, t^{-1}]$ such that $\pi_\omega(\xi)$ acting on $\mathbf{C}[t, t^{-1}]$ as the multiplication by $\hat{\xi}_\omega$. The map $\xi \mapsto \hat{\xi}_\omega = (\hat{\phi}_\omega)_{\omega \in \Omega(R^\times)}$ defines a homomorphism of algebras

$$\mathscr{F} : \mathscr{D}_c(F^\times) \to \Gamma(\Omega(F^\times), \mathscr{O}) \tag{4.9}$$

For every compact open subgroup K of $R^\times$, for every $\xi \in e_K \star \mathscr{D}_c(G)$, we have $\hat{\xi}_\omega = 0$ unless ω lies in the finite subgroup $\Omega(R^\times/K)$ of $\Omega(R^\times)$. It follows that the restriction of (4.9) to $\mathscr{H}(R^\times)$ has image in the subalgebra $\Gamma_0(\Omega(F^\times), \mathscr{O})$ of regular algebraic functions on $\Omega(F^\times)$ whose restrictions to all but finitely many components vanish.

For every $\omega \in \Omega(R^\times)$ we construct an element $e_\omega \in \mathscr{H}(R^\times)$ as follows:

$$e_\omega = \omega \mu(R^\times)^{-1} \mu \tag{4.10}$$

where $\omega : F^\times \to \mathbf{C}$ is the extension by zero of the character $\omega : R^\times \to \mathbf{C}$ for any invariant distribution $\mu \in \mathscr{D}(F^\times)^{F^\times}$, . It is easy to see that the element e_ω are idempotents and mutually orthogonal i.e. $e_\omega \star e_{\omega'} = 0$ if $\omega \neq \omega'$.

One can check that

$$\mathscr{H}(F^\times) = \bigoplus_{\omega \in \Omega(R^\times)} e_\omega \star \mathscr{H}(F^\times)$$

and for every ω, the restriction of (4.9) to $e_\omega \star \mathscr{H}(F^\times)$ induces an isomorphism of this algebra on the algebra $\Gamma(\Omega(F^\times,\omega),\mathscr{O})$ of regular algebraic functions on $\Omega(F^\times,\omega)$. By passing to direct sum we get an isomorphism $\mathscr{H}(F^\times) \to \Gamma_0(\Omega(F^\times),\mathscr{O})$. □

The homomorphism (4.9) is injective. Indeed if $\xi \in \mathscr{D}_c(F^\times)$ such that $\hat{\xi} = 0$ then for every compact open subgroup K of $F^\times$ we have $\mathscr{F}(e_K \star \xi) = \hat{e}_K \star \hat{\xi} = 0$. Since $\mathscr{F}$ is injective on $\mathscr{H}(F^\times)$ this implies that $e_K \star \xi = 0$ for all K. We infer $\xi = 0$.

The homomorphism (4.9) is however not surjective. If ξ is supported on $\bigcup_{i=-n}^{n} \mathrm{u}_F^i R^\times$ then for every $\omega \in \Omega(R^\times)$, $\hat{\xi}_\omega \in \mathbf{C}[t,t^{-1}]$ is a Laurent polynomial of degree no more than n. It follows that if $\xi \in \mathscr{D}_c(G)$, the degree of the Laurent polynomials $\hat{\xi}_\omega$ as ω varies, is uniformly bounded.

A distribution $\xi \in \mathscr{D}(G)$ is said to be essentially compact if for every $\phi \in \mathscr{H}(G)$, we have $\xi \star \phi \in \mathscr{H}(G)$. We note $\mathscr{D}_{ec}(F^\times)$ the space of all essentially compact distributions. From its very definition $\mathscr{D}_{ec}(F^\times)$ is an algebra containing $\mathscr{D}_c(G)$.

Proposition 4.7. *The homomorphism* (4.9) *extends to an isomorphism of algebras*

$$\mathscr{D}_{ec}(F^\times) \to \Gamma(\Omega(F^\times),\mathscr{O}) \tag{4.11}$$

Proof. For every $\xi \in \mathscr{D}_{ec}(F^\times)$, for every $\omega \in \Omega(R^\times)$, as $\xi \star e_\omega \in \mathscr{D}_c(G)$, then $\pi_\omega(\xi \star e_\omega)$ acts on the representation $\mathbf{C}[t,t^{-1}]$ as homomorphism of $\mathbf{C}[t,t^{-1}]$-modules, with π_ω being the representation (4.8). Thus it acts as by the multiplication by a Laurent polynomial $\hat{\xi}_\omega$. The map $\xi \mapsto \hat{\xi} = (\hat{\xi}_\omega)$ defines a homomorphism of algebras $\mathscr{D}_{ec}(F^\times) \to \Gamma(\Omega(F^\times),\mathscr{O})$ extending (4.9).

Conversely, we need to prove that the map $\xi \mapsto (\xi \star e_\omega)$ defines an isomorphism

$$\mathscr{D}_{ec}(G) \to \prod_{\omega \in \Omega(R^\times)} \mathscr{D}_c(F^\times) \star e_\omega. \tag{4.12}$$

We define an inverse map to (4.12). For every compact open subgroup K of $R^\times$ we have $e_K = \sum_{\omega \in \Omega(R^\times/K)} e_\omega$ and hence

$$\mathscr{D}_c(F^\times) \star e_K = \bigoplus_{\omega \in \Omega(R^\times/K)} \mathscr{D}_c(F^\times) \star e_\omega. \tag{4.13}$$

It follows that the right hand side of (4.12) can be identified with the projective limit of $\mathscr{D}_c(F^\times) \star e_K$ as K ranging over all compact open subgroups of $R^\times$. Let (ξ_K) be a system of compatible elements $\xi_K \in \mathscr{D}_c(F^\times) \star e_K$. Since $\mathscr{C}_c(F^\times)$ is the union of $\mathscr{C}_c(F^\times) \star e_K$, the system of compatible elements (ξ_K) defines a linear form on $\mathscr{C}_c(F^\times)$ thus a element $\xi \in \mathscr{D}(G)$. Moreover, as ξ satisfies the property $\xi \star e_K$ has compact support for every compact open subgroup K, it is an essentially compact distribution. □

In the course of the above argument, we proved that $\mathscr{D}_{ec}(F^\times)$ can be identified with the projective limit

$$\mathscr{D}_{ec}(F^\times) = \varprojlim_K \mathscr{D}_c(F^\times) \star e_K \tag{4.14}$$

with K ranging over all compact open subgroups of $F^\times$. The subalgebra $\mathscr{D}_c(F^\times)$ consists in a system of compatible elements $\xi_K \in \mathscr{D}_c(F^\times) \star e_K$ whose supports are contained in a compact subset C of $F^\times$ that can be chosen independently of K.

Although there are obviously a lot more essentially compact distributions than distribution with compact support, it isn't obvious to construct an explicit example. We may ask a more general question: if X is a *td*-space acted on by a *td*-group G, find a distribution $\xi \in \mathscr{D}(X)$ such that for every compact open subgroup K of G, $e_K \star \xi$ is of compact support. Here is at least an example. Let us consider the space F acted on by $F^\times$. Let $\psi : F \to \mathbf{C}^\times$ be a nontrivial smooth character of the additive group F. It can be proven by a direct calculation that although the function $\psi : F \to \mathbf{C}$ is not of compact support, for every compact open subgroup K of $F^\times$, $e_K \star \psi$ is of compact support.

The case of the additive group

We now consider the case of the additive group F where F is a nonarchimedean local field. As in the previous paragraph, we denote R the ring of integers of F, u_F a generator of the maximal ideal of R and q the cardinal of the residue field $\mathbf{k}$. Let $\Omega(F)$ denote the group of all smooth characters of F. For every nontrivial character $\psi : F \to \mathbf{C}^\times$, the **conductor** of ψ is the maximal R-submodule of F, necessarily of the form $\mathrm{u}_F^n R$ for some $n \in \mathbf{Z}$, which is contained in the kernel of ψ.

Proposition 4.8. *There exists an additive character $\psi_1 \in \Omega(F)$ of conductor R. For every $x \in R$, if $\psi_x : F/R \to \mathbf{C}^\times$ is the smooth character given by $y \mapsto \psi_1(xy)$, then the map $x \mapsto \psi_x$ induces an isomorphism of groups $R \to \Omega(F/R)$ that can be extended to an isomorphism of topological groups*

$$\psi : F \to \Omega(F). \tag{4.15}$$

Proof. The quotient F/R is an union of finite groups

$$F/R = \bigcup_{n \in \mathbf{N}} \mathrm{u}_F^{-n} R/R \tag{4.16}$$

so that

$$\Omega(F/R) = \varprojlim_n \Omega(\mathrm{u}_F^{-n} R/R). \tag{4.17}$$

It is known that if A and B are finite groups with $A \subset B$ then the induced map on their group of characters $\Omega(B) \to \Omega(A)$ is surjective. It follows that the projective limit $\Omega(F/R)$ is nonempty. Moreover, there exists $\psi_1 \in \Omega(F/R)$ whose restriction to $\mathrm{u}_F^{-1} R/R$ is nontrivial.

For every $x \in R$, if we restrict the character $\psi_x(y) = \psi_1(xy)$ to $\mathrm{u}_F^{-1}R/R$, we obtain a nonzero homomorphism $\omega_1 : R/\mathrm{u}_F R \to \Omega(\mathrm{u}_F^{-1}R/R)$. We observe that both $R/\mathrm{u}_F R$ and $\Omega(\mathrm{u}_F^{-1}R/R)$ are naturally equipped with a structure of $\mathbf{k}$-vector space, and as such they both have dimension one. One can easily check that the map ω_1 is a nonzero $\mathbf{k}$-linear, and therefore it is an isomorphism.

Next, we prove by induction on $n \in \mathbf{N}$ that the restriction of ψ_x to $\mathrm{u}_F^{-n}R/R$ induces an isomorphism $\omega_n : R/\mathrm{u}_F^n R \to \Omega(\mathrm{u}_F^{-n}R/R)$. Assuming this statement is true for n, we will prove that it is true for $n+1$. For that we consider morphism between short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{u}_F^n R/\mathrm{u}_F^{n+1}R \arrow[r] \arrow[d] & R/\mathrm{u}_F^{n+1}R \arrow[r] \arrow[d, "\omega_{n+1}"] & R/\mathrm{u}_F^n R \arrow[r] \arrow[d, "\omega_n"] & 0 \\
0 \arrow[r] & \Omega(\mathrm{u}_F^{-n-1}R/\mathrm{u}_F^{-n}R) \arrow[r] & \Omega(\mathrm{u}_F^{-n-1}R/R) \arrow[r] & \Omega(\mathrm{u}_F^{-n}R/R) \arrow[r] & 0
\end{tikzcd} \tag{4.18}$$

Since ω_n is known to be an isomorphism, in order to prove that ω_{n+1} is an isomorphism, it is enough to prove that the restriction of ω_{n+1} to $\mathrm{u}_F^n R/\mathrm{u}_F^{n+1}R$ induces a isomorphism $\mathrm{u}_F^n R/\mathrm{u}_F^{n+1}R \to \Omega(\mathrm{u}_F^{-n-1}R/\mathrm{u}_F^{-n}R)$. For that, we can use again the argument for ω_1 by checking that the restriction of ω_{n+1} to $\mathrm{u}_F^n R/\mathrm{u}_F^{n+1}R$

$$\omega_{n+1}|_{\mathrm{u}_F^n R/\mathrm{u}_F^{n+1}R} : \mathrm{u}_F^n R/\mathrm{u}_F^{n+1}R \to \Omega(\mathrm{u}_F^{-n-1}R/\mathrm{u}_F^{-n}R)$$

is a nonzero $\mathbf{k}$-linear map between one-dimensional $\mathbf{k}$-vector spaces.

By passing to the projective limit as $n \to \infty$, we see that the map $x \mapsto \psi_x$ induces an isomorphism of profinite groups $R \to \Omega(F/R)$. For every $m \in \mathbf{N}$, the same argument as above shows that the map $x \mapsto \psi_x$ induces an isomorphism $\mathrm{u}_F^{-m}R \to \Omega(F/\mathrm{u}_F^m R)$. By passing to the inductive limit as $m \to \infty$, we obtain an isomorphism $\psi : F \to \Omega(F)$. □

Proposition 4.8 can be found in Tate's thesis [14] Lemma 2.2.1.

Compact *td*-groups

It is well known that, for finite and more generally compact groups, all irreducible representations are finite dimensional, and all finite dimensional representations are isomorphic to a direct sum of irreducible representations. In the literature on representation of finite and compact groups, for instant [**Serre**] or [**Simon**], the semisimplicity property of representations derived from the existence an invariant Hermitian form. For the purpose of this section is to establish similar semisimplicity property for a special class of representation of non necessarily compact *td*-groups, the compact representations, it will be useful to review the compact *td*-groups case in keeping the arguments purely algebraic, in other words, in avoiding using Hermitian forms.

Proposition 4.9. *If G is a compact td-group, then every smooth representation of G is a union of finite-dimensional representations. In particular, every smooth irreducible representation of G is finite-dimensional.*

Proof. Let (π, V) be an irreducible smooth representation of G and $v \in V$ a nonzero vector. The fixer of v is a compact open subgroup K of G. If $g_1, \ldots, g_n$ is a set of representatives of right cosets of K in G, then the finite-dimensional subspace V_1 of V generated by $\pi(g_1)v, \ldots, \pi(g_n)v$ is stable under G. It follows that V is a union of finite dimensional representations of G. □

Let G be a compact *td*-group. For every an irreducible smooth representation (π, V) of G, V is equipped with a structure of $\mathscr{H}(G)$-modules i.e we have a homomorphism of algebras

$$\pi : \mathscr{H}(G) \to \mathrm{End}(V). \tag{4.19}$$

Under the assumption of compactness of G, we will construct a section of h_π that is a homomorphism of nonunital algebras so that we can split off $\mathrm{End}(V)$ as a multiplicative direct factor of $\mathscr{H}(G)$.

After Prop. 4.9, we know that V is finite dimensional. For every $v \in V$ and $v^* \in V^*$ we will consider the matrix coefficient m_{v,v^*} which is a smooth function $m_{v,v^*} : G \to \mathbf{C}$ given by

$$m_{v,v^*}(g) = \langle v, \pi^*(g)v^* \rangle. \tag{4.20}$$

A priori, one may ask why shouldn't we use the similar formula $\langle \pi(g)v, v^* \rangle$ instead. The formula (4.20) is the only one reasonable in the sense that it produces a map

$$m_\pi : V \otimes V^* \to \mathscr{H}(G) \tag{4.21}$$

that is $G \times G$-equivariant i.e. m satisfies

$$m_\pi(\pi(g_1)v \otimes \pi^*(g)v^*) = l_{g_1} r_{g_2} m_\pi(v \otimes v^*). \tag{4.22}$$

Each Haar measure μ on G defines a $G \times G$-equivariant isomorphism $\mu : \mathscr{C}^\infty(G) \to \mathscr{H}(G)$ given by $f \mapsto f\mu$. Thus $\mu \circ m : V \otimes V^* \to \mathscr{H}(G)$ is a $G \times G$-equivariant map and so is

$$\pi \circ \mu \circ m_\pi : V \otimes V^* \to V \otimes V^*. \tag{4.23}$$

Since V is an irreducible representation of G, $V \otimes V^*$ is an irreducible representation of $G \times G$. By the Schur lemma, $\pi \circ \mu \circ m_\pi$ must be a scalar to be denoted by $c_\mu(\pi)$.

Proposition 4.10. *We have $c_\mu(\pi) = \dim(V)^{-1}\mathrm{vol}_\mu(G)$. In particular, $c_\mu(\pi) \neq 0$.*

Proof. For every vector $v \otimes v^* \in V \otimes V^*$, we have

$$\pi \circ \mu \circ m_\pi(v \otimes v^*) = c_\mu(V)(v \otimes v^*)$$

by the very definition of $d_\mu(V)$. If we denote $f_{v\otimes v^*}$ the element of $\mathrm{End}(V)$ corresponding to $v \otimes v^* \in V \otimes V^*$ then for every $u \in V$ we have

$$f_{v\otimes v^*} u = v\langle v^*, u\rangle. \tag{4.24}$$

It follows that for every $u \in V$ and $u^* \in V^*$, we have

$$\langle f_{v\otimes v^*} u, u^*\rangle = \langle v, u^*\rangle\langle v^*, u\rangle. \tag{4.25}$$

What we have to prove is that for every $u \in V$ and $u^* \in V^*$, we have

$$\langle \pi \circ \mu \circ m_\pi(v \otimes v^*)u, u^*\rangle = \dim(V)^{-1}\mathrm{vol}_\mu(G)\langle v, u^*\rangle\langle v^*, u\rangle. \tag{4.26}$$

By the very definition of m and h_π, we have

$$\pi \circ \mu \circ m_\pi(v \otimes v^*)u = \int_{G,\mu} \langle v, \pi^*(g)v^*\rangle \pi(g)u \tag{4.27}$$

so that for every $u^* \in V^*$, the left hand side of (4.26) is

$$\int_G \mu\langle v, \pi^*(g)v^*\rangle\langle u^*, \pi(g)u\rangle = \langle v \otimes u^*, \int_{G,\mu} (\pi^*(g) \otimes \pi(g))(v^* \otimes u)\rangle. \tag{4.28}$$

The calculation of the right hand side of (4.28) takes several steps. First we evaluate the integral

$$\mathrm{av}(w) = \int_{G,\mu} (\pi^*(g) \otimes \pi(g))w$$

depending on $w \in V^* \otimes V$. Because μ is an invariant measure on G, $\mathrm{av}(w)$ is a G-invariant vector of $V^* \otimes V$. Since V is irreducible, the space of G-invariant vectors of $V^* \otimes V$ is one dimensional and generated by the vector $1_V \in V^* \otimes V$ that corresponds to the identity endomorphism of V. Thus there is a unique linear form $\ell : V^* \otimes V \to \mathbf{C}$ such that

$$\mathrm{av}(w) = c(w)1_V.$$

More over the linear form $\ell : V^* \otimes V \to \mathbf{C}$ is also G-invriant and thus belongs to the one dimensional space of G-invariant linear form on $V^* \otimes V$ generated by $\mathrm{ev} : V^* \otimes V \to \mathbf{C}$ defined by $\mathrm{ev}(v^* \otimes v) = \langle v^*, v\rangle$. There exists a constant $\alpha \in \mathbf{C}$ such that

$$\mathrm{av}(w) = \alpha \mathrm{ev}(w)1_V.$$

It remains to compute the constant c. For this purpose, we can pick $w = 1_G$. On the one hand, we have $\mathrm{av}(1_G) = \mathrm{vol}_\mu(G)1_G$, and on the other we have $\mathrm{ev}(1_G) = \dim(V)$, and therefore $\alpha = \dim(V)^{-1}\mathrm{vol}_\mu(G)$. We infer the formula

$$\mathrm{av}(w) = \dim(V)^{-1}\mathrm{vol}_\mu(G)\mathrm{ev}(w)1_V$$

holds for all $w \in V^* \otimes V$.

The right hand side of (4.28) is now equal to:

$$\langle v \otimes u^*, \dim(V)^{-1}\mathrm{vol}_\mu(G)\mathrm{ev}(v^* \otimes u)1_V\rangle = \dim(V)^{-1}\mathrm{vol}_\mu(G)\langle v, u^*\rangle\langle v^*, u\rangle$$

that implies (4.26). □

With $c_\mu(V) \neq 0$ granted, we obtain a section of $\pi : \mathscr{H}(G) \to V \otimes V^*$

$$h_\pi : V \otimes V^* \to \mathscr{H}(G). \tag{4.29}$$

given by the formula

$$h_\pi = c_\mu(V)^{-1}\mu \circ m_\pi = \dim(\pi)\mathrm{vol}_\mu(G)^{-1}\mu m_\pi. \tag{4.30}$$

We derive a decomposition of $\mathscr{H}(G)$ as a direct sum of algebras

$$\mathscr{H}(G) = \mathscr{H}(G)_\pi \oplus \mathscr{H}(G)_\pi^\perp \tag{4.31}$$

where $\mathscr{H}(G)_\pi = V \otimes V^*$ and $\mathscr{H}(G)_\pi^\perp = \ker(h_\pi)$. The idempotent of $\mathscr{H}(G)$ corresponding to the unit of the component $\mathscr{H}(G)_\pi$ is

$$e_\pi = h_\pi(1_V) = \dim(\pi)\mathrm{vol}_\mu(G)^{-1}\mu\chi_\pi \tag{4.32}$$

where $\chi_\pi = m(1_G) \in \mathscr{C}^\infty(G)$ is the **character of** π.

Proposition 4.11. *The elements e_π are central idempotents of $\mathscr{H}(G)$. If π and π' are not isomorphic, then $e_\pi \star e_{\pi'} = 0$.*

Proof. □

We can now derive the Peter-Weyl theorem for compact *td*-groups.

Proposition 4.12. *Let G be a compact td-group. The direct sum of $h_\pi : V_\pi \otimes V_\pi^* \to \mathscr{H}(G)$ over the set of isomorphism classes of irreducible representations π of G*

$$\bigoplus_\pi h_\pi : \bigoplus_\pi V_\pi \otimes V_\pi^* \to \mathscr{H}(G) \tag{4.33}$$

is an isomorphism of algebras.

Proof. First we prove that the map (4.33) is injective. Let $\pi_1, \dots, \pi_n$ be a finite set of non isomorphic irreducible representations of G and $w_i \in V_{\pi_i} \otimes W_{\pi_i}^*$ such that $\sum_{i=1}^n n_{\pi_i}(w_i) = 0$. Since $e_{\pi_i} = n_{\pi_i}(1_{V_i})$ we have $n_{\pi_i}(w_i) = e_{\pi_i} \star n_{\pi_i}(w_i)$ for all i. Since $e_{\pi_i} \star e_{\pi_j} = 0$ for $i \neq j$, we derive $n_{\pi_i}(w_i) = e_{\pi_i} \star n_{\pi_i}(w_i) = 0$ by operating the left convolution by e_{π_i} on $\sum_{j=1}^n e_{\pi_j} \star n_{\pi_j}(w_j)$. It follows that $w_i = 0$ for all i since n_{π_i} is injective.

Now we prove that (4.33) is surjective. For every $\phi \in \mathscr{H}(G)$ there exists a compact open subgroup K of G such that $e_K \star \phi = \phi \star e_K = \phi$. Since G is compact, we may assume that K is a normal subgroup of G. Unless $\pi : G \to \mathrm{GL}(V_\pi)$ factorizes through the finite quotient G/K, we have $e_\pi \star e_K = 0$. It follows that unless $\pi : G \to \mathrm{GL}(V_\pi)$ factorizes through the finite quotient G/K, we have $e_\pi \star \phi = e_\pi \star e_K \star \phi = 0$. It now makes sense to consider the difference

$$\phi' = \phi - \sum_\pi e_\pi \star \phi \tag{4.34}$$

for the sum $\sum_\pi e_\pi \star \phi$ has only finitely many nonzero terms. For every irreducible representation π, we have $e_\pi \star \phi' = 0$. Since $\pi(e_\pi \star \phi') = \pi(e_\pi)\pi(\phi') = \pi(\phi')$ we derive $\pi(\phi') = 0$ for all π. By the following separation lemma, we infer then $\phi' = 0$ and therefore (4.33) is surjective. □

Proposition 4.13. *Let G be a compact td-group. For every smooth irreducible representation π of G, $e_\pi = h_\pi(1_{V_\pi})$ is a central idempotent of $\mathscr{H}(G)$. For every smooth representation W of G, we have a decomposition*

$$W = \bigoplus_\pi e_\pi W \tag{4.35}$$

for π ranging over the set of isomorphism classes of smooth irreducible representations of G. More over for every π, $e_\pi W$ is isomorphic to a direct sum of copies of V_π.

Proof. Because e_π are central elements of $\mathscr{H}(G)$, $e_\pi W$ are a $\mathscr{H}(G)$-submodule of W. We consider the $\mathscr{H}(G)$-linear map

$$\bigoplus_\pi e_\pi W \to W \tag{4.36}$$

given by the addition in W. We claim that this map is an isomorphism.

First we prove that it is injective. Let $w_1, \dots, w_n \in W$ and $\pi_1, \dots, \pi_n$ non isomorphic irreducible representations of G such that $\sum_{i=1}^n e_{\pi_i} w_i = 0$. Using the relation $e_{\pi_i} \star e_{\pi_j} = 0$ if $i \neq j$, and $e_{pi_i} \star e_{pi_i} = e_{pi_i}$, this implies that $e_{\pi_i} w_i = 0$ for all i. This implies that (4.36) is injective.

Let us prove that (4.36) is surjective. For every $w \in W$ there exists a compact open subgroup K of G such that $e_K w = w$. Since G is compact, we can assume that K is a normal subgroup. Then we have

$$e_K = \sum_\pi e_\pi \star e_K \tag{4.37}$$

where $e_\pi \star e_K = 0$ unless π factorizes through G/K in which case $e_\pi \star e_K$. We derive $w = e_K w = \sum_\pi e_\pi w$ where π ranges over the irreducible representations π factorizing through G/K. It follows that w lies in the image of (4.36).

It remains to prove that $e_\pi W$ is isomorphic to a direct sum of copies of V_π. This derives from the G-equivariant surjective map

$$e_\pi \mathscr{H}(G) \times W \to e_\pi W \tag{4.38}$$

where G acts on $e_\pi \mathscr{H}(G) \times W$ by $g(\phi \otimes w) = g(\phi) \otimes w$. Since we have an G-equivariant isomorphism $e_\pi \mathscr{H}(G) \simeq V_\pi \otimes V_\pi^*$ where g acts on $e_\pi \mathscr{H}(G)$ be left translation and on $V_\pi \otimes V_\pi^*$ through its action on the first factor. It follows that $e_\pi \mathscr{H}(G) \times W$ is isomorphic to a direct sum of copies of V_π. Thus $e_\pi W$ is a sum, not necessarily direct, of copies of V_π. Since V_π are simple modules, this implies that W is isomorphic to a direct sum of copies of V_π (see [**Lang-Algebra**]). □

Matrix coefficients of compact representations

The concept of matrix coefficients already used in the study of compact groups can be generalized to not necessarily compact groups. Let (π, V) be an irreducible smooth representation of a *td*-group G, V^* its dual and V' its contragredient consisting of smooth vectors in V^*. For given vectors $v \in V$ and $v^* \in V^*$, we can define the matrix coefficient function

$$m_{v,v^*}(g) = \langle v, \pi^*(g) v^* \rangle. \tag{4.39}$$

Since v is a smooth vector, v is invariant under a compact open subgroup K of G. It follows that m_{v,v^*} is invariant under the right action of K. It is thus a smooth function on G i.e. $m_{v,v^*} \in \mathscr{C}^\infty(G)$, and moreover it is smooth vector in this space with respect to the action of G by right translation:

$$m_{v,v^*} \in \mathscr{C}^\infty(G)^{\mathrm{sm}(r_G)}. \tag{4.40}$$

If now $v^* \in V'$ is a smooth vector in V^* then the matrix coefficient m_{v,v^*} is a smooth vector in $\mathscr{C}^\infty(G)$ with respect to the action of $G \times G$ by left and right translation i.e. $m_{v,v^*} \in \mathscr{C}^\infty(G)^{\mathrm{sm}(l_G \times r_G)}$. We have thus a morphism of smooth representations of $G \times G$:

$$m_\pi : V \otimes V' \to \mathscr{C}^\infty(G)^{\mathrm{sm}(l_G \times r_G)} \tag{4.41}$$

defined by the matrix coefficient.

An irreducible smooth representation V is said to be **compact** if $m(V \otimes V')$ is contained in the space $\mathscr{C}_c^\infty(G)$ of smooth functions with compact support. If V is an irreducible representation of G then $V \otimes V'$ is an irreducible representation of $G \times G$. It follows that $m(V \otimes V') \subset \mathscr{C}_c^\infty(G)$ if and only if $m(V \otimes V') \cap \mathscr{C}_c^\infty(G) \neq \emptyset$. In other words, V is compact if and only if there exist nonzero vectors $v \in V$ and $v' \in V'$ such that $m_{v,v'}$ is compactly supported.

Proposition 4.14. *Let (π, V) be a smooth representation of a td-group G. For every $v \in V$ and compact open subgroup K of G, we consider the function $\phi_{K,v} : G \to V$ given by $\phi_{K,v}(g) = \pi(e_K \star \delta_g \star e_K)v$. If π is compact, then the image of $\phi_{K,v}$ lies in a finite dimensional subspace of V.*

Proof. If the subvector space of V generated by $\phi_{K,v}(g)$ isn't finite dimensional, there exists an infinite sequence $g_1, g_2, \ldots \in G$ such that the vectors $v_i = \phi_{K,v}(g_i) \in V^K$ are linearly independent. We observe that the set $\{g_i\}$ is not contained in any compact subset of G, because any compact subset C would have nonempty intersection with only finitely many double cosets KgK. Now we construct a vector $v^* \in V^*$ such that $\langle v', v_i \rangle = 1$ for all i.[2] By replacing $v' = e_K v^*$ we obtain $v' \in V'$ satisfying the same equalities. Then $m_{v,v'}(g_i) = 1$ for all i and therefore the support of the matrix coefficient $m_{v,v'}$ contains g_i, and therefore it is not compact. □

Proposition 4.15. *Every compact irreducible smooth representation is admissible.*

Proof. Let (π, V) be a compact irreducible representation of G. For every compact open subgroup K and $v \in V^K$ a nonzero vector, we have defined the function $\phi_{K,v} : G \to V^K$ by the rule $\phi_{K,v}(g) = \pi(e_K \star \delta_g \star e_K)v$. The subspace U of V^K generated by the vectors of the form $\phi_{K,v}(g)$, is stable under the action of $\mathscr{H}_K = e_K \star \mathscr{D}_c(G) \star e_K$. Since V is irreducible, V^K is an irreducible $\mathscr{H}_K$-module, and hence we must have $U = V^K$. By Prop. 4.14, U is finite dimensional. It follows that V is finite dimensional and hence π is admissible. □

Proposition 4.16. *Let (π, V) be a smooth irreducible representation of a td-group G. Then the following properties are equivalent:*

(1) The matrix coefficients of π are compactly supported functions,

(2) The function $\phi_{K,v} : G \to V$ given by $\phi_{K,v}(g) = \pi(e_K \star \delta_g \star e_K)v$ is compactly supported.

Proof. First we prove (1) ⇒ (2): Assume that for every $v \in V$ and $v' \in V'$, the support of the matrix coefficient $m_{v,v'}$ is compact. We claim that the subvector space of V generated by $\phi_{K,v}(g)$ is finite dimensional. Indeed, if it is not, there exists an infinite sequence $g_1, g_2, \ldots \in G$ such that the vectors $v_i = \phi_{K,v}(g_i) \in V^K$ are linearly independent. We observe that the set $\{g_i\}$ is not contained in any compact subset of G, because any compact subset C would have nonempty intersection with only finitely many double cosets KgK. Now we construct a vector $v' \in V'^K$ such that $\langle v', v_i \rangle = 1$ for all i. Then $m_{v,v'}(g_i) = 1$ for all i and therefore the support of the matrix coefficient $m_{v,v'}$ contains g_i, and therefore it is not compact.

Assume that the subvector space U of V generated by the vectors $\pi_{K,v}(g)$ when g varies, is finite dimensional. We can choose finitely many vectors $v'_1, \ldots, v'_n \in V'$ such that a vector $u \in U$ is zero if and only if $\langle v'_i, u \rangle = 0$ for all $i \in \{1, \ldots, n\}$. It follows that the support of

[2] Hahn-Banach?

the function $\phi_{K,v}$ is contained in the union of the supports of matrix coefficients $m_{v'_i,v}$ as i ranging the finite set $\{1,\ldots,n\}$. Thus, the support of $\phi_{K,v}$ is compact.

Next we prove (2) $\Rightarrow$ (1): Assume that for every compact open subgroup K, the function $G \to V$ defined by $g \mapsto \pi(e_K \star \delta_g \star e_K)v$ is nonzero and compactly supported. For every $v \in V$, and $v' \in V'$, if K is a compact open subgroup K fixing both v and v', then $m_{v,v'}(g) = \langle v', \phi_{K,v}(g)\rangle$. In particular, the support of $m_{v,v'}$ in contained in the support of the function $\phi_{K,v}(g)$, and hence is compact. □

The formal degree

The action of a *td-G* on every irreducible smooth representation (π, V) can be extended as a homomorphism of algebras $\pi : \mathscr{H}(G) \to \mathrm{End}_C(V)$. If π is admissible, $\pi(\mathscr{H}(G))$ is contained in the subalgebra $\mathrm{End}_{\mathrm{fin}}(V)$ of endomorphism of V with finite dimensional image. This homomorphism is $G\times G$-equivariant. We have also identified the space of $G\times G$-smooth vectors in $\mathrm{End}_{\mathrm{fin}}(V)$ with $V \otimes V'$ so that π factorizes through a $G\times G$-equivariant homomorphism of algebras $\mathscr{H}(G) \to V \otimes V'$ that we will also denote by π.

If (π, V) is a compact irreducible representation then we have seen that π is admissible (Prop. 4.15). Moreover, by the very definition of compactness, the matrix coefficient gives rise to a $G \times G$-equivariant map

$$m_\pi : V \otimes V' \to \mathscr{C}_c^\infty(G). \tag{4.42}$$

Given a choice of a Haar measure μ, we can identify the space $\mathscr{C}_c^\infty(G)$ and the space $\mathscr{H}(G)$ of smooth measures with compact support: the map $\phi \mapsto \phi\mu$ defines a $G\times G$-equivariant isomorphism $\mathscr{C}_c^\infty(G) \to \mathscr{H}(G)$. The composition of π and m_π gives rise to a $G\times G$-equivariant map

$$\pi \circ m_\pi : V \otimes V' \to V \otimes V'. \tag{4.43}$$

For $V \otimes V$" is an irreducible representation of $G\times G$, this map must be a scalar after the Schur lemma. We will denote the scalar by $c_\mu(\pi)$. This constant depends proportionally on the Haar measure μ.

Proposition 4.17. *The constant $c_\mu(\pi)$ is non zero.*

Proof. Let $w = V \otimes V'$ be a non zero vector of $V \times V'$ and $\phi = \mu m_\pi(w) \in \mathscr{H}(G)$ its image in $\mathscr{H}(G)$. We want to prove that $h_\pi(\phi) \neq 0$. For every irreducible representation $(\pi', V_{\pi'})$ not isomorphic to π, the composition

$$\pi' \circ m_\pi : V \otimes V' \to \mathrm{End}(V_{\pi'}) \tag{4.44}$$

is $G \times G$-equivariant. For every $u \in V_{\pi'}$, the map $V \otimes V' \to V_{\pi'}$ given by $w \mapsto \pi'(m_\pi(w))(u)$ is G-equivariant with respect to the left action of G on $V \otimes V'$. As left G-module, $V \otimes V'$ is isomorphic to a direct sum of copies of V, and as $V_{\pi'}$ is an simple G-module non isomorphic to V, the G-equivariant map $V \otimes V' \to V_{\pi'}$ is necessarily zero. It follows that $\pi'(m_\pi(w)) = 0$ for every irreducible representation π' non isomorphic to π. Assuming $w \neq 0$, and hence $m_\pi(w) \neq 0$, we have $\pi(\phi) \neq 0$ by the separation lemma (Prop. 4.4). □

We will denote $d_\mu(\pi) = c_\mu(\pi)^{-1}$ and call it the **formal degree**. As the constant $c_\mu(\pi)$ depends proportionally on the choice of the Haar measure μ, its inverse depends on it in the inversely proportional way.

We now have a $G \times G$-equivariant map

$$h_\pi = d_\mu(\pi)\mu m_\pi : V \otimes V' \to \mathscr{H}(G) \tag{4.45}$$

which is a section of $\pi : \mathscr{H}(G) \to V \otimes V'$ by the very definition of the formal degree. We also observe that h_π doesn't depend on the choice of the Haar measure μ as $d_\mu(\pi)$ varies inverse proportionally with μ. As a section of $\pi : \mathscr{H}(G) \to V \otimes V'$, h_π is necessarily injective.

Proposition 4.18. *The section $h_\pi : V \otimes V' \to \mathscr{H}(G)$ is a homomorphism of algebras. Its image $\mathscr{H}(G)_\pi$ is a two sided ideal of $\mathscr{H}(G)$.*

Proof. As we have seen in the proof of Prop. 4.17, the map (4.44) is zero. In other words, for every $w \in V \otimes V'$, for every irreducible representation $(\pi', V_{\pi'})$ non isomorphic to π, we have $\pi'(h_\pi(w)) = 0$.

For $w_1, w_2 \in V \otimes V'$, both equalities $\pi'(h_\pi(w_1) \star h_\pi(w_2)) = 0$ and $\pi'(h_\pi(w_1 w_2)) = 0$ hold for for every irreducible representation π' non isomorphic to π. On the other hand we have

$$\pi(h_\pi(w_1) \star h_\pi(w_2)) = \pi(h_\pi(w_1 w_2)) = w_1 w_2$$

for $\phi : \mathscr{H}(G) \to V \otimes V'$ is a homomorphism of algebras and h_π is a section of π. By the separation lemma (Prop, 4.4), we have $h_\pi(w_1 w_2) = h_\pi(w_1) \star h_\pi(w_2)$. In other words $h_\pi : V \otimes V' \to \mathscr{H}(G)$ is a homomorphism of algebras. In particular, $\mathscr{H}(G)_\pi$ is a subalgebra of $\mathscr{H}(G)$.

For every $w \in V \otimes V'$, every $\phi \in \mathscr{H}(G)$ we have

$$\pi'(h_\pi(w) \star \phi) = \pi'(h_\pi(w))\pi'(\phi) = 0 \tag{4.46}$$

and similarly $\pi'(\phi \star h_\pi(w)) = 0$. Using again the separation lemma we have

$$h_\pi(w) \star \phi = h_\pi(w\pi(\phi)) \text{ and } \phi \star h_\pi(w) = h_\pi(\pi(\phi)w). \tag{4.47}$$

We infer that $\mathscr{H}(G)_\pi$ is a two sided ideal of $\mathscr{H}(G)$. □

We will denote $\mathscr{H}(G)_\pi^\perp$ the kernel of the homomorphism $\pi : \mathscr{H}(G) \to V \otimes V'$. By construction, $\mathscr{H}(G)_\pi^\perp$ is a two sided ideal of $\mathscr{H}(G)$. Since h_π is a section of π, we have a decomposition in direct sum

$$\mathscr{H}(G) = \mathscr{H}(G)_\pi \oplus \mathscr{H}(G)_\pi^\perp \tag{4.48}$$

of two sided ideals. For both $\mathscr{H}(G)_\pi$ and $\mathscr{H}(G)_\pi^\perp$ are two sided ideals, it is easy to see that the above decomposition in direct sum respects the structure of $\mathscr{D}_c(G) \times \mathscr{D}_c(G)$-module of $\mathscr{H}(G)$. In particular, it is a decomposition of $G \times G$-modules.

In the case of a compact group, the algebra $\mathscr{H}(G)_\pi$ has a unit given by $d_\mu(\pi)\mu\chi_\pi$ where χ_π is the character of π. In the present situation, for the representation π is of infinite dimension, it is not obvious to make sense of its character χ_π. Although, a posteriori, we will make sense of χ_π, it won't be a smooth function on G. In fact, the algebra $\mathscr{H}(G)_\pi$ doesn't have a unit in general. Nevertheless, at a finite level, the decomposition (4.48) is given by a central idempotent element.

Proposition 4.19. *Let (π, V) be a compact irreducible representation of G. For every compact open subgroup K of G, $e_{\pi,K} = h_\pi(\pi(e_K))$ is a central idempotent of $\mathscr{H}_K(G)$. Moreover, we have*

$$\mathscr{H}(G)_\pi \cap \mathscr{H}_K(G) = e_{\pi,K}\mathscr{H}_K(G) \tag{4.49}$$

and

$$\mathscr{H}(G)_\pi^\perp \cap \mathscr{H}_K(G) = (e_K - e_{\pi,K})\mathscr{H}_K(G) \tag{4.50}$$

If K' is a compact open subgroup contained in K then we have $e_K \star e_{\pi,K'} = e_{\pi,K}$.

Proof. Since h_π is a homomorphism of algebras, $e_{\pi,K} = h_\pi(\pi(e_K))$ is an idempotent. Using the formula (4.47) with $\phi = e_K$ and $w = \phi(e_K)$, we have $e_{\pi,K} \in \mathscr{H}_K(G)$. Applying the same formula for an arbitrary element $\phi \in \mathscr{H}_K(G)$ and $w = \pi(e_K)$, we obtain $\phi \star e_{\pi,K} = e_{\pi,K} \star \phi$. Therefore, $e_{\pi,K}$ is a central idempotent of $\mathscr{H}_K(G)$. □

Splitting property with respect to a compact representation

Proposition 4.20. *Let (π, V_π) be an irreducible compact representation of a td-group G. Then for every smooth representation W of G, there is a decomposition in direct sum $W = W_\pi \oplus W_\pi^\perp$, depending functorially on W, such that W_π is isomorphic to a direct sum of copies of V_π, and $W_\pi^\perp$ has no subquotient isomorphic to V_π.*

Proof. We define $W_\pi = \mathscr{H}(G)_\pi W$ and $W_\pi^\perp = \mathscr{H}(G)_\pi^\perp W$. We claim that $W = W_\pi \oplus W_\pi^\perp$. For every compact open subgroup K, $e_K W$ is a module over the unital algebra $\mathscr{H}_K(G)$. By Prop. 4.19, we have a central idempotent $e_{\pi,K} \in \mathscr{H}_K(G)$. Thus $e_K W$ decomposes as a direct sum of $\mathscr{H}_K(G)$-modules

$$e_K W = e_{\pi,K} W \oplus (e_K - e_{\pi,K})W \tag{4.51}$$

We also have $e_{\pi,K}W = \mathscr{H}(G)_\pi W \cap e_K W$ and $(e_K - e_{\pi,K})W = \mathscr{H}(G)_\pi^\perp W \cap e_K W$. Since the splitting (4.51) holds for all K, we have $W = W_\pi \oplus W_\pi^\perp$. □

Compact modulo the center

Compact representations are the simplest from the analytic point of view, but also the most interesting from the number theoretic point view. We will come back to investigate extensively compact representations later.

We observe however that there are compact representations only if the center Z of G is compact. Indeed, if the center Z of G is not compact, by the Schur lemma, for every smooth irreducible representation (π, V) of G, there exists a smooth character $\chi : G \to \mathbf{C}^\times$ such that $\pi(z)v = \chi(z)v$ for all $z \in Z$ and $v \in V$. It follows that the matrix coefficient satisfies the formula $m_{v,v'}(zg) = \chi(z)m_{v.v'}(g)$ and therefore can't be of compact support unless Z is compact. For a *td*-group G whose center is not compact, there is a more useful notion of **compact representation modulo the center**. An irreducible smooth representation of G is said to be compact modulo the center Z if its matrix coefficients have supports contained in ZC where C is some compact open subset of G. We will see later, if G is a reductive p-adic group, an irreducible representation of G is compact modulo center if and only if it is cuspidal in the sense of Harish-Chandra.

Proposition 4.21. *Every compact modulo the center irreducible representation of G is admissible.*

Proof. We will argue in an almost identical way as in the proof of Prop. 4.15. In fact, Prop. 4.14 is valid under the hypothesis of compactness modulo the center. The deduction of Prof. 4.15 from Prop. 4.14 is the same. □

Bibliographical comments

5 Sheaves on *td*-spaces

Let X be a *td*-space. We consider the category Top_X whose objects are open subsets of X and morphisms are inclusion maps. A presheaf on X is a contravariant functor

$$\mathscr{F} : \mathrm{Top}_X \to \mathrm{Vec}_C.$$

A sheaf of $\mathbf{C}$-vector spaces on X is a presheaf satisfying the glueing condition: if U is an open subset of X and $\mathscr{U} = \{U_\alpha | \alpha \in \alpha_{\mathscr{U}}\}$ is a covering family of open subset of U then the restriction map defines the equalizer sequence:

$$\mathscr{F}(U) \to \prod_{\alpha \in \alpha_{\mathscr{U}}} \mathscr{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta \in \alpha_{\mathscr{U}}} \mathscr{F}(U_\alpha \cap U_\beta). \tag{5.1}$$

We will call *td*-sheaf a sheaf a $\mathbf{C}$-vector space on a *td*-space X. For every *td*-space, we denote by Sh_X the category of sheaves of $\mathbf{C}$-vector spaces on a *td*-space X.

One can form a sheaf $\mathscr{F}$ associated to each presheaf F: for every open subset U of X, we define $\mathscr{F}(U)$ to be the limit of the inductive system consisting of equalizers

$$\mathrm{eq}\left[\prod_{\alpha\in\alpha_{\mathscr{U}}}\mathscr{F}(U_\alpha)\rightrightarrows\prod_{\alpha,\beta\in\alpha_{\mathscr{U}}}\mathscr{F}(U_\alpha\cap U_\beta)\right]$$

as $\mathscr{U}=\{U_\alpha|\alpha\in\alpha_{\mathscr{U}}\}$ ranging over all families of open covering of of U. For instant, for the constant presheaf $U\mapsto\mathbf{C}$ then the associated sheaf will be $U\mapsto\mathscr{C}^\infty(U)$. We will denote this sheaf $\mathscr{C}^\infty_X$.

The category of *td*-sheaves on a *td*-space X is an abelian category. If $\alpha:\mathscr{F}\to\mathscr{G}$ is a $\mathbf{C}$-linear map of *td*-sheaves that $\ker(\alpha)$ is the sheaf $U\mapsto\ker(\mathscr{F}(U)\to\mathscr{G}(U))$ and $\mathrm{coker}(\alpha)$ is the associated sheaf of the presheaf $U\mapsto\mathrm{coker}(\mathscr{F}(U)\to\mathscr{G}(U))$.

Fibers of *td*-sheaves

If $\mathscr{F}$ is a *td*-sheaf on a *td*-space X, and $x\in X$ is an element of X, we define the **fiber** $\mathscr{F}_x$ as the inductive limit

$$\mathscr{F}_x=\varinjlim_{x\in U}\mathscr{F}(U) \tag{5.2}$$

ranging over all neighborhoods U of $x\in X$.

Proposition 5.1. *A sequence $0\to\mathscr{A}\to\mathscr{B}\to\mathscr{C}\to 0$ is a short exact sequences of td-sheaves on X if and only if for every $x\in X$, the sequence $0\to\mathscr{A}_x\to\mathscr{B}_x\to\mathscr{C}_x\to 0$ is an exact sequence of* $\mathbf{C}$*-vector spaces.*

Proof. □

If $\phi\in\Gamma(X,\mathscr{F})$ is a global section of $\mathscr{F}$, we define $\phi(x)$ to be the element $\phi(x)\in\mathscr{F}_x$ consisting of the system of restrictions $\phi|U$ of ϕ to every neighborhood U of x. In particular if $\phi(x)=0$ then there exists a open neighborhood U of x such that $\phi|U=0$. In particular, the subset V of X consisting of elements $x\in X$ such that $\phi(x)=0$ is an open subset. The complement of V, which is a closed subset of X, is called the support of ϕ.

Sections with compact support

Let $\mathscr{F}$ be a *td*-sheaf on a *td*-space X and $\phi\in\Gamma(X,\mathscr{F})$ a global section of $\mathscr{F}$. We will denote $\Gamma_c(X,\mathscr{F})$ the subspace of $\phi\in\Gamma(X,\mathscr{F})$ consisting of global sections with compact support. If U is an open subset of X, there is a natural map $\Gamma_c(U,\mathscr{F})\to\Gamma_c(X,F)$ defined by the extension by zero: indeed if $\phi\in\Gamma_c(U,\mathscr{F})$ is a section with compact support C, then one can glue it with the zero section on $X\backslash C$.

If U_1 and U_2 are open subsets of X, with $U_1 \subset U_2$, then we have the restriction map $\alpha_{U_1}^{U_2} : \Gamma(U_2, \mathscr{F}) \to \Gamma(U_1, \mathscr{F})$ and the extension by zero map $\beta_{U_1}^{U_2} : \Gamma_c(U_1, \mathscr{F}) \to \Gamma_c(U_2, \mathscr{F})$. If $\iota_U : \Gamma_c(U, \mathscr{F}) \to \Gamma(U, \mathscr{F})$ is the map consisting of forgetting the compact support condition then we have a commutative diagram

$$\begin{tikzcd} \Gamma_c(U_1, \mathscr{F}) \arrow[r, "\iota_{U_1}"] \arrow[d, "\beta_{U_1}^{U_2}"'] & \Gamma(U_1, \mathscr{F}) \\ \Gamma_c(U_2, \mathscr{F}) \arrow[r, "\iota_{U_2}"'] & \Gamma(U_2, \mathscr{F}) \arrow[u, "\alpha_{U_1}^{U_2}"'] \end{tikzcd} \tag{5.3}$$

In particular, $\beta_{U_1}^{U_2}$ is injective for ι_{U_1} is.

Proposition 5.2. *For every open subset U of X, $\Gamma_c(U, \mathscr{F})$ can be identified with the inductive limit of $\Gamma(C, \mathscr{F})$ for C ranging over all compact open subsets of U. Similarly, $\Gamma(U, \mathscr{F})$ can be identified with the projective limit of $\Gamma(C, \mathscr{F})$ for C ranging over all compact open subsets of U.*

Proof. By means of extension by zero maps, we have a map

$$\varinjlim_{C} \Gamma(C, \mathscr{F}) \to \Gamma_c(U, \mathscr{F}) \tag{5.4}$$

where the injective limit ranges over all compact open subsets C of U. This map is injective for every extension by zero map $\Gamma(C, \mathscr{F}) \to \Gamma_c(U, \mathscr{F})$ is injective. It is also surjective because if the support $\phi \in \Gamma_c(U, \mathscr{F})$ being a compact subset of U, is contained in some compact open subset of U.

By means of restriction maps, we have a map

$$\Gamma(U, \mathscr{F}) \to \varprojlim_{C} \Gamma(C, \mathscr{F}). \tag{5.5}$$

This map is an isomorphism because of the equalizer property of a sheaf. $\square$

Every sheaf $\mathscr{F}$ of $\mathbf{C}$-vector spaces on X is automatically a sheaf on $\mathscr{C}_X^\infty$-modules i.e. for every open subset U, there is a canonical map $\mathscr{C}^\infty(U) \otimes \mathscr{F}(U) \to \mathscr{F}(U)$ endowing $\mathscr{F}(U)$ with a structure of $\mathscr{C}^\infty(U)$-module, and this module structure is compatible with restriction.

Let $\mathscr{F}$ be a *td*-sheaf on a *td*-space X and $\phi \in \Gamma(X, \mathscr{F})$ a global section of $\mathscr{F}$. We will denote $\Gamma_c(X, \mathscr{F})$ the subspace of $\phi \in \Gamma(X, \mathscr{F})$ consisting of global sections with compact support. We have maps $\mathscr{C}_c^\infty(X) \otimes \Gamma_c(X, \mathscr{F}) \to \Gamma_c(X, \mathscr{F})$ so that $\Gamma_c(X, \mathscr{F})$ is equipped with a structure of $\mathscr{C}_c^\infty(X)$-module.

Proposition 5.3. *The functor $\mathscr{M} \mapsto M = \Gamma_c(X, \mathscr{M})$ defines an equivalence between the category* $\mathrm{Sh}(X)$ *of sheaves of* $\mathbf{C}$*-modules on X and the category of nondegenerate $\mathscr{C}_c^\infty(X)$-modules.*

Proof. For every $\phi \in M = \Gamma_c(X, \mathscr{M})$, there exists a compact open subset U of X such that the support of ϕ is contained in U. In this case we have $e_U \phi = \phi$. We infer that M is a nondegenerate module over $A = \mathscr{C}_c^\infty(X)$.

Conversely, if M is a nondegenerate module over A, for every open compact open subset C of X, we set $\Gamma(C, \mathscr{M}) = e_C M$. For an arbitrary open subset U, we define $\Gamma(U, \mathscr{M})$ by Formula (5.5). We claim that the presheaf $U \mapsto \Gamma(U, \mathscr{M})$ is a sheaf i.e it satisfies the equalizer property (5.1). Because every open subset of X is a union of compact open subsets, it is enough to check (5.1) in the case U is a compact open subset covered by a family of compact open subsets. By compactness, it is enough to restrict ourselves to the case of a compact open subset covered by a finite family of compact open subsets. By subdivision, it is in fact enough to restrict ourselves further to the case of a compact open subset U covered by a finite disjoint union of compact open subset $U_1, \ldots, U_n$. If e_U and e_{U_i} denote the characteristic functions of U and U_i respectively, then e_U, e_{U_i} are idempotents elements of $\mathscr{C}^\infty(X)$ satisfying $e_{U_i} e_{U_j} = 0$ for $i \neq j$ and $e_U = e_{U_1} + \cdots + e_{U_n}$. Now the equalizer property (5.1) is reduced to the equality

$$e_U M = e_{U_1} M \oplus \cdots \oplus e_{U_n} M, \tag{5.6}$$

which is a particular case of the Chinese remainder theorem. □

Operations on *td*-sheaves

If $f : X \to Y$ is a continuous map between *td*-spaces, and $\mathscr{F}$ is a sheaf over X, we define its direct image $f_* \mathscr{F}$ by

$$f_* \mathscr{F}(U) = \mathscr{F}(f^{-1}(U)) \tag{5.7}$$

for all compact open subset U of Y. The functor $U \mapsto f_* \mathscr{F}(U)$ satisfies the equalizer equation (5.1): the equalizer equation for $f_* \mathscr{F}$ with respect to the covering $\bigsqcup_{\alpha \in \alpha_U} U_\alpha \to U$ is identical to the equalizer equation for $\mathscr{F}$ with respect to the covering $\bigsqcup_{\alpha \in \alpha_U} f^{-1}(U_\alpha) \to f^{-1}(U)$. The operation $\mathscr{F} \mapsto f_* \mathscr{F}$ defines a functor $f_* : \mathrm{Sh}_X \to \mathrm{Sh}_Y$ from the category of sheaves on X to the category of sheaves on Y. For instant, if Y is a point, then for every sheaf $\mathscr{F}$ on X, $f_* \mathscr{F}$ consists in the vector space $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$.

If $f : X \to Y$ is a continuous map between *td*-spaces, and if $\mathscr{G}$ is a sheaf on Y, we define $f^* \mathscr{G}$ to be the sheaf associated to the presheaf $f^\beta \mathscr{G}$ assigning to every open subset U of X the inductive limit of $\mathscr{G}(V)$ for V ranging over all open subsets of Y such that $f(U) \subset V$. The operation $\mathscr{G} \mapsto f^* \mathscr{G}$ defines a functor $f_* : \mathrm{Sh}_Y \to \mathrm{Sh}_X$ from the category of sheaves on Y to the category of sheaves on X. For instant, if Y is a point, $\mathscr{G} = \mathbf{C}$ is the constant sheaf on Y of value $\mathbf{C}$, then $f^\beta \mathbf{C}$ is the constant presheaf $U \mapsto f^\beta \mathscr{G}(U) = \mathbf{C}$. Its associated sheaf $f^* \mathbf{C}$ is then the sheaf $\mathscr{C}_X^\infty$ of smooth functions on X.

If $f : X \to Y$ is a continuous map of *td*-spaces, the functors f^* and f_* form a pair of adjoint functors, f^* being a left adjoint to f_*. For instant, if Y is just a point, $\mathscr{G} = \mathbf{C}$ is the constant sheaf of value $\mathbf{C}$ on Y, $f^* \mathscr{C}$ is the sheaf of $\mathscr{C}_X^\infty$ of smooth functions on X and $f_* f^* \mathbf{C}$

is the space $\mathscr{C}^\infty(X)$ of all smooth functions on X, the adjunction map $\mathbf{C} \to f_* f^* \mathbf{C} = \mathscr{C}^\infty(X)$ consists in the inclusion of the space of constant functions in the space of all smooth functions on X.

Proposition 5.4. *Let $f : X \to Y$ is a continuous map between td-spaces. Then there exists an isomorphism of functors from $\mathrm{Sh}_X^{\mathrm{op}} \times \mathrm{Sh}_Y$ to $\mathrm{Vec}_{\mathbf{C}}$:*

$$\mathrm{Hom}_{\mathrm{Sh}_X}(f^*\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathscr{G}, f_*\mathscr{F}) \tag{5.8}$$

In other words, the functor $f^* : \mathrm{Sh}_Y \to \mathrm{Sh}_X$ is a left adjoint to the functor $f_* : \mathrm{Sh}_X \to \mathrm{Sh}_Y$.

Proof. First, we define a morphism of functors $f^* f_* \to \mathrm{id}_{\mathrm{Sh}_X}$ from Sh_X to Vec_C i.e. a map $f^* f_* \mathscr{F} \to \mathscr{F}$ depending functorially on $\mathscr{F}$. Since $f^*\mathscr{G}$ is the sheaf to the presheaf $f^\beta \mathscr{G}$, it is equivalent to define a morphism of sheaves $f^* f_* \mathscr{F} \to \mathscr{F}$ or a morphism of presheaves $f^\beta f_* \mathscr{F} \to \mathscr{F}$. By definition $f^* f_* \mathscr{F}$ is the sheaf associated to the presheaf f^β assigning to every open subset U of X the inductive limit of $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ on open subsets V of Y such that $f(U) \subset V$ or equivalently $U \subset f^{-1}(V)$. For every such V, we have the restriction map $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ that can be organized in a compatible system giving rise to a map $(f_\beta f^* \mathscr{F})(U) \to \mathscr{F}(U)$ depending functorially on U.

Second, we define a morphism of functors $\mathrm{id}_{\mathrm{Sh}_Y} \to f_* f^*$ from Sh_Y to $\mathrm{Vec}_{\mathbf{C}}$ i.e. a map $\mathscr{G} \to f_* f^* \mathscr{G}$ depending functorially on $\mathscr{G}$. For every open subset V of Y we have $(f_* f^* \mathscr{G})(V) = (f^*\mathscr{G})(f^{-1}(V))$. For $f^*\mathscr{G}$ is the sheaf associated to the presheaf f^β, there is a canonical map $f^\beta \mathscr{G}(f^{-1}(V)) \to f^*\mathscr{G}(f^{-1}(V))$. From the very definition of f^β, we have $f^\beta \mathscr{G}(f^{-1}(V)) = \mathscr{G}(V)$ for every open subset V of Y. We infer a map $\mathscr{G}(V) \to f_* f^* \mathscr{G}(V)$ depending functorially on V. □

If $\mathscr{F}$ is a sheaf of $\mathbf{C}$-vector spaces on X, we define $f_!\mathscr{F}$ to be the sheaf of $\mathbf{C}$-vector spaces on Y such that for all open subset U of Y, $f_!\mathscr{F}(U)$ is the subspace of $\mathscr{F}(f^{-1}(U))$ consisting of section whose support is proper over Y.

If the fibers of $f : X \to Y$ are discrete, the functors $f_!$ and f^* form a pair ,of adjoint functors, $f_!$ being a left adjoint to f^*. For instant if Y is a point, X is a discrete space, $\mathscr{G} = \mathbf{C}$ is the constant sheaf of value $\mathbf{C}$ on Y, then $f^*\mathbf{C}$ is the sheaf of all functions $X \to \mathbf{C}$, $f_! f^* \mathbf{C}$ is the space of functions $\phi : X \to \mathbf{C}$ of finite support, and the adjunction map $f_! f^* \mathbf{C} \to \mathbf{C}$ assigns to each function $\phi : X \to \mathbf{C}$ of finite support the number $\sum_{x \in X} \phi(x)$.

Proposition 5.5. *Let $f : X \to Y$ be a continuous map of td-spaces with discrete fibers. Then there is a isomorphism of functors from $\mathrm{Sh}_X^{\mathrm{op}} \times \mathrm{Sh}_Y$ to $\mathbf{C}$:*

$$\mathrm{Hom}_{\mathrm{Sh}_Y}(f_!\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathrm{Sh}_X}(\mathscr{F}, f^*\mathscr{G}). \tag{5.9}$$

In other words, $f_!$ is a left adjoint to f^*.

Proof. □

Base change and excision

The operations f^* and $f_!$ have convenient description fiberwise.

Proposition 5.6. *Let $f : X \to Y$ be a continuous map of td-spaces. For every sheaf $\mathscr{G}$ on Y, and $x \in X$, we have $(f^*\mathscr{G})_x = \mathscr{G}_{f(x)}$. For every sheaf $\mathscr{F}$ on X, and $y \in Y$, we have $(f_!\mathscr{F})_y = \Gamma_c(f^{-1}(y), \mathscr{F})$.*

Proof. Let $\mathscr{G}$ is a sheaf on Y, $x \in X$ and $y = f(x)$. The fiber of $f^*\mathscr{G}$ at x is the same as the fiber of the presheaf $f^\beta\mathscr{G}$. Now, the fiber of $f^\beta\mathscr{G}$ at x is the limit inductive of $\mathscr{F}^\beta\mathscr{G}(U,V) = \mathscr{G}(V)$ over all pairs (U,V) consisting of a neighborhood U of x and an open neighborhood V of y such that $f(U) \subset V$. Since $\mathscr{F}^\beta\mathscr{G}(U,V)$ only depends on V, the limit doesn't change when we restrict to the subsystem consists in pairs (U,V) where $U = f^{-1}(V)$, and V is a neighborhood of y. The limit on this subsystem is by definition $\mathscr{G}_y$.

Now let $\mathscr{F}$ be a sheaf on X, $y \in Y$ and $X_y = f^{-1}(y)$ is the fiber of f over y. We want to prove that there exists an isomorphism between the fiber $(f_!\mathscr{F})_y$ of $f_!\mathscr{F}$ at y and $\Gamma_c(X_y, \mathscr{F}|_{X_y})$.

Let U be a neighborhood of y and ϕ a section $\phi \in \Gamma(f^{-1}(U), \mathscr{F})$ whose support C_ϕ is proper over Y. Restricting to X_y, the section $\phi|_{X_y} \in \Gamma(X_y, \mathscr{F})$ has compact support $C_\phi \cap X_y$. We infer a map $\Gamma(U, f_!\mathscr{F}) \to \Gamma_c(X_y, \mathscr{F}|_{X_y})$, and by passing to the inductive limit, we have a map

$$\varinjlim_U \Gamma(U, f_!\mathscr{F}) \to \Gamma_c(X_y, \mathscr{F}|_{X_y}) \tag{5.10}$$

which depends functorially on $\mathscr{F}$.

Pick a section $\phi \in \Gamma(f^{-1}(U), \mathscr{F})$ such that $\phi|_{X_y} = 0$. Then we have $C_\phi \cap X_y$ where C_ϕ is the support of ϕ. As C_ϕ is proper over U, its image $f(C_\phi)$ is a closed subset of U that doesn't contain y. It follows that the restriction on ϕ to $f^{-1}(U - f(C_\phi))$ vanishes, and a fortiori, the image of ϕ in $\varinjlim_U \Gamma(U, f_!\mathscr{F})$ must also vanish. We infer that (5.10) is injective.

Pick a section $\psi \in \Gamma_c(X_y, \mathscr{F}|_{X_y})$ of $\mathscr{F}|_{X_y}$ with compact support C_ψ. As a global section of $\mathscr{F}|_{X_y}$, there exists an open subset V of X containing X_y and a section $\psi_V \in \Gamma(V, \mathscr{F})$ such that $\psi_V|_{X_y} = \psi$. For every point $x \in C_\psi$ in the support C_ψ of ψ, there exists a compact open neighborhood $K_x \subset V$ of x in V. Because C_ψ is compact, there are finitely many points $x_1, \ldots, x_n$ such that $C_\psi \subset K_{x_1} \cup \cdots \cup K_{x_n} = K \subset V$. Let $\psi_K = \psi_V|K$ and $\psi_X \in \Gamma_c(X, \mathscr{F})$ the global section obtained by extending ψ_K by zero. As the support ψ_X is compact, a fortiori proper over Y, we obtain a section $\psi_X \in \Gamma(Y, f_!\mathscr{F})$ whose restriction to X_y is ψ. It follows that (5.10) is surjective. □

Proposition 5.7. *The base change theorem for proper morphism is satisfied for sheaves over* *td-spaces: for every continuous map $f : X \to Y$ a $g : Y' \to Y$, and the cartesian diagram:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \tag{5.11}$$

we have an isomorphism $g^* f_! \mathscr{F} = f'_! g'^* \mathscr{F}$ depending functorially on $\mathscr{F}$.

Proof. □

Proposition 5.8. *Let X be a td-space, Y a closed subset of X, and U its complement which is an open subset. Let $i : Y \to X$ and $j : U \to X$ denote the inclusion maps. For every td-sheaf $\mathscr{F}$ on X, we have the excision exact sequence*

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_! i^* \mathscr{F} \to 0 \tag{5.12}$$

depending functorially on $\mathscr{F}$. By applying the functor $\mathscr{M} \mapsto \Gamma_c(X, \mathscr{M})$, we get the exact sequence

$$0 \to \Gamma_c(U, \mathscr{F}) \to \Gamma_c(X, \mathscr{F}) \to \Gamma_c(Y, \mathscr{F}|_Y) \to 0 \tag{5.13}$$

also depending functorially on $\mathscr{F}$.

Proof. □

Equivariant sheaves

Let G be a *td*-group acting on a *td*-space X. The action map is given by a map $\alpha : G \times X \to X$ that we may simply write as $\alpha(g, x) = g^{-1}x$ for the notation saving purposes. A G-equivariant sheaf on X is a sheaf $\mathscr{F}$ on $\mathbf{C}$-vector spaces on X equipped with an isomorphism $\alpha_F : \alpha^* \mathscr{F} \to \mathrm{pr}_X^* \mathscr{F}$ satisfying a certain cocycle equation. Fiberwise, $\alpha_{\mathscr{F}}$ is consists of a collection of isomorphisms

$$\alpha_{\mathscr{F}}(g, x) : \mathscr{F}_{g^{-1}x} \to \mathscr{F}_x \tag{5.14}$$

depending on (g, x), and the cocycle equation is

$$\alpha_{\mathscr{F}}(g, x) \circ \alpha_{\mathscr{F}}(h, g^{-1}x) = \alpha_{\mathscr{F}}(gh, x) \tag{5.15}$$

as an isomorphism $\mathscr{F}_{h^{-1}g^{-1}x} \to \mathscr{F}_x$. In particular, if $g = e_G$ is the neutral element of G, $\alpha(e_G, x)$ is the identity of $\mathscr{F}_x$.

Proposition 5.9. *Over a point $X = \{x\}$, there is an equivalence between the category of G-equivariant sheaves on $\{x\}$ and the category of smooth representations of G defined by $\mathscr{F} \mapsto \mathscr{F}_x$.*

Proof. Let $\mathscr{F}$ be a G-equivariant sheaf on $X = \{x\}$. For every $g \in G$, we have $\mathscr{F}_{g^{-1}x} = \mathscr{F}_x$ and hence $\alpha(g,x)$ defines a $\mathbf{C}$-linear automorphism of $\mathscr{F}_x$. The cocycle equation (5.15) implies that $g \mapsto \alpha(g,x)$ defines a homomorphism of groups $\rho : G \to \mathrm{GL}(\mathscr{F}_x)$. For the fiber of the morphism of *td*-sheaves $\alpha_{\mathscr{F}} : \alpha^*\mathscr{F} \to \mathrm{pr}_X^*\mathscr{F}$ over $e_G \in G$ is identity, so that $\alpha_{\mathscr{F}}$ is identity on a neighborhood of e_G. It follows that ρ is smooth.

If $\rho : G \to \mathrm{GL}(\mathscr{F}_x)$ is a smooth representation, we can just reverse the above process to obtain a G-equivariant structure on the constant sheaf over $X = \{x\}$ of fiber $\mathscr{F}_x$. □

Proposition 5.10. *Let X and Y be td-spaces acted on by a td-group G and $f : X \to Y$ a G-equivariant map.*

If $\mathscr{F}$ is a G-equivariant sheaf on Y then $f^*\mathscr{F}$ is a G-equivariant sheaf on X. On the other hand, if $\mathscr{F}$ is a G-equivariant sheaf on X, then $f_!\mathscr{F}$ is a G equivariant sheaf on Y by the proper base change theorem 5.8. In particular, if $\mathscr{F}$ is a G-equivariant sheaf on X, then $\Gamma_c(X,\mathscr{F})$ is a smooth representation of G.

Proposition 5.11. *Let X be a td-space acted on by a td-group G. Let $\mathscr{F}$ be a G-equivariant sheaf. Then $\Gamma(X,\mathscr{F})$ is a (continuous) representation, and $\Gamma_c(X,\mathscr{F})$ is a smooth representation of G.*

Proof. □

Restriction and induction

Let G be a *td*-group. If H is a closed subgroup of a *td*-group G, then H is also a *td*-group. If (π, V) is a smooth representation of G then the restriction of π to a closed subgroup H of G is also a smooth representation. Indeed, for every $v \in V$, if the the function $g \mapsto \pi(g)v$ is smooth, then so is its restriction to H. The restriction defines an exact functor

$$\mathrm{Res}_G^H : \mathrm{Rep}(G) \to \mathrm{Rep}(H). \tag{5.16}$$

The functor Res_G^H is additive and exact because it does not alter the vector space underlying the representations.

We will construct the right adjoint

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G) \tag{5.17}$$

to the restriction functor (5.16). The functor Ind_H^G will also be additive and exact.

Let (σ, V_σ) be a smooth representation of H. We consider the space $\beta\mathrm{Ind}_H^G(V_\sigma)$ [3] of all smooth functions $f : G \to V_\sigma$ satisfying $f(hg) = \sigma(h)f(g)$. The group G acts on $\beta\mathrm{Ind}_H^G(V_\sigma)$ by right translation. For this representation of G is not smooth in general, we define $\mathrm{Ind}_H^G(V_\sigma)$

[3] In this sentence, the letter β has the similar meaning to the β in software's β-release.

to be the subspace of smooth vectors in $\beta\mathrm{Ind}_H^G(V_\sigma)$ with respect to the action of G by right translation, and $\mathrm{Ind}_H^G(\sigma)$ the smooth representation of G on $\mathrm{Ind}_H^G(V_\sigma)$. For instant, if $H = \{e_G\}$ is the trivial group, $\mathbf{C}$ is the trivial representation of $\mathbf{H}$, then $\beta\mathrm{Ind}_H^G = \mathscr{C}^\infty(G)$ is the space of all smooth functions on G, and $\mathrm{Ind}_H^G(\mathbf{C})$ is the subspace of $\mathscr{C}^\infty(G)$ consisting of smooth functions on G that are smooth vectors with respect to the right translation of G. That subspace, denoted by $\mathscr{C}^\infty(G)^{\mathrm{sm}(r_G)}$ was described by the formula (2.5).

Proposition 5.12 (Frobenius reciprocity). *If H is a closed subgroup of a td-group G, there is a natural isomorphism of functors from the category* $\mathrm{Rep}(G)^{\mathrm{op}} \times \mathrm{Rep}(H)$ *to the category of* $\mathbf{C}$*-vector spaces* $\mathrm{Vec}_{\mathbf{C}}$

$$\mathrm{Hom}_H(\mathrm{Res}_G^H(\pi), \sigma) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)). \tag{5.18}$$

In other words, $\mathrm{Ind}_H^G(\sigma)$ *is a right adjoint to* Res_G^H.

Proof. We will first construct a map $\epsilon(V_\sigma) : \mathrm{Res}_G^H\mathrm{Ind}_H^G(V_\sigma) \to V_\sigma$ depending functorially on representation (σ, V_σ) of H. As $\beta\mathrm{Ind}_H^G(V_\sigma)$ consists of all smooth functions $f : G \to V_\sigma$ satisfying $f(hg) = \sigma(h)f(g)$, we have a map $\beta\mathrm{Ind}_H^G(V_\sigma) \to V_\sigma$ by assigning to f its value $f(e_G) \in V_\sigma$. This map is H-equivariant. By restricting it to $\mathrm{Ind}_H^G(V_\sigma)$, we obtained the desired map $\epsilon(V_\sigma) : \mathrm{Res}_G^H\mathrm{Ind}_H^G(V_\sigma) \to V_\sigma$.

Next we construct a map $\eta(V_\pi) : V_\pi \to \mathrm{Ind}_H^G\mathrm{Res}_G^H(V_\pi)$ depending functorially on representation (π, V_π) of G. It is enough to construct a functorial map $V_\pi \to \beta\mathrm{Ind}_H^G\mathrm{Res}_G^H(V_\pi)$ because as all vectors in V_π are smooth, that map would necessarily factorize through $\mathrm{Ind}_H^G\mathrm{Res}_G^H(V_\pi)$. Now for every $v \in V_\pi$, we define a function $f : G \to V_\pi$ be setting $g \mapsto \pi(g)v$.

In order to prove that Ind_H^G is the right adjoint to Res_G^H, it remains only to check that the composition maps

$$\mathrm{Res}_G^H(V_\pi) \xrightarrow{\mathrm{Res}_G^H(\eta(V_\pi))} \mathrm{Res}_G^H\mathrm{Ind}_H^G\mathrm{Res}_G^H(V_\pi) \xrightarrow{\epsilon(\mathrm{Res}_G^H(V_\pi))} \mathrm{Res}_G^H(V_\pi) \tag{5.19}$$

and

$$\mathrm{Ind}_H^G(V_\sigma) \xrightarrow{\eta(\mathrm{Ind}_H^G(V_\sigma))} \mathrm{Ind}_H^G\mathrm{Res}_G^H\mathrm{Ind}_H^G(V_\sigma) \xrightarrow{\mathrm{Ind}(\epsilon(V_\sigma))} \mathrm{Ind}_H^G(V_\sigma) \tag{5.20}$$

are identity maps, see [**MacLane**]. This can achieve this by chasing through the definitions of ϵ and η. □

There is a variation on the induction theme. If H is a closed subgroup of a *td*-group G, compact induction is an additive and exact functor

$$c\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G). \tag{5.21}$$

If (σ, V_σ) is a smooth representation of H, we have defined $\beta\mathrm{Ind}(V_\sigma)$ to be the space of all smooth functions $f : G \to V_\sigma$ such that $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and $g \in G$. We define $c\mathrm{Ind}_H^G(V_\sigma)$ to be the subspace of $\beta\mathrm{Ind}(V_\sigma)$ consisting of functions $f : G \to V_\sigma$ such that there exists a compact open set C such that f is supported in HC. This space is clearly stable under the right translation of G. We have the inclusions

$$\mathrm{cInd}_H^G(V_\sigma) \subset \mathrm{Ind}_H^G(V_\sigma) \subset \beta\mathrm{Ind}_H^G(V_\sigma). \tag{5.22}$$

In other words, the representation of G on $c\mathrm{Ind}_H^G(V_\sigma)$ is smooth, and we denote it by $c\mathrm{Ind}_H^G(\sigma)$. For instant, if $H = \{e_G\}$ then the compact induction $\mathrm{cInd}_H^G(\mathbf{C})$ of the trivial representation is the space $\mathscr{C}_c^\infty(G)$ of smooth functions with compact support.

To establish the inclusion $\mathrm{cInd}_H^G(V_\sigma) \subset \mathrm{Ind}_H^G(V_\sigma)$, we will show that if $\phi \in \mathrm{cInd}_H^G(V_\sigma)$, then ϕ is a smooth vector of $\beta\mathrm{Ind}_H^G(V_\sigma)$ with respect to the right translation of G. For every $x \in X$, there exists an open compact subgroup K_x of G such that ϕ is constant on xK_x. As the support of ϕ is compact modulo the left action of H, there exists finitely many $x_1, \dots, x_n$ such that the support of ϕ is contained in $H(\bigcup_{i=1}^n x_i K_{x_i})$. It follows that ϕ is invariant under the right translation of the compact open subgroup $K = \bigcap_{i=1}^n K_{x_i}$.

In the case H is an open subgroup of G then $c\mathrm{Ind}_H^G$ is a left adjoint to the functor of restriction (5.16):

Proposition 5.13. *If H is an open subgroup of a td-group G, then there is a natural isomorphism of functors from* $\mathrm{Rep}(H)^{\mathrm{op}} \times \mathrm{Rep}(G)$ *to* $\mathrm{Vec}_{\mathbf{C}}$

$$\mathrm{Hom}_G(c\mathrm{Ind}_H^G(\sigma), \pi) = \mathrm{Hom}_H(\sigma, \mathrm{Res}_G^H \pi). \tag{5.23}$$

In other words, $c\mathrm{Ind}_H^G$ is a left adjoint to Res_G^H.

Proof. Since H is an open subgroup of G, $H\backslash G$ is a discrete set. We choose a set of representatives $\{x_i | i \in I\} \subset G$ of left H-cosets in G. For every smooth representation σ of H, an element $\phi \in c\mathrm{Ind}_H^G(V_\sigma)$ is a smooth function $\phi : G \to V_\sigma$ of compact support modulo H acting on the left, and satisfying $\phi(hg) = \sigma(h)\phi(g)$. The condition $\phi(hg) = \sigma(h)\phi(g)$ implies that ϕ is completely determined by the values $\phi(x_i)$, and the condition of compact support implies that $\phi(x_i) = 0$ for all but finitely many x_i. Conversely, as $H\backslash G$ is a discrete, every function $\{x_i | i \in I\} \to \mathbf{C}$, which is zero away from a finite set, gives rise to a unique function $\phi \in c\mathrm{Ind}_H^G(V_\sigma)$.

First, we define a G-linear map $\epsilon : c\mathrm{Ind}_H^G \mathrm{Res}_G^H(V_\pi) \to V_\pi$ depending functorially on π. For every $\phi \in c\mathrm{Ind}_H^G \mathrm{Res}_G^H(\pi)$ we set

$$\epsilon(\phi) = \sum_i \pi(x_i)^{-1}\phi(x_i) \in V_\pi.$$

This is a finite sum as $\phi(x_i) = 0$ for all but finitely many x_i. Moreover, this sum does not depend on the choice of representatives x_i. Indeed If $x_i' = hx_i$ for some $h \in H$, then we have

$$\pi(x_i')^{-1}\phi(x_i') = \pi(x_i')^{-1}\pi(h)^{-1}\pi(h)\phi(x_i') = \pi(x_i)^{-1}\phi(x_i).$$

We can also check that $\phi \mapsto \epsilon(\phi)$ is G-linear.

Next, we define a functorial H-linear map $\eta(\sigma) : \sigma \to \mathrm{Res}_G^H(c\mathrm{Ind}_H^G(\sigma))$. Recall that the representation $c\mathrm{Ind}_H^G(\sigma)$ consists in the vector space $c\mathrm{Ind}_H^G(V_\sigma)$ of smooth functions $\phi : G \to V_\sigma$ of compact support modulo H acting on the left, and satisfying $\phi(hg) = \sigma(h)\phi(g)$, with G acting by the right translation. With every $v \in V_\sigma$, we can associate the unique smooth $\phi_v \in c\mathrm{Ind}_H^G(V_\sigma)$ such that $\phi_v(e_G) = v$ and $\phi_v(g) = 0$ if $v \notin H$.

An easy calculation shows that η and ϵ establish the adjoint property between the functors $c\mathrm{Ind}_H^G$ and Res_G^H. $\square$

Sheaf theoretic induction

Shift theoretic descriptions of the functors restriction, induction an induction with compact support would be completely straightforward if we have in our disposition a theory of *td*-stack, sheaves on *td*--stacks with usual operations. For developing a full fledged theory of *td*-stacks would implicate too much digressions for the scope of this document, we will simply attempt to figure out how to the functors restriction, induction, and induction with compact support, along with their properties of adjunction would naturally fit in such a theory.

As in Prop. 5.9, the category of smooth representations of G is equivalent to the category of G-equivariant sheaf over a point with respect to the trivial action of G. We will imagine that the quotient BG of the point by the trivial action of G exists in the theory of *td*-stacks so that the category of sheaves on BG is equivalent to the category of G-equivariant sheaves over a point and thus also equivalent to the category of smooth representations of G.

If H is a closed subgroup of G, the we would have a morphism $f : BH \to BG$. Following this thread, the functor Res_H^G would correspond to the functor f^*, the functor Ind_H^G would correspond to the functor f_* and the functor $c\mathrm{Ind}_H^G$ would correspond to the functor $f_!$. The Frobenius reciprocity i.e. the adjunction of the pair $(\mathrm{Res}_G^H, \mathrm{Ind}_H^G)$ as in Prop. 5.12 would then correspond to the adjunction of the pair as in Prop. 5.4. In the case H is an open subgroup i.e. $H\backslash G$ is discrete, the adjunction of the pair $(c\mathrm{Ind}_H^G, \mathrm{Res}_G^H)$ as in Prop. 5.13 would correspond to the adjunction of the pair $(f_!, f_*)$ as Prop. 5.5.

To circumvent the theory of *td*-stacks, we will use the following property of descent.

Proposition 5.14. *Let $f : X \to Y$ be a G-torsor. Then there exists an equivalence between the category of G-equivariant td-sheaves on X and the category of td-sheaves on Y.*

Proof. If $\mathscr{G}$ is a *td*-sheaf on Y, $f^*\mathscr{G}$ is a G-equivariant *td*-sheaf on X.

Conversely, let $\mathscr{F}$ be a G-equivariant sheaf on X. We cover Y by open subsets U_α with $\alpha \in \alpha_{\mathscr{U}}$ such that for each α, there exists an isomorphism $f^{-1}(U_\alpha) = G \times U_\alpha$. For each $\alpha \in \alpha_{\mathscr{U}}$, we choose a section $s_\alpha : U_\alpha \to f^{-1}(U_\alpha)$. For $\alpha, \beta \in \alpha_{\mathscr{U}}$, and $U_{\alpha\beta} = U_\alpha \cap U_\beta$, there exists a unique map

$$\gamma_\alpha^\beta : U_{\alpha\beta} \to G \text{ such that } \gamma_\alpha^\beta s_\beta|_{U_{\alpha\beta}} = s_\alpha|_{U_{\alpha\beta}} \tag{5.24}$$

We construct a sheaf $\mathscr{G}$ on Y by the following requirements:

- on each U_α, we set $\mathscr{G}|_{U_\alpha} = s_\alpha^* \mathscr{F}$;
- on $U_{\alpha\beta}$, we glue the restrictions of $s_\alpha^*\mathscr{F}$ and $s_\beta^*\mathscr{F}$ by means of the equation (5.24) and the G-equivariant structure of $\mathscr{F}$.

One can check that the sheaf $\mathscr{G}$ constructed as above doesn't depend neither on the open covering U_α, nor on the choice of sections s_α, by the usual refining covering argument. □

We now consider the Cartesian diagram

$$\begin{tikzcd} H\backslash G \arrow[r, "f'"] \arrow[d, "g'"'] & \mathrm{pt} \arrow[d, "g"] \\ BH \arrow[r, "f"'] & BG \end{tikzcd} \tag{5.25}$$

where the map $g' : H\backslash G \to BH$ corresponds to the H-torsor $G \to H\backslash G$. We will now describe the functors Res_H^G, Ind_H^G and $c\mathrm{Ind}_H^G$ using f'^*, f'_* and $f'_!$ respectively.

Proposition 5.15. *There is an equivalence between the category of smooth representations of H and the category of G-equivariant sheaves on $H\backslash G$ where G acts on $H\backslash G$ by right translation. The equivalence assigns to every smooth representation σ of H a G-equivariant sheaf $\mathscr{V}_\sigma$ on $H\backslash G$.*

Proof. We consider the cartesian diagram

$$\begin{tikzcd} G \arrow[r, "h'"] \arrow[d, "p_G"'] & H\backslash G \arrow[d, "g'"] \\ \mathrm{pt} \arrow[r, "h"'] & BH \end{tikzcd} \tag{5.26}$$

where $p_G : G \to \mathrm{pt}$ is a G-torsor with respect to the right translation of G on it self and $h' : G \to H\backslash G$ is a H-torsor after Prop. 2.4. We won't use the right left corner of the diagram except for memory.

Let (σ, V_σ) be a smooth representation of H. After Prop. 5.9, (σ, V_σ) defines a H-equivariant sheaf on the point $\{1\}$. As the map $p_G : G \to \{1\}$ is obviously $H \times G$-equivariant with respect to action of $H \times G$ on G, $p_G^* V_\sigma$ is equipped with a $H \times G$-equivariant structure. By Prop. 2.5, $p_G^* V_\sigma$ descends to a G-equivariant sheaf $\mathscr{V}_\sigma$ on $H\backslash G$ which depends functorially on σ. This gives rise to a functor from the category of smooth representations of H to the category of G-equivariant sheaves on $H\backslash G$.

Inversely, if $\mathscr{V}$ is a G-equivariant sheaf on $H \times G$, $h'^*\mathscr{V}$ is a $H \times G$-equivariant sheaf on G. By Prop. 2.5, $h'^*\mathscr{V}$ descends to a H-equivariant sheaf over the point, thus a smooth representation of H. We have defined a functor from the category of G-equivariant sheaves on $H\backslash G$ to the category of smooth representations of H. The two functors we have constructed are inverse of each other. □

For every smooth representation σ of H, we can easily check upon the definitions of Ind_H^G and $c\mathrm{Ind}_H^G$ that the equalities

$$c\mathrm{Ind}_H^G(\sigma) = \Gamma_c(H\backslash G, \mathscr{V}_\sigma) \text{ and } \mathrm{Ind}_H^G(\sigma) = \Gamma(H\backslash G, \mathscr{V}_\sigma)^{\mathrm{sm}}.$$

holds. Moreover, the Frobenius reciprocity (Prop. 5.12) and the adjunction property of $c\mathrm{Ind}_H^G$ in the case H is an open subgroup can be derived from Prop. 5.4 and Prop. 5.5.

Invariant measure on homogenous space and the canonical pairing

The construction of invariant measure on homogenous space is based on the following fact.

Proposition 5.16. *Let G be a unimodular td-group, and H a closed subgroup of G. Let $\chi : H \to \mathbf{C}^\times$ be a smooth character and $\mu_{H,\chi}$ a nonzero vector of $\mathscr{D}(H)^{l(H,\chi)}$. Then the map $\phi \mapsto \mu_{H,\chi} \star \phi$ defines a surjective G-equivariant map $\mathscr{C}_c^\infty(G) \to c\mathrm{Ind}_H^G(\chi)$ with respect to the action of G on $\mathscr{C}_c^\infty(G)$ by right translation.*

Proof. For every $\phi \in \mathscr{C}_c^\infty(G)$ and $h \in H$ we have

$$\delta_h \star \mu_{H,\chi} \star \phi = \chi(h)(\mu_{H,\chi} \star \phi). \tag{5.27}$$

On the other hand, $\mu_{H,\Delta_H} \star \phi$ is compactly supported modulo the left action of H by its very construction. It follows that $\mu_{H,\chi} \star \phi \in c\mathrm{Ind}_H^G(\chi)$.

We claim that $\phi \mapsto \mu_{H,\chi} \star \phi$ defines a surjective map

$$\mathscr{C}_c^\infty(G) \to c\mathrm{Ind}_H^G(\chi).$$

Indeed every vector in $c\mathrm{Ind}_H^G(\Delta_H)$ is represented by a smooth function $f : G \to \mathbf{C}$ satisfying $\delta_h \star f = \chi(h) f$ which is compactly supported modulo H acting by left translation. There exists a compact open subgroup K and finitely many elements $x_1, \ldots, x_n \in G$ such that

$$\mathrm{Supp}(f) \subset \bigsqcup_{i=1}^n H x_i K. \tag{5.28}$$

It follows that f is a linear combination of the functions $\mu_{H,\chi} \star \mathbb{I}_{x_i K}$. □

Proposition 5.17. *Let G be a unimodular td-group, and H a closed subgroup of G. Let $\Delta_H : H \to \mathbf{C}^\times$ denote the modulus character of H. Then there is a nonzero G-invariant linear form*

$$\nu_H^G : c\mathrm{Ind}_H^G(\Delta_H) \to \mathbf{C} \tag{5.29}$$

depending only on the choice of an invariant measure μ_G of G.

Proof. We choose a nonzero element $\mu_{H,\Delta_H} \in \mathscr{D}(H)^{I(H,\Delta_H)}$ a distribution on H such that for every $h \in H$. We have a map $\mathscr{C}_c^\infty(G) \to c\mathrm{Ind}_H^G(\Delta_H)$ given by $\phi \mapsto \mu_{H,\Delta_H} \star \phi$.

Now we claim the every left G-invariant linear form $\mu : \mathscr{C}_c^\infty(G) \to \mathbf{C}$ factors through a linear form $\nu_H^G : c\mathrm{Ind}_H^G(\Delta_H) \to \mathbf{C}$. It is enough to prove that the kernel of the map $\mathscr{C}_c^\infty(G) \to c\mathrm{Ind}_H^G(\Delta_H)$ given by $\phi \mapsto \mu_{H,\Delta_H} \star \phi$ is contained in the kernel of μ, in other words $\mu_{H,\Delta_H} \star \phi = 0$ implies $\mu(\phi) = 0$.

For this purpose we consider the bilinear form $\mathscr{C}_c^\infty(G) \otimes \mathscr{C}_c^\infty(G) \to 0$ defined by

$$\langle \phi_1, \phi_2 \rangle = \mu(\phi_1 \phi_2). \tag{5.30}$$

Let $\phi_1 \in \mathscr{C}_c^\infty(G)$ such that $\mu_{H,\Delta_H} \star \phi_1 = 0$. For all $\phi_2 \in \mathscr{C}_c^\infty(G)$, we have

$$\langle \mu_{H,\Delta_H} \star \phi_1, \phi_2 \rangle = \langle \phi_1, \check{\mu}_{H,\Delta_H} \star \phi_2 \rangle. \tag{5.31}$$

where $\mu_H = \check{\mu}_{H,\Delta_H}$ is a left invariant measure on H, see (3.24). Since $\phi \mapsto \mu_H \circ \phi$ defines a surjective map $\mathscr{C}_c^\infty(G) \to c\mathrm{Ind}_H^G(1)$, there exists $\phi_2 \in \mathscr{C}_c^\infty(G)$ such that $\mu_H \circ \phi_2$ is the characteristic function of HC where C is a compact open subset containing the support of ϕ_1. In this case, we have

$$\langle \phi_1, \mu_H \star \phi_2 \rangle = \mu(\phi_1). \tag{5.32}$$

Thus we have derived $\mu(\phi_1) = 0$ from $\mu_{H,\Delta_H} \star \phi_1 = 0$.

Now we have constructed a linear map $\nu_H^G : c\mathrm{Ind}_H^G(\Delta_H) \to \mathbf{C}$ such that for every $\phi \in \mathscr{C}_c^\infty(G)$ we have $\mu(\phi) = \nu_H^G(\mu_{H,\Delta_H} \star \phi)$. We claim that ν_H^G is G-invariant. For every $f \in c\mathrm{Ind}_H^G(\Delta_H)$, there exists $\phi \in \mathscr{C}_G^\infty$ such that $f = \mu_{H,\Delta_H} \star \phi$. For every $g \in G$, we have indeed

$$\nu_H^G(\mu_{H,\Delta_H} \star \phi \star \delta_g) = \mu(\phi \star \delta_g) = \mu(\phi) = \nu_H^G(\mu_{H,\Delta_H} \star \phi).$$

In the above equalities, we have exploited the assumption G being unimodular i.e. the left invariant measure μ is also right invariant. □

Proposition 5.18. *Let G be a unimodular td-group, and H a closed subgroup of G. For every smooth representation (σ, V) of H with contragredient (σ', V'), there is a nonzero G-invariant bilinear form*

$$\mathrm{cInd}_H^G(\sigma) \otimes \mathrm{Ind}_H^G(\sigma' \otimes \Delta_H) \to \mathbf{C} \tag{5.33}$$

depending only on the choice of an invariant measure μ_G of μ. Moreover, this pairing induces an isomorphism between $\mathrm{Ind}_H^G(\sigma' \otimes \Delta_H)$ and the contragredient of $\mathrm{cInd}_H^G(\sigma)$.

Proof. We use the canonical pairing $\mathrm{cInd}_H^G(\sigma) \otimes \mathrm{Ind}_H^G(\sigma' \otimes \Delta_H) \to \mathrm{cInd}_H^G(\Delta_H)$ and apply the previous proposition to obtain (5.33).

In order to prove that the contragredient of $\mathrm{cInd}_H^G(\sigma)$ and $\mathrm{Ind}_H^G(\sigma' \otimes \Delta_H)$, we will prove that for every compact open subgroup K of G, we have

$$\mathrm{Hom}(\mathrm{cInd}_H^G(\sigma)^K, \mathbf{C}) = \mathrm{Ind}_H^G(\sigma' \otimes \Delta_H)^K. \tag{5.34}$$

For this we decompose $X = H\backslash G$ into orbits $X = \bigsqcup_{\alpha\in\alpha_X} X_\alpha$ under the action of K by translation. Each orbit X_α is a compact open subset of X. For each $\alpha \in \alpha_X$, the map $\mathrm{cInd}_H^G(\sigma) \otimes \mathrm{Ind}_H^G(\sigma' \otimes \Delta_H) \to \mathrm{cInd}_H^G(\Delta_H, \mathscr{V}_\sigma)$ induces a pairing

$$\Gamma(X_\alpha, \mathscr{V}_\sigma) \otimes \Gamma(X_\alpha, \mathscr{V}_{\sigma'\otimes\Delta_H}) \to \Gamma(X_\alpha, \mathscr{V}_{\Delta_H}) \to \mathbf{C} \tag{5.35}$$

... (to be completed) □

Induction from an open subgroup

Proposition 5.19. *Let G be a td-group and H an open subgroup of G. For every smooth representation σ of H, $\mathrm{End}_G(\mathrm{cInd}_H^G(\sigma))$ can be identified with the space $\mathscr{H}(G,\sigma)$ of spherical functions i.e smooth functions $\phi : G \to \mathrm{End}_{\mathbf{C}}(V_\sigma)$ satisfying*

1. *$\phi(hgh') = \sigma(h)\phi(g)\sigma(h')$ for all $h, h' \in H$ and $g \in G$,*
2. *$\mathrm{Supp}(\phi)$ is a finite union of H-double cosets.*

Proof. As a particular case of Proposition 5.13, we have

$$\mathrm{Hom}_G(c\mathrm{Ind}_H^G(\sigma), c\mathrm{Ind}_H^G(\sigma)) = \mathrm{Hom}_H(\sigma, \mathrm{Res}_G^H c\mathrm{Ind}_H^G(\sigma)).$$

□

Proposition 5.20 (Mautner)**.** *Let G be a td-group, and H an open subgroup of G which is compact modulo the center of G. Let σ be a smooth representation of H such that $c\mathrm{Ind}_H^G(\sigma)$ is irreducible. Then the representation $c\mathrm{Ind}_H^G(\sigma)$ is a compact modulo the center.*

Proof. □

Bibliographical comments

The concept of *td*-sheaves has been introduced by Bernstein and Zelevinski in [2] . We follow the presentation of [1].

6 Structure of p-adic reductive groups

The purpose of this section is to recall some basic facts on split reductive groups $\mathbf{G}$ over a nonarchimedean local field. Various double coset decompositions will be stated for an arbitrary split reductive groups. For classical groups, these facts can be proved by means of elementary linear algebras. We will do it only in the case of GL_n as uniform proofs will be provided in a later section.

Root datum

We first recall how to construct root data out of a reductive group over an algebraically closed field. Let $\mathbf{G}$ be a reductive group over an algebraically closed field $\mathbf{k}$. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, $\Lambda = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ and $\Lambda^\vee = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$. The adjoint action of $\mathbf{G}$ on its Lie algebra $\mathfrak{g}$, when restricted to $\mathbf{T}$, gives rise to a decomposition of $\mathfrak{g}$ as a direct sum of eigenspaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{6.1}$$

where Φ is a finite set of $\Lambda^\vee - \{0\}$, the set of **roots** of G. For every $\alpha \in \Phi$, the eigenspace $\mathfrak{g}_\alpha$ consisting of vector $x \in \mathfrak{g}$ such that $\mathrm{ad}(t)x = \alpha(t)x$, is one dimensional. The eigenspace for the trivial character of T coincides with its Lie algebra $\mathfrak{t}$.

If $\mathbf{B}^+$ is a Borel subgroup containing T, $\mathfrak{b}$ its Lie algebra, then

$$\mathbf{b}^+ = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha \tag{6.2}$$

where Φ^+ is a subset of Φ such that for every $\alpha \in \Phi$, $\Phi^+ \cap \{\alpha, -\alpha\}$ has exactly one element.

The Weyl group $W = \mathrm{Nor}_G(T)/T$ acts simple transitively on the set of Borel subgroups containing $\mathbf{T}$. The Weyl group is generated by the reflections s_α attached to every root $\alpha \in \Phi$ given by

$$s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee \in W. \tag{6.3}$$

where $\alpha^\vee \in \Phi^\vee$ are the coroots, $\Phi^\vee$ is a finite subset of Λ in bijection with Φ by $\alpha \mapsto \alpha^\vee$. We have $\langle \alpha^\vee, \alpha \rangle = 2$. For every simple root $\alpha \in \Delta^+$, $s_\alpha \in W$ is the unique element of W such that

$$s_\alpha(\Phi^+) = \Phi^+ \backslash \{\alpha\} \cup \{-\alpha\}. \tag{6.4}$$

For every $\alpha \in \Phi$, we define H_α to be the hyperplane of $\Lambda_{\mathbf{R}}$ defines by the linear form $\alpha : \Lambda_{\mathbf{R}} \to \mathbf{R}$. These hyperplane cuts out $\Lambda_{\mathbf{R}}$ as a union of cones

$$\Lambda_{\mathbf{R}} = \bigsqcup_{C \in C(\Phi)} C \tag{6.5}$$

where each $C \in C(\Phi)$ is defined by a system (in)equality of the form $\langle x, \alpha \rangle = 0$, or $\langle x, \alpha \rangle < 0$, or $\langle x, \alpha \rangle > 0$. We have in particular a cone of maximal dimension defined by $\langle x, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$.

For each cocharater $\lambda : \mathbf{G}_m \to \mathbf{T}$, the induced action of $\mathbf{G}_m$ on $\mathfrak{g}$ gives rise to a decomposition

$$\mathfrak{g} = \mathfrak{n}_\lambda^- \oplus \mathfrak{m}_\lambda \oplus \mathfrak{n}_\lambda \tag{6.6}$$

where $\mathfrak{n}_\lambda^- = \bigoplus_{\langle \lambda, \alpha \rangle < 0} \mathfrak{g}_\alpha$, $\mathfrak{n}_\lambda = \bigoplus_{\langle \lambda, \alpha \rangle > 0} \mathfrak{g}_\alpha$ and $\mathfrak{m}_\lambda$ is the maximal subspace of $\mathfrak{g}$ where $\mathbf{G}_m$ acts trivially. This decomposition of the Lie algebra can be exponentiated to the level of group: we have corresponding subgroups $\mathbf{N}_\lambda^-, \mathbf{N}_\lambda$ and $\mathbf{M}_\lambda$ whose Lie algebras are $\mathfrak{n}_\lambda^-, \mathfrak{n}_\lambda$ and $\mathfrak{m}_\lambda$ respectively. We will denote $\mathbf{P}_\lambda = \mathbf{M}_\lambda \mathbf{N}_\lambda$, and $\mathbf{P}_\lambda^- = \mathbf{M}_\lambda \mathbf{N}_\lambda^-$.

The parabolic subgroup $\mathbf{P}_\lambda$ only depends on the cone $C \in C(\Phi)$ in which λ lies. We can assigns to each $C \in C(\Phi)$ a parabolic subgroup P_C and that assignment defines a bijection between $C(\Phi)$ in (6.5) and the set $\mathscr{P}(\mathbf{T})$ of parabolic subgroups containing $\mathbf{T}$. The cones of maximal dimension correspond to the Borel subgroups containing $\mathbf{T}$. The Weyl group acts compatibly on $C(\Phi)$ and $\mathscr{P}(\mathbf{T})$; it acts simply transitively on the set of cones of maximal dimension as well as the set of Borel subgroups containing $\mathbf{T}$.

The set $\mathscr{P}(\mathbf{T})$ of subgroups of the form $\mathbf{P}_\lambda$ is the finite set of parabolic group containing $\mathbf{T}$. For every $P \in \mathscr{P}(\mathbf{T})$, and $P = \mathbf{P}_\lambda$ for some $\lambda \in \Lambda$, then $\mathbf{P}_\lambda^-$ is then independent of the choice of λ. We will denote it P^-, the opposite parabolic of $P \in \mathscr{P}(\mathbf{T})$. The opposite parabolic defines an involution of the finite set $\mathscr{P}(\mathbf{T})$.

Given a choice of $\mathbf{B}^+ \in \mathscr{B}(\mathbf{T})$, we define a partial order in Λ: we say that $\lambda \geq \lambda'$ if

$$\lambda = \lambda' + \sum_{\alpha \in \Delta^+} n_\alpha \alpha^\vee \tag{6.7}$$

where n_α are nonnegative integers. We call it the **coroot partial order**.

We note that the abelian group generated by the coroots $\alpha^\vee$ is more often than not a strict subgroup of Λ so that the relation $\lambda \geq \lambda'$ implies that λ and λ' have the same image in $\Lambda / \sum_{\alpha \in \Phi} \mathbf{Z}\alpha^\vee$. The latter is called the the algebraic fundamental group of $\mathbf{G}$, it vanishes if $\mathbf{G}$ is simply connected.

Galois action on root datum

We are now considering a reductive group $\mathbf{G}$ over a non algebraically closed field F. We assume that $\mathbf{G}$ is quasi-split i.e there exists a maximal torus $\mathbf{T}$ and a Borel subgroup $\mathbf{B}^+ \in \mathscr{B}(\mathbf{T})$ defined over F. Over the algebraic closure $\bar{F}$ of F, we have the root data defined as in the precedent subsection $(\Lambda, \Lambda^\vee, \Phi^\vee, \Phi)$.

The Galois $\Gamma = \mathrm{Gal}(\bar{F}/F)$ acts on $\Lambda^\vee$, and that action preserves the set of roots Φ, the set of positive roots Φ^+ and the set of simple roots Δ^+. It also acts on Λ, preserving the cone decomposition and the positive chamber C^+ corresponding to $\mathbf{B}^+$. As W acts simply transitively on the set of chambers, this induces an action of Γ on W so that the action of W and Γ on Λ and $\Lambda^\vee$ combine into an action of the semidirect product $W \rtimes \Gamma$.

Let $\Lambda_F = \Lambda^\Gamma$ the subgroup of elements of Λ that fixed under the action of Γ. It is the group of cocharacters of the maximal split torus $\mathbf{A}$ of $\mathbf{T}$

$$\Lambda_F = \mathrm{Hom}_F(\mathbf{G}_m, \mathbf{T}) = \mathrm{Hom}_{\bar{F}}(\mathbf{G}_m, \mathbf{A}) \tag{6.8}$$

The coroot partial order on Λ induces a partial order on on Λ_F. Let us denote $\Lambda_{F,\mathbf{R}} = \Lambda_F \otimes \mathbf{R}$ and $\Lambda_{F,\mathbf{R}}^\vee$ its dual vector space. For every $\alpha \in \Phi$, we denote α_F its image in $\Lambda_{F,\mathbf{R}}^\vee$, and we denote Φ_F the image of the root system in that vector space.

For every $\alpha_F \in \Phi_F$, the preimage in Φ is a Γ-orbit. The finite set $\Phi_F \subset \Lambda_{F,\mathbf{R}}^\vee$ is called the **relative root system**. It defines a cone decomposition in $\Lambda_{F,\mathbf{R}}$

$$\Lambda_{F,\mathbf{R}} = \bigsqcup_{C_F \in C(\Phi_F)} C_F.$$

This is obviously the induced cone decomposition (6.5) on the subspace $\Lambda_{F,\mathbf{R}}$. There is a canonical bijection between $\mathscr{C}^\infty(\Phi_F)$ and the set of parabolic subgroups P containing the centralizer $L_{\mathbf{A}}$ of $\mathbf{A}$. These parabolic subgroups are all defined over F.

Bruhat decomposition

Let G be split reductive group over $\mathbf{k}$. By choosing a representative $\dot{w} \in N_G(T)$ for each $w \in W$, we have a stratification of G into a union of locally closed subset:

$$G = \bigsqcup_{w \in W} B\dot{w}B \tag{6.9}$$

where $B\dot{w}B/B$ is isomorphic to an affine space $\mathbb{A}^{\ell(w)}$ of dimension $\ell(w)$ where $\ell : W \to \mathbf{N}$ is the length function. The orbit corresponding to the longest element $w_0 \in W$ is the open orbit.

We have a similar decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B^- \tag{6.10}$$

of G in to $(B \times B^-)$-orbits. The orbit U corresponding to the unit element e_W of W is the open orbit. This orbit is isomorphic to $N \times T \times N^-$ as an algebraic variety. In other words, the morphism

$$N \times T \times N^- \to G \tag{6.11}$$

defined by $(n, t, n^-) \mapsto ntn^-$ induces an isomorphism from $N \times T \times N^-$ on the open subset U of G.

More generally, for all parabolic subgroups P, Q containing T, we have a decomposition

$$G = \bigsqcup_{w \in W_P \backslash W / W_Q} P\dot{w}Q \tag{6.12}$$

where we have to choose a representative of each $W_P \times W_Q$-orbit in W.

Iwahori factorization

The morphism (6.11): $\mathbf{N}\times\mathbf{T}\times\mathbf{N}^- \to \mathbf{G}$ induces an isomorphism from $N\times T\times N^-$ to a Zariski open neighborhood U of e_G. Although (6.11) is not a homomorphism of group, U is not a subgroup of $\mathbf{G}$, it induces a factorization of a certain compact open subgroups of $G = \mathbf{G}(F)$, when $\mathbf{G}$ is defined over a nonarchimedean local field F. A compact open subgroup K of G is said to satisfy the **Iwahori factorization** if the map

$$(K\cap N)\times(K\cap T)\times(K\cap N^-)\to K \tag{6.13}$$

induced from (6.11), is a homeomorphism.

Proposition 6.1. *Let $Q \subset B(R_F/\mathrm{u}_F^n)$ be a subgroup of $G(R_F/\mathrm{u}_F^n)$. Let K be the subgroup of $G(R_F)$ defined as the preimage of Q via the reduction modulo u_F^n homomorphism*

$$G(R_F)\to G(R_F/\mathrm{u}_F^n). \tag{6.14}$$

Then K has satisfies the Iwahori factorization (6.13).

Proof. An element $g\in G(R_F)$ consists in a morphism $g:\mathrm{Spec}(R_F)\to\mathbf{G}$. If $g\in K$ then its restriction it maps the closed point of $\bar{g}:\mathrm{Spec}(\mathbf{k}_F)$ to B. In particular $\bar{g}$ factorizes through the open subset U of $\mathbf{G}$, image of (6.11). It follows that $g:\mathrm{Spec}(R_F)\to\mathbf{G}$ factorizes through U. We have then a unique factorization $g = ntn^-$ where $n\in\mathbf{N}(R_F), t\in\mathbf{T}(R_F)$ and $n^-\in\mathbf{N}^-(R_F)$ as (6.11) is an isomorphism from $\mathbf{N}\times\mathbf{T}\times\mathbf{N}^-$ on U. By using the uniqueness of the decomposition, we can prove that $n\in K\cap N$, $t\in K\cap T$ and $n^-\in K\cap N^-$. □

Using this proposition, one can easily construct a system of neighborhoods of e_G consisting of compact open subgroups

$$K_0\supset K_1\supset K_2\supset\cdots \tag{6.15}$$

such that:

- for $i\le j$, K_j is a normal subgroup of K_i,
- for each standard parabolic $P = MN$ with opposite parabolic $P^- = MN^-$, we have unique decompositions:

$$K_i = N_i^- M_i N_i = N_i M_i N_i^- \tag{6.16}$$

  where $N_i = N\cap K_i$, $N_i^- = N^-\cap K_i$ and $M_i = M\cap K_i$.

For instant, if one takes K_i to be the "principal congruence" compact open subgroup defined as the kernel of $G(R_F)\to G(R_F/\mathrm{u}_F^i)$, then K_i satisfies the Iwahori factorization for all $i>0$.

The Cartan decomposition

Let F be a non-archimedean local field, R_F its ring of integers. We will denote $\mathbf{k}_F$ the residue field and choose a generator u_F of the maximal ideal of R_F. If $\mathbf{G}$ is a split reductive group over F, then it has a reductive model over R_F that we will also denote by $\mathbf{G}$. In this case $G = \mathbf{G}(F)$ is a *td*-group and $\mathbf{K} = \mathbf{G}(\mathscr{O})$ is a maximal compact subgroup of $G = \mathbf{G}(F)$.

We consider the decomposition of G as union of $\mathbf{K} \times \mathbf{K}$-orbits. Each $\mathbf{K} \times \mathbf{K}$-orbit can be represented by a unique element of the form u_F^λ where $\mathrm{u}_F^\lambda \in \mathbf{T}(F)$ is the image of $\mathrm{u}_F \in F^\times$ by the cocharacter $\lambda : \mathbf{G}_m \to \mathbf{T}$ with $\lambda \in \Lambda^+$. In short, we have the Cartan decomposition:

$$G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{K}\mathrm{u}_F^\lambda \mathrm{K}. \tag{6.17}$$

In the case $\mathbf{G} = \mathrm{GL}_n$, the Cartan decomposition is equivalent to the well known theorem of elementary divisors. In this case, G acts transitively on the set of lattices $\mathscr{L} \subset F^n$ and K is the stabilizer of the standard lattice $\mathscr{L}_0 = R_F^n$. It follows that G/K can be identified with the set of lattices $\mathscr{L} \subset F^n$ by mapping the coset $g\mathrm{K} \in G/\mathrm{K}$ to the lattice $\mathscr{L} = g\mathscr{L}_0$. According to the theorem of elementary divisors, for every lattice $\mathscr{L} \subset \mathbf{F}^n$, there exists vectors $v_1, \dots, v_n \in F^n$ such that $\mathscr{L}_0 = \bigoplus_{i=1}^n v_i\mathscr{O}$ and $\mathscr{L} = \bigoplus_{i=1}^n \mathrm{u}_F^{\lambda_i} v_i \mathscr{O}$ where $(\lambda_1 \geq \cdots \geq \lambda_n)$ is a decreasing sequence of integers completely determined by $\mathscr{L}$. If $u_1, \dots, u_n$ form the standard basis of F^n, if $k \in G$ denotes the base change matrix $k(u_i) = v_i$, then $k \in \mathrm{K}$ as $\mathscr{L}_0 = \bigoplus_{i=1}^n v_i \mathscr{O} = k\mathscr{L}_0$ and $\mathscr{L} = k\mathrm{u}_F^\lambda \mathscr{L}_0$ for $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \Lambda^+$.

We will now explain a simple way to determine in which $\mathrm{K} \times \mathrm{K}$-orbit lies a given element $g \in G(F)$ in the case $G = \mathrm{GL}_n$.

First, for $G = \mathrm{GL}_2$, a matrix $g \in G(F)$ lies in the double coset

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{K} \begin{pmatrix} \mathrm{u}_F^{\lambda_1} & 0 \\ 0 & \mathrm{u}_F^{\lambda_2} \end{pmatrix} \mathrm{K} \tag{6.18}$$

with $\lambda_1 \geq \lambda_2$ if and only if $\mathrm{ord}(\det(g)) = \lambda_1 + \lambda_2$ and if $\mathrm{u}_F^{\lambda_2}$ is the generator of the fractional ideal of F generated by by the entries a, b, c, d of g.

In general, if g is a nonzero matrix, we will denote

$$\mathrm{ord}(g) = \min_{i,j} \{\mathrm{ord}(g_{i,j})\} \tag{6.19}$$

the order of g being the minimum of the order of its entries. In other words, $\mathrm{ord}(g)$ is the integer such that $\mathrm{u}_F^{\mathrm{ord}(g)}$ is a generator of the fractional ideal of F generated by the entries of g.

For $G = \mathrm{GL}_n$, a matrix $g \in G(F)$ lies in the double coset $\mathrm{K}\mathrm{u}_F^\lambda \mathrm{K}$ for $\lambda = (d_1 \geq \cdots \geq d_n) \in \Lambda^+$ if and only if for every $m \in \{1, \dots, n\}$, we have

$$\mathrm{ord}(\wedge^m g) = \lambda_{n-m+1} + \cdots + \lambda_n. \tag{6.20}$$

The Iwasawa decomposition

Let $\mathbf{N}$ denote the unipotent radical of $\mathbf{B}^+$, and $N = \mathbf{N}(F)$. We consider the decomposition of G as union of $N \times \mathrm{K}$-orbits. Each $N \times \mathrm{K}$-orbit contains an unique element of the form u_F^ν for $\nu \in \Lambda$. In other words, we have the Iwasawa decomposition

$$G = \bigsqcup_{\nu \in \Lambda} N \mathrm{u}_F^\nu \mathrm{K}. \tag{6.21}$$

We will explain an elementary way to determine in which $N \times \mathrm{K}$-orbit lies a given matrix $g \in \mathbf{G}$ where $\mathbf{G} = \mathrm{GL}_n$ the group of all linear transformations of a vector space V given with a basis $e_1, \ldots, e_n$. The Borel subgroup $\mathbf{B}^+$ is the stabilizer of the standard flag

$$0 = V_0 \subset V_i \subset \cdots \subset \cdots V_n = V \tag{6.22}$$

where V_i is the subvector space generated by the first vectors $e_1, \ldots, e_i$ in the standard basis. An element $g \in G$ belongs to $N \mathrm{u}_F^\nu \mathrm{K}$ if and only if the lattice $\mathcal{V} = g\mathcal{V}_0$ satisfies the condition

7 Parabolic inductions and cuspidal representations

Let $\mathbf{G}$ be a linear algebraic group over a F. If $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ then $\mathbf{G}/\mathbf{P}$ is a projective variety. If F is a nonarchimedean local field, we know by Proposition 1.2 that $(\mathbf{G}/\mathbf{P})(P)$ is a compact *td*-space. Since $(\mathbf{G}/\mathbf{P})(P) = \mathbf{G}(F)/\mathbf{P}(F)$, the quotient $\mathbf{G}(F)/\mathbf{P}(F)$ is a compact *td*-space. In this section, we will study induced representations from parabolic subgroups.

Finiteness of parabolic induction

Let G be a *td*-group, H a closed subgroup of G and $X = H\backslash G$. For every smooth representation (σ, V_σ) of H, we have defined in Prop. 5.15 a G-equivariant sheaf $\mathcal{V}_\sigma$ on X such that

$$\mathrm{cInd}_H^G(\sigma) = \Gamma_c(X, \mathcal{V}_\sigma) \text{ and } \mathrm{Ind}_H^G(\sigma) = \Gamma(X, \mathcal{V}_\sigma)^{\mathrm{sm}}. \tag{7.1}$$

If X is assumed to be **compact** then $\Gamma_c(X, \mathcal{V}_\sigma) = \Gamma(X, \mathcal{V}_\sigma)$. In particular $\Gamma(X, \mathcal{V}_\sigma)$ is a smooth representation of G and we have $\mathrm{cInd}_H^G(\sigma) = \mathrm{Ind}_H^G(\sigma)$. The compactness of $H\backslash G$ guarantees different finiteness properties of the induced representation.

Proposition 7.1. *Let G be a unimodular td-group and H a closed subgroup of G such that $H\backslash G$ is compact.*

1. *If (σ, V_σ) if a finitely generated smooth representation of H, then* $\mathrm{Ind}_H^G(\sigma)$ *is a finitely generated smooth representation of G.*

2. *If (σ, V_σ) is an admissible representation of H, then $\mathrm{Ind}_H^G(\sigma)$ is an admissible representation of G.*

Proof. Let K be a compact open subgroup of G. We decompose X as a disjoint union $X = \bigsqcup_{\alpha\in\alpha_X} X_\alpha$ of orbits of K which are compact open subsets of X. As X is compact, the set of orbits α_X is finite. For every $\alpha \in \alpha_X$, we choose a representative x_α is the corresponding $H \times K_\alpha$ double coset in G. Then we have

$$\mathrm{Ind}_H^G(\sigma)^K = \Gamma(X, \mathscr{V}_\sigma)^K = \bigoplus_{\alpha\in\alpha_X} \Gamma(X_\alpha, \mathscr{V}_\sigma)^K = \bigoplus_{\alpha\in\alpha_X} V_\sigma^{H\cap x_\alpha K x_\alpha^{-1}}. \tag{7.2}$$

Each summand $V_\sigma^{H\cap x_\alpha K x_\alpha^{-1}}$ is finite dimensional as σ is admissible, their direct sum is finite dimensional. Therefore $\mathrm{Ind}_H^G(\sigma)$ is admissible. □

Normalized parabolic induction

Let $G = \mathbf{G}(F)$ be a reductive p-adic group, P a standard parabolic subgroup of G, and $P = MN$ its Levi decomposition where N the unipotent radical of P and M its standard Levi factor. We define the **normalized parabolic induction** functor $i_M^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ by the formula

$$i_M^G(\sigma) = \mathrm{Ind}_P^G(\mathrm{Inf}_M^P(\sigma) \otimes \Delta_P^{1/2}) \tag{7.3}$$

where $\mathrm{Inf}_M^P(\sigma)$ is the representation of P obtained by inflating the representation σ of M: for every $p = mn \in P$, we set $\mathrm{Inf}_M^P(\sigma)(p) = \sigma(m)$ as a linear transformation of V_σ.

The character $\Delta_P^{1/2} : P \to \mathbf{C}^\times$ is a square root of the modulus character $\Delta_P : P \to \mathbf{C}^\times$. We can make this choice uniform by choosing beforehand a squareroot $q^{1/2}$ of the cardinal of the residue field of F. The normalization by a square of the modulus character has the advantage to commute with the contragredient functor. The following proposition is an immediate consequence of Prop. 5.33.

Proposition 7.2. *if σ' is the contragredient of σ then $i_M^G(\sigma')$ is the contragredient of $i_M^G(\sigma)$.*

For normalization by twisting by a character doesn't affect the finiteness properties of representations, the following proposition is an immediate consequence of Prop. 7.1.

Proposition 7.3. *If σ is a finitely generated smooth representation of M, then $i_M^G(\sigma)$ is a finitely generated smooth representation of G. If σ is an admissible representation of M, then $i_M^G(\sigma)$ is an admissible representation of G.*

Restriction to parabolic subgroups and Jacquet modules

Let P be a standard parabolic subgroup of G and $P = MN$ its standard Levi decomposition. The compactness of the quotient $P\backslash G$ also implies a finiteness property of the restriction functor Res_P^G.

Proposition 7.4. *The restriction* $\mathrm{Res}_G^P(\pi)$ *of a finitely generated smooth representation* π *of* G *is also finitely generated.*

Proof. Let $v_1, \dots, v_n \in V_\pi$ be a system of generators of the representation π i.e. the vectors of the form $\pi(g)v_i$ with $i \in \{1, \dots, n\}$ and $g \in G$ generate V_π as vector space. Let K be a compact open subgroup of G which fixes $v_1, \dots, v_n$. Since $P\backslash G$ is compact, G decomposes as a finite union of $P \times K$-orbits

$$G = \bigsqcup_{j=1}^{m} P g_j K \tag{7.4}$$

It follows that the vectors of the form $\pi(p)\pi(g_j)v_i$ generate V_π as vector space. In other words, the restriction of π to P is finitely generated. $\square$

We already know that $\mathrm{Res}_P^G : \mathrm{Rep}_G \to \mathrm{Rep}_P$ is a left adjoint to the function $\mathrm{Ind}_P^G : \mathrm{Rep}_P \to \mathrm{Rep}_G$. We will now define the Jacquet functor $J_P^M : \mathrm{Rep}_P \to \mathrm{Rep}_M$ which is a left adjoint to the functor $\mathrm{Inf}_M^P : \mathrm{Rep}_M \to \mathrm{Rep}_P$. Let (π, V) be a smooth representation of P. We define the **Jacquet module** $J_P^M(\pi) = \pi_N$ to be representation of M on the maximal quotient of V on which N acts trivially. In other words,if $V(N)$ is the subspace of V generated by vectors of the form $\pi(n)v - v$ with $n \in N$ and $v \in V$, then

$$V_N = V/V(N). \tag{7.5}$$

As P normalizes N, the subspace $V(N)$ is P-stable. The action of P on the quotient $V_N = V/V(N)$ factorizes through M as N acts trivially by construction.

We observe that the functor $J_P^M : \mathrm{Rep}_P \to \mathrm{Rep}_M$ is a left adjoint to the functor $\mathrm{Inf}_M^P : \mathrm{Rep}_M \to \mathrm{Rep}_P$. There is indeed an isomorphism of functors from $\mathrm{Rep}_P^{\mathrm{op}} \times \mathrm{Rep}_M \to \mathrm{Vec}_C$:

$$\mathrm{Hom}_M(J_P^M(\pi), \sigma) = \mathrm{Hom}_P(\pi, \mathrm{Inf}_M^P(\sigma)). \tag{7.6}$$

Since N acts trivially on $\mathrm{Inf}_M^P(\sigma)$ every P-equivariant map $V_\pi \to V_\sigma$ factorizes through the maximal N-invariant quotient $V_{\pi,N}$ and gives rise to a M-equivariant map $V_{\pi,N} \to V_\sigma$, and vice versa.

If π is a smooth representation of G, σ a smooth representation of M then we have a sequence of canonical bijections:

$$\mathrm{Hom}_M(J_P^M(\mathrm{Res}_G^P(\pi)), \sigma) = \mathrm{Hom}_P(\mathrm{Res}_G^P(\pi), \mathrm{Inf}_M^P(\sigma)) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\mathrm{Inf}_M^P(\sigma))) \tag{7.7}$$

It follows that $J_P^M \circ \mathrm{Res}_G^P : \mathrm{Rep}_G \to \mathrm{Rep}_M$ is a left adjoint to the functor $\mathrm{Ind}_P^G \circ \mathrm{Inf}_M^P : \mathrm{Rep}_M \to \mathrm{Rep}_G$. We now have a functor $r_G^M : \mathrm{Rep}_G \to \mathrm{Rep}_M$ which is left adjoint to the normalized parabolic induction functor $i_M^G(\sigma) = \mathrm{Ind}_P^G(\mathrm{Inf}_M^P(\sigma) \otimes \Delta_P^{1/2})$. If we set

$$r_G^M(\pi) = J_P^M(\mathrm{Res}_G^P(\pi) \otimes \Delta_P^{-1/2}) \tag{7.8}$$

then we have a canonical isomorphism of functors $\mathrm{Rep}_G^{\mathrm{op}} \times \mathrm{Rep}_M \to \mathrm{Vec}_{\mathbf{C}}$:

$$\mathrm{Hom}_M(r_G^M(\pi), \sigma) = \mathrm{Hom}_G(\pi, i_M^G(\sigma)). \tag{7.9}$$

In other words, r_G^M is a left adjoint to the functor i_M^G.

Proposition 7.5. *If π is a finitely generated smooth representation of G then $r_G^M(\pi)$ is a finite generated M-modules.*

Proof. If π is a finite generated representation of G, then its restriction $\mathrm{Res}_G^P(\pi)$ is a finitely generated representation of P. For every representation V of P, as V_N is the maximal quotient of V on which N acts trivially, if V is a finitely generated representation of P, then so is V_N. Since N acts trivially on V_N, V_N is also finitely generated as representation of M. It follows that if π is a finitely generated smooth representation of G then $r_G^M(\pi)$ is a finite generated representation of M. □

It is also true that if π is an admissible representation of G, then $r_G^M(\pi)$ is an admissible representation of M. For this inference will require more preparation, we will come back to it in Prop. 7.12.

Proposition 7.6. *Let N be a td-group which is a union of its compact open subgroups*

$$N = \bigcup_{s \in s_N} N_s,$$

$\{N_s | s \in s_N\}$ being the set of compact open subgroups of N. Let (π, V) be a smooth representation of N. Let $V(N)$ be the subspace of V generated by vectors of the form $\pi(n)v - v$. Then we have

$$V(N) = \bigcup_{s \in s_N} \mathrm{im}(\pi(e_{N_s} - 1)) = \bigcup_{s \in s_N} \ker(\pi(e_{N_s})). \tag{7.10}$$

Proof. Since $e_{N_s} \in \mathscr{D}_c(N)$ is an idempotent element, we have $\mathrm{im}(\pi(e_{N_s} - 1)) = \ker(\pi(e_{N_s}))$. It is enough to prove

$$V(N) \subset \bigcup_{s \in s_N} \ker(\pi(e_{N_s}))$$

and

$$V(N) \supset \bigcup_{s \in s_N} \mathrm{im}(\pi(e_{N_s} - 1)).$$

Let $u \in V(N)$, we will prove that $u \in \ker(\pi(e_{N_s}))$ for some large enough compact open subgroup N_s of N. Since u is a linear combination of vectors of the form $\pi(x)v - v$ with $x \in N$ and $v \in V$, we can assume $u = \pi(x)v - v$. If N_s is a compact open subgroup on N containing x, then we have

$$\pi(e_{N_s})(\pi(x)v - v) = \pi(e_{N_s} \star \delta_x - e_{N_s})v = 0$$

for $e_{N_s} \star \delta_x = e_{N_s}$ in $\mathscr{D}_c(N)$.

Now assume $u = \pi(e_{N_1} - 1)v$ for some compact open subgroup N_1 of N and vector $v \in V$. Let N_0 be a compact open subgroup of N such that $\pi(e_{N_0})v = v$ so that $u = \pi(e_{N_1} - e_{N_0})v$. We can assume that $N_0 \subset N_1$. Next, we decompose N_1 as disjoint finite union of N_0 right cosets $N_1 = \bigsqcup_{\alpha \in \alpha_{N_1}} x_\alpha N_0$ so that

$$e_{N_1} = \frac{1}{\#(N_1/N_0)} \sum_{\alpha \in \alpha_{N_1}} \delta_{x_\alpha} \star e_{N_0}.$$

It follows that

$$u = \frac{1}{\#(N_1/N_0)} \sum_{\alpha \in \alpha_{N_1}} \pi(x_\alpha)v - v.$$

In particular, u lies in $V(N)$. □

Generators of Hecke algebras

Thanks to Bernstein, we have a convenient description of the Hecke algebra $\mathscr{H}_K(G)$ under the assumption that K satisfies the Iwahori factorization.

Let A be a maximal split torus of G, B^+ a minimal parabolic subgroup of G containing A, $B^+ = TU$ its Levi decomposition where A is the maximal split torus contained in the center of M. Let $B^- = TU^-$ be opposite parabolic. We denote $\Lambda = \mathrm{Hom}(\mathbf{G}_m, A)$ and Λ_+ the group of cocharacters $\lambda : \mathbf{G}_m \to A$ such that $\lambda(\mathbf{G}_m)$ acts with only nonnegative exponents on the Lie algebra of U and with only nonpositive exponents on the Lie algebra of U^-.

We recall that a compact open subgroup K_i of G is said to satisfy the Iwahori decomposition if we have the unique decomposition $K_i = U_i T_i U_i^-$ where $U_i = K_i \cap U$, $T_i = K \cap T$ and $U_i^- = K_i \cap U^-$. In this case, according to Fubini's theorem, the identity

$$e_{K_i} = e_{U_i} \star e_{T_i} \star e_{U_i^-} \tag{7.11}$$

holds in $\mathscr{D}_c(G)$. The essence of the Bernstein presentation consists in construct a large commutative subalgebra of $\mathscr{H}_K(G)$.

Proposition 7.7. *With notations as above, we denote $h_\lambda = e_{K_i} \star \delta_{\mathrm{u}_F^\lambda} \star e_{K_i}$ for every $\lambda \in \Lambda_+$. Then the linear map $\mathbf{C}[\Lambda_+]$ given by $\lambda \to h_\lambda$ is a homomorphism of algebras.*

Proof. For every $\lambda \in \Lambda_+$, since $\lambda(\mathbf{G}_m)$ acts on $\mathrm{Lie}(N)$ with nonnegative exponents, and on $\mathrm{Lie}(N^-)$ with nonpositive exponents, we have

$$\mathrm{u}_F^{\lambda} U_i \mathrm{u}_F^{-\lambda} \subset U_i \text{ and } \mathrm{u}_F^{-\lambda} U_i^- \mathrm{u}_F^{\lambda} \subset U_i^- \tag{7.12}$$

According to (7.11), we have $e_{K_i} = e_{U_i} \star e_{T_i} \star e_{U_i^-}$. For every $\lambda, \lambda' \in \Lambda_+$, we have

$$\delta_{\mathrm{u}_F^{\lambda}} \star e_{K_i} \star \delta_{\mathrm{u}_F^{\lambda'}} = e_{\mathrm{u}_F^{\lambda} U_i \mathrm{u}_F^{-\lambda}} \star e_{T_i} \star \delta_{\mathrm{u}_F^{\lambda+\lambda'}} \star e_{\mathrm{u}_F^{-\lambda'} U_i^- \mathrm{u}_F^{\lambda'}}.$$

It follows then from (7.12) that

$$e_{K_i} \star \delta_{\mathrm{u}_F^{\lambda}} \star e_{K_i} \star \delta_{\mathrm{u}_F^{\lambda'}} \star e_{K_i} = e_{K_i} \star \delta_{\mathrm{u}_F^{\lambda+\lambda'}} \star e_{K_i}. \tag{7.13}$$

It follows that $\lambda \mapsto h_\lambda$ defines a homomorphism of algebras $h_{K_i} : \mathbf{C}[\Lambda_+] \to \mathscr{H}_{K_i}$. $\square$

Now we assume G is a split reductive group and $K_0 = \mathbf{G}(R_F)$ is a maximal compact open subgroup of G. Then we have the Cartan decomposition

$$G = \bigsqcup_{\lambda \in \Lambda_+} K_0 \mathrm{u}_F^{\lambda} K_0 \tag{7.14}$$

We further assume that K_i is a normal subgroup of K_0 and choose a system of representatives

$$K_0 = \bigsqcup_{\alpha \in \alpha_{K_i}} x_\alpha K_i = \bigsqcup_{\alpha \in \alpha_{K_i}} K_i x_\alpha. \tag{7.15}$$

Since x_α normalizes K_i we have the commutation relation:

$$\delta_{x_\alpha} \star e_{K_i} = e_{K_i} \star \delta_{x_\alpha}. \tag{7.16}$$

Proposition 7.8. *The elements*

$$e_{K_i} \star \delta_{x_\alpha \mathrm{u}_F^{\lambda} x_\beta} \star e_{K_i} = \delta_{x_\alpha} \star h_\lambda \star \delta_{x_\beta} \tag{7.17}$$

form a system of generators of $\mathscr{H}_{K_i}(G)$ as α and β range over α_{K_i} and $\lambda \in \Lambda_+$.

Proof. From (7.14) and (7.15) we infer a decomposition of G as $(K_i \times K_i)$-double cosets:

$$G = \bigcup_{\alpha,\beta \in \alpha_{K_i}} \bigcup_{\lambda \in \Lambda_+} K_i x_\alpha \mathrm{u}_F^{\lambda} x_\beta K_i, \tag{7.18}$$

some double cosets may appear more than once. Up to a nonzero scalar, (7.17) is the unique $K_i \times K_i$-invariant distribution on the double coset $K_i x_\alpha \mathrm{u}_F^{\lambda} x_\beta K_i$. As α and β range over α_{K_i} and $\lambda \in \Lambda_+$, they form a system of generators of $\mathscr{H}_{K_i}(G)$. $\square$

Jacquet's lemmas

We will use Bernstein's presentation of Hecke algebras to study Jacquet's module of smooth representations. Let (π, V) be a smooth representation of G. Let $P = MN$ the Levi decomposition of a standard parabolic subgroup of G, $P^- = MN^-$ the opposite parabolic. Let K_i be a compact open subgroup of G satisfying the Iwahori factorization $K_i = N_i M_i N_i^-$ where $N_i = K_i \cap N$, $M_i = K_i \cap M$ and $N_i^- = K_i \cap N^-$. If (π_N, V_N) is the Jacquet module of (π, V) with respect to P, then we have a map

$$V^{K_i} \to V_N^{M_i}. \tag{7.19}$$

We will use Bernstein's presentation of the Hecke algebras $\mathscr{H}_{K_i}(G)$ and $\mathscr{H}_{M_i}(M)$ to study this map. In particular we will characterize its kernel and, under the admissible assumption, prove that it is surjective. The arguments necessary to carry out this work are known as Jacquet's lemmas.

We recall that we have defined a homomorphism of algebras $h_{K_i} : \mathbf{C}[\Lambda^+] \to \mathscr{H}_{K_i}(G)$ with

$$\lambda \mapsto h_{K_i,\lambda} = e_{K_i} \star \delta_{\mathrm{u}_F^\lambda} \star e_{K_i}. \tag{7.20}$$

Let $\Lambda_M = \mathrm{Hom}(\mathbf{G}_m, A_M)$ denote the subgroup of Λ consisting of cocharacter $\lambda : \mathbf{G}_m \to A$ that factors through the split center A_M of M, and

$$\Lambda_M^+ = \Lambda_M \cap \Lambda^+. \tag{7.21}$$

The submonoid Λ_M^+ of Λ is characterized by the property $\lambda(\mathbf{G}_m)$ acts trivially on M and acts with only nonnegative exponents on the Lie algebra of N. We will denote Λ_P the submonoid of Λ is characterized by the property $\lambda(\mathbf{G}_m)$ acts trivially on M and acts with only positive exponents on the Lie algebra of N. We know that Λ_M^+ is a closed face of the cone Λ^+, and Λ_P is an open face. For the purpose of studying the map (7.19), we will focus on the restriction of the algebra homomorphism $h_{K_i} : \mathbf{C}[\Lambda^+] \to \mathscr{H}_{K_i}(G)$ to subalgebras $\mathbf{C}[\Lambda_M^+]$ and $\mathbf{C}[\Lambda_P]$.

We consider the map $h_{M_i} : \mathbf{C}[\Lambda_M^+] \to \mathscr{H}_{M_i}(M)$ given by

$$\lambda \mapsto h_{M_i,\lambda} = e_{M_i} \star \delta_{\mathrm{u}_F^\lambda} \star e_{M_i} \tag{7.22}$$

For the Levi factor M, we have a similar homomorphism

$$\lambda \mapsto h_{M,\lambda} = e_{M_i} \star \delta_{\mathrm{u}_F^\lambda} = \delta_{\mathrm{u}_F^\lambda} \star e_{M_i}$$

that defines a homomorphism $\mathbf{C}[\Lambda_M] \to \mathscr{H}_{M_i}(M)$. This is easy to check because for $\lambda \in \Lambda_M$, u_F^λ lies in the center of M. By restriction, we have a homomorphism of algebras

$$\mathbf{C}[\Lambda_P] \to \mathscr{H}_{M_i}.$$

Proposition 7.9. *For every smooth representation (π, V) of G, the quotient map $V \to V_N$ induces for each $i \in N$, a homomorphism of $\mathbf{C}[\Lambda_P]$-modules $V^{K_i} \to V_N^{M_i}$. Here $\mathbf{C}[\Lambda_P]$ acts on $V^{K_i} \to V_N^{M_i}$ through $\lambda \mapsto h_\lambda$ and $\lambda \mapsto h_{M,\lambda}$ respectively.*

Proof. First, to prove that $V^{K_i} \to V_N^{M_i}$ is a $\mathbf{C}[\Lambda_P]$-linear, we only need to check that for every $v \in V^{K_i}$ and $\lambda \in \Lambda_P$, the vectors $\pi(h_\lambda)v$ and $\pi(h_{M,\lambda})v$ have the same image in V_N. Let us evaluate $\pi(h_\lambda)v$. Since $v \in V^{K_i}$, we have $\pi(h_\lambda)v = \pi(e_{K_i} \star \delta_{\mathrm{u}_F^\lambda})v$. By the Iwahori decomposition (6.16), we have $e_{K_i} \star \delta_{\mathrm{u}_F^\lambda} = e_{N_i} \star e_{M_i} \star e_{N_i^-} \star \delta_{\mathrm{u}_F^\lambda}$. By passing $\delta_{\mathrm{u}_F^\lambda}$ to the left by conjugation, we have

$$e_{N_i} \star e_{M_i} \star e_{N_i^-} \star \delta_{\mathrm{u}_F^\lambda} = e_{N_i} \star h_{M,\lambda} \star e_{\mathrm{u}_F^{-\lambda} N_i^- \mathrm{u}_F^\lambda}.$$

Since $\mathrm{u}_F^{-\lambda} N_i^- \mathrm{u}_F^\lambda \subset N_i^-$, we have $e_{\mathrm{u}_F^{-\lambda} N_i^- \mathrm{u}_F^\lambda} v = v$, and therefore

$$\pi(h_\lambda)v = \pi(e_{N_i})\pi(h_{M,\lambda})v.$$

Now the difference

$$u = \pi(h_\lambda)v - \pi(h_{M,\lambda})v = \pi(e_{N_i} - 1)\pi(h_{M,\lambda})v$$

lies in $V(N)$ because $\pi(e_{N_i})u = 0$. $\square$

We will also choose an element $\lambda_+ \in \Lambda_M^+$ such that $\lambda(\mathbf{G}_m)$ acts on $\mathrm{Lie}(N)$ with only positive exponents. For compact open subgroup N_s of N we have

$$N = \bigcup_{n \in \mathbf{N}} \mathrm{u}_F^{-n\lambda_+} N_s \mathrm{u}_F^{n\lambda_+}. \tag{7.23}$$

Proposition 7.10. *For each $\lambda_+ \in \Lambda_P$ such that $\mathbf{G}_m(\lambda_+)$ acts on* $\mathrm{Lie}(N)$ *with only positive exponents, we consider the maximal subspace* $\mathrm{Nil}(V^{K_i}, \lambda_+)$ *of V^{K_i} where h_{λ_+} acts nilpotently. Then we have*

$$\mathrm{Nil}(V^{K_i}, \lambda_+) = \ker(V^{K_i} \to V_N^{M_i}).$$

In particular, $\mathrm{Nil}(V^{K_i}, \lambda_+)$ *is independent of the choice of λ_+.*

Proof. Since the action of $\mathbf{C}[\Lambda_P]$ on $V_N^{M_i}$ factors through $\mathbf{C}[\Lambda_M]$, for every $\lambda \in \Lambda_M^+$, the action of $\pi_M(h_{M,\lambda})$ on $V_N^{M_i}$ is invertible. It follows that $\mathrm{Nil}(V^{K_i}, \lambda_+)$ is contained in the kernel of $V^{K_i} \to V_N^{M_i}$.

We now check conversely every vector v lying in the kernel of $V^{K_i} \to V_N^{M_i}$ is annihilated by $\pi(h_\lambda)$ for some $\lambda \in \Lambda_{P^-}$. Since $v \in V(N)$ there exists a compact open subgroup N_s of N such that $\pi(e_{N_s})v = 0$. There exists $n \in \mathbb{N}$ such that $e_{N_s} \subset \mathrm{u}_F^{-n\lambda_+} N_i \mathrm{u}_F^{n\lambda_+}$. It follows that $\pi(e_{\mathrm{u}_F^{-n\lambda_+} N_i \mathrm{u}_F^{n\lambda_+}})v = 0$. By developing $e_{\mathrm{u}_F^{-n\lambda_+} N_i \mathrm{u}_F^{n\lambda_+}} = \delta_{\mathrm{u}_F^{-n\lambda_+}} \star e_{N_i} \star \delta_{\mathrm{u}_F^{-\lambda_+}}$, we see that $\pi(e_{N_i} \star \delta_{\mathrm{u}_F^{n\lambda_+}})v = 0$ since the operator $\delta_{\mathrm{u}_F^{-n\lambda_+}}$ is invertible. For $\pi(e_{K_i})v = v$, it follows that $\pi(e_{K_i} \star \delta_{\mathrm{u}_F^{n\lambda_+}} \star e_{K_i})v = 0$ and therefore $\pi(h_{n\lambda_+})v = 0$. $\square$

Proposition 7.11. *If (π, V) is an admissible representation of G then the map $V^{K_i} \to V_N^{M_i}$ is surjective.*

Proof. For the kernel of $V^{K_i} \to V_N^{M_i}$ is $\mathrm{Nil}(V^{K_i}, \lambda_+)$, the maximal subspace of V^{K_i} where h_{λ_+} acts nilpotently, the action of h_{λ_+} on $\mathrm{im}(V^{K_i} \to V_N^{M_i})$ is injective. With the admissibility assumption, we know that V^{K_i} is finite dimensional, and so is its image in $V_N^{M_i}$. It follows that the action of h_{λ_+} on $\mathrm{im}(V^{K_i} \to V_N^{M_i})$ is invertible. To prove that $\mathrm{im}(V^{K_i} \to V_N^{M_i})$ is equal to $V_N^{M_i}$, it is now enough to prove that for every $u \in V_N^{M_i}$, there exists $n \in \mathbb{N}$ such that $u' = h_{M,n\lambda_+} u \in \mathrm{im}(V^{K_i} \to V_N^{M_i})$.

Let $v \in V$ be an arbitrary vector whose image in V_N is $u \in V_N^{M_i}$. By replacing v by $e_{M_i} v$, we can assume that $v \in V^{M_i}$. By the smoothness assumption, there exists a compact open subgroup N_s^- of N^- such that $v \in V^{N_s^-}$. For $n \in \mathbb{N}$ big enough, we have $\mathrm{u}_F^{-n\lambda_+} N_i^- \mathrm{u}_F^{n\lambda_+} \subset N_s^-$. If we denote $v' = \pi(\mathrm{u}_F^{n\lambda_+})v$ then its image in $V_N^{M_i}$ is $u' = \pi_N(\mathrm{u}_F^{n\lambda_+})u = h_{M,n\lambda_+} u$. We will prove that u' lies in $\mathrm{im}(V^{K_i} \to V_N^{M_i})$.

By construction we have $v' \in V^{N_i^-}$ and therefore $v' \in V^{M_i N_i^-}$ i.e. $\pi(e_{M_i})\pi(e_{N_i^-})v' = v'$. If $v'' = \pi(e_{N_i})v'$ then on the one hand, we have $v'' = \pi(e_{K_i})v'' \in V^{K_i}$ and on the other hand, $v'' - v' \in V(N)$ or in other words the image of v'' in V_N is u'. It follows that u' lies in $\mathrm{im}(V^{K_i} \to V_N^{M_i})$ and so does u. □

Here is an immediate consequence of the surjectivity of $V^{K_i} \to V_N^{M_i}$:

Proposition 7.12. *If (π, V) is an admissible representation of G, then the Jacquet module V_N is also an admissible representation of M.*

Harish-Chandra's theorem on cuspidal representations

A smooth representation (π, V) of G is said to be **cuspidal** if for every proper parabolic subgroup $P = MN$, the Jacquet module V_N is zero.

Proposition 7.13 (Harish-Chandra). *Let (π, V) be a smooth irreducible representation of G, (π', V') its contragredient. A smooth representation (π, V) of G is cuspidal if for every vector $v \in V$ and $v' \in V'$, the matrix coefficient $m_{v,v'}(g) = \langle v', \pi(g)v\rangle$ is compactly supported modulo the center of G.*

Proof. Assume that $V_N = 0$ for all parabolic subgroups $P = MN$, we will prove that the matrix coefficient $m_{v,v'}(g) = \langle v, \pi'(g)v'\rangle$ is compactly supported modulo the center of G for all $v \in V$ and $v' \in V'$.

First, *we claim* it is enough to prove that for every compact open subgroup K of G, for every $v \in V^K$ and $v' \in V'^K$, the set of $\lambda \in \Lambda_{P_0}$ such that $\langle v', \pi(\mathrm{u}_F^\lambda)v\rangle \neq 0$ is finite modulo $\Lambda_Z = \mathrm{Hom}(\mathbf{G}_m, Z)$, Z being the center of G.

Indeed, for every $v \in V$ and $v' \in V'$, we may assume that $v \in V^K$ and $v' \in V'^K$ for a certain compact open subgroup K that is a normal subgroup of K_0. Since K is a normal subgroup of K_0, K_0 stabilizes V^K and $(V')^K$. We can choose a finite set of representatives $x_1, \ldots, x_n$ of cosets

$$K_0 = \bigsqcup_{j=1}^{n} x_i K = \bigsqcup_{j=1}^{n} K x_i.$$

We derive from the Cartan decomposition $G = K_0 \Lambda_{P_0} K_0$ a decomposition in K double cosets:

$$G = \bigsqcup_{i,j=1}^{n} \bigsqcup_{\lambda \in \Lambda_{P_0}} K x_i \mathrm{u}_F^{\lambda} x_j K$$

where $K x_i \mathrm{u}_F^{\lambda} x_j K = x_i K \mathrm{u}_F^{\lambda} K x_j$. For $v \in V^K$ and $v' \in V'^K$ the matrix coefficient $g \mapsto m_{v,v'}(g)$ is left and right invariant under K. Moreover, $m_{v,v'}$ vanishes on the double coset $K x_i \mathrm{u}_F^{\lambda} x_j K$ if and only if $\langle \pi'(x_i)^{-1} v', \pi(\mathrm{u}_F^{\lambda}) \pi(x_j) v \rangle = 0$. For every $i, j \in \{1, \ldots, n\}$, we have

$$\langle \pi'(x_i)^{-1} v', \pi(\mathrm{u}_F^{\lambda}) \pi(x_j) v \rangle = 0$$

for λ lying in the complement of a certain subset of Λ_{P_0} which is invariant and finite modulo Λ_Z. It follows that $m_{v,v'}$ is compactly supported modulo Z.

Now we come to the proof of the claim: if $V_N = 0$ for every proper parabolic subgroup $P = MN$, then for every compact open subgroup K of G, for every $v \in V^K$ and $v' \in V'^K$, the set of $\lambda \in \Lambda_{P_0}$ such that $\langle v', \pi(\mathrm{u}_F^{\lambda}) v \rangle \neq 0$ is finite modulo Λ_Z.

Recall that we have a decomposition

$$\Lambda_{P_0} = \bigsqcup_{P} \Lambda_P^+$$

for P ranging over the set of parabolic subgroup $P \supset P_0$, and Λ_P^+ being the set of $\lambda \in \Lambda_P^+$ such that $\mathrm{Lie}(N) = \bigoplus_{\langle \lambda, \alpha \rangle > 0} \mathfrak{g}_\alpha$, N being the unipotent radical of P. It is enough to prove that for every compact open subgroup K of G, for every $v \in V^K$ and $v' \in V'^K$, for each parabolic subgroup $P \supset P_0$, the set of $\lambda \in \Lambda_P^+$ such that $\langle v', \pi(\mathrm{u}_F^{\lambda}) v \rangle \neq 0$ is finite modulo Λ_Z.

It is also enough to restrict ourselves to the case $K = K_i$ satisfying the Iwahori factorization (6.16). As in Proposition 7.10, for every $\lambda \in \Lambda_P^+$, $h_\lambda = e_{K_i} \star \delta_{\mathrm{u}_F^{\lambda}} \star e_{K_i}$ acts nilpotently on the kernel of $V^{K_i} \to V_N^{M_i}$. As $V_N = 0$, we know that h_λ acts nilpotently on V^{K_i}. Since Λ_P^+ / Λ_Z is finitely generated as monoid, for every $v \in V^{K_i}$, there exists a subset C_v of Λ_P^+, invariant under Λ_Z and finite modulo Λ_Z, such that $\pi(h_\lambda) v = 0$ unless $\lambda \in C_v$. For $v' \in V'^{K_i}$, we have $\langle v', \pi(\mathrm{u}_F^{\lambda}) v \rangle = \langle v', \pi(h_\lambda) v \rangle = 0$ unless $\lambda \in C_v$.

We have thus completed the proof of half the proposition stating that if π is cuspidal, then its matrix coefficients are compactly supported modulo Z. The proof of the other half consists essentially in following the above proof in the backward direction. Assume that matrix coefficients of π are compactly supported modulo the center, then for every compact open subgroup K of G, for every $v \in V^K$ and $v' \in V'^K$, the set of $\lambda \in \Lambda_{P_0}$ such that $\langle v', \pi(\mathrm{u}_F^\lambda)v\rangle \neq 0$ is finite modulo Λ_Z. If follows that for every $\lambda \in \Lambda_P \backslash \Lambda_Z$, $h_\lambda = e_{K_i} \star \delta_{\mathrm{u}_F^\lambda} \star e_{K_i}$ acts on V^{K_i} nilpotently. It follows from Proposition 7.10 that the map $V^{K_i} \to V_N^{M_i}$ is zero. Assuming V admissible, then by Proposition 7.12, $V^{K_i} \to V_N^{M_i}$ is surjective. We infer then $V_N^{M_i} = 0$ for all i, and therefore $V_N = 0$. In fact, we can prove $V_N = 0$ without assuming V being admissible by using an argument in the proof of Proposition 7.12. In the proof of Proposition 7.12 we proved, without assuming V admissible, for every $v \in V_N^{M_i}$, there exists $\lambda \in \Lambda_P^+$ such that $\pi_M(\mathrm{u}_F^\lambda)v$ lies in $\mathrm{im}(V^{K_i} \to V_N^{M_i})$. Now in the present context the map $V^{K_i} \to V_N^{M_i}$ is zero, and so is $\pi_M(\mathrm{u}_F^\lambda)v$. This implies $v = 0$ as the operator $\pi_M(\mathrm{u}_F^\lambda)$ is invertible. □

Proposition 7.14. *Let (π, V) be an irreducible smooth representation of G. Then (π, V) is isomorphic to a subrepresentation of the parabolic induction of an irreducible cuspidal representation of a Levi subgroup.*

Proof. Let $P = MN$ be a minimal standard parabolic subgroup such that $r_G^M(\pi) \neq 0$. By minimality $r_G^M(\pi)$ is a cuspidal representation of M. For π is finitely generated, so is $r_G^M(\pi)$ by Prop. 7.4. Since a finitely generated representation, $r_G^M(\pi)$ admits an irreducible quotient $\pi_N \to \sigma$. Since the Jacquet functor is exact, σ is also a cuspidal representation of M. By the Frobenius reciprocity, we have a nonzero G-linear map $\pi \to i_P^G(\sigma)$. Since π is irreducible, π is a submodule of $i_P^G(\sigma)$. □

Proposition 7.15. *All irreducible smooth representations of G are admissible.*

Proof. Every irreducible cuspidal representations is compact modulo the center, and thus admissible. Every irreducible representation can be realized as a subrepresentation of a parabolic induction of an irreducible cuspidal representation. It remains only to apply Prop. 7.1 □

Uniform admissibility theorem

Proposition 7.16. *Let G be a reductive p-adic group and K a compact open subgroup of G. There exists a constant $c = c(G, K)$ depending only on G and K such that for every irreducible representation (π, V) of G, we have* $\dim V^K \leq c$.

For simplicity, we will assume that $G = \mathbf{G}(F)$ where $\mathbf{G}$ is is a split reductive group. The group $K_0 = \mathbf{G}(R_F)$ of R_F-points of $\mathbf{G}$ is a maximal compact open subgroup of G whose double cosets can be described by the Cartan decomposition:

$$G = \bigsqcup_{\lambda \in \Lambda_+} K_0 \mathrm{u}_F^\lambda K_0. \tag{7.24}$$

By replacing $K = K_i$ by a smaller open compact subgroup, we may assume that K is a normal subgroup of K_0 satisfying the Iwahori factorization $K_i = N_i M_i N_i^-$ with respect to the standard Borel subgroup $B^+ = MN$. Under this assumption, we will prove that for every irreducible smooth representation (π, V) of G we have

$$\dim(V^K) \leq \#(K_0/K)^{2^r} \tag{7.25}$$

where r is the rank of G.

This estimate relies on an elementary problem of linear algebras. Let A be a commutative $\mathbf{C}$-algebra embedded in $\mathrm{End}(V)$ where V is an n-dimensional $\mathbf{C}$-vector space. We would like an upper bound of the dimension of A as $\mathbf{C}$-linear vector space. After Schur we have a general bound:

$$\dim(A) \leq 1 + \lfloor n^2/4 \rfloor. \tag{7.26}$$

Without assumption on A, this upper bound is optimal for it is reached when A is the algebra generated by the scalar matrices and the nilradical of the subalgebra of $\mathrm{End}(V)$ consisting of matrices stabilizing a given subspace of dimension $\lfloor n/2 \rfloor$. This bound is not optimal when A has a few number of generators. For instant, if A is generated by one element, then the Cayley-Hamilton theorem implies that $\dim_{\mathbf{C}}(A) \leq n$. If A is generated by two elements, the same upper bound holds after Gerstenhaber and Taussky-Todd. We don't seem to know a good upper bound depending on the number r of generators of A as $r \geq 3$. It seems reasonable to expect that

$$\dim(A) \leq a(r)n \tag{7.27}$$

where $a(r)$ is a constant depending on r, in other words as r fixed, $\dim(A)$ should be bounded by a linear function on n. Unfortunately, this elementary looking problem does not have an answer so far. Bernstein and Zelevinski proved the uniform admissibility by using an upper bound which slightly improves the exponent 2 in the Schur inequality (7.26).

Proposition 7.17. *Let A be a commutative $\mathbf{C}$-algebra embedded in $\mathrm{End}(V)$ where V is an n-dimensional $\mathbf{C}$-vector space. Assume that A is generated by r elements. Then*

$$\dim_{\mathbf{C}}(A) \leq n^{2-1/2^{r-1}}. \tag{7.28}$$

Proof. □

Proof of the uniform admissibility theorem. Let (π, V) be an irreducible smooth representation of G. If $V^K = 0 = 0$, then there is nothing to be proven. If $V^K \neq 0$, then V^K is an irreducible $\mathscr{H}_K(G)$-module by the admissibility theorem. Let us denote $n = \dim_{\mathbf{C}}(V)$.

Let A denote the image of commutative subalgebra $\mathbf{C}[\Lambda_+]$ of $\mathscr{H}_K(V)$ in $\mathrm{End}(V)$. Since A is generated by r elements, r being the rank of G, we have

$$\dim_{\mathbf{C}}(A) \leq n^{2-1/2^{2-1}} \tag{7.29}$$

By Burnside's theorem, the algebra homomorphism

$$\pi : \mathscr{H}_K(G) \to \mathrm{End}(V) \tag{7.30}$$

is surjective. It follows that

$$\dim(\mathscr{H}_K(K_0))^2 \dim(A) \geq n^2. \tag{7.31}$$

It follows that $n \leq \dim(\mathscr{H}_K(K_0))^{2^r}$. ☐

There are finitely many cuspidal representations with K-fixed vectors

Proposition 7.18. *For every reductive p-adic group G and compact open subgroup K of G, there are only finitely many cuspidal irreducible representation (π, V) of G such that $V^K \neq 0$ with a given central character.*

Proof. We assume that **G** is a semisimple group. We claim that there exists a compact subset of C of G, depending only on K, such that for every irreducible cuspidal representation V of contragredient V', every $v \in V^K$ and $v' \in V'^K$, the matrix coefficient $m_{v,v'}$ is supported by C. Since matrix coefficients of irreducible cuspidal representations are linearly independent, it follows that there are finitely many irreducible cuspidal representations with nonzero K-fixed vectors with a given central character.

Assume that K is a compact open subgroup of G satisfying the Iwahori factorization which is a normal subgroup of K_0. After the uniform admissibility theorem, there exists an integer n_K such that for every irreducible representation (π, V) we have $\dim(V^K) \leq n_K$.

Let (π, V) be an irreducible cuspidal representation of G, then for every $\lambda \in \Lambda_+ \backslash \{0\}$, the operator $h_{\lambda,K} = e_K \star \delta_{u_F^\lambda} \star e_K$ acts on V^K as a nilpotent operator. Since $\dim(V^K) \leq n_K$, we have

$$\pi(h_{n\lambda,K}) = 0 \tag{7.32}$$

☐

We have just derived the finiteness of the number of cuspidal irreducible representations of G with K-fixed vectors from the uniform admissibility. Conversely, one can also derive the uniform bound for admissibility from the finiteness of cuspidal irreducible representations of G and its Levi subgroups. Indeed, for every irreducible representation π of G, there exists a standard Levi subgroup M and a cuspidal irreducible representation σ of M such that π is a submodule of $i_M^G(\sigma)$. We have

$$\dim(\pi^K) \leq \dim(i_M^G \sigma) = \sum_x \dim \sigma^{xKx^{-1} \cap P} \tag{7.33}$$

where x ranges over a finite set of representative of $(P \times K)$-cosets on G. For a given central character, and $x \in P \backslash G/K$ there are finitely many cuspidal irreducible representations of M

with $xKx^{-1} \cap P$-fixed vectors so that we have a uniform upper bound on $\dim \sigma^{xKx^{-1}\cap P}$. It follows a uniform upper bound of $\dim(\pi^K)$.

With deeper result from harmonic analysis, one can in fact obtain a more accurate upper bound on the number of cuspidal representations with K-fixed vectors as well as the dimension of their space of K-fixed vectors. After a deep theorem of Harish-Chandra, which we will later prove, there exists a Haar measure μ on G^0 such that for all compact representations π of G^0, the formal degrees $d_\mu(\pi)$ are positive integers for all compact representations π of G^0.

For $\pi_1, \ldots, \pi_n$ are non isomorphic compact representations of G^0 with K^0-fixed vectors, we have a direct orthogonal decomposition of algebras

$$\mathscr{H}_{K^0}(G^0) = \mathscr{H}_{K^0}(G^0)_{\pi_1} \oplus \cdots \oplus \mathscr{H}_{K^0}(G^0)_{\pi_n} \oplus \mathscr{H}_{K^0}(G^0)'. \tag{7.34}$$

For every $i = 1, \ldots, n$ we have an algebra homomorphism

$$h_\mu(\pi) : \mathrm{End}_{\mathbf{C}}(V_{\pi_i}^{K_0}) \to \mathscr{H}_{K^0}(G^0)_{\pi_i} \tag{7.35}$$

given by

$$v \otimes v^* \mapsto d_\pi(\mu)\mu m_{v\otimes v^*}. \tag{7.36}$$

In particular the image of the identity element $\mathrm{id}_{V_i^{K^0}} \in \mathrm{End}_{\mathbf{C}}(V_{\pi_i}^{K_0})$ is the the unit e_{π_i,K^0} of $\mathscr{H}_{K^0}(G^0)_{\pi_i}$.

We have a decomposition of e_{K_0} as a sum of orthogonal commuting idempotents:

$$e_{K_0} = e_{\pi_1,K_0} + \cdots + e_{\pi_n,K_0} + e' \tag{7.37}$$

By dividing by μ, we obtain an equality of functions

$$\mathbb{I}_{K^0}\mathrm{vol}_\mu(K^0)^{-1} = \sum_{i=1}^n d_\pi(\mu) m_{\mathrm{id}_{V_{\pi_i}^{K^0}}} + f \tag{7.38}$$

By evaluating at the unit element e_G of G, we have

$$\mathrm{vol}_\mu(K^0)^{-1} = \sum_{i=1}^n d_\pi(\mu)\dim(V_{\pi_i}^{K^0}) + f(e_G). \tag{7.39}$$

Using the fact $f\mu$ is an idempotent element of $\mathscr{H}(G)$ and $\check{f} = \bar{f}$ we have $f(e_G) \geq 0$. It follows that

$$\mathrm{vol}_\mu(K^0)^{-1} \geq \sum_{i=1}^n d_\pi(\mu)\dim(V_{\pi_i}^{K^0}). \tag{7.40}$$

Since $d_\pi(\mu)$ are positive integers, we derive both the uniform bound $\dim(V_{\pi_i}^{K^0}) \leq \mathrm{vol}_\mu(K^0)^{-1}$ and that the number of compact representations of G^0 is no more than $\mathrm{vol}_\mu(K^0)^{-1}$.

Bibliographical comments

8 Composition series of parabolic inductions

An important chapter of representation theory of reductive groups is the study of the composition series of parabolic inductions. The case of $G = \mathrm{GL}_2(F)$ is very instructive for the understanding of the general picture.

Parabolic induction in GL_2

The usual choice for Borel subgroup is the subgroup B of upper triangular matrices. We have $B = AN$ where A is the subgroup of diagonal matrices and N the subgroup of unipotent upper triangular matrices. The Weyl group $W = \mathrm{Nor}(T)/T$ has two elements among those the non trivial element can be represented by the permutation matrix

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.1}$$

The Bruhat decomposition has two double cosets

$$G = BwB \cup B. \tag{8.2}$$

The group G acts on the set of vector lines inside F^2, and B can be characterized as the stabilizer of the line generated by the vector of coordinates $v_1(1,0)$. The coset BwB consists of elements $g \in G$ such that gv_1 and v_1 aren't colinear.

A character $\chi : A \to \mathbf{C}^\times$ is given by

$$\chi \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \chi_1(a_1)\chi_2(a_2) \tag{8.3}$$

where χ_1, χ_2 are characters of $F^\times$. The parabolic induction $i_A^G(\chi)$ consists of all $f : G \to \mathbf{C}$ satisfying

$$f(nag) = \Delta_B^{1/2}(a)\chi(a)f(g) \tag{8.4}$$

for all $n \in N$, $a \in A$ and $g \in G$. We recall that the modulus character Δ_B of B can be given explicitly by the formula

$$\Delta_B(a) = |a_1/a_2|^{1/2}. \tag{8.5}$$

Proposition 8.1. *Let χ be a character of A and χ^w the character given by $\chi^w(a) = \chi(w(a))$ for any $w \in W$. Then we have an exact sequence of A-modules:*

$$0 \to \Delta_B^{1/2}\chi^w \to i_A^G(\chi)_N \to \Delta_B^{1/2}\chi \to 0. \tag{8.6}$$

In other words:

$$0 \to \chi^w \to r_G^A i_A^G(\chi) \to \chi \to 0. \tag{8.7}$$

Proof. If $\mathscr{F}$ denotes the sheaf on $X = B\backslash G$ associated to the character of B given by $b = an \mapsto \Delta_B^{1/2}(a)\chi(a)$ for $a \in A$ and $n \in N$, N being the unipotent radical of B, then we have

$$i_A^G(\chi) = \Gamma(X, \mathscr{F}) \tag{8.8}$$

by definition. The Bruhat decomposition of $G = \mathrm{GL}_2$, consisting in the partition $G = BwB \sqcup B$, induces on the partition $X = X_1 \sqcup X_0$ where $X_1 = F$ and $X_0 = \{x_0\}$, x_0 being the base point of $B\backslash G$ of stabilizer B. This is also the partition of X in to B-orbits, B acting on the right, with X_1 being the open B-orbit and X_0 the closed orbit. It gives rise to the exact sequence of B-modules:

$$0 \to \Gamma_c(X_1, \mathscr{F}) \to \Gamma_c(X, \mathscr{F}) \to \Gamma(X_0, \mathscr{F}) \to 0. \tag{8.9}$$

Since the Jacquet functor $V \mapsto V_N$ is exact, we derive the exact sequence:

$$0 \to \Gamma_c(X_1, \mathscr{F})_N \to \Gamma_c(X, \mathscr{F})_N \to \Gamma(X_0, \mathscr{F})_N \to 0. \tag{8.10}$$

It remains to prove that $\Gamma(X_0, \mathscr{F}) = \Delta_B^{1/2}\chi$ and $\Gamma_c(X_1, \mathscr{F})_N = \Delta_B^{1/2}\chi^w$ as A-modules.

The closed orbit being a point, we have $\Gamma(X_0, \mathscr{F}) = \mathscr{F}_{x_0}$ and the map $\Gamma_c(X, \mathscr{F}) \to \Gamma(X_0, \mathscr{F})$ consists in evaluation of a section ϕ of $\mathscr{F}$ at the point x_0. By definition a global section $\phi \in \Gamma(X, \mathscr{F})$ consists of a smooth function $\phi : G \to \mathbf{C}$ satisfying

$$l_{(an)^{-1}}\phi(x) = \phi(anx) = \Delta_B^{1/2}(a)\chi(a)\phi(x)$$

for all $a \in A$ and $n \in N$, and the evaluation at x_0 consists in $\phi \mapsto \phi(e_G)$. For every $a \in A$ we have

$$r_a\phi(e_G) = l_{a^{-1}}\phi(e_G) = \Delta_B^{1/2}(a)\chi(a)\phi(e_G).$$

We infer $\Gamma(X_0, \mathscr{F}) = \Delta_B^{1/2}\chi$ as A-modules.

On the open orbit $X_1 = B\backslash BwB$, we define the linear functional $\ell_1 : \Gamma_c(X_1, \mathscr{F}) \to \mathbf{C}$ by

$$\ell_1(\phi) = \int_N \phi(wx)\mathrm{d}x \tag{8.11}$$

where $\mathrm{d}x$ is a Haar measure on N. [4] For every $n \in N$ we have $\ell_1(r_n\phi) = \ell_1(\phi)$ for $\mathrm{d}x$ is invariant under right translation by n. Now we compute the integral

$$\ell_1(r_a\phi) = \int_N \phi(wxa)\mathrm{d}x \tag{8.12}$$

for $a \in A$. If $x' = a^{-1}xa$, we have $\mathrm{d}x' = \Delta_B(a)\mathrm{d}x$ and thus

$$\ell_1(r_a\phi) = \Delta_B(a)\int_N \phi(wax')\mathrm{d}x'. \tag{8.13}$$

[4] Thus ℓ_1 depends on the choice of a Haar measure on N.

On the other hand we have

$$\phi(wax') = \phi(w(a)wx') = \Delta_B(w(a))^{1/2}\chi(w(a))\phi(wx')$$

where $w(a) = waw^{-1}$ and therefore

$$\ell_1(r_a\phi) = \Delta_B^{1/2}(a)\chi(w(a))\ell_1(\phi) \tag{8.14}$$

for $\Delta_B(w(a))^{1/2} = \Delta_B^{-1/2}(a)$. Therefore ℓ_1 defines a homomorphism of A-modules

$$\ell_1 : \Gamma_c(X_1, \mathscr{F}) \to \Delta_B^{1/2}\chi^w.$$

It remains to prove that it is an isomorphism. □

If $\chi : A \to \mathbf{C}^\times$ is a character such that $\chi \neq \chi^w$ then the exact sequence (8.7) splits:

$$r_G^A i_A^G(\chi) = \chi^w \oplus \chi. \tag{8.15}$$

It follows then from the Frobenius reciprocity

$$\dim \mathrm{End}_G(i_B^G(\chi), \chi) = \dim \mathrm{Hom}_A(r_G^A i_A^G(\chi), \chi) = \dim \mathrm{Hom}_A(\chi^w \oplus \chi, \chi) = 1. \tag{8.16}$$

In other words under the assumption $\chi \neq \chi^w$ all endomorphisms of $i_B^G(\chi)$ are scalar multiplications. Under the same assumption, we infer from the canonical projection $r_G^A i_A^G(\chi) \to \chi^w$ a G-map

$$A_w(\chi) : i_A^G(\chi) \to i_A^G(\chi^w). \tag{8.17}$$

Since $\dim \mathrm{End}(i_A^G(\chi)) = 1$, the composition

$$A_w(\chi^w) \circ A_w(\chi) : i_A^G(\chi) \to i_A^G(\chi) \tag{8.18}$$

must be the scalar multiplication by $c(\chi) \in \mathbf{C}$.

Proposition 8.2. *Assume that $\chi \neq \chi^w$. Then $i_B^G(\chi)$ is reducible if and only if $c(\chi) = 0$.*

The proof Proposition 8.2 relies on the following assertion.

Proposition 8.3. *If π be an irreducible subquotient of $i_A^G(\sigma)$ then $r_G^A(\pi) \neq 0$.*

Proof. We will argue by reductio ad absurdum. If $r_G^A(\pi) = 0$, then π is cuspidal. It follows that π being an irreducible subquotient of $i_A^G(\sigma)$, is in fact an irreducible submodule. The nonzero map $\pi \to i_A^G(\sigma)$ induces then by the Frobenius reciprocity a nonzero map $r_G^A(\pi) \to \sigma$ and therefore $r_G^A(\pi) \neq 0$. □

Proof of Proposition 8.2. If $i_A^G(\chi)$ is reducible then there exists an exact sequence

$$0 \to \pi \to i_A^G(\chi) \to \pi' \to 0$$

where π, π' are nonzero G-modules. We infer an exact sequence

$$0 \to r_G^A \pi \to r_G^A i_A^G(\chi) \to r_G^A \pi' \to 0$$

where both $r_G^A \pi$ and $r_G^A \pi'$ are nonzero. Since $r_G^A i_A^G(\chi) = \chi \oplus \chi^w$, we have either $r_G^A \pi = \chi$ or $r_G^A \pi = \chi^w$. For the nonzero map $\pi \to i_B^G(\chi)$ induces by the Frobenius reciprocity a nonzero map $r_G^A \pi \to \chi$, we have $r_G^A \pi = \chi$. As the map $A_w(\chi) : i_A^G(\chi) \to i_A^G(\chi^w)$ is defined as the Frobenius reciprocity of $r_G^A i_A^G(\chi) \to \chi^w$, it follows that the restriction of $A_w(\chi)$ to π is zero. This implies that $A_w(\chi)$ is not injective and therefore $A_w(\chi^w) \circ A_w(\chi)$ is not injective. It follows that $c(\chi) = 0$.

Assume that $i_A^G(\chi)$ is irreducible. Since $A_w(\chi) : i_A^G(\chi) \to i_A^G(\chi^w)$ is nonzero, it ought be injective. We have an exact sequence

$$0 \to i_A^G(\chi) \to i_A^G(\chi^w) \to \pi \to 0.$$

We infer an exact sequence

$$0 \to r_G^A i_A^G(\chi) \to r_G^A i_A^G(\chi^w) \to r_G^A \pi \to 0.$$

Since $\dim_{\mathbf{C}}(r_G^A i_A^G(\chi)) = \dim_C(r_G^A i_A^G(\chi^w)) = 2$, we must have $r_G^A \pi = 0$. After Proposition 8.3, this implies that $\pi = 0$. It follows that $i_A^G(\chi^w)$ is also irreducible. Since $A_w(\chi^w) : i_A^G(\chi^w) \to i_A^G(\chi)$ is non zero, it ought be injective. It follows that the composition $A_w(\chi^w) \circ A_w(\chi) : i_A^G(\chi) \to i_A^G(\chi)$ is injective and therefore $c(\chi) \neq 0$. $\square$

Let us consider examples when $i_A^G(\chi)$ is reducible. We consider the character $\chi : A \to \mathbf{C}^\times$ given by

$$\chi(a) = \Delta^{-1/2}(a).$$

Then $i_A^G(\Delta^{-1/2}) = \mathrm{Ind}_B^G(\mathbf{C})$ is the induction of the trivial representation. The space of $\mathrm{Ind}_B^G(\mathbf{C})$ consists of all smooth function $\phi : G \to \mathbf{C}$ such that $\phi(bg) = \phi(g)$, with G acting by translation on the right. It contains as G-invariant subspace the space of constant functions. We obtain the exact sequence

$$0 \to \mathbf{C} \to i_A^G(\Delta^{-1/2}) \to \mathrm{St} \to 0 \tag{8.19}$$

where the quotient of $\mathrm{Ind}_B^G(\mathbf{C})$ by $\mathbf{C}$ is called the Steinberg representation. It follows from Propositions 8.1 and 8.3 that the Steinberg representation is irreducible with $r_G^A \mathrm{St} = \Delta^{1/2}$.

We may also consider the character $\chi : A \to \mathbf{C}^\times$ given by

$$\chi(a) = \Delta^{1/2}(a).$$

Then $i_A^G(\Delta^{1/2}) = \mathrm{Ind}_B^G(\Delta)$ is the induction of the modulus character. In this case we have a canonical G-equivariant map $\mathrm{Ind}_B^G(\Delta) \to \mathbf{C}$ and thus an exact sequence

$$0 \to \mathrm{St} \to i_A^G(\Delta^{1/2}) \to \mathbf{C} \to 0 \tag{8.20}$$

which is dual to the sequence (8.19).

Stratification of flag varieties and filtration

Let $P = MN$ and $P' = M'N'$ be two standard parabolic subgroups of G. We let P' acts on $P\backslash G$ by the translation on the right. There are only finitely many orbits which are parametrized by a certain subset $\{w_1, \dots, w_n\}$ of the Weyl group:

$$P\backslash G = \bigsqcup_{i=1}^{n} P\dot{w}_i P'. \tag{8.21}$$

The elements $w_1, \dots, w_n$ form a set of representatives of $W_M \times W_{M'}$-cosets in W. They are to be ordered in such a way that for every $m \leq n$, $\bigsqcup_{i=1}^m P\backslash P w_i P'$ is an open subset of $P\backslash G$. There are no preferred total order on $\{w_1, \dots, w_n\}$. If we denote $M_i = M \cap w_i M' w_i^{-1}$ and $M'{}_i = w_i^{-1} M w_i \cap M'$, then M_i and M_i' are Levi subgroups of M and M' respectively, and $\mathrm{ad}(w_i)^{-1}$ induces an isomorphism from M_i to M_i'.

Proposition 8.4. *With $P = MN$ and $P' = M'N'$ and $w_1, \dots, w_n \in W$ as above, for every smooth representation σ of M, the representation $r_G^{M'} i_M^G(\sigma)$ has a filtration, depending functorially on σ*

$$\mathrm{F}_1 r_G^{M'} i_M^G(\sigma) \subset \mathrm{F}_2 r_G^{M'} i_M^G(\sigma) \subset \cdots \subset \mathrm{F}_n r_G^{M'} i_M^G(\sigma) = r_G^{M'} i_M^G(\sigma) \tag{8.22}$$

such that

$$\mathrm{F}_i r_G^{M'} i_M^G(\sigma)/\mathrm{F}_{i-1} r_G^{M'} i_M^G(\sigma) = i_{M_i'}^{M'} \mathrm{ad}(w_i) r_M^{M_i}(\sigma).$$

Proof. The choice of a total order $w_1, \dots, w_n$ compatible with the topological order, allows us to filter the flag variety $X = P\backslash G$ as

$$X_1 \subset X_2 \subset \cdots \subset X_n = X$$

where $X_i = \bigsqcup_{j=1}^i P\backslash P w_j P'$ are open subsets of X. For every M-module (σ, V_σ), we have constructed a sheaf $\mathscr{F}$ on $P\backslash G$ such that

$$\Gamma_c(X, \mathscr{F}) = i_M^G(\sigma) = \mathrm{Ind}_P^G(\mathrm{Inf}_M^P(\sigma) \otimes \Delta_P^{1/2})$$

is the space of all smooth functions $\phi : G \to V_\sigma$ such that $\phi(pg) = \Delta_P^{1/2}(p)\sigma(p)\phi(g)$. We define the filtration

$$\mathrm{F}_1 i_M^G(\sigma) \subset \mathrm{F}_2 i_M^G(\sigma) \subset \cdots \subset \mathrm{F}_n i_M^G(\sigma)$$

by setting

$$\mathrm{F}_i i_M^G(\sigma) = \Gamma_c(X_i, \mathscr{F}).$$

We then have

$$\mathrm{F}_i i_M^G(\sigma)/\mathrm{F}_{i-1} i_M^G(\sigma) = \Gamma_c(P\backslash Pw_iP', \mathscr{F})$$

which is the space of smooth functions $\phi : Pw_iP' \to V_\sigma$ satisfying $\phi(pg) = \Delta_B^{1/2}(p)\chi(p)\phi(g)$ whose support is contained in BC for a certain compact subset C of Pw_iP'.

In order to analyze $\Gamma_c(P\backslash Pw_iP', \mathscr{F})$ we observe that there is a canonical identification

$$P\backslash Pw_iP' = (P' \cap w_i^{-1}Pw_i)\backslash P' \tag{8.23}$$

The restriction of $\mathscr{F}$ to $(P' \cap w_i^{-1}Pw_i)\backslash P'$ correspond to the representation of $P' \cap w_i^{-1}Pw_i$ obtained from the representation (σ, V_σ) of P via the homomorphism $\mathrm{ad}(w_i) : P' \cap w_i^{-1}Pw_i \to P$. If σ^{w_i} denotes the resulted representation of $P' \cap w_i^{-1}Pw_i$ then we have

$$\Gamma_c(P\backslash Pw_iP', \mathscr{F}) = c\mathrm{Ind}_{P'\cap w_i^{-1}Pw_i}^{P'}((\sigma \otimes \Delta_P^{1/2})^{w_i}) \tag{8.24}$$

It remains to calculate the Jacquet module $c\mathrm{Ind}_{P'\cap w_i^{-1}Pw_i}^{P'}(\sigma^{w_i})_{N'}$ as M'-module.

We will perform this calculation first in the case $P = P' = B$ where B is the standard Borel subgroup, and $B = AN$ its Levi decomposition, for less root combinatorics get involved in this basic case. For every $w \in W$, we will calculate the Jacquet module

$$c\mathrm{Ind}_{B\cap w^{-1}Bw}^{B}(\chi^w \otimes (\Delta_B^{1/2})^w)_N. \tag{8.25}$$

To simplify notations we set $B_w = B \cap w^{-1}Bw$ and $\tau = \chi^w \otimes (\Delta_B^{1/2})^w$. We know that the contragredient of $c\mathrm{Ind}_{B_w}^B(\tau)$ is

$$c\mathrm{Ind}_{B_w}^B(\tau)' = \mathrm{Ind}_{B_w}^B(\tau' \otimes \Delta_{B_w} \otimes \Delta_B^{-1}).$$

The Jacquet module $c\mathrm{Ind}_{B_w}^B(\tau)_N$ can be then identified with the space of N-invariant vectors in $\mathrm{Ind}_{B_w}^B(\tau' \otimes \Delta_{B_w} \otimes \Delta_B^{-1})$.

We have a decomposition $N = N_wN^w$ where $N_w = N \cap w^{-1}Bw$ and $N^w = N \cap w^{-1}B^-w$ where B^- is the opposite Borel subgroup where N^w acts simply transitively on $B_w\backslash B$.

□

Associated cuspidal data

A cuspidal datum of a p-adic reductive group is a pair (M, σ) consisting of a standard Levi subgroup M and an cuspidal irreducible representation σ of M. Cuspidal data (M, σ) and (M', σ) are said to be associated if there exists $w \in W$ such that $\mathrm{ad}(w)(M) = M'$ and $\mathrm{ad}(w)(\sigma) = \sigma'$.

Proposition 8.5. *For every smooth representation π of G, there exists a cuspidal datum (M,σ), such that π is a subquotient of $i_M^G(\sigma)$. Moreover, if (M,σ) and (M',σ') are cuspidal data such that π is a subquotient of both $i_M^G(\sigma)$ and $i_{M'}^G(\sigma')$, then (M,σ) and (M',σ') are associated.*

Proof. Since association is a transitive relation, one may assume that (M',σ') is a cuspidal data such that π is a subrepresentation of $i_M^G(\sigma')$, or by the Frobenius reciprocity, σ' is a quotient of $r_M^G(\pi)$. If π is a subquotient of $i_M^G(\sigma)$, then σ' is a subquotient of $r_{M'}^G i_M^G(\sigma)$. By Proposition 8.4, σ' has to be a subquotient of

$$\sigma' \angle\, i_{M_i'}^{M'} \circ \mathrm{ad}(w_i)(r_{M_i}^M(\sigma))$$

for some i. Now as σ is cuspidal, $r_{M_i}^M(\sigma)$ is nonzero if and only if $M = M_i$.

We claim that a cuspidal irreducible representation π of G is a subquotient of an induced representation $i_M^G(\sigma)$ then $M = G$. Indeed, a cuspidal representation is compact modulo center. If it is a subquotient of $i_M^G(\sigma)$, it has to be a submodule. We derive from the nonzero map $\pi \to i_M^G(\sigma)$ a nonzero map $r_M^G(\pi) \to \sigma$. It follows that $r_M^G(\pi) \neq 0$ and thus σ is not cuspidal.

By the following proposition, if the cuspidal irreducible representation σ' is a subquotient of an induced representation $i_{M_i'}^{M'}$ then $M = M_i'$. It follows that (M,σ) and (M',σ') are associated. □

Bernstein's second adjunction theorem

Proposition 8.6.

9 The Bernstein decomposition and center

Cuspidal components

For a connected reductive group $\mathbf{G}$ over F we define $\Lambda_G^* = \mathrm{Hom}(\mathbf{G}, \mathbf{G}_m)$ to be the group of all algebraic characters of $\mathbf{G}$. This is a free abelian group of finite type of rank equal to the rank of the split center of $\mathbf{G}$. For every $\alpha \in \Lambda_G^*$ and $s \in \mathbf{C}$, we define $|\alpha|^s : \mathbf{G} \to \mathbf{C}^\times$ to be the character

$$g \mapsto |\alpha(g)|^s. \tag{9.1}$$

Characters of this form are called **unramified characters** of G. For every algebraic character $\alpha \in \Lambda_G^*$, we have a homomorphism $\mathrm{ord}(\alpha) : G \to \mathbf{Z}$ defined by

$$g \mapsto \mathrm{ord}(\alpha(g)). \tag{9.2}$$

By duality, we obtain a homomorphism $\mathrm{ord} : G \to \Lambda_G$ where Λ_G is the dual abelian group of G. We denote

$$G^0 = \ker(\mathrm{ord} : G \to \Lambda_G) \tag{9.3}$$

its kernel. This is a normal subgroup of G that can also be defined as the intersection of all unramified characters $|\alpha|^s$ for $\alpha \in \Lambda_G^*$ and $s \in \mathbf{C}$, or of all homomorphism $\mathrm{ord}(\alpha)$ for $\alpha \in \Lambda_G^*$.

Proposition 9.1. *If A_G denotes the split center of G, then $A_G G^0$ is a normal subgroup of G of finite index.*

Proof. □

Proposition 9.2. *If π be an irreducible representation of G, then the restriction of π to G^0 is isomorphic to a direct sum of irreducible representation of G^0 which are G-conjugate with respect to the action of G on G^0 by conjugation.*

If π_1, π_2 are irreducible representations of G such that

$$\mathrm{Hom}_{G^0}(\mathrm{Res}_G^{G^0} \pi_1, \mathrm{Res}_G^{G^0} \pi_2) \neq 0$$

then there exists an unramified character $\chi : G \to \mathbf{C}^\times$ such that $\pi_1 \simeq \pi_2 \otimes \chi$. In this case, $\mathrm{Res}_G^{G^0} \pi_1$ and $\mathrm{Res}_G^{G^0} \pi_2$ are isomorphic as representations of G^0. In this case, we will say that π_1 and π_2 are inertially equivalent.

Proof. This is an instance of the Clifford theory for smooth representations of *td*-groups. □

Let π be a cuspidal irreducible representation of G. A smooth representation σ of G is said to be in the **cuspidal component** of π if every irreducible subquotient of σ is inertially equivalent to π. Let $\mathrm{Rep}(G)_\pi$ denote the full subcategory of $\mathrm{Rep}(G)$ consisting of representations in the cuspidal component of π. Let $\mathrm{Rep}(G)^\pi$ be the category consisting of smooth representations σ with no irreducible quotient inertially equivalent to π. Consider $\Pi_\pi = \mathrm{Ind}_{G^0}^G \pi^0$ where π^0 is the restriction of π to G^0.

Proposition 9.3. *We have a direct decomposition $\mathrm{Rep}(G) = \mathrm{Rep}(G)_\pi \oplus \mathrm{Rep}(G)^\pi$. The category $\mathrm{Rep}(G)_\pi$ admits $\Pi = c\mathrm{Ind}_{G^0}^G \pi^0$ as progenerator and thus is isomorphic to the category of modules over $\mathrm{End}_G(\Pi)$.*

Induced components

Let π be a non cuspidal irreducible representation of G. There exists a standard Levi subgroup M and a cuspidal irreducible representation σ of M such that π is a subquotient of $i_M^G \sigma$, and moreover the pair (M, σ) is uniquely determined up to association. We consider the full subcategory $\mathrm{Rep}(G)_{(M,\sigma)}$ of all smooth representations of $\mathrm{Rep}(G)$ whose every irreducible subquotient is a subquotient of $i_M^G \sigma'$ with σ' inertially equivalent to σ. We have constructed a progenerator $\Pi_\sigma = c\mathrm{Ind}_{G^0}^G(\sigma^0)$ of the category $\mathrm{Rep}(M)_\sigma$.

Proposition 9.4. *$i_M^G(M_\sigma)$ is a progenerator of the abelian category $\mathrm{Rep}(G)_{(M,\sigma)}$. In particular, $\mathrm{Rep}(G)_{(M,\sigma)}$ is equivalent to the category of modules over $\mathrm{End}_G(i_M^G(M_\sigma))$.*

Proposition 9.5.

Center of a category of modules

The center $Z(\mathscr{A})$ of an abelian category $\mathscr{A}$ is the ring of endomorphism of its identical functor. An element $z \in Z(\mathscr{A})$ is a collection of $\mathscr{A}$-morphisms $z_M : M \to M$ for each object $M \in \mathscr{A}$ compatible with all $\mathscr{A}$-morphisms $\phi : M \to M'$ i.e. $\phi \circ z_M = z_{M'} \circ \phi$.

Let A be an unital associative ring and $\mathscr{A}$ the category of A-modules. We claim that the center of $Z(\mathscr{A})$ is just the center Z_A of A. Indeed, there is an obvious map $Z_A \to Z(\mathscr{A})$ by letting an element $z \in Z_A$ acts on every A-module M by multiplication. The multiplication by z in M is a morphism of A-modules as z belongs to the center of A. The compatibility with A-linear morphism $\phi : M \to M'$ is satisfied by the multiplication by any element of A, and in particular, elements belonging to its center. Conversely, if $z \in Z(\mathscr{A})$ is an element of the center of the category of A-modules, then its action on A, as an A-module, defines an element z_A of A^{op} as the opposite ring A^{op} is the ring of A-linear morphisms of A. Moreover, the compatibility with all A-linear morphisms implies that A belongs to the center of A^{op} that is also the center of A. Therefore $z \mapsto z_A$ defines a map $Z(\mathscr{A}) \to Z_A$ that is inverse to the map $Z_A \to Z(\mathscr{A})$ we defined just above. We have thus proved that the center $Z(\mathscr{A})$ of the category $\mathscr{A}$ of A-modules is nothing but the center of A, assuming that A is an unital associative ring.

This assertion does not hold for non-unital ring as $Z(\mathscr{A})$ is by construction unital but Z_A is not. If A does not possess a unit, we can however formally add an unit to A by considering the ring $A^+ = \mathbf{Z} \oplus A$ with the multiplication law $(1 \oplus a_1)(1 \oplus a_2) = 1 \oplus (a_1 + a_2 + a_1 a_2)$. As the category of A-modules $\mathscr{A}$ is obviously equivalent to the category of A^+-modules, the center of $\mathscr{A}$ can be identified with the center of A^+. However, we are not very often interested in the whole category of A-modules if A is non unital as it contains degenerate objects M in which every element $a \in A$ acts as 0. We are more often interested in certain subcategory of non-degenerate modules instead. Before explaining the concept of non-degenerate modules, let us consider an instructive example.

An easy example

Let A_i be a family of unital associative ring indexed by a certain set I, $\mathscr{A}_i$ the category of A_i-modules. Let $A^I = \prod_{i \in I} A_i$ denote the direct product of A_i, elements of $\prod_{i \in I} A_i$ are collections $(a_i)_{i \in I}$, A^I is a unital associative ring in an obvious way. Any projection $p_i : A^I \to A_i$ has an obvious section $e_i : A_i \to A^I$ that is compatible with addition and multiplication but not with units. The image of the unite of A_i in A^I will be denoted also by $e_i \in A^I$, which is an idempotent element of A i.e. $e_i^2 = e_i$. We have

$$e_i A^I = A^I e_i = A_i. \tag{9.4}$$

More generally, for every subset $J \subset I$, the projection $A^I \to A^J$ has a non-unital section $e_J : A^J \to A^I$ and we denote also by e_J the image of the unit of A^J in A^I, which is an idempotent element of A^I.

Let $\mathscr{A}^I = \prod_{i\in I} \mathscr{A}_i$ denote the direct product of the abelian categories $\mathscr{A}_i$, its objets are collections $(M_i)_{i\in I}$ where M_i are A_i-modules. There is an equivalence of categories between $\prod_{i\in I} \mathscr{A}_i$ and the category of $\prod_{i\in I} \mathscr{A}_i$-modules. Indeed, if $(M_i)_{i\in I}$ is an object of $\prod_{i\in I} \mathscr{A}_i$, $\prod_{i\in I} M_i$ is a $\prod_{i\in I} \mathscr{A}_i$-module in the obvious way. Conversely, if M is a $\prod_{i\in I} \mathscr{A}_i$-module, $M_i = e_i M$ is a $A_i = A^I e_i$-module, and $M \mapsto (M_i)_{i\in I}$ is inverse to the functor $(M_i)_{i\in I} \mapsto \prod_{i\in I} M_i$. The center of $\mathscr{A}^I$ can thus be identified with the center of A^I and therefore

$$Z(\mathscr{A}^I) = Z_{A^I} = \prod_{i\in I} Z_{A_i}. \tag{9.5}$$

Let $A_I = \bigoplus_{i\in I} A_i$ denote the direct sum of A_i as abelian groups, elements of $\bigoplus_{i\in I} A_i$ are collections $(a_i)_{i\in I}$ with $a_i \in A_i$ being zero for all but finitely many $i \in I$. For all $i \in I$, the idempotent $e_i \in A_I$ but for every subset $J \subset I$, the idemptotent $e_J \in A^I$ belongs to A_I only if J is finite. If the index set I is infinite, A_I is a non-unital subring of A^I.

Let $\mathscr{A}_I = \bigoplus_{i\in I} \mathscr{A}_i$ denote the direct sum of the abelian categories $\mathscr{A}_i$, its objets are collections $(M_i)_{i\in I}$ where M_i are A_i-modules such that $M_i = 0$ for all but finitely many i. If $(M_i)_{i\in I}$ is an object of $\mathscr{A}_I$, $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$ has obvious structures of A^I-module, and consequently it is also an A_I-module. We claim that the center of $\mathscr{A}_I$ is the center of A^I.

For every M is an A_I-module, if we denote $M_i = e_i M$ then we have natural maps

$$\bigoplus_{i\in I} M_i \to M \to \prod_{i\in I} M_i. \tag{9.6}$$

Both maps may be strict inclusion as shown by the following example: let the set of indices I be $\mathbf{N}$, $A_i = \mathbf{C}$ for all $i \in \mathbf{N}$. In this case A_I is the algebra of sequences $(a_i)_{i\in\mathbf{N}}$ whose members a_i vanish for all i but finitely many. Let M denote the space of all sequences $a_i \in \mathbf{C}$ converging to 0, M is an A_I-module. We have $M_i = A_i$ for all i, $\bigoplus_{i\in I} M_i = A_I$, $\prod_{i\in I} M_i = A^I$, and the strict inclusions $A_I \subset M \subset A^I$.

An A_I-module M is said to be non-degenerate if the morphism $\bigoplus_{i\in I} M_i \to M$ is an isomorphism of A_I-modules. The category $\mathscr{A}_I$ is then equivalent to the category of non-degenerate A_I-modules.

We claim that the center of the category of non-degenerate A_I-modules, or the category $\mathscr{A}_I$, is the center of A^I. An element $z \in Z(A^I)$ with $z = (z_i)$, $z_i \in Z(A_i)$ acts on $M = \bigoplus_{i\in I} M_i$ by letting z_i acts on M_i. This gives rise to a homomorphism $Z(A^I) \to Z(\mathscr{A}_I)$. In the other direction, if $z \in Z(\mathscr{A}^I)$, it acts on A_i as an object of $\mathscr{A}_I$ as the multiplication by an element $z_i \in Z(A_i)$. The collection $(z_i)_{i\in I}$ defines an element of $Z(A^I)$.

Idempotented algebra

Let A be an associative algebra. Let E denote the set of idempotents in an algebra A. We consider the partial order on E by setting $e \le f$ if $eAe \subset fAf$. This is equivalent to saying that $e \in fAf$ or $e = fef$. For every $e \in E$, $A(e) = eAe$ is a unital subalgebra of A, e being its unit. For every pair of idempotents $e \leq f$, we have an inclusion of algebras $A(e) \subset A(f)$, both $A(e)$ and $A(f)$ are unital algebras but the inclusion does not sent the unit of $A(e)$ on the unit of $A(f)$. The algebra A is said to be *idempotented* A is the inductive limit of algebras $A(e)$ for $e \in E$ ranging over the ordered set of idempotents of A.

If A is unital, the unit of A is obviously the maximal element of E, and therefore A is obviously idempotented. The algebra A_I considered in the previous section is obviously idempotented, with the e_J forming a filtering system of idempotents.

Let A be an idempotented algebra. For every A-module M and $e \in E$, eM is a module over the unital algebra $A(e)$. An A-module M is said to be *non-degenerate* if for every $x \in M$, there exists an idempotent $e \in A$ such that $ex = x$, in other words, M is the filtered union of eM as $e \in E$ ranging ranging over the ordered set of idempotents of A.

Theorem 9.6. *Let $\mathscr{M}_A$ denote the abelian category of non-degenerate A-modules. Let $Z(\mathscr{M}_A)$ denote its center. For every idempotent $e \in E$, we denote $Z(e)$ the center of $A(e)$. If $e \leq f$, we have an inclusion of algebras $A(e) \leq A(f)$ that induces a morphism between their centers $Z(f) \to Z(e)$ given by $z \mapsto ze = ez$. Then the center $Z(\mathscr{M}_A)$ of the category $\mathscr{M}_A$ can be identified with the projective limit $Z(A)$ of $Z(e)$ as e ranging over the ordered set of idempotents of A.*

Proof. Let M be a non-degenerate A-module, that is an filtered union of subgroup $M(e) = eM$, $M(e)$ being an $A(e)$-module. For every $z \in Z(A)$, let $z(e)$ denote the image of z in $Z(e)$, the center of $A(e)$. As $Z(e)$ is the center of the category of $A(e)$-modules, $z(e)$ acts on $M(e)$. These actions are compatible in the sense that for every pair of idempotent $e \leq f$, the restriction of the action of $z(f)$ on $M(f)$ to $M(e)$ coincides with the action of $z(e)$. We infer an action of $Z(A)$ on M.

Conversely, let z be an element in the center $Z(\mathscr{M}_A)$ of the category $\mathscr{M}_A$ of non-degenerate A-modules. Since A is itself a non-degenerate A-module, $z_A : A \to A$ is an endomorphism homomorphism of A as a A-module, for the left multiplication, and commuting with all endomorphisms of A as an A-module, in particular commuting with the right multiplication by all elements of A, in other words the application $z_A : A \to A$ commutes with both left and right multiplication by elements of A. It follows that for every idempotent e, z_A stabilizes $A(e) = eAe$ and the restriction of z_A to $A(e)$ defines an application $z_e : A(e) \to A(e)$ that commutes with both left and right multiplication by $A(e)$. Since $A(e)$ is a unital algebra, the application $z_e : A(e) \to A(e)$ must be given by the multiplication by an element in the center $Z(e)$ of $A(e)$. It is thus legitimate to write $z_e \in Z(e)$. The elements $z_e \in Z(e)$ are compatible with transition maps $Z(f) \to Z(e)$ for every pair of idempotents $e \leq f$, and define an element in the projective limit $Z(A)$. $\square$

Completion of idempotented algebra

Let A be an idempotented algebra. We consider the projective system Ae for e ranging over the ordered set of idempotents, if $e \leq f$ we have a map $Af \to Ae$ given by $x \mapsto xe$. We will denote A^c the limit of this projective system, A^c stands for the right completion (by left side ideals), A^c is a left A-module. We claim that A^c is equipped with a multiplication. Let $x, y \in A^c$ representing projective systems $(x_e)_{e\in E}$ and $(y_e)_{e\in E}$ with $x_e, y_e \in Ae$. For every e, there exists $f \in E$ such that $f y_e = y_e$, and we set $z_e = x_f y_e$. The element z_e does not depend on the choice of the idempotent f. Indeed if $f_1 \geq f$ then $f_1 y_e = f_1 f y_e = f y_e = y_e$, and moreover we have $x_{f_1} y_e = x_{f_1} f y_e = x_f y_e = z_e$. We can also easily check that the elements $z_e \in Ae$ satisfies $z_f e = z_e$ for all pair of idempotents $f \geq e$.

Similarly, we define ${}^cA$ the right completion of A as the limit of the projective systems eA ranging over the ordered set of idempotents of A. The right completion ${}^cA$ is a right A-module. We can similarly define an algebra structure on ${}^cA$. It is clear that ${}^cA$ is the right completion of the opposite algebra A^{op}.

The algebra A^c defined in this way is a unital associative algebra containing A, its unit consists in the system of elements $e \in Ae$. The action of A on every non-degenerate module M can be extended uniquely to an action of A^c. For every $x \in A^c$ and $v \in M$, we define $xv = x(e)v$ for an idempotent e such that $em = m$. This definition is independent of the choice of e as if $f \geq e$ we have $x(f)v = x(f)ev = x(e)v$.

Proposition 9.7. *The right completion A^c is the ring of endomorphism of the "fiber" functor mapping a non-degenerate A-module to its underlying vector space. The left completion ${}^cA$ is the ring of endomorphism of A as a left non-degenerate module.*

It can be checked that the center $Z(A)$ of the category of non-degenerate A-modules is canonical isomorphic to the center of $\hat{A}$.

Center of the category of representations of finite groups

Let G be a finite group. Let R denote the space of complex valued functions on G. The multiplication map $G \times G \to G$ induces a comultiplication on $R \to R \otimes R$. Let A denote the vector dual of R, elements of A are linear forms on R. The comultiplication on R induces a multiplicative structure on A. For every $g \in G$, let δ_g denote the the linear form $R \to \mathbf{C}$ given by evaluation at g i.e $\delta_g(\phi) = \phi(g)$ for all $\phi \in R$. The elements δ_g form a basis of A so that every element of $a \in A$ can be uniquely written as a linear combination of the form $a = \sum_{g\in G} a_g \delta_g$ with $a_g \in \mathbf{C}$. The multiplication law of A is the uniquely defined by the requirement $\delta_h \delta_{h^{-1}g} = \delta_g$ for all $h, g \in G$. If 1 denotes the neutral element of G, δ_1 is the unit of A.

A Haar measure on G is a left (and right) G-invariant linear form on R, in other words a G-invariant element of A. For instant, the counting measure on G corresponds to the element $\mu = \sum_{g\in G} \delta_g \in A$. As R is a $\mathbf{C}$-algebra, its linear dual A has a structure of R-module. The element $\mu \in A$ gives rise to a unique R-linear map $R \to A$, $\phi \mapsto \phi\mu$. This map is is clearly a $G \times G$-equivariant R-linear isomorphism. Although R and A are isomorphic, up to the choice of a Haar measure, it is best to keep them separate to make a clear difference between their elements.

Let $\rho_V : G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation of G, V^* its dual. One can attach a function on G to every vector $v \in V$ and $v^* \in V^*$

$$r_{v,v^*}(g) = \langle \rho_V(g)v, v^* \rangle. \tag{9.7}$$

This function is called the matrix coefficient. The map $(v, v^*) \mapsto r_{v,v^*}$ is bilinear and induces a linear map $r : V \otimes V^* \to R$ which is $G \times G$-equivariant.

The dual of the matrix coefficient map is a linear map $A \to V \otimes V^*$. By using the canonical isomorphism $V \otimes V^* = \mathrm{End}(V)$, we obtain a linear map $A \to \mathrm{End}(V)$ which is in fact a morphism of algebras. The action $\rho(a)$ of element $a = \sum_{g\in G} a_g \delta_g$ on v is given by

$$\rho_V(a) = \sum_{g\in G} a_g \rho_V(g) \in \mathrm{End}(V). \tag{9.8}$$

There is an equivalence of categories between the category $\mathscr{M}_G$ of complex representations of G and the category of A-modules. Since A is unital, the center of the category of A-modules is the center A. The center of A consists of elements $a = \sum_{g\in G} a_g \delta_g \in A$ such that for every $h \in G$, we have $\delta_h a = a\delta_h$. This is equivalent to saying that a is invariant under the conjugation action of G i.e. $a \sum_{g\in G} a_g \delta_g$ with $a_{hgh^{-1}} = a_g$ for all $g, h \in G$.

On the other hand, the category of complex representations of G is semisimple, with finitely many classes of irreducible objects. There is thus an isomorphism

$$Z_A = Z(\mathscr{M}_G) = \prod_V \mathbf{C}_V \tag{9.9}$$

V ranging over the set of isomorphism classes of irreducible representation of G and $\mathbf{C}_V$ is a copy of $\mathbf{C}$ indexed by V. Each irreducible representation V defines thus an idempotent $e_V \in Z_A$. Moreover these idempotents sum up to the unit of Z_A:

$$\delta_1 = \sum_V e_V. \tag{9.10}$$

One can obtain a more explicit expression of the idempotent e_V by means of the theory of characters. The trace map $\mathrm{tr} : \mathrm{End}(V) \to \mathbf{C}$ defines a linear form $\mathrm{tr}_\rho : A \to \mathbf{C}$

$$\mathrm{tr}_{\rho_V}(a) = \mathrm{tr}(\rho_V(a)) = \sum_{g\in G} a_g \mathrm{tr}(\rho_V(g)). \tag{9.11}$$

Its dual is a map $\mathrm{End}(V)^* \to R$. We will denote $\chi_V \in R$ the image of $\mathrm{id}_V \in \mathrm{End}(V)$, that is the character of V:

$$\chi_V(g) = \mathrm{tr}(\rho_V(g)). \tag{9.12}$$

As the character χ_V is a function on G equivariant under conjugation, the measure $\chi_V(\mu) = \chi_V\mu$, μ being the counting measure, is an element of Z_A. The orthonality relation of characters implies that $\chi_V(\mu)e_{V'} = 0$ for every irreducible representation V' non isomorphic to V. It follows that $\chi_V(\mu)$ is proportional to the idempotent e_V. To determine the proportionality constant, we recall the relation

$$\chi_V(\mu)\chi_V(\mu) = \frac{\#G}{\dim(V)}\chi_V(\mu). \tag{9.13}$$

It follows that

$$e_V = \frac{\dim(V)}{\#G}\chi_V(\mu). \tag{9.14}$$

We now derive from (9.10) the Plancherel formula for finite group:

$$\delta_1 = \sum_V \frac{\dim(V)}{\#G}\chi_V(\mu). \tag{9.15}$$

Center of the category of representations of compact *td*-groups

Representation theory of a compact *td*-group can be developed following a pattern similar to finite groups. Irreducible representations are finite-dimensional and discrete although there are infinitely many of them.

Let G be a compact *td*-group. We only consider $\mathscr{C}^\infty(G)$ the space of smooth functions on G and $\mathscr{D}(G)$ the dual space of distributions for both smooth functions and distributions on G are automatically of compact support.

References

[1] Joseph Bernstein. "Representations of p-adic groups". In: (Sept. 2008), pp. 1–110.

[2] Joseph Bernstein and A V Zelevinskii. "Representations of the group $GL(n, F)$, where F is a local non-Archimedean field". In: *Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk* 31.3(189) (1976), pp. 5–70.

[3] Francois Bruhat. "Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes wp -adiques". In: *Bulletin de la Société Mathématique de France* (1961).

[4] Francois Bruhat and Jacques Tits. "Groupes réductifs sur un corps local, I". In: *Publications Mathématiques de l'IHES* (1972).

[5] Pierre Cartier. "Representations of p-adic groups: a survey". In: (Aug. 2002), pp. 1–45.

[6] Stephen Debacker. *Representations theory of p-adic groups*. URL: `http://www.math.lsa.umich.edu/~smdbackr/MATH/notes.pdf`.

[7] Hervé Jacquet. "Representations des Groupes Lineaires p-Adiques". In: *Theory of Group Representations and Fourier Analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 119–220.

[8] Robert Kottwitz. "Harmonic analysis on reductive p-adic groups and Lie algebras". In: *Harmonic analysis* (2005).

[9] F I Mautner. "Spherical Functions Over p-Adic Fields, I". In: *American Journal of Mathematics* 80.2 (Apr. 1958), p. 441.

[10] Fiona Murnaghan. "Representations of reductive p-adic groups". In: (Nov. 2013), pp. 1–128.

[11] David Renard. *Représentations des groupes réductifs p-adiques.* 2010.

[12] Ichiro Satake. "Theory of spherical functions on reductive algebraic groups over $germp$-adic fields". In: *Institut des Hautes Etudes Scientifiques. Publications Mathématiques* (1963).

[13] Gordan Savin. "Lectures on representations of p-adic groups". In: *Representations of Real and P-adic Groups* (2004).

[14] John Tate. "Fourier analysis in number fields, and Hecke's zeta-functions". In: (1967).

[15] André Weil. "Intégration dans les groupes topologiques". In: (2008).