ON A CERTAIN SUM OF AUTOMORPHIC L-FUNCTIONS

NGÔ BAO CHÂU

In Tate's thesis [20], the characteristic function of $\mathbf{Z}_p$ has been used in the integral representation of the local L-factor associated to an unramified quasicharacter. This construction has been generalized by Tamagawa, Godement and Jacquet in [19, 8] to principal L-functions of GL_n with the characteristic function of the space of integral matrices as test function. The general automorphic L-function depends on a representation of the dual group. For each representation of the dual group, one can construct a function depending on a complex parameter s whose trace on an unramified representation is the associated local L-factor. The main contribution of this paper is a conjectural geometric description of this test function by means of Vinberg's theory of algebraic monoids [21]. Prior to us, efforts have been made in this direction, notably by Braverman and Kazhdan [4] and Lafforgue [11]. By inserting into the trace formula the product of these functions at almost every place, we would get, at least formally, the sum of L-functions considered in [6, 7]. We will explain a conjectural geometric interpretation of this sum of L-functions.

This paper is intended for an expository purpose, necessary details and proofs will be published elsewhere.

I would like to thank W. Casselman, D. Jiang and the referee for their comments, and especially Y. Sakellaridis for enlightening exchange of views.

I wasn't fortunate enough to know Professor Piatetski-Shapiro personally, though I met him once or twice. In a conference in Luminy some time around 2004, his wife told me that Professor Piatetski-Shapiro appreciated my works. I keep the fond memory of this comment as one of my proudest mathematical prizes. I dedicate this work to his memory as an expression of a deep admiration for his ideas and courage.

1. AUTOMORPHIC L-FUNCTIONS AS A TRACE

Let G be a split reductive group defined over a global field K. Let π be a discrete automorphic representation of G i.e. an irreducible subrepresentation of $L^2(G(K)\backslash G(\mathbf{A}_K),\chi)$ where χ is a central character. It is known that π can factorized as tensor product $\pi = \bigotimes_v \pi_v$. For every nonarchimedean place v of K, π_v is an irreducible admissible representation of $G(K_v)$ which is unramified for almost all v.

Let me recall that π_v is unramified if it contains a non-zero fixed vector of the compact open subgroup $G(\mathcal{O}_v)$. The space of all such fixed vectors is an irreducible module over the algebra

$$\mathcal{H}_v = C_c^\infty(G(\mathcal{O}_v)\backslash G(K_v)/G(\mathcal{O}_v))$$

of compactly supported functions on $G(K_v)$ that are left and right $G(\mathcal{O}_v)$-invariant. The multiplication is given by the convolution product with respect to the Haar measure on $G(K_v)$ normalized in the way that $G(\mathcal{O}_v)$ has volume one. This is a commutative algebra whose structure is best described with the aid of the Satake isomorphism [18]. Let $\hat{G}$ denote the Langlands dual group of G that is the complex reductive group whose root datum is obtained from the root datum of G by exchanging the role of roots and coroots as in [12]. With this definition, the algebra $\mathcal{H}_v$ can be identified with the algebra $\mathbf{C}[\hat{G}]^{\hat{G}}$ of regular algebraic functions on $\hat{G}$ that are invariant with respect to the adjoint action. For every Hecke function $\phi \in \mathcal{H}_v$, we will denote $\tilde{\phi} \in \mathbf{C}[\hat{G}]^{\hat{G}}$ the corresponding regular algebraic function on $\hat{G}$.

As an $\mathcal{H}_v$-irreducible module, $\pi_v^{G(\mathcal{O}_v)}$ has dimension at most one as $\mathbf{C}$-vector space. If it is non zero, it defines an algebra homomorphism $\sigma_v : \mathcal{H}_v \to \mathbf{C}$ that can be identified with a semi-simple conjugacy class in $\hat{G}$ according to the Satake isomorphism. This identification was made so that for every $\phi \in \mathcal{H}_v$, the equality

$$\mathrm{tr}_{\pi_v}(\phi) = \tilde{\phi}(\sigma_v)$$

holds.

Let $\rho : \hat{G} \to \mathrm{GL}(V_\rho)$ be an irreducible algebraic representation of $\hat{G}$. Following Langlands, we attach to each pair (ρ, π) an L-function with its local factor at any unramified place

$$L(s, \rho, \pi_v) = \det(1 - \rho(\sigma_v) q_v^{-s})^{-1}$$

where q_v is the cardinal of the residue field κ_v and s is a complex number.

We will consider $\det(1 - \rho(\sigma) q_v^{-s})^{-1}$ as a rational function of $\sigma \in \hat{G}$. We set

$$\tilde{\psi}_s(\sigma) = \det(1 - \rho(\sigma) q_v^{-s})^{-1}$$

and ask the question whether $\tilde{\psi}_s$ is the Satake transform of some function ψ_s on $G(K_v)$. This function would satisfy the equality

$$\mathrm{tr}_{\pi_v}(\psi_s) = L(s, \rho, \pi_v)$$

and would lead to the possibility of expressing L-function as the trace of a certain operator. We can expand $\det(1 - \rho(\sigma) q_v^{-s})^{-1}$ as a formal series

$$\det(1 - \rho(\sigma) q_v^{-s})^{-1} = \sum_{n=0}^{\infty} \mathrm{tr}(\mathrm{Sym}^n \rho(\sigma)) q_v^{-ns}$$

which is absolutely convergent for large $\Re(s)$. Let us denote ψ_n the element of $\mathscr{H}_v$ whose Satake transform is the regular invariant function on $\hat{G}$

$$\tilde{\psi}_n(\sigma) = \mathrm{tr}(\mathrm{Sym}^n \rho(\sigma)).$$

Thus the function ψ we seek to define must have the form

$$\psi_s = \sum_{n=0}^{\infty} \psi_n q_v^{-ns}$$

which is absolutely convergent for large $\Re(s)$.

It will be convenient to restrict ourselves to particular cases in which the sum $\sum_{n=0}^{\infty} \psi_n q_v^{-ns}$ is well defined for all s and in particular, the sum $\sum_{n=0}^{\infty} \psi_n$ is well defined. This restriction will not deprive us much of generality.

2. Geometric construction of ψ

We seek to define a function ψ on $G(K_v)$ such that

$$\mathrm{tr}(\psi_s, \pi) = \mathrm{tr}(\psi, \pi \otimes |\det|^s).$$

where $\det : G \to \mathbb{G}_m$ generalizes the determinant in the case of GL_n. We will assume that the kernel G' of $\det : G \to \mathbb{G}_m$ is a semi-simple group. Dualizing the exact sequence

$$0 \to G' \to G \to \mathbb{G}_m \to 0$$

we get an exact sequence

$$0 \to \mathbb{G}_m \to \hat{G} \to \hat{G}' \to 0$$

of groups defined over the field of complex numbers. We will assume that the representation $\rho : \hat{G} \to \mathrm{GL}(V)$ induces the identity map from the central $\mathbb{G}_m$ of $\hat{G}$ to the central $\mathbb{G}_m$ of $\mathrm{GL}(V)$. Under this assumption, the n-th symmetric power $\mathrm{Sym}^n \rho : \hat{G} \to \mathrm{GL}(\mathrm{Sym}^n V)$ induces the n-th power map from the central $\mathbb{G}_m$ of $\hat{G}$ to the central $\mathbb{G}_m$ of $\mathrm{GL}(\mathrm{Sym}^n V)$. It follows that the Hecke function ψ_n whose Satake transform $\psi_n(\sigma) = \mathrm{tr}(\mathrm{Sym}^n \rho(\sigma))$ is supported on the subset of $G(K_v)$ of elements satisfying $\mathrm{val}(\det(g)) = n$. In particular, different functions ψ_n have disjoint support and therefore, the infinite sum $\psi = \sum_{n=0}^{\infty} \psi_n$ is well defined as function on $G(K_v)$.

All this sounds of course very familiar because it is modeled on the case of principal L-function of Tamagawa, Godement and Jacquet [19, 8]. In this case $G = \mathrm{GL}_n$ and ρ is the standard representation of $\hat{G} = \mathrm{GL}_n(\mathbf{C})$. According to Tamagawa, Godement and Jacquet, the function ψ is very simple to describe: it is the restriction of the characteristic function $1_{\mathrm{Mat}_n(\mathscr{O}_v)}$ of $\mathrm{Mat}_n(\mathscr{O}_v)$ from $\mathrm{Mat}_n(K_v)$ to $\mathrm{GL}_n(K_v)$, which generalizes the function $1_{\mathscr{O}_v}$ appearing in Tate's thesis.

We may ask whether it is possible to describe our function ψ in a similar way. To put the question differently, we may ask what would play the role of Mat_n for an arbitrary representation of $\hat{G}$. We will show that Vinberg's theory of algebraic monoids provides a nice conjectural answer to this question.

In Vinberg's theory, the semi-simple group G' is going to be fixed and G is allowed to be any reductive group of which G' is the derived group. A monoid is an open embedding $G \hookrightarrow M$ into a normal affine scheme M such that the actions of G on itself by left and right translation can be extended as actions on M. It can be proved that these two commuting actions can be merged into a multiplicative structure on M, under the assumption that M is affine and normal, or in other words M is an algebraic monoid. Following Vinberg, we call the GIT quotient $A = M//(G' \times G')$ the abelianization of M. The monoid M is said to be flat if the quotient map $M \to A$ is flat and its geometric fibers are reduced. The upshot of Vinberg's theory is that there is a universal flat monoid for a given derived group G' and every flat monoid with the same derived group G' can be obtained from the universal one by base change over its abelianization [16, 21].

The universal flat monoid M^+ is an affine embedding of G^+ where G^+ is an entension of a torus T^+ by G', r being the rank of G'

$$0 \to G' \to G^+ \to T^+ \to 0.$$

Let T' be a maximal torus of G'. Folowing Vinberg, we set $G^+ = (G' \times T')/Z'$ where Z' is the center of G' acting diagonally on G' and T'. It follows that $T^+ = T'/Z'$ is the maximal torus of the adjoint group that can be identified with $\mathbb{G}_m^r$ with aid of the set of simple roots $\{\alpha_1, \dots, \alpha_r\}$ associated with the choice of a Borel subgroup of G' containing T'. The universal abelianization A^+ is the obvious toric variety $\mathbb{A}^r$ of the torus $\mathbb{G}_m^r$.

For simplicity, we will assume that G' is simply connected from now on. Let $\omega_1, \dots, \omega_r$ denote the fundamental weights dual to the simple coroots $\alpha_1^\vee, \dots, \alpha_r^\vee$ and let $\rho_i : G \to \mathrm{GL}(V_i)$ denote the irreducible representation of highest weight ω_i. This can be extended to G^+

$$\rho_i^+ : G^+ \to \mathrm{GL}(V_i)$$

by the formulae $\rho_i^+(t, g) = \omega_i(w_0 t^{-1})\rho_i(g)$ where w_0 is the long element in the Weyl group W of G. The root $\alpha_i : T \to \mathbb{G}_m$ will also be extended to G^+

$$\alpha_i^+ : G^+ \to \mathbb{G}_m$$

given by $\alpha_i^+(t,g) = \alpha_i(t)$. Altogether, these maps define a homomorphism

$$(\alpha^+, \rho^+) : G^+ \to \mathbb{G}_m^r \times \prod_{i=1}^r \mathrm{GL}(V_i).$$

In good characteristics, Vinberg's universal monoid is defined as the closure of G^+ in $\mathbb{A}^r \times \prod_{i=1}^r \mathrm{End}(V_i)$. In small characteristic, it is defined to be the normalization of this closure.

Back to our group G which is an extension of $\mathbb{G}_m$ by G'. We are looking for a flat monoid M whose group of invertibles is G and abelianization is the affine line $\mathbb{A}^1$ as toric variety of $\mathbb{G}_m$. By universal property, this amounts to the same as a homomorphism $\lambda : \mathbb{G}_m \to T^+$ which can be extended as a regular map $\mathbb{A}^1 \to \mathbb{A}^r$ such that

$$G = G^+ \times_{T^+} \mathbb{G}_m. \tag{2.1}$$

The co-character is the highest weight of the representation $\rho : \hat{G}' \to \mathrm{GL}(V)$. The relation $G = G^+ \times_{T^+} \mathbb{G}_m$ derives from the hypothesis that ρ restricted on the center, induces the identity map from the central $\mathbb{G}_m$ of $\hat{G}$ on the central $\mathbb{G}_m$ of $\mathrm{GL}(V)$.

The set M_n of elements $g \in M(\mathcal{O}_v) \cap G(K_v)$ such that $\mathrm{val}(\det(g)) = n$ is a compact subset of $G(K_v)$ which is invariant under left and right actions of $G(\mathcal{O}_v)$. One can check that the support of the Hecke function ψ_n whose Satake transform is $\tilde{\psi}_n(\sigma) = \mathrm{tr}(\mathrm{Sym}^n \rho(\sigma))$, is contained in M_n.

Assume from now on that K is the field of rational functions of certain smooth projective curve X over a finite field κ. For every closed point $v \in |X|$, we denote K_v the completion of K at v, $\mathcal{O}_v$ its ring of integers and κ_v its residue field. In this case, the set M_n can be seen as the set of κ_v-points of a certain infinite dimensional algebraic variety $\mathcal{M}_n$. The geometric Satake theory, due to Ginzburg, Mirkovic and Vilonen [14], allows us to define a perverse sheaf $\mathcal{A}_n$ on $\mathcal{M}_n$ for which ψ_n is the function on κ_v-points given by the Frobenius trace. In an appropriate sense, $\mathcal{A}_n$ is the n-th symmetric convolution power of $\mathcal{A}_1$ just as its Satake transform is the n-th symmetric tensor power of ρ.

The union $M = \bigcup_{n=0}^{\infty} M_n$ is the set of κ_v-points of some algebraic variety $\mathcal{M}$ which is an open subset of the loop space LM of M. It is tempting to glue the perverse sheaves $\mathcal{A}_n$ on different strata M_n into a single object and to ask whether the outcome is the intersection complex of LM. Hints in this direction have been given in [17]. In that paper, Sakellaridis also pointed out links between local L-factors and geometry of spherical varieties. Although Vinberg's monoids are special instances of spherical varieties, the relation between his construction and ours is not clear for the moment. We observe that for the moment, the very definition of the intersection complex on loop space isn't available in published form but some preparatory results have been obtained [5, 9].

3. A CERTAIN SUM OF L-FUNCTIONS

Let X be a smooth projective curve defined over the finite field $\kappa = \mathbf{F}_q$, K its field of rational functions. For every closed point v of X, let K_v denote the completion of K at v and $\mathcal{O}_v$ its ring of integers. Let us choose a point $\infty \in X$, a uniformizing parameter $\epsilon_\infty \in K_\infty^\times$. Let G be a split reductive group defined over κ. Let us also choose a nontrivial central homomorphism $\mathbb{G}_m \to G$. The uniformizing parameter ϵ_∞ defines a central element of $G(\mathbf{A}_K)$, still denoted by ϵ_∞, which satisfies $|\det(\epsilon_\infty)| = q^{-m}$, for some positive integer m.

We are interested in automorphic representations $\pi = \otimes_v \pi_v$ as irreducible subrepresentation of

$$L^2(G(F)\epsilon_\infty^{\mathbf{Z}} \backslash G(\mathbf{A}_K)/G(\mathcal{O}_K))$$

with $\mathcal{O}_K = \prod_{v \in |X|} \mathcal{O}_v$. Its partial L-function can be represented as a trace

$$L'(s,\rho,\pi) = \prod_{v \neq \infty} L(s,\rho,\pi_v) = \mathrm{tr}(\prod_v \psi_v, \pi \otimes |\det|^s).$$

If we were willing to ignore the continuous spectrum, we would have the equality

$$\sum_\pi L'(s,\rho,\pi) + \cdots = \mathrm{tr}(\prod_{v\neq\infty} \psi_v, L^2(G(F)\epsilon_\infty^{\mathbf{Z}} \backslash G(\mathbf{A}_K)/G(\mathcal{O}_K)) \otimes |\det|^s).$$

We expect that the right hand side can be expanded geometrically as integration of $\prod_{v\neq\infty}\psi_v$ over a diagonal by an appropriate form of the trace formula, and we hope that the geometric side would provide us insights for understanding this sum of L-functions. The intended trace formula is beyond the application range of Arthur's trace formula [2] as our test functions do not have compact support.

Assume that π is tempered and that it corresponds in the sense of Langlands' reciprocity to a homomorphism

$$\sigma_\pi : W_K \to \hat{G}$$

where W_K denotes the Weil group of K. If we denote $H(\pi)$ the closure of the image of σ_π, the order of the pole at $s=1$ of $L(s,\rho,\pi)$ will be equal to the multiplicity of the trivial representation in the representation of $H(\pi)$ obtained by restricting ρ. Following [13, 6], we hope that an investigation of the sum

$$\sum_\pi L'(s,\rho,\pi) + \cdots \tag{3.1}$$

will eventually allow us to break the set of π into subsets in which $H(\pi)$ is a given reductive subgroup of $\hat{G}$, up to conjugacy, and therefore will lead us towards Langlands' functoriality conjecture. It seems reasonable to seek to understand this phenomenon from the geometric expansion of the trace formula for the test function $\prod_{v\neq\infty}\psi_v$. This strategy has been carried out successfully by S. Ali Altug in his PhD thesis [1] for $G = \mathrm{GL}_2$ and ρ the symmetric square.

The geometric expansion can be given a geometric interpretation with aid of a certain moduli space of bundles with additional structures. Recall that the moduli stack of principal G-bundles over X is an algebraic stack locally of finite type which will be denoted by $\mathrm{Bun}(G)$. Let $\epsilon_\infty^{\mathbf{Z}}\backslash\mathrm{Bun}(G)$ be the moduli stack of G-bundles over X modulo the equivalence relation generated by the relation $V \sim V(\epsilon_\infty)$ where $V(\epsilon_\infty)$ is the central twisting of a G-bundle V by the line bundle $\mathcal{O}_X(\infty)$ via the chosen central homomorphism $\mathbb{G}_m \to G$. We will denote $[V]$ the point in $\epsilon_\infty^{\mathbf{Z}}\backslash\mathrm{Bun}(G)$ represented by a G-bundle V.

An M-morphism between two G-bundles V and V' is a global section of the twisted space obtained by twisting M by V and V' via the left and right action of G on M. We will denote the set of M-morphisms from V to V' by $M(V,V')$. The multiplication structure on M allows us to define a composition $M(V,V')\times M(V',V'')\to M(V,V'')$ and thus a category $\mathrm{Bun}(G,M)$ of G-bundles with M-morphisms. If $G=\mathrm{GL}_n$ and $M=\mathrm{Mat}_n$, $M(V,V')$ is nothing but the space of linear morphisms of vector bundles from V to V'.

We observe that the identity transformation of $V|_{X-\infty}$ can be extended to an M-morphism $i_d : V \to V(d\epsilon_\infty)$ for all $d\geq 0$. The groupoid $\epsilon_\infty^{\mathbf{Z}}\backslash\mathrm{Bun}(G)$ can be constructed by formally inverting the M-morphisms i_d in the category $\mathrm{Bun}(G,M)$. In the localized category, we have

$$M([V],[V]) = \varinjlim_{d\to\infty} M(V, V(d\epsilon_\infty)).$$

Since the determinant $M\to\mathbf{A}^1$ is invariant with respect to G-conjugation, the determinant of $\phi\in M([V],[V])$ is well defined as a meromorphic function on X which is regular at $X-\{\infty\}$. For every d, let M_d denote the moduli stack of pairs $([V],\phi)$ where

$$[V]\in\epsilon_\infty^{\mathbf{Z}}\backslash\mathrm{Bun}(G) \text{ and } \phi\in M([V],[V])$$

with $\det(\phi)\in \mathrm{H}^0(X,\mathcal{O}_X(d\infty))$. This is a algebraic stack locally of finite type. As d varies, these stacks form an injective system

$$\mathsf{M}_0 \hookrightarrow \mathsf{M}_1 \hookrightarrow \cdots \qquad (3.2)$$

over the injective system of finite dimensional vector spaces

$$\cdots \hookrightarrow \mathrm{H}^0(X, \mathscr{O}_X(d\infty)) \hookrightarrow \mathrm{H}^0(X, \mathscr{O}_X((d+1)\infty) \hookrightarrow \cdots$$

Let M denote the limit of the inductive system M_d.

We expect that the sum of L-functions (3.1) can be expanded as formal series $\sum_{d=0}^{\infty} a_d q^{-ds}$, the "number" a_d is approximately

$$\operatorname{tr}(\mathrm{Frob}_q, \mathrm{R}\Gamma_c(\mathsf{M}_d - \mathsf{M}_{d-1}, \mathrm{IC}) \tag{3.3}$$

where IC is denote the intersection complex on $\mathsf{M}_d - \mathsf{M}_{d-1}$. The word number has been put into quotation marks because it may be infinite as stated. Since the support of ψ_n can be expressed in terms of M, the "number" a_d can be formally expressed as a sum over $\mathbf{F}_q$-points of $\mathsf{M}_d - \mathsf{M}_{d-1}$. In order to prove that the trace of Frobenius of the fiber of IC over a $\mathbf{F}_q$-point is equal to the term indexed by that point, we need to prove that singularities of $M_d - M_{d-1}$ are equivalent to singularities in Schubert cells of the affine Grassmannian. This should be true under the assumption that d is large enough with respect to the invariant δ defined as in [15]. Similar results have been obtained by Bouthier in his PhD thesis [3]. It is also very interesting to understand how the IC of the strata $M_d - M_{d-1}$ be glued together in comparison with local picture. To summarize, we expect that the sum of L-functions (3.1) can be calculated from the injective system (3.2).

Under the assumption that the derived group G' is simply connected, the ring of invariant functions $k[M]^G$ is a polynomial algebra. Using the generators of $k[M]^G$, we can construct a morphism

$$f_d : \mathsf{M}_n \to \mathsf{B}_d$$

similar to the Hitchin fibration [10]. It seems to be possible to study the direct image $(f_d)_* \mathbb{Q}_\ell$ in a similar way as [15]. Whether the direct images $(f_d)_* \mathbb{Q}_\ell$ form an inductive system as d varies, depends on the possibility of glueing the IC on $\mathsf{M}_d - \mathsf{M}_{d-1}$ together. At any rate, it is desirable to study the asymptotic of the sum of L-functions (3.1) via $(f_d)_* \mathbb{Q}_\ell$ as $d \to \infty$.

References

[1] S. Altug, Princeton PhD thesis 2012.

[2] J. Arthur. *An introduction to the trace formula* in Harmonic analysis, the trace formula, and Shimura varieties edited by J.Arthur, D. Ellwood and R. Kottwitz.

[3] A. Bouthier, Orsay-Chicago thesis in progress.

[4] A. Braverman, D. Kazhdan. *γ-functions of representations and lifting* in Visions in Mathematics 2000 pp. 237-278.

[5] V. Drinfeld. *On the Grinberg - Kazhdan formal arc theorem.* Preprint 2002.

[6] E. Frenkel, R. Langlands, B.C. Ngô. *Formule des traces et fonctorialité: le début d'un programme*, Ann. Sci. Math. Québec 34 (2010), no. 2, 199–243.

[7] E. Frenkel, B.C. Ngô. *Geometrization of trace formulas*, Bull. Math. Sci. 2011 vol. 1 (1) pp. 129-199.

[8] R. Godement, H. Jacquet. *Zeta functions of simple algebras*, Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972. ix+188 pp.

[9] M. Grinberg, D. Kazhdan. *Versal deformations of formal arcs* Geom. Func. Anal. 2000 vol. 10 (3) pp. 543-555.

[10] N. Hitchin. *Stable bundles and integrable systems*, Duke Math. J. 1987 vol. 54 (1) pp. 91-114.

[11] L. Lafforgue. *Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires*. Preprint 2013.

[12] R. Langlands. *Problems in the Theory of Automorphic Forms*, Lectures in Modern Analysis and Applications III, C. T. Taam, ed., Springer Lecture Notes in Mathematics 170, 1970, pp. 18–61.

[13] R. Langlands. *Beyond endoscopy*, in Contributions to automorphic forms, geometry, and number theory, 611–697, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[14] I. Mirkovic, K. Vilonen. *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. 2007 vol. 166 (1) pp. 95-143.

[15] B.C. Ngô. *Le lemme fondamental pour les algèbres de Lie*, Publ. math. de l'IHES 2010 (111) pp. 1-169.

[16] A. Rittatore. *Very flat monoids*, Publ. Mat. Urug. Publ. Mat. Urug. 9 (2001), 93–121 (2002)

[17] Y. Sakellaridis. *Spherical varieties and integral representations of L-functions*, Algebra Number Theory 2012 vol. 6 (4) pp. 611-667.

[18] I. Satake. *Theory of spherical functions on reductive algebraic groups over p-adic fields*, Inst. Hautes Etudes Sci. Publ. Math. 1963 (18) pp. 5-69.

[19] T. Tamagawa. *On the ζ-functions of a division algebra*, Ann. of Math 1963 vol. 77 pp. 387-405.

[20] J. Tate. *Fourier analysis in number fields and Hecke's zeta functions*, in Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965) pp. 305–347 Thompson, Washington, D.C.

[21] E. Vinberg On reductive algebraic monoids, Amer. Math. Soc. Transl., Serie 2, 169 (1994), pages 145 – 182. Lie Groups and Lie Algebras. E.B. Dynkin seminar.