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Perverse sheaves and fundamental lemmas

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The Langlands program is a rich supply of deep and beautiful problems. Some of those problems are very abstract while others are rather concrete. In these lectures, I will discuss some problems in harmonic analysis, issued from the Langlands's program, which require, in spite of their concrete and elementary appearance, sophisticated machinery in algebraic geometry to be fully understood and eventually solved.

After recalling the Grothendieck dictionary between ℓ-adic sheaves on algebraic varieties over finite fields and functions on their sets of rational points, I will attempt to enunciate a vague notion of perverse continuation principle that seems to be useful to construct a solution for these problems. I will then go on to work out this principle in three cases: the Jacquet-Ye fundamental lemma, the Jacquet-Rallis fundamental lemma (due to Z. Yun), the Langlands-Shelstad standard and the Walsdspurger nonstandard endoscopic fundamental lemmas.

These lecture notes aim to be complementary to other expository papers in this topics, including [6], [28], [29], [31], [14], [24]. In particular, I won't discuss the motivation behind the fundamental lemmas, that has been discussed in [6] and [14].

We will instead focus on the construction of global moduli spaces for which one can establish the perverse continuation principle. We emphasize that the construction of global moduli spaces, in all the cases, follows essentially the same pattern. The crucial proof of the perverse continuation principle is however very different in each of these cases. We won't dive into the details of this part as for instance the expository paper [31] has been devoted to this purpose in the endoscopic case.

The paper is divided into five sections. The first section contains standard materials on Grothendieck's dictionary of sheaves and functions. In the second section, the principle of perverse continuation is enunciated. We also discuss, in this section, various techniques that may be used in establishing this principle in different geometric situations. The three last sections are devoted to the construction of moduli spaces related to the fundamental lemma of Jacquet-Ye, Jacquet-Rallis, Langlands-Shelstad and Waldspurger respectively, and for which one can establish the principle of perverse continuation by various techniques presented in section 2.

The construction of moduli spaces presented in the last three sections are completely parallel. We start with certain morphisms of algebraic stacks, intrinsically related to invariant theory, and then consider the space of maps from either the formal disc or a proper smooth curve into those algebraic stacks. Fibers of the morphism obtained on the level of space of maps from the formal disc should be seen as the geometric incarnation of local orbital integrals. Fibers of the morphism obtained on the level of space of maps from the projective curve should be seen as the geometric incarnation of certain global orbital integrals that are related to local orbital integrals by a product formula. These global orbital integrals are those that appear in the geometric side of the relevant trace formula. The perverse continuation principle can be established for the global moduli space and the fundamental lemma can be derived as a local consequence.

1. Grothendieck's dictionary of sheaves and functions

1.1. The dictionary Let $k = \mathbf{F}_q$ denote the finite field with q elements. According to Grothendieck, a scheme X over k can be identified with its functor of points attaching to each k-algebra A the set $X(A)$ of A-points on X. For many purposes, instead of all k-algebra A, we may restrict ourselves to field extensions of k.

Let $\bar{k}$ be an algebraic closure of k, $\sigma(\alpha) = \alpha^q$ the Frobenius elements in $\mathrm{Gal}(\bar{k}/k)$, and for every integer $r \geqslant 1$, let $k_r = \mathrm{Fix}(\sigma^r, \bar{k})$ be the extension of degree r of k contained in $\bar{k}$. For every k-scheme X, the set $X(\bar{k})$ of $\bar{k}$-points on X is equipped with action of σ such that the set $X(k_r)$ can be identified with the set of fixed points of σ^r in $X(\bar{k})$. In other words, the set $X(\bar{k})$ equipped with the action of σ determines the set of k_r-points on X for every finite extension k_r of k, including k itself.

For instance, if $X = \mathrm{Spec}(R)$ where R is the quotient of the polynomial ring $k[x_1, \ldots, x_n]$ by the ideal generated by a finite set of polynomials

$$P_1, \ldots, P_m \in k[x_1, \ldots, x_n],$$

then $X(k)$ is the set of solutions $(x_1, \ldots, x_n) \in k^n$ of the system of polynomial equations

$$P_1(x_1, \ldots, x_n) = 0, \ldots, P_m(x_1, \ldots, x_n) = 0 \tag{1.1.1}$$

which is the fixed points set of σ in $X(\bar{k})$, the set of solutions in $\bar{k}$.

We won't recall the definition of ℓ-adic sheaves but will limit ourselves to their functorial properties permitting the definition of the associated trace function. For every constructible ℓ-adic sheaves $\mathcal{F}$ on an algebraic variety X, there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ such that the restriction of $\mathcal{F}$ to X_α is a local system. A local system $\mathcal{F}$ on X_α is given by a continuous representation of the fundamental group of X_α.

If $X = \mathrm{Spec}(k)$ is a point, an ℓ-adic sheaf $\mathcal{F}$ over X is a continuous representation ℓ-adic representation of $\mathrm{Gal}(\bar{k}/k)$. In this case, we define

$$\mathrm{tr}(\mathcal{F}) = \mathrm{tr}(\sigma, \mathcal{F}_{\bar{x}}).$$

where σ is the Frobenius element in $\mathrm{Gal}(\bar{k}/k)$ and $\mathcal{F}_{\bar{x}}$ is the geometric fiber of $\mathcal{F}$ over a given geometric point $\bar{x} = \mathrm{Spec}(\bar{k})$ of X. More generally, if $\mathcal{F}$ is an ℓ-adic sheaf over a k-scheme X and if $x : \mathrm{Spec}(k) \to X$ is a k-point, then $x^*\mathcal{F}$ gives rise to a continuous representation of $\mathrm{Gal}(\bar{k}/k)$ on $\mathcal{F}_{\bar{x}}$, and thus an ℓ-adic number

$$\mathrm{tr}_{\mathcal{F}}(x) = \mathrm{tr}(x^*\mathcal{F}).$$

Therefore, the ℓ-adic sheaf $\mathcal{F}$ gives rise to a function

$$\mathrm{tr}_{\mathcal{F}} : X(k) \to \bar{\mathbf{Q}}_\ell.$$

The construction of the trace function can be extended to every object $\mathcal{F}$ of the derived category of bounded complex of constructible sheaves $D_b^c(X, \bar{\mathbf{Q}}_\ell)$ by setting

$$\mathrm{tr}_{\mathcal{F}} = \sum^{i} (-1)^i \mathrm{tr}_{H^i(\mathcal{F})}$$

where $H^i(\mathcal{F})$ are cohomology sheaves of $\mathcal{F}$. Among Grothendieck's six operations on the derived categories of ℓ-adic sheaves, tensor product, inverse image, and direct image with compact support have translations in terms of function trace, see [20][1.1.1.1-4].

Proposition 1.1.2. (1) *Let $\mathcal{F}, \mathcal{G} \in D_c^b(X, \bar{\mathbf{Q}}_\ell)$, then we have*

$$\mathrm{tr}_{\mathcal{F}}(x)\mathrm{tr}_{\mathcal{G}}(x) = \mathrm{tr}_{\mathcal{F}\otimes\mathcal{G}}(x)$$

for all $x \in X(k)$.

(2) *Let $f : X \to Y$ be a morphism of k-schemes and $\mathcal{F} \in D_c^b(Y, \bar{\mathbf{Q}}_\ell)$, then we have*

$$\mathrm{tr}_{f^*\mathcal{F}}(x) = \mathrm{tr}_{\mathcal{F}}(f(x))$$

for all $x \in X(k)$.

(3) *If X is a k-scheme of finite type and $\mathcal{F} \in D_c^b(X, \bar{\mathbf{Q}}_\ell)$, then for every $y \in Y(k)$, X_y the fiber of f over y, we have*

$$\sum_{x \in X_y(k)} \mathrm{tr}_{\mathcal{F}}(x) = \mathrm{tr}_{f_!\mathcal{F}}(y).$$

These rules constitute the basic dictionary between ℓ-adic sheaves and functions. The two first rules derive directly from the definition of tensor product and inverse image of ℓ-adic sheaves. The last rule derives from the base change theorem for proper morphisms and the Grothendieck-Lefschetz fixed points formula, see [SGA5, exp. XII]:

Theorem 1.1.3. *If X is a k-scheme of finite type and $\mathcal{F}$ is an ℓ-adic sheaf on X, we have*

$$\sum_{x \in X(k)} \mathrm{tr}_{\mathcal{F}}(x) = \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{tr}(\sigma \otimes_k \mathrm{id}_{\bar{k}}, H_c^i(X \otimes_k \bar{k}, \mathcal{F})).$$

1.2. Character sums Important examples of local systems arise from the Lang isogeny of commutative algebraic groups. Let G be a commutative algebraic group defined over k, and $\sigma : G \to G$ its geometric Frobenius endomorphism. The Lang isogeny of G defined as the morphism $L_G(x) = \sigma(x)x^{-1}$ is a finite, étale homomorphism of groups whose kernel is the discrete subgroup $G(k)$. We have an exact sequence:

$$0 \to G(k) \to G \xrightarrow{L_G} G \to 0.$$

Every ℓ-adic representation $\phi : G(k) \to GL(V)$ gives rise to a ℓ-adic sheaf $\mathcal{F}_\phi$ on G, by means of the Lang isogeny. Its trace function theoretic shadow can be described as follows, see [20, 1.3.3.3]:

Lemma 1.2.1. *The trace function* $\mathrm{tr}_{\mathcal{F}_\phi} : G(k) \to \bar{\mathbf{Q}}_\ell$ *is equal to the trace function of the representation* ϕ*:*

$$\mathrm{tr}_{\mathcal{F}_\phi}(g) = \mathrm{tr}(\phi(g)).$$

When $G = \mathbb{G}_a$ is the additive group, the Lang isogeny can be expressed as

$$L_{\mathbb{G}_a}(x) = x^q - x$$

in the additive notation. For its kernel is $\mathbb{G}_a(k) = k$, every character $\psi : k \to \bar{\mathbf{Q}}_\ell^\times$ gives rise to a local system of rank one $\mathcal{L}_\psi$ on $\mathbb{G}_a$, the Artin-Schreier sheaf attached to ψ. For every $x \in k$, we have $\mathrm{tr}_{\mathcal{L}_\psi}(x) = \psi(x)$.

When $G = \mathbb{G}_m$ is the multiplicative group, the Lang isogeny can be expressed as

$$L_{\mathbb{G}_m}(x) = x^{q-1}$$

in the multiplicative notation. For its kernel is $\mathbb{G}_m(k) = k^\times$, every character $\mu : k^\times \to \bar{\mathbf{Q}}_\ell^\times$ gives rise to a local system of rank one $\mathcal{L}_\mu$ on $\mathbb{G}_m$, the Kummer sheaf attached to μ. For every $x \in k^\times$, we have $\mathrm{tr}_{\mathcal{L}_\mu}(x) = \mu(x)$.

One can thus derive from the dictionary 1.1.2 between ℓ-adic sheaves and functions the cohomological interpretation of character sums. For instance, the sum

$$\sum_{x \in k} \psi(0) = 0$$

which is zero for non trivial additive character ψ can be interpreted as

$$\sum_{x \in k} \psi(x) = \sum_{i=0}^{2} (-1)^i \mathrm{tr}(\sigma \otimes_k \mathrm{id}_{\bar{k}}, H_c^i(\mathbb{G}_a \otimes_k \bar{k}, \mathcal{L}_\psi)).$$

One can prove that in fact the group $H_c^i(\mathbb{G}_a \otimes_k \bar{k}, \mathcal{L}_\psi))$ vanishes for all $i \in \{0, 1, 2\}$, see [10, 2.7].

The Gauss sum attached to a multiplicative character $\mu : k^\times \to \bar{\mathbf{Q}}_\ell^\times$ and an additive character $\psi : k \to \bar{\mathbf{Q}}_\ell^\times$:

$$G(\mu, \psi) = \sum_{x \in k^\times} \mu(x)\psi(x)$$

is equal to

$$G(\mu,\psi) = \sum_{i=0}^{2} (-1)^i \mathrm{tr}(\sigma \otimes_k \mathrm{id}_{\bar{k}}, H_c^i(\mathbb{G}_m \otimes_k \bar{k}, \mathcal{L}_\mu \otimes j^*\mathcal{L}_\psi))$$

where $j^*\mathcal{L}_\psi$ is the restriction to $\mathbb{G}_m$ of the Artin-Schreier sheaf on $\mathbb{G}_a$.

Similarly, the Kloosterman sum attached to $a \in k^\times$ and $\psi : k \to \bar{\mathbb{Q}}_\ell^\times$:

$$\mathrm{Kl}(a,\psi) = \sum_{x \in k^\times} \psi(x + ax^{-1})$$

is equal to

$$\mathrm{Kl}(a,\psi) = \sum_{i=0}^{2} (-1)^i \mathrm{tr}(\sigma \otimes_k \mathrm{id}_{\bar{k}}, H_c^i(K_a \otimes_k \bar{k}, l^*\mathcal{L}_\psi))$$

where K_a is the curve in the plane $\mathbb{A}^2 = \mathrm{Spec} k[x,y]$ defined by the equation $xy = a$, and l is the restriction to K_a of the map $l : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x,y) \mapsto x+y$.

1.3. The Swan conductors and Euler-Poincaré characteristics When X is an affine scheme, we have $H_c^0(X \otimes_k \bar{k}, \mathcal{L}) = 0$ for every local system $\mathcal{L}$ as $\mathcal{L}$ has no nonzero compactly supported section over X. If X is a smooth curve, we infer from the Poincaré duality the equality:

$$H_c^2(X \otimes_k \bar{k}, \mathcal{L})^\vee = H^0(X \otimes_k \bar{k}, \mathcal{L}^\vee(1))$$

whose right hand side is trivial unless the restriction of $\mathcal{L}^\vee$ to $X \otimes_k \bar{k}$ contains the constant sheaf. Thus if X is a smooth affine curve and $\mathcal{L}$ is a geometrically non constant and irreducible local system of X, we have

$$H_c^0(X \otimes_k \bar{k}, \mathcal{L}) = H_c^2(X \otimes_k \bar{k}, \mathcal{L}) = 0.$$

The dimension of H_c^1 can then be derived from the Euler-Poincaré characteristic:

$$\dim H_c^1(X \otimes_k \bar{k}, \mathcal{L}) = -\chi_c(X \otimes_k \bar{k}, \mathcal{L})$$

that can be calculated by the Grothendieck-Ogg-Shafarevich formula, see [32].

Theorem 1.3.1. *Let $\bar{X}$ be an proper smooth algebraic curve over k, X an open subset of $\bar{X}$ and $\mathcal{F}$ a local system over X. Then the formula*

$$\chi_c(X \otimes_k \bar{k}, \mathcal{F}) = \chi_c(X \otimes_k \bar{k})\mathrm{rk}(\mathcal{F}) - \sum_{x \in \bar{X} - X} \mathrm{Sw}_x(\mathcal{F})$$

holds.

In the above formula, the Swan conductor $\mathrm{Sw}_x(\mathcal{F})$ of $\mathcal{L}$ at x is a certain non negative integer which depends only on the restriction of $\mathcal{L}$ to the punctured formal disc $X_x^\bullet$. We know that:

- $\mathrm{Sw}_x(\mathcal{F}) = 0$ if and only if the representation of local Galois group attached to the restriction of $\mathcal{L}$ to the punctured disc at x is tame.
- If $\mathcal{G}$ is a tame ℓ-adic local system at $X_x^\bullet$, then

$$\mathrm{Sw}_x(\mathcal{F} \otimes \mathcal{G}) = \mathrm{Sw}_x(\mathcal{F})\mathrm{rk}(\mathcal{G}) \tag{1.3.2}$$

Using Theorem 1.3.1 and the formula (1.3.2), one can calculate the Euler-Poincaré characteristic occurring in the cohomological interpretation of the Gauss sums and Kloosterman sums. By applying Theorem 1.3.1 to the case $X = \mathbb{G}_a$ and $\mathcal{F} = \mathcal{L}_\psi$ the Artin-Schreier sheaf associated to a certain non trivial additive character, we derive

$$\mathrm{Sw}_\infty(\mathcal{L}_\psi) = 1.$$

Using (1.3.2) and Theorem 1.3.1 one can show that

$$\dim \mathrm{H}^1_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\mu) = 1.$$

in the case of the Gauss sum and

$$\dim \mathrm{H}^1_c(K_a, l^*\mathcal{L}_\psi) = 2$$

in the case of the Kloosterman sum.

1.4. The Hasse-Davenport identity Let k' denote the quadratic extension of k, $\mathrm{Tr}_{k'/k}; k' \to k$ and $\mathrm{Nm}_{k'/k} : k'^\times \to k^\times$ the trace and norm maps. For every $a \in k^\times$, we define the "twisted" Kloosterman sum

$$\mathrm{Kl}'(a, \psi) = \sum_{x \in k'^\times, \mathrm{Nm}(x) = a} \psi(\mathrm{Tr}(a))$$

for every non trivial additive character $\psi : k \to \bar{\mathbf{Q}}_\ell^\times$.

Proposition 1.4.1. *The equality*

$$\mathrm{Kl}(a, \psi) = -\mathrm{Kl}'(a, \psi) \tag{1.4.2}$$

holds for all $a \in k^\times$ *and non trivial additive character* $\psi : k \to \bar{\mathbf{Q}}_\ell^\times$.

We consider the affine curve

$$K_a = \mathrm{Spec}(k[x, y]/(xy - a))$$

equipped with the morphism $l : K_a \to \mathbb{G}_a$ given by $l(x, y) = x + y$. Let τ denote the involution of K_a defined by $\tau(x, y) = (y, x)$. There exists k-scheme K'_a with an isomorphism

$$K_a \otimes_k \bar{k} = K'_a \otimes_k \bar{k}$$

such that the Frobenius $\sigma' = \sigma_{X'_a}$ induces on $X'_a \otimes_k \bar{k}$ the endomorphism

$$(\sigma' \otimes_k \mathrm{id}_{\bar{k}}) = (\sigma \otimes_k \mathrm{id}_{\bar{k}}) \circ \tau.$$

We can then check that

$$\mathrm{Fix}(\sigma \circ \tau, K_a(\bar{k})) = \{x \in k'^\times | \mathrm{Nm}(x) = a\}.$$

and thus the Grothendieck-Lefschetz fixed points formula yields the identity

$$\mathrm{Kl}'(a, \psi) = \sum_{i=0}^{2} (-1)^i \mathrm{Tr}((\sigma \otimes_k \bar{k}) \circ \tau, \mathrm{H}^i_c(K_a \otimes_k \bar{k}, l^*\mathcal{L}_\psi)).$$

Now to prove (1.4.2), it is enough to prove the following:

Lemma 1.4.3. *The involution* σ *acts on* $H^i_c(K_a \otimes_k \bar{k}, l^*\mathcal{L}_\psi)$ *as* -1.

The morphism $l : K_a \to \mathbb{G}_a$ is a finite covering of degree 2. By the projection formula, we have

$$l_* l^* \mathcal{L}_\psi = l_* \bar{\mathbf{Q}}_\ell \otimes \mathcal{L}_\psi.$$

We observe that τ acts on K_a as a deck transformation with respect to the covering morphism $h : K_a \to \mathbb{G}_a$, and in particular, one can decompose $h_*\bar{\mathbf{Q}}_\ell$ as a direct sum according to the action of τ:

$$h_*\bar{\mathbf{Q}}_\ell = (h_*\bar{\mathbf{Q}}_\ell)_+ \oplus (h_*\bar{\mathbf{Q}}_\ell)_-,$$

with τ acting on $(h_*\bar{\mathbf{Q}}_\ell)_+$ as 1 and on $(h_*\bar{\mathbf{Q}}_\ell)_-$ as -1. We derive

$$H^i_c(K_a, l^*\mathcal{L}_\psi) = H^i_c(\mathbb{G}_a, (h_*\bar{\mathbf{Q}}_\ell)_+ \otimes \mathcal{L}_\psi) \oplus H^i_c((h_*\bar{\mathbf{Q}}_\ell)_- \otimes \mathcal{L}_\psi)$$

with τ acting on $H^i_c(\mathbb{G}_a, (h_*\bar{\mathbf{Q}}_\ell)_+ \otimes \mathcal{L}_\psi)$ as 1 and on $H^i_c(\mathbb{G}_a, (h_*\bar{\mathbf{Q}}_\ell)_- \otimes \mathcal{L}_\psi)$ as -1. For $(h_*\bar{\mathbf{Q}}_\ell)_+$ is isomorphic to the constant sheaf $\bar{\mathbf{Q}}_\ell$ over $\mathbb{G}_a$, we have

$$H^i_c(\mathbb{G}_a, (h_*\bar{\mathbf{Q}}_\ell)_+ \otimes \mathcal{L}_\psi) = H^i_c(\mathbb{G}_a, \mathcal{L}_\psi) = 0.$$

It follows that τ acts on $H^i_c(K_a, l^*\mathcal{L}_\psi)$ as -1.

2. Purity and perversity

2.1. Deligne's theorem on weights By choosing an isomorphism $\iota : \bar{\mathbf{Q}}_\ell \to \mathbf{C}$ once for all, we can assign an Archimedean absolute value to every element of $\bar{\mathbf{Q}}_\ell$. An ℓ-adic sheaf $\mathcal{F}$ on X is said to be mixed of weight $\leqslant 0$ if for every point $x \in X(k')$ of X with value in a finite extension k' of k, all eigenvalues of σ_x on $\mathcal{F}_{\bar{x}}$, $\bar{x}$ being a geometric point over x, have Archimedean absolute value at most 1. We recall now the celebrated theorem of Deligne on weights [9], formely known as the Weil conjecture.

Theorem 2.1.1. *Let* X *be a* k*-scheme of finite type,* $\mathcal{F}$ *an* ℓ*-adic sheaf on* X *which is mixed of weight* $\leqslant 0$*, then all eigenvalues of* $\sigma \otimes_k \mathrm{id}_{\bar{k}}$ *acting on* $H^i_c(X \otimes_k \bar{k}, \mathcal{F})$ *have Archimedean absolute values* $\leqslant q^{\frac{i}{2}}$.

Applying this theorem to the case of Gauss sums and Kloosterman sums we get the estimates

$$|G(\mu, \psi)| \leqslant q^{1/2}$$

and

$$|\mathrm{Kl}(a, \psi)| \leqslant 2q^{1/2}.$$

In fact, we know more:

$$|G(\mu, \psi)| = q^{1/2}$$

and

$$\mathrm{Kl}(a, \psi) = \alpha + \beta$$

where α, β are ℓadic numbers having Archimedean absolute values $q^{1/2}$. This further information can be derived from the Poincaré duality as follows.

We consider the derived category $D^c_b(X)$ of bounded constructible complex of ℓ-adic sheaves on X. A complex of ℓ-adic sheaves $\mathcal{F} \in D^c_b(X)$ is said to be mixed of weight $\leqslant 0$ if for every $n \in \mathbf{Z}$, $H^i(\mathcal{F})$ is mixed of weight $\leqslant i$. The derived category $D^c_b(X)$ is equipped with the Verdier duality functor $D : D^c_b(X) \to D^c_b(X)$ such that for every $\mathcal{F} \in D^c_b(X)$, the Poincaré duality holds between the vector spaces $H^i_c(X \otimes_k \bar{k}, \mathcal{F})$ and $H^i(X \otimes_k \bar{k}, D(\mathcal{F}))$ as vector spaces equipped with action of the Frobenius endomorphism.

A complex of ℓ-adic sheaves $\mathcal{F} \in D^c_b(X)$ is said to be mixed of weight $\geqslant 0$ if $D(\mathcal{F})$ is mixed of weight $\leqslant 0$. It is said to be pure of weight 0 if it is both mixed of weight $\leqslant 0$ and mixed of weight $\geqslant 0$.

It follows from Theorem 2.1.1 and the Poincaré duality that if $\mathcal{F}$ is mixed of weight $\geqslant 0$ then the eigenvalues of $\sigma \otimes_k \mathrm{id}_{\bar{k}}$ acting cohomology groups without support condition $H^i(X \otimes_k \bar{k}, \mathcal{F})$ have Archimedean absolute value $\geqslant q^{i/2}$. In the case of Gauss sums and Kloosterman sums, one can show that, on the one hand, the sheaves $\mathcal{L}_\mu \otimes \mathcal{L}_\chi$ on $\mathbb{G}_m$ and $h^*\mathcal{L}_\psi$ on K_a are pure, and on the other hand, the natural maps from cohomology with compact support to cohomology without the support condition

$$H^i_c(\mathbb{G}_m \otimes_k \bar{k}, \mathcal{L}_\mu \otimes j^*\mathcal{L}_\psi) \to H^i(\mathbb{G}_m \otimes_k \bar{k}, \mathcal{L}_\mu \otimes j^*\mathcal{L}_\psi)$$

and

$$H^i_c(K_a \otimes_k \bar{k}, h^*\mathcal{L}_\psi) \to H^i(K_a \otimes_k \bar{k}, h^*\mathcal{L}_\psi)$$

are isomorphisms, see [10, 4.3, 7.4]. It follows that the eigenvalues of $\sigma \otimes_k \mathrm{id}_{\bar{k}}$ acting on these vector spaces have Archimedean absolute values equal to $q^{i/2}$.

In [11], Deligne proved the following relative variant of the Weil conjecture. The relative version is much more powerful than its absolute version, especially while combined with the theory of perverse sheaves. In particular, it is a crucial ingredient in the implementation of our perverse continuation principle in many cases.

Theorem 2.1.2. *Let $f : X \to Y$ be a proper morphism between k-schemes of finite type. Let $\mathcal{F}$ be a complex of ℓ-adic sheaves on X which is pure of weight 0. Then $f_*\mathcal{F}$ is also pure of weight 0.*

2.2. Perverse sheaves and the decomposition theorem Deligne's purity theorem 2.1.2 is greatly reinforced by the theory of perverse sheaves [1]. Let X be a smooth algebraic variety over k and $\mathcal{F}$ is a local system such that for all $x \in |X|$, all eigenvalues of the Frobenius σ_x acting on $\mathcal{F}_{\bar{x}}$ have Archimedean absolute value 1, then the same is true for the dual local system $\mathcal{F}^\vee$ that is also the Verdier dual $D(\mathcal{F})$, up to a degree shift. In particular $\mathcal{F}$ is a pure sheaf of weight 0. If X is singular, it is more difficult to construct pure sheaves as the Verdier dual of local systems are no longer local systems in general.

This difficulty is solved by Goreski-Macpherson's construction of the intermediate extension. If U is a smooth dense open subset of X with $j : U \to X$ denoting the open embedding, and $\mathcal{F}$ is a pure local system on U, then the intermediate extension [1, 1.4.22]

$$\mathcal{K} = j_{!*}\mathcal{F}[\dim X]$$

is a pure perverse sheaf. The purity of the intermediate extension follows from the fact that this functor commutes with the Verdier duality.

The intermediate extensions of local systems are objects of an Abelian subcategory $\mathcal{P}(X)$ of the derived category $D^b_c(X)$, namely the category of perverse sheaves, [1]. The category of perverse sheaves form a heart of a t-structure on $D^b_c(X)$, in the sense that there are cohomological functors $D^b_c(X) \to \mathcal{P}(X)$, denoted by

$$\mathcal{F} \to {}^p\mathrm{H}^i(\mathcal{F})$$

transforming triangles into long exact sequences.

Theorem 2.2.1. [1, 5.1.15(iii)] *Every pure complex of sheaves $\mathcal{F} \in D^b_c(X)$ is isomorphic over $X \otimes_k \bar{k}$ to a direct sum of simple perverse sheaves with shift.*

If $\mathcal{F}$ is a pure complex of sheaves on a k-scheme X, then there exists a non-canonical isomorphism over $X \otimes_k \bar{k}$.

$$\mathcal{F} \otimes_k \bar{k} \simeq \bigoplus_i {}^p\mathrm{H}^i(\mathcal{F} \otimes_k \bar{k})[-i],$$

and moreover each perverse sheaf ${}^p\mathrm{H}^i(\mathcal{F} \otimes_k \bar{k})$ is isomorphic to a direct sum of simple perverse sheaves

$${}^p\mathrm{H}^i(\mathcal{F} \otimes_k \bar{k}) = \bigoplus_{\alpha \in \mathfrak{A}_i(\mathcal{F})} K_\alpha.$$

where $\mathfrak{A}_i$ is a certain finite set of indices. We denote $\mathfrak{A}(\mathcal{F}) = \bigsqcup_{i \in \mathbf{Z}} \mathfrak{A}_i(\mathcal{F})$.

According to [1, 4.3.1], for every simple perverse sheave K_α, there exists an irreducible closed subset Z_α of $X \otimes_k \bar{k}$, a dense smooth open subset U_α of Z_α, an irreducible local system L_α on U_α such that

$$K_\alpha = i_{\alpha,*} j_{\alpha,!*} L_\alpha[\dim(Z_\alpha)]$$

where j_α is the open embedding $U_\alpha \to Z_\alpha$ and i_α is the closed embedding $Z_\alpha \to X \otimes_k \bar{k}$.

We consider the finite set

$$\mathrm{Supp}(\mathcal{F}) = \{Z_\alpha \mid \alpha \in \mathfrak{A}(\mathfrak{F})\} \tag{2.2.2}$$

of irreducible closed subsets of $X \otimes_k \bar{k}$. We may see this set as the set of loci where $\mathcal{F}$ undergoes significant changes in the perverse perspective.

Let $f : X \to Y$ be a proper morphism of k-schemes of finite type. If $\mathcal{F}$ is a pure complex of sheaves on X, the derived direct image $f_*\mathcal{F}$ is also pure after Theorem 2.1.2. Thus the definition (2.2.2) applies to $f_*\mathcal{F}$. If moreover X is smooth, then the constant sheaf $\bar{\mathbf{Q}}_\ell$ is pure, and we set

$$\mathrm{Supp}(f) = \mathrm{Supp}(f_*\bar{\mathbf{Q}}_\ell).$$

In many circumstances, the determination of $\mathrm{Supp}(\mathcal{F})$, $\mathrm{Supp}(f)$ is an important ingredient for the application of Grothendieck's dictionary to problems in number theory and harmonic analysis.

2.3. Perverse continuation method The determination of supports of pure sheaves pertains to a method of proving certain equalities by perverse continuation. The typical argument can often run as follows.

Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be proper morphisms between k-schemes of finite type. Assume that both X_1 and X_2 are smooth and Y is irreducible. Assume also that:

(1) there exists a non empty open subset U of Y such that for every point $y \in U(k')$ of U with values in some finite extension k' of k, there are the same number of k'-points in the fibers $f_1^{-1}(y)$ and $f_2^{-1}(y)$;
(2) both $\mathrm{Supp}(f_1)$ and $\mathrm{Supp}(f_2)$ are the singleton of element $Y \otimes_k \bar{k}$.

In this situation, one can conclude that for every point $y \in Y(k')$ of Y with value in some finite extension k' of k, the equality

$$\# f_1^{-1}(y)(k') = \# f_2^{-1}(y)(k')$$

of numbers of k'-points in the fibers of f_1 and f_2, holds.

Let us denote $\mathcal{F}_1 = f_{1*}\bar{\mathbb{Q}}_\ell$ and $\mathcal{F}_2 = f_{2*}\bar{\mathbb{Q}}_\ell$. For $\mathrm{Supp}(f_1) = \mathrm{Supp}(f_2) = \{Y \otimes_k \bar{k}\}$, by restricting further the open subset U, we may assume that there exist local systems L_1^i, L_2^i on U such that

$${}^p\mathrm{H}^i(\mathcal{F}_1) = j_{!*}L_1^i \quad \text{and} \quad {}^p\mathrm{H}^i(\mathcal{F}_2) = j_{!*}L_2^i$$

with j being the open embedding $U \to X$.

The first assumption implies that for every point $y \in U(k')$, if σ_y is the Frobenius conjugacy class in $\pi_1(U)$ associated to y, then

$$\sum_i (-1)^i \mathrm{tr}(\sigma_y, L_1^i) = \sum_i (-1)^i \mathrm{tr}(\sigma_y, L_2^i).$$

For L_1^i and L_2^i are pure local systems of weight i, one can separate the above identity in to a family of identities for each i

$$\mathrm{tr}(\sigma_y, L_1^i) = \mathrm{tr}(\sigma_y, L_2^i).$$

After the Chebotarev density theorem, we infer that L_1^i and L_2^i are isomorphic up to semisimplification.

Although the intermediate extension functor is not exact in general, we can still derive that ${}^p\mathrm{H}^i(\mathcal{F}_1) = j_{!*}L_1^i$ and ${}^p\mathrm{H}^i(\mathcal{F}_2) = j_{!*}L_2^i$ are isomorphic up to semisimplification under the assumption $\mathrm{Supp}(\mathcal{F}_1) = \mathrm{Supp}(\mathcal{F}_2) = \{X \otimes_k \bar{k}\}$. Indeed, if $0 \to A \to B \to C \to 0$ is an exact sequence of local systems of on U, then we have a sequence

$$0 \to j_{!*}A \to j_{!*}B \to j_{!*}C \to 0 \tag{2.3.1}$$

where $\alpha : j_{!*}A \to j_{!*}B$ is injective and $\beta : j_{!*}B \to j_{!*}C$ is surjective, however $\mathrm{im}(\alpha)$ may be strictly smaller than $\ker(\beta)$, see [8, 2.7]. Nevertheless, under the assumption that B is geometrically semisimple and $\mathrm{Supp}(B) = \{X \otimes_k \bar{k}\}$, the subquotient $\ker(\beta)/\mathrm{im}(\alpha)$, being supported on $X - U$, ought to vanish, and therefore the sequence (2.3.1) is exact.

We have proved that for all i there exists an isomorphism ${}^p\mathrm{H}^i(\mathcal{F}_1) \simeq {}^p\mathrm{H}^i(\mathcal{F}_2)$. The conclusion on equalities of number of points on fibers of f_1 and f_2 follows from the Grothendieck-Lefschetz formula.

The above argument demonstrated the power of the assumption on the support. This is the reason why the determination of the support is usually a hard problem. For this purpose, there are different methods which may be applied in various situations: the Fourier-Deligne transform, the Goresky-MacPherson theorem for small maps, and the support theorem for abelian fibrations. One should also mention the recent theorem of Migliorini-Shende asserting that the set of supports is a subset of the set of higher discriminants [22] in characteristic zero case. It would be very interesting to generalize their theorem to the case of positive characteristic.

We will present three problems of harmonic analysis that can be solved with help of this method: the fundamental lemma of Jacquet-Ye, Jacquet-Rallis and Langlands-Shelstad where those methods can be respectively applied.

2.4. The Fourier-Deligne transform Let S be an algebraic variety over k. Let $p_V : V \to S$ be a vector bundle of rank n, $p_{V^\vee} : V^\vee \to S$ the dual vector bundle. We consider the cartesian product $V \times_S V^\vee$ equipped with the projection $\mathrm{pr}_V : V \times_S V^\vee \to V$, $\mathrm{pr}_{V^\vee} : V \times_S V^\vee \to V^\vee$ and the canonical pairing

$$\mu : V \times_S V^\vee \to \mathbb{G}_a.$$

The Fourier-Deligne transform is the functor

$$\mathrm{FD} : D^b_c(V) \to D^b_c(V^\vee)$$

defined by

$$\mathrm{FD}(\mathcal{F}) = \mathrm{pr}_{V^\vee,!}(\mathrm{pr}_V^* \mathcal{F} \otimes \mu^* \mathcal{L}_\psi)[n]$$

The following theorem on the Fourier-Deligne transform has been proven by Katz and Laumon in [18] and [20, 1.3.2.3].

Theorem 2.4.1. *The Fourier-Deligne transform preserve perversity:*

$$\mathrm{FD} : \mathcal{P}(V) \to \mathcal{P}(V^\vee).$$

Moreover, if S' is an open subset of S, $j_V : V_{S'} \to V$ and $j_{V^\vee} : V^\vee_{S'} \to V^\vee$ the open subsets of V and $V^\vee$ induced by base change, if $\mathcal{F} = j_{V,!*} j_V^* \mathcal{F}$ then

$$\mathrm{FD}(\mathcal{F}) = j_{V^\vee,!*} j_{V^\vee}^* \mathrm{FD}(\mathcal{F})$$

2.5. Small maps Let X, Y be algebraic varieties over k. Let $f : X \to Y$ be a proper surjective morphism. We say that f is a semismall map if

$$\dim(X \times_Y X) \leqslant X.$$

We say it is small if moreover for every irreducible component Z of $X \times_Y X$ of dimension equal to dimension of X, the induced map $Z \to X$ is surjective. The following theorem, due to Goresky and MacPherson, can be proven by simple dimension counting, see [13, p. 120] .

Theorem 2.5.1. *Let $f : X \to Y$ be a small map. Assume that X is smooth, then $\mathcal{F} := f_*\mathbf{Q}_\ell[\dim(Y)]$ is a perverse sheaf. Moreover, for every dense open subset U with embedding $j : U \to X$, we have $\mathcal{F} = j_{!*}j^*\mathcal{F}$.*

Important instances of small maps are the Grothendieck-Springer simultaneous resolution [34], and the Hilbert scheme of zero-dimensional subschemes of a smooth surface, as small resolution of the Chow scheme [25]. Both have important applications in representation theory.

2.6. Support theorem for abelian fibrations We define *abelian fibration* to be the following data:

- $f : M \to S$ is a proper morphism of relative dimension d,
- $g : P \to S$ is a smooth commutative group scheme acting on M i.e. equipped with a morphism $P \times_S M \to M$ satisfying all the usual axioms of group action,
- for generic points $s \in S$, P_s acts simply transitively on M_s,
- for every geometric point $s \in S$ and $m \in M_s$, the stabilizer $\mathrm{Stab}_m(P_s)$ is an affine subgroup of P_s.

For every geometric point $s \in S$, the group fiber P_s can be decomposed as follows. First, we have the exact sequence

$$0 \to P_s^0 \to P_s \to \pi_0(P_s) \to 0$$

where $\pi_0(P_s)$ is the group of connected components of P_s and P_s^0 its neutral component. The neutral component itself can be further decomposed into an exact sequence:

$$0 \to R_s \to P_s \to A_s \to 0$$

where A_s is an abelian variety and R_s is a connected affine group, according to a theorem of Chevalley and Rosenlicht, see [33], [5]. We refer to [5] for a modern treatment of the Chevalley-Rosenlicht theorem.

We can stratify S by the dimension of the affine part $\delta_s = \dim(R_s)$:

$$S = \bigsqcup_{\delta} S_\delta$$

with

$$S_\delta(\bar{k}) = \{s \in S(\bar{k}) | \delta_s = \delta\}$$

We say that P is δ-*regular* if for every $\delta \geqslant 0$, we have

$$\mathrm{codim}(S_\delta) \geqslant \delta.$$

The condition of δ-regularity is automatically satisfied by all algebraic integrable system over $\mathbf{C}$, see [31]. For certain families of Jacobians, the δ-regularity has been proven by Melo, Rapagneta and Viviani [21].

There is yet another technical condition in the statement of the support theorem. We say that P/S is polarizable if there exists an alternating form λ on the sheaf of Tate modules

$$H_1(P/S) = H^{2d-1}(g_! \mathbf{Q}_\ell)$$

such that over every geometric point $s \in S$, λ annihilates the Tate module $H_1(R_s)$ of the affine part, and induces a non-degenerate form on the Tate module $H_1(A_s)$ of the abelian part.

Theorem 2.6.1. *Let $f : M \to S$ and $g : P \to S$ form an abelian fibration that is δ-regular and polarizable. Assume that M is smooth.*

(1) *If the geometric fibers M_s are irreducible then all irreducible perverse sheaves that are direct factors of $f_* \mathbf{Q}_\ell$ have support S.*
(2) *More generally, M_s no longer being assumed irreducible, if Z is the support of a simple perverse sheaf occurring as a direct factor of $f_* \mathbf{Q}_\ell$, if $i_Z : Z \to X$ denotes the closed embedding of Z in X, then in a étale neighborhood of the generic point of Z, $i_{Z*} \mathbf{Q}_{\ell Z}$ is a direct factor of the sheaf $H^{2d} f_* \bar{\mathbf{Q}}_\ell$ of cohomology of top degree of f.*

For more information about the support theorem for abelian fibrations, the readers may consult [31], and for a generalization in the characteristic zero case [22].

3. Double unipotent action

3.1. Invariant functions Let A denote the subgroup of diagonal matrices and U the subgroup of unipotent upper triangular matrices in $G = GL_n$. We denote $\mathfrak{g} = \mathfrak{gl}_n$ the space of $n \times n$-matrices. We consider the action of $U \times U$ on $\mathfrak{g}$ given by $x \mapsto {}^\top u^- x u^+$ with $x \in \mathfrak{g}$, $u^-, u^+ \in U$, ${}^\top u^-$ being the transpose matrix of u^-.

Let V denote the standard n-dimensional k-vector space, and $v_1, \ldots, v_n$ its standard basis. Let $V^\vee$ denote the dual vector space, and $v_1^\vee, \ldots, v_n^\vee$ the dual basis. We consider the function $\mathbf{e}_i \in k[\mathfrak{g}]$ given by

$$\mathbf{e}_i(x) = \langle v_1^\vee \wedge \cdots \wedge v_i^\vee, x(v_1 \wedge \cdots \wedge v_i) \rangle.$$

In terms of matrices, $\mathbf{e}_i(x)$ is the determinant of the $(i \times i)$-square matrix located in the upper left corner of x. We observe that for $u^-, u^+ \in U$, we have

$$({}^\top u^-)^{-1}(v_1^\vee \wedge \cdots \wedge v_i^\vee) = v_1^\vee \wedge \cdots \wedge v_i^\vee \quad \text{and} \quad u^+(v_1 \wedge \cdots \wedge v_i) = v_1 \wedge \cdots \wedge v_i$$

It follows that

$$\mathbf{e}_i({}^\top u^- x u^+) = \langle ({}^\top u^-)^{-1}(v_1^\vee \wedge \cdots \wedge v_i^\vee), x u^+(v_1 \wedge \cdots \wedge v_i)\rangle = \mathbf{e}_i(x),$$

and therefore $\mathbf{e}_i$ is an $U \times U$-invariant function on $\mathfrak{g}$ i.e. $\mathbf{e}_i \in k[\mathfrak{g}]^{U\times U}$. We obtain a $U \times U$-invariant morphism

$$\mathbf{e} : \mathfrak{g} \to \mathfrak{e} = \mathrm{Spec}(k[\mathbf{e}_1, \dots, \mathbf{e}_n])$$

It is clear that $G = GL_n$ is the inverse image of the open subset of $\mathfrak{e}$ defined by $\mathbf{e}_n \neq 0$.

Let $\mathfrak{e}^\circ$ denote the open subset of $\mathfrak{e}$ defined by the conditions $\mathbf{e}_i \neq 0$ for all i:

$$\mathfrak{e}^\circ = \mathrm{Spec}(k[\mathbf{e}_1^{\pm 1}, \dots, \mathbf{e}_n^{\pm 1}]).$$

The restriction of $\mathbf{e}$ to the diagonal torus A defines an isomorphism $\mathbf{e}|_A : A \to \mathfrak{e}^\circ$ mapping the diagonal matrix of entries $(a_1, \dots, a_n)$ to the point in $\mathfrak{e}$ of coordinates $(e_1, \dots, e_n)$ with $e_1 = a_1, e_2 = a_1 a_2, \dots, e_n = a_1 \dots a_n$. The inverse map $\mathbf{a} : \mathfrak{e}^\circ \to A$ defines a section of $\mathbf{e}$ over $\mathfrak{e}^\circ$. Using this section, we define a map

$$\mathfrak{e}^\circ \times U \times U \to f^{-1}(\mathfrak{e}^\circ) = \mathfrak{g}^\circ \tag{3.1.1}$$

given by $(d, u^-, u^+) \mapsto {}^\top u^- a(d) u^+$ that is an isomorphism.

It will be convenient to repackage the above discussion in the language of algebraic stacks.

Proposition 3.1.2. *The invariant functions* $\mathbf{e}_1, \dots, \mathbf{e}_n$ *define a morphism:*

$$[\mathbf{e}] : [\mathfrak{g}/(U \times U)] \to \mathfrak{e}$$

which is an isomorphism over $\mathfrak{e}^\circ$*.*

One can derive from this proposition that the algebra $k[\mathfrak{g}]^{U\times U}$ of functions on $\mathfrak{g}$ invariant under the action of $U \times U$ is the polynomial algebra:

$$k[\mathfrak{g}]^{U\times U} = k[\mathbf{e}_1, \dots, \mathbf{e}_n].$$

Indeed the $k[\mathfrak{g}]^{U\times U}$ consists of regular functions on $\mathfrak{g}$ whose restriction to fg° is $U \times U$-invariant. In other words, we have the equality

$$k[\mathfrak{g}]^{U\times U} = k[\mathbf{e}_1^{\pm 1}, \dots, \mathbf{e}_n^{\pm 1}] \cap k[\mathfrak{g}]$$

whose right hand side is $k[\mathbf{e}_1, \dots, \mathbf{e}_n]$. We won't need this fact in the sequel.

3.2. The Klossterman orbital integrals Let F be a nonarchimedean local field, $\mathcal{O}$ its ring of integers, $k = \mathcal{O}/\mathfrak{m}$ its residue field. We will denote $\mathrm{val} : F^\times \to \mathbf{Z}$ the valuation. We also choose a generator ϖ of the maximal ideal $\mathfrak{m}$ with help of which we can define a residue map $\mathrm{res} : F \to k$. We fix a non trivial character $\psi : k \to \bar{\mathbf{Q}}_\ell^\times$ and by composing it with the residue map we obtain a character $\psi_F : F \to \bar{\mathbf{Q}}_\ell^\times$ of conductor $\mathcal{O}$.

We equip $U(F)$ with the Haar measure such that $U(\mathcal{O})$ is of volume one. We consider the character

$$\psi_U(u) = \sum_{i=1}^{n-1} \psi_F(u_{i,i+1})$$

where $u_{i,i+1}$ are the entries of u that are located just above the diagonal.

For every function $\phi \in C_c^\infty(\mathfrak{g}(F))$, we consider the integral

$$\mathrm{Kl}_\phi(a) = \int_{U(F)\times U(F)} \phi({}^\top u^- a u^+)\psi_U(u^-)\psi_U(u^+)du^-du^+ \tag{3.2.1}$$

depending of $a \in A(F)$. The orbit of $U(F)\times U(F)$ passing through a is the fiber of $f : \mathfrak{g}(F) \to \mathfrak{c}(F)$ over $f(a)$, and in particular it is a closed subset of $\mathfrak{g}(F)$. Since the action $U \times U$ is free at a, this orbit is isomorphic with $U(F)\times U(F)$. The restriction of ϕ to the fiber of $f : \mathfrak{g}(F) \to \mathfrak{c}(F)$ over $f(a)$ defines thus a locally constant function with compact support in $U(F)\times U(F)$. These integrals appear in the geometric side of the Kuznetsov trace formula for $\mathrm{GL}(n)$.

We will restrict to the case $\phi = \mathbb{I}_{\mathfrak{g}(\mathcal{O})}$:

$$\mathrm{Kl}(a) = \int_{U(F)\times U(F)} \mathbb{I}_{G(\mathcal{O})}({}^\top u^- a u^+)\psi_U(u^-)\psi_U(u^+)du^-du^+. \tag{3.2.2}$$

The local harmonic analysis of the Kuznetsov trace formula consists in understanding the space of all functions $\mathrm{Kl}_\phi(a)$ along with its basic function $\mathrm{Kl}(a)$.

Jacquet and Ye have introduced a twisted version of these integrals. Let F' denote the unramified quadratic extension of F, $\mathcal{O}'$ its ring of integers whose residue field k' is the quadratic extension of k. Let us denote $x \mapsto \bar{x}$ the Galois conjugation in k'/k and F'/F. We consider the Haar measure on $U(F')$ such that $U(\mathcal{O}')$ is of measure one, and the character $\psi'_U : U(F') \to \bar{\mathbf{Q}}_\ell^\times$ given by

$$\psi'_U(u) = \sum_{i=1}^{n-1} \psi_F(\mathrm{tr}_{F'/F} u_{i,i+1}).$$

For every $a \in A(F)$, Jacquet and Ye consider the integral:

$$\mathrm{Kl}'(a) = \int_{N(F')} \mathbb{I}_{\mathfrak{g}(\mathcal{O}')}({}^\top \bar{u} a u)\psi'_U(u)dx.$$

In [17] , Jacquet and Ye have conjectured the following identity. It has first been proved in [26] in the case where F is a local field of Laurent formal series, and later transferred to p-adic fields for for large p by Cluckers and Loeser [4]. Jacquet has also proved the p-adic case by a completely different method [16]. We also note that Do Viet Cuong has proved a similar fundamental lemma in the metaplectic case [12] by a method similar to [26].

Theorem 3.2.3. *If we denote*

$$r(a) = \mathrm{val}(a_1) + \mathrm{val}(a_1a_2) + \cdots + \mathrm{val}(a_1 \ldots a_{n-1})$$

where $(a_1, \ldots, a_n)$ *are the entries of the diagonal matrix* $a \in A(F)$, *then the equality*

$$\mathrm{Kl}(a, \psi_U) = (-1)^{r(a)}\mathrm{Kl}'(a, \psi_U)$$

holds.

The matrix calculation in the case $G = GL_2$:

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_1 y \\ a_1 x & a_1 x y + a_2 \end{bmatrix}$$

shows that we need to determine the set of $x, y \in F/\mathcal{O}$ such that the matrix in the right hand side has integral coefficients. For instant, if we assume that $\mathrm{val}(a_1) = 1$, then $x, y \in \varpi^{-1}\mathcal{O}$. Let $u, v \in k$ denote the free coefficients of the formal series $a_1 x$ and $a_1 y$ respectively, and α the free coefficient of $a_1 a_2$, then

$$\mathrm{Kl}(a, \psi_U) = \sum_{uv=\alpha} \psi(u+v)$$

is an usual Kloosterman sum.

3.3. Cohomological interpretation of the Kloosterman integral Computing the Kloosterman integral (3.2.2) boils down to counting the set

$$K_a = \{(u^-, u^+) \in U(F)/N(\mathcal{O}) \times U(F)/N(\mathcal{O}) \mid {}^\top u^- a u^+ \in \mathfrak{g}(\mathcal{O}) \}$$

For $d_i({}^\top x a y) = d_i(a)$, this set is empty unless $d_1(a), \dots, d_n(a) \in \mathcal{O}$.

Assume that F is the field of Laurent formal series of variable ϖ with coefficients in the finite field k. It is not difficult to see that K_a can be given a structure of finite dimensional algebraic variety over k, which is equipped with a morphism $l : K_a \to \mathbb{G}_a$ such that

$$l(u^-, u^+) = \sum_{i=1}^{n-1} \mathrm{res}(u^-_{i,i+1} + u^+_{i,i+1})$$

so that

$$\mathrm{Kl}(a) = \sum_{m \in K_a(k)} \psi(l(m)).$$

By applying the Grothendieck-Lefschetz formula, we get

$$\mathrm{Kl}(a, \psi_U) = \sum_i (-1)^i \mathrm{tr}(\sigma, H^i_c(K_a \otimes_k \bar{k}), l^* \mathcal{L}_\psi).$$

The twisted Kloosterman integral can also be interpreted similarly. If $\tau : M_a \to M_a$ denote the involution $\tau(x, y) = (y, x)$, then we have

$$\mathrm{Kl}'(a, \psi_U) = \sum_{m' \in \mathrm{Fix}(\sigma \circ \tau, K_a)} \psi(l(m)).$$

By applying the Grothendieck-Lefschetz formula, we get

$$\mathrm{Kl}'(a, \psi_U) = \sum_i (-1)^i \mathrm{tr}(\sigma \circ \tau_a, H^i_c(K_a \otimes_k \bar{k}), l^* \mathcal{L}_\psi).$$

The identity (3.2.3) follows from the following:

Theorem 3.3.1. *For every* $a \in A(F)$ *such that* $a_1, a_1 a_2, \dots, a_1 a_2 \dots a_n \in \mathcal{O}$, *the involution* τ *acts on* $H^i_c(K_a \otimes_k \bar{k}, l^* \mathcal{L}_\psi)$ *as* $(-1)^{r(a)}$.

We observe that K_a is empty unless $e_1, e_2, \ldots, e_n \in \mathcal{O}$ where $e_i = a_1 \ldots a_i$. Assuming that $\mathrm{val}(d_j) = 0$ for $j \neq i$ and $\mathrm{val}(d_i) = 1$ for a given index $1 \leqslant i \leqslant n-1$, one can show that M_a is of the form

$$\{(u, v) \mid uv + \alpha = 0\}$$

for some $\alpha \in k^\times$, $h(u, v) = u + v$. In this case, the involution τ acts on $H^i_c(M_a \otimes_k \bar{k}, l^*\mathcal{L}_\psi)$ as it was shown in the case of usual Kloosterman sums.

3.4. Arc spaces and families of Kloosterman integrals To study geometrically families of local orbital integrals, it is best to use the concept of formal arcs on varieties and algebraic stack. Although this approach still lacks rigorous foundation, and has not been used in practice so far, it often offers an useful viewpoint.

We will denote $\mathbb{D} = \mathrm{Spec}(k[[t]])$ and $\mathbb{D}^\bullet = \mathrm{Spec}(k((t))$. Let X be an algebraic variety over k, X° an open subset of X. We consider the formal arc space $\mathcal{L}X$ of X that is an infinite dimensional over k such that

$$\mathcal{L}X(k) = X(k[[t]]),$$

in other words, k-points of $\mathcal{L}X$ are maps $x : \mathbb{D} \to X$. We are mostly interested in the open subset $\mathcal{L}^\circ X$ of maps $\mathbb{D} \to X$ whose restriction to $\mathbb{D}^\bullet$ has image in X°.

If we want to put the varieties K_a in family, it will more convenient to use, instead of $a = (a_1, \ldots, a_n)$, the parameter $e = (e_1, \ldots, e_n)$ where $e_i = a_1 \ldots a_i$. We will write K_e instead of K_a. By construction $e_i \in F^\times$ and we have seen that K_e is empty unless $e_i \in \mathcal{O}$. Thus $e = (e_1, \ldots, e_n)$ is a k-point of $\mathcal{L}^\circ\mathfrak{e}$ where $\mathfrak{e} = \mathbb{A}^n$ and $\mathfrak{e}^\circ = \mathbb{G}_m{}^n$.

For every $e \in \mathcal{L}^\circ\mathfrak{e}$, the variety K_e can be identified with the space of maps

$$x : \mathbb{D} \to [\mathfrak{g}/U \times U]$$

lying over the map $e : \mathbb{D} \to \mathfrak{e}$. In particular the restriction of x to $\mathbb{D}^\bullet$ has image in $[\mathfrak{g}^\circ/U \times U]$ where $\mathfrak{g}^\circ$ is the inverse image of $\mathfrak{c}^\circ$.

Formally at least, we can consider the "arc stack" $\mathcal{L}([\mathfrak{g}/(U \times U)])$ of maps

$$x : \mathbb{D} \to [\mathfrak{g}/(U \times U)]$$

and its open part $\mathcal{L}^\circ([\mathfrak{g}/(U \times U)])$ consisting in maps x as above whose restriction to $\mathbb{D}^\bullet$ has image in $[\mathfrak{g}^\circ/U \times U]$. As opposed to the whole arc stack $\mathcal{L}([\mathfrak{g}/(U \times U)])$, we expect its open part $\mathcal{L}^\circ([\mathfrak{g}/(U \times U)])$ to be a space instead of stack. The varieties K_a can now be seen as the fibers of the map

$$\mathcal{L}^\circ([\mathfrak{g}/(U \times U)]) \to \mathcal{L}^\circ\mathfrak{e}.$$

3.5. Global family For only few information about the varieties K_a is available, it is almost hopeless to calculate $H^i_c(K_a \otimes_k \bar{k}, l^*\mathcal{L}_\psi)$ explicitly. It may be tempting to try to put K_a and $H^i_c(K_a \otimes_k \bar{k}, h^*\mathcal{L}_\psi)$ in family, and prove Theorem 3.3.1 by the perverse continuation principle , similar to the argument outlined in Subsection 2.3.

Experiences show that instead of local orbital integrals, it is often more effective to consider families of global orbital integrals. Such a family of global orbital integrals appears naturally in the geometric side of the trace formula, and in the present case, it appears in the geometric side of the Kuznetsov trace formula.

Global Kloosterman integral is just a product of local Kloosterman integrals. In a family of global Kloosterman integrals, a generic member is product of many local integrals which are all very simples. As proving 3.3.1 is not difficult for a generic member in the family of global Kloosterman integrals, the perverse continuation principle would allow us to derive the statement like 3.3.1 for the special members from the generic members.

The construction of the global family $f : \mathcal{K} \to \mathcal{E}$ will depend on some auxiliary data: a smooth projective curve C and a divisor D. We will denote $\mathcal{T}_D$ the $\mathbb{G}_m$-torsor associated to the line bundle $\mathcal{O}_C(D)$. We observe that the scalar action of $\mathbb{G}_m$ on $\mathfrak{g}$ commutes with the action of $U \times U$, and induces an action of $\mathbb{G}_m$ on $\mathfrak{e}$:

$$t(e_1, \ldots, e_n) = (te_1, \ldots, t^n e_n).$$

The morphism $[\mathfrak{g}/U \times U] \to \mathfrak{e}$ is then $\mathbb{G}_m$-equivariant. Over $\mathcal{T}$, we can twist by means of the $\mathbb{G}_m$-torsors $\mathcal{T}$:

$$[\mathfrak{g}/U \times U] \wedge^{\mathbb{G}_m} \mathcal{T}_D \to \mathfrak{e} \wedge^{\mathbb{G}_m} \mathcal{T}_D.$$

We define $\mathcal{K}$ to be the space of maps

$$x : C \to [\mathfrak{g}/U \times U] \wedge^{\mathbb{G}_m} \mathcal{T}_D$$

whose restriction to the generic point of C has image lying in $[\mathfrak{g}^\circ/U \times U]$. One can prove that $\mathcal{K}$ is an algebraic space of finite type.

We define $\mathcal{E}(C, D)$ to be the space of morphisms

$$e : C \to \mathfrak{e} \wedge^{\mathbb{G}_m} \mathcal{T}_D$$

that map the generic point of C in $\mathfrak{e}^\circ$. It is easy to see that

$$\mathcal{E} = \prod_{i=1}^{n} (H^0(C, \mathcal{O}_C(iD)) - \{0\})$$

In order to describe the fibers of $\mathcal{K} \to \mathcal{E}$ in terms of local Kloosterman varieties, we will choose uniformizing parameters at every point $v \in D$ so that $\mathcal{O}_C(D)$ has a trivialization at each formal disc C_v. Let C' denote the biggest open subset of C being mapped to $\mathfrak{e}^\circ$. Then the restriction of x to C' is completely determined by the restriction of e to C'. It follows that

$$\mathcal{K}_e = \prod_{v \in C - C'} K_{e,v}.$$

In order to define the family of global Kloosterman integrals, we need to fix some additional data. We choose a nonzero rational 1-form ω and denote $\mathrm{div}(\omega)$ its associated divisor. Let $\mathcal{E}'$ denote the open subset of $\mathcal{E}$ of points $e = (e_1, \ldots, e_n)$

such that

$$\mathrm{div}(e) \cap \mathrm{div}(\omega) = \emptyset.$$

where $\mathrm{div}(e) = \mathrm{div}(e_1) + \cdots + \mathrm{div}(e_n)$. For every $e \in \mathcal{E}'$, we have an embedding

$$\mathcal{K}_e = \prod_{v \in \mathrm{div}(e)} K_{e,v} \hookrightarrow \prod_{v \in \mathrm{div}(e)} (U(F_v)/U(\mathcal{O}_v))^2.$$

For every $v \notin \mathrm{div}(\omega)$, we have a map

$$U(F_v)/U(\mathcal{O}_v) \to \mathbb{G}_a$$

given by

$$u \mapsto \sum_{i=1}^{n-1} \mathrm{res}(u_{i,i+1}\omega).$$

This induces a morphism $l : \mathcal{K}' \to \mathbb{G}_a$ where $\mathcal{K}'$ is the pre image of $\mathcal{E}'$ by the morphism $f : \mathcal{K} \to \mathcal{E}$. We denote $f' : \mathcal{K}' \to \mathcal{E}'$ the restriction of f to $\mathcal{K}'$.

Conjecture 3.5.1. *Up to a shift, $f_! l^* \mathcal{L}_\psi$ is a perverse sheaf which is the intermediate extension of a local system of rank $2^{\frac{(n-1)n}{2}\deg(D)}$ over the open subset $\mathcal{E}''$ of $\mathcal{E}'$ defined by the condition $\mathrm{div}(e)$ being multiplicity free.*

3.6. Coordinate calculation in a special case This conjecture has been proven in [26] in one special case where $f : \mathcal{K}' \to \mathcal{E}'$ and $l : \mathcal{K}' \to \mathbb{G}_a$ can be described explicitly in terms of coordinates. Let C be the projective line $\mathbb{P}^1$ of coordinate t given as a global section of $\mathcal{O}(1)$ vanishing at ∞, and the meromorphic 1-form dt with a double pole at ∞. In this case

$$\mathcal{E} = \prod_{i=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i\infty)).$$

The open subset $\mathcal{E}'$ of $\mathcal{E}$ consisting of $e = (e_1, \ldots, e_n)$ with $e_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i\infty))$ such that the $\mathrm{div}(e_i)$ is prime to ∞, in other words e_i is a polynomial of degree i in the variable t.

The inverse image $\mathcal{K}'$ of $\mathcal{E}'$ in $\mathcal{K}$ can be described in coordinats as follows. First we observe that every U-torsor over $\mathcal{P}^1$ is trivial because $H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$. It follows that the stack Bun_U of principal U-torsor over $\mathbb{P}^1$ has only the trivial object whose automorphism group is U. We also observe that the space

$$H^0(\mathfrak{g} \wedge^{\mathbb{G}_m} \mathcal{T}_D) = \{x_0 + x_1 t \mid x_0, x_1 \in \mathfrak{g}\}$$

and thus the space of maps

$$C \to [\mathfrak{g}/U \times U] \wedge^{\mathbb{G}_m} \mathcal{T}_D$$

is the quotient of $\mathfrak{g} \times \mathfrak{g}$ by $U \times U$ where $U \times U$ acts diagonally on $\mathfrak{g} \times \mathfrak{g}$. The condition that $\mathfrak{e}_i(x_0 + x_1 t)$ is a polynomial of degree i, for every i, implies that $x_1 \in G^0$. It follows that a point $\mathcal{K}'$ can be uniquely represented under the form $x + \alpha t$ with x being an arbitrary matrix, and $\alpha \in A$ is the invertible diagonal matrix:

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_1^{-1}\alpha_2, \ldots, \alpha_{n-1}^{-1}\alpha_n).$$

where $\alpha_i \neq 0$ denote the coefficient of t^i in the polynomial e_i. In other words, we have

$$\mathcal{K}' = \mathfrak{g} \times A.$$

One can show by direct calculation that the function l is given by the formula

$$l(x, \alpha) = \sum_{i=1}^{n-1} \alpha_{i-1}^{-1}\alpha_i (x_{i,i+1} + x_{i+1,i})$$

In this very special case, with help of these coordinates, we proved the conjecture 3.5.1 in [26] by an induction argument based on the Fourier-Deligne transform. Although this special case of 3.5.1 is enough to deduce the Jacquet-Ye fundamental lemma in positive characteristic, it is of interest to find a proof of 3.5.1 in general.

4. Action of GL_{n-1} on $\mathfrak{gl}_n$ by conjugation

4.1. Invariant functions Let V be a n-dimensional vector space, $V^\vee$ its dual. We pick $v \in V$ and $v^\vee \in V^\vee$ a vector and a covector such that $\langle v^\vee, v\rangle = 1$. Let $G = \mathrm{GL}(V)$ and H the subgroup of G consisting of elements $g \in G$ such that $gv = v$ and $gv^\vee = v^\vee$. We have $H \simeq \mathrm{GL}_{n-1}$. Since $G = \mathrm{GL}(V)$ acts transitively on the set of pairs of vectors $(v, v^\vee)$ satisfying $\langle v^\vee, v\rangle = 1$, one can identify the classifying stack BH of H with the stack of triples $(V, v, v^\vee)$ satisfying $\langle v^\vee, v\rangle = 1$. It follows that one can identify $[\mathfrak{g}/H]$ with the algebraic stack classifying the quadruples $(V, x, v, v^\vee)$ where V is a n-dimensional vector space, $x \in \mathrm{End}(V)$, $v \in V$, $v^\vee \in V^\vee$ are vector and covector such that $\langle v^\vee, v\rangle = 1$.

We will also consider the algebraic stack $\mathcal{Y}$ classifying quadruple $(V, v, v^\vee, x)$ as above but without the equation $\langle v^\vee, v\rangle = 1$. If $y : \mathcal{Y} \to \mathbb{G}_a$ is the map $(V, x, v, v^\vee) \mapsto \langle v^\vee, v\rangle$, then $[\mathfrak{g}/H]$ can be identified with the fiber $\mathcal{Y}_1$ of $\mathcal{Y}$ over the point $1 \in \mathbb{G}_a$. We are going to investigate regular algebraic functions on $\mathcal{Y}$ and $\mathcal{Y}_1$ that are, in the latter case, H-invariant functions on $\mathfrak{g}$.

We note that $[\mathfrak{g}/G]$ is the classifying stack of pairs (V, x) where V is a n-dimensional vector space and $x \in \mathrm{End}(V)$. Let $\mathbf{a}_i : \mathfrak{g} \to \mathbb{G}_a$ denote the functions

$$\mathbf{a}_i(x) = \mathrm{tr}(\wedge^i x)$$

for $i = 1, \ldots, n$. The functions $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are G-invariants. In other words, we have a map

$$\mathbf{a} : [\mathfrak{g}/G] \to \mathfrak{a} = \mathrm{Spec}(k[\mathbf{a}_1, \ldots, \mathbf{a}_n]). \tag{4.1.1}$$

We also denote $\mathbf{a} : \mathcal{Y} \to \mathfrak{a}$ the induced map obtained by composing the forgetting morphism $\mathcal{Y} \to [\mathfrak{g}/G]$ with $\mathbf{a} : [\mathfrak{g}/G] \to \mathfrak{a}$.

We consider the vector bundle of rank n over $\mathfrak{a}$ whose fiber over $a \in \mathfrak{a}(k)$ is

$$\mathfrak{r}_a = k[\mathbf{x}]/(\mathbf{x}^n - a_1\mathbf{x}^{n-1} + \cdots + (-1)^n a_n). \tag{4.1.2}$$

We observe that $\mathfrak{r}$ can be trivialized, as a vector bundle over $\mathfrak{a}$, by means of the sections $1, \mathbf{x}, \ldots, \mathbf{x}^{n-1}$.

Let $\mathfrak{b}$ denote the dual vector bundle of $\mathfrak{r}$. For every $a \in \mathfrak{a}(k)$, the fiber of $\mathfrak{b}$ over a is $\mathfrak{b}_a = \mathrm{Hom}(\mathfrak{r}_a, k)$. We observe that $\mathfrak{b}_a$ is equipped with a structure of $\mathfrak{r}_a$-module, and since $\mathfrak{r}_a$ is Gorenstein, $\mathfrak{b}_a$ is a free $\mathfrak{r}_a$-module of rank one. As a vector bundle over $\mathfrak{a}$, $\mathfrak{b}$ can also be trivialized

$$\mathfrak{b} = \mathrm{Spec}(k[\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1}]) = \mathbb{A}^{2n}$$

by means of the basis that is dual to the basis $1, \mathbf{x}, \ldots, \mathbf{x}^{n-1}$ of $\mathfrak{r}$. With these coordinates we will identify a point of $\mathfrak{b}$ with a pair (a, b) where $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ and $b = (b_0, \ldots, b_{n-1}) \in \mathbb{A}^n$.

We consider the map of vector bundles over $\mathfrak{b}$

$$\gamma : \mathfrak{r} \times_{\mathfrak{a}} \mathfrak{b} \to \mathfrak{b} \times_{\mathfrak{a}} \mathfrak{b} \tag{4.1.3}$$

given by $(r, b) \mapsto (rb, b)$ given by the structure of $\mathfrak{r}_a$-module of $\mathfrak{b}_a$ for every $a \in \mathfrak{a}$. Its determinant defines a divisor of $\mathfrak{b}$. Since both $\mathfrak{r}$ and $\mathfrak{b}$ are trivial vector bundles, the determinant divisor is principal and can be given by a regular function

$$\det(\gamma) \in k[\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1}].$$

Let $\mathfrak{b}^{\mathrm{reg}}$ denote the complement of this divisor. By construction $(a, b) \in \mathfrak{b}^{\mathrm{reg}}$ if and only if b is a generator of $\mathfrak{b}_a$ as $\mathfrak{r}_a$-module.

We have a morphism

$$\mathbf{b} : \mathcal{Y} \to \mathfrak{b} \tag{4.1.4}$$

mapping $\mathbf{b}(V, x, v, v^\vee) = (a, b)$ defined as follows. We set $a = \mathbf{a}(V, x)$. We set $b \in \mathfrak{b}_a$ to be the linear form $b : \mathfrak{r}_a \to k$ defined by

$$r \mapsto \langle v^\vee, rv \rangle.$$

In terms of coordinates, we have

$$\mathbf{b}_j(V, x, v, v^\vee) = \langle v^\vee, x^j v \rangle \tag{4.1.5}$$

for all $j = 0, 1, \ldots, n-1$.

Next we will need next a piece of linear algebra:

Lemma 4.1.6. *Over $\mathcal{Y}$, the morphism of vector bundles $\gamma \times_{\mathfrak{b}} \mathcal{Y}$ factors as the composition of linear maps of vector bundles over $\mathcal{Y}$:*

$$\mathfrak{r} \times_{\mathfrak{a}} \mathcal{Y} \xrightarrow{c_v} \mathcal{V} \xrightarrow{c_{v^\vee}} \mathfrak{b} \times_{\mathfrak{a}} \mathcal{Y}$$

where

- *$\mathcal{V}$ is the vector bundle over $\mathcal{Y}$ whose fiber over a point $(V, x, v, v^\vee)$ is V;*
- *over a point $y = (V, x, v, v^\vee) \in \mathcal{Y}$ with $a = \mathbf{a}(y)$, $c_v : \mathfrak{r}_a \to V$ is the linear map given by $r \mapsto rv$;*

- *over a point* $y = (V, x, \nu, \nu^\vee) \in \mathcal{Y}$ *with* $a = \mathbf{a}(y)$, $c_{\nu^\vee} : V \to \mathfrak{b}_a$ *is dual to the map* $c_{\nu^\vee} : \mathfrak{r}_a \to V^\vee$ *given by* $r \mapsto r\nu^\vee$.

In particular, the divisor $\mathbf{b}^{-1}(\mathrm{div}(\det(\gamma)))$ *is the union of the divisors of* $\det(c_\nu)$ *and* $\det(c_{\nu^\vee})$ *on* $\mathcal{Y}$.

Proof. Let $y = (V, x, \nu, \nu^\vee) \in \mathcal{Y}$ and $r \in \mathfrak{r}_a$ with $a = \mathbf{a}(V, x)$. We need to prove that the equality

$$r\mathbf{b}(y) = c_{\nu^\vee}(c_\nu(r))$$

holds in $\mathfrak{b}_a$. By inspecting formulas, one can see that both sides represent the linear form on $\mathfrak{r}_a$:

$$r' \mapsto \langle \nu^\vee, r' r \nu \rangle$$

therefore define the same element of $\mathfrak{b}_a$. □

We now observe that there is a section of (4.1.4)

$$\mathbf{k} : \mathfrak{b} \to \mathcal{Y}$$

mapping $(a, b) \in \mathfrak{b}$ to the quadruple $(V, x, \nu, \nu^\vee)$ with $V = \mathfrak{r}_a$ as in (4.1.2), $x \in \mathrm{End}_k(\mathfrak{r}_a)$ is the multiplication by $\mathbf{x}$, $\nu_0 = 1$ and $\nu_0^\vee = b \in \mathfrak{b}_a$.

Proposition 4.1.7. *The morphism* (4.1.4) *induces an isomorphism over* $\mathfrak{b}^{\mathrm{reg}}$.

Proof. We only need to prove that if $(V, x, \nu, \nu^\vee)$ is points of $\mathcal{Y}$ over $(a, b) \in \mathfrak{b}^{\mathrm{reg}}$ then there exists a unique isomorphism between $\mathbf{k}(a, b)$ and $(V, x, \nu, \nu^\vee)$.

Since $\det(\gamma_{a,b}) \neq 0$, $\gamma_{a,b} : \mathfrak{r}_a \to \mathfrak{b}_a$ is an isomorphism. Since $\gamma_{a,b} : \mathfrak{r}_a \to \mathfrak{b}_a$ factors through $c_\nu : \mathfrak{r}_a \to V$, c_ν is injective. For dimension reason, we infer that c_ν is bijective. By construction c_ν maps $a \in \mathfrak{r}_a$ to $\nu \in V$, and the inverse of its dual maps b to $\nu^\vee$. This map induces the unique isomorphism between $\mathbf{k}(a, b)$ and $(V, x, \nu, \nu^\vee)$. □

We also observe that by restriction, we obtain a morphism

$$\mathbf{b} : \mathcal{Y}_1 \to \mathfrak{b}_1 \tag{4.1.8}$$

where $\mathfrak{b}_1$ is the subscheme of $\mathfrak{b}$ defined by the equation $\mathbf{b}_0 = 1$. By restricting $\mathbf{k}$ to $\mathfrak{b}_1$, we obtain a section of (4.1.8). We will denote by $f : \mathfrak{g} \to \mathfrak{b}_1$ the morphism induced from $\mathbf{b} : [\mathfrak{g}/H] \to \mathfrak{b}_1$. Let $\mathfrak{g}^{H-\mathrm{reg}}$ the open subset of $\mathfrak{g}$ defined as the pre image of $\mathfrak{b}_a^{\mathrm{reg}}$, a matrix $x \in \mathfrak{g}^{H-\mathrm{reg}}$ will be said H-regular. By the above proposition, the morphism

$$f^{\mathrm{reg}} : \mathfrak{g}^{H-\mathrm{reg}} \to \mathfrak{b}_1^{\mathrm{reg}}$$

is a H-principal bundle.

4.2. Untwisted integrals Let $\mathcal{O} = k[[t]]$ denote the ring of formal series in the variable t, $F = k((t))$ its quotient field. For every $(a, b) \in \mathfrak{b}^{\mathrm{reg}}(F)$, the fiber of $\mathfrak{g} \to \mathfrak{b}$ over (a, b) is a H-principal homogenous space over F. Since $H^1(F, H) = 0$, $f^{-1}(a, b)(F)$ is non empty.

Let $x \in \mathfrak{g}^{H-\mathrm{reg}}(F)$ mapping to $(a,b) \in \mathfrak{b}^{\mathrm{reg}}(F)$. For every function $\phi \in C_c^\infty(\mathfrak{g}(F))$, we consider the integral

$$\mathrm{JR}(x) = \int_{H(F)} \phi(h^{-1}xh)dh. \tag{4.2.1}$$

where dh is the Haar measure on $H(F)$ for which $H(\mathcal{O})$ has volume one. Since the map $h \mapsto h^{-1}xh$ induces a homeomorphism from $H(F)$ to the fiber of $f : \mathfrak{g}(F) \to \mathfrak{b}_1(F)$ over (a,b), the preimage of every compact subset of $\mathfrak{g}(F)$ is a compact subset of $H(F)$. We infer the convergence of the integral (4.2.1).

We will restrict ourselves to the case $\phi = \mathbb{I}_{\mathfrak{g}(\mathcal{O})}$. In that case the integral (4.2.1) is the cardinal of the set

$$N_x = \{h \in H(F)/H(\mathcal{O}) \mid h^{-1}xh \in \mathfrak{g}(\mathcal{O})\}$$

which is necessarily finite by the above compactness argument. We also observe that this set is empty unless $(a,b) \in \mathfrak{b}_1(\mathcal{O})$.

Because H acts freely on $\mathfrak{g}^{H-\mathrm{reg}}$, the map $h \mapsto h^{-1}xh$ defines a canonical bijection from N_x on the set $N_{a,b}$ of $H(\mathcal{O})$-orbits in the set of $y \in \mathfrak{g}(\mathcal{O})$ such that $f(y) = (a,b)$. Given $(a,b) : \mathrm{Spec}(\mathcal{O}) \to \mathfrak{b}$ whose restriction to $\mathrm{Spec}(F)$ has image in $\mathfrak{b}^{\mathrm{reg}}$, $N_{a,b}$ is the space of maps

$$y : \mathbb{D} \to \mathcal{Y}$$

lying over $(a,b) : \mathbb{D} \to \mathfrak{b}$.

Now we will give a more concrete description of $\mathcal{N}_{a,b}$.

Proposition 4.2.2. *Let $(a,b) \in \mathfrak{b}(\mathcal{O}) \cap \mathfrak{b}^{\mathrm{reg}}(F)$. Then there is a canonical bijection between $N_{a,b}$ with $\mathfrak{r}_a$-submodule $\mathcal{V}$ of $\mathfrak{b}_a$ such that*

$$\gamma_{a,b}(\mathfrak{r}_a) \subset \mathcal{V} \subset \mathfrak{b}_a. \tag{4.2.3}$$

Proof. Let $(a,b) \in \mathfrak{b}(\mathcal{O}) \cap \mathfrak{b}^{\mathrm{reg}}(F)$. By definition, the $\mathcal{O}$-linear morphism $\gamma_{a,b} : \mathfrak{r}_a \to \mathfrak{r}_a^\vee$ becomes after tensorization with F. It follows that $\gamma_{a,b} : \mathfrak{r}_a \to \mathfrak{b}_a^\vee$ is injective.

A morphism $y : \mathbb{D} \to \mathcal{Y}$ consists of a quadruple $(\mathcal{V}, x, v, v^\vee)$ with $\mathcal{V}$ being a $\mathcal{O}$-module free of rank n, $x \in \mathrm{End}(\mathcal{V})$, $v \in \mathcal{V}$ and $v^\vee \in \mathcal{V}^\vee$. Assume that y maps to $(a,b) \in \mathfrak{b}(\mathcal{O}) \cap \mathfrak{b}^{\mathrm{reg}}(F)$. By Lemma 4.1.6, $\gamma_{a,b} : \mathfrak{r}_a \to \mathfrak{b}_a^\vee$ factorizes as the composition of two maps

$$\mathfrak{r}_a \xrightarrow{c_v} \mathcal{V} \xrightarrow{c_{v^\vee}} \mathfrak{b}_a.$$

Because the $\gamma_{a,b} \otimes F$ is an isomorphism, and $\mathfrak{r}_a$, V, and $\mathfrak{b}_a$ are all of rank n, both $c_V \otimes F$ and $c_{v^\vee} \otimes F$ are isomorphism. Thus $\mathcal{V}$ can be identified with its image in $\mathfrak{b}_a$ that is a $\mathfrak{r}_a$-lattice satisfying the relation of inclusion (4.2.3).

Conversely let $\mathcal{V}$ be a $\mathfrak{r}_a$-lattice satisfying the relation of inclusion (4.2.3). We note $\alpha : \mathfrak{r}_a \to \mathcal{V}$ the map induced by the inclusion $\gamma_{a,b}(\mathfrak{r}_a) \subset \mathcal{V}$. Then we can construct the quadruple $(\mathcal{V}, x, v, v^\vee)$ by setting x to be the endomorphism of $\mathcal{V}$ given by the action of $\mathbf{x} \in \mathfrak{r}_a$, $v = \alpha(1)$ the image of $a \in \mathfrak{r}_a$, and $v^\vee = \alpha^\vee(b)$ the image of $b \in \mathfrak{b}_a$. □

4.3. Twisted integrals There are several twisted versions of the Jacquet-Rallis integrals (4.2.1). On the one hand, we consider the twisted orbital integral with respect to the unramified quadratic character $F^\times \to \{\pm 1\}$:

$$\mathrm{JR}^\eta(x) = \int_{H(F)} \mathbb{I}_{\mathfrak{g}(\mathcal{O})}(h^{-1}xh)\eta(\det(h))dh. \tag{4.3.1}$$

On the other hand, we can consider the unramified quadratic extension F'/F, and the associated quasi-split unitary group U_{n-1} acting on the space of $n \times n$ Hermitian matrices $\mathfrak{s}_n$:

$$\mathrm{JR}'(x') = \int_{U_{n-1}(F)} \mathbb{I}_{\mathfrak{s}_n(\mathcal{O})}(g^{-1}x'g)dg. \tag{4.3.2}$$

When x and x' match in the sense that they have the same image in $\mathfrak{b}(F)$, Jacquet and Rallis conjectured the equality

$$\mathrm{JR}^\eta(x) = \pm \mathrm{JR}'(x')$$

holds. This equality has been proven by Z. Yun, see [35] . We refer to loc. cit for more precision about the sign appearing in the equality.

4.4. Global model Prior to the construction of global model, we observe that there is an action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathcal{Y}$ defined by

$$(\alpha, \beta)(V, x, v, v^\vee) = (V, \alpha x, v, \beta v^\vee).$$

By formula (4.1) and (4.1.5), we have a compatible action of $\mathbb{G}_m{}^2$ on $\mathfrak{b}$ defined by the formula:

$$(\alpha, \beta)(a, b) = (\alpha a_1, \alpha^2 a_2, \ldots, \alpha^n a_n, \beta b_0, \alpha\beta b_1, \ldots, \alpha^{n-1}\beta b_{n-1}).$$

We derive a morphism of algebraic stacks:

$$[\mathcal{Y}/\mathbb{G}_m \times \mathbb{G}_m] \to [\mathfrak{b}/\mathbb{G}_m \times \mathbb{G}_m].$$

The global data will consist of a smooth projective curve C over k and two divisors D and E of large degrees. Let $\mathcal{O}(D)$ and $\mathcal{O}(E)$ denote the associated line bundles on C. We consider the space $\mathcal{B}$ of maps

$$(a, b) : C \to [\mathfrak{b}/\mathbb{G}_m \times \mathbb{G}_m]$$

lying over the map $C \to B(\mathbb{G}_m \times \mathbb{G}_m)$ given by the line bundles $\mathcal{O}(D)$ and $\mathcal{O}(E)$. By definition, $\mathcal{B}$ is the finite dimensional vector space:

$$\mathcal{B} = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}(iD)) \oplus \bigoplus_{j=0}^{n-1} H^0(C, \mathcal{O}(jD) \otimes \mathcal{O}(E)).$$

We consider the space $\mathcal{N}$ of maps

$$y : C \to [\mathcal{Y}/\mathbb{G}_m \times \mathbb{G}_m]$$

lying over the map $C \to B(\mathbb{G}_m \times \mathbb{G}_m)$ given by the line bundles $\mathcal{O}(D)$ and $\mathcal{O}(E)$. We have a a morphism

$$f : \mathcal{N} \to \mathcal{B}$$

induced by the $\mathbb{G}_m \times \mathbb{G}_m$-equivariant morphism $\mathcal{Y} \to \mathfrak{b}$. Let $\mathcal{B}'$ denote the open subset of $\mathcal{B}$ of those morphisms $C \to [\mathfrak{b}/\mathbb{G}_m \times \mathbb{G}_m]$ that map the generic point of C in $\mathcal{B}^{\text{reg}}$. We denote $\mathcal{N}'$ the preimage of $\mathcal{B}'$.

For every geometric point $(a, b) \in \mathcal{B}'(\bar{k})$, we will describe the fiber $\mathcal{N}_{a,b}$ of f over (a, b). Let C' denote the preimage of $\mathcal{B}^{\text{reg}}$ by the morphism $(a, b) : C \to [\mathfrak{b}/\mathbb{G}_m \times \mathbb{G}_m]$. Let $y : C \to [\mathcal{Y}/\mathbb{G}_m \times \mathbb{G}_m]$ be a point of $\mathcal{N}_{a,b}$. For the morphism $\mathcal{Y} \to \mathfrak{b}$ is an isomorphism over $\mathfrak{b}^{\text{reg}}$, y is completely determined over C'. It follows that y is completely determined by its restrictions y_v to the completions C_v of C at the points $v \in C - C'$. In other words, we have the product formula

$$\mathcal{N}_{a,b} = \prod_{v \in C - C'} \mathcal{N}_{a_v, b_v}.$$

Here $\mathcal{N}_{a_v,b_v}$ is the space of maps $C_v \to \mathcal{Y}$ lying over $(a_v, b_v) : C_v \to \mathfrak{b}$, where (a_v, b_v) are restriction of (a, b) to $\mathcal{V}$. The map (a_v, b_v) is well defined after we choose trivializations of the line bundles $\mathcal{O}(C)$ and $\mathcal{O}(D)$ restricted to C_v.

A more concrete description of $\mathcal{N}_{a,b}$ can be obtained by means of Lemma 4.1.6. Pulling back the vector bundle $\mathfrak{r}$ and $\mathfrak{b}$ by the map $a : C \to [\mathfrak{a}/\mathbb{G}_m]$, we get a vector bundles $\mathfrak{r}_a$ and $\mathfrak{b}_a$ over C. By pulling back γ by $(a, b) : C \to \mathfrak{b}$, we obtain a $\mathfrak{r}_a$-linear morphism

$$\gamma_{a,b} : \mathfrak{r}_a \to \mathfrak{b}_a.$$

which is generically an isomorphism. In particular $\gamma_{a,b}$ is injective as morphism of $\mathcal{O}_C$-modules with finite quotient $\mathfrak{b}_a/\gamma_{a,b}(\mathfrak{r}_a)$.

By the same argument as in Lemma 4.2.2, we can identify points of $\mathcal{N}_{a,b}$ with rank n vector bundle $\mathcal{V}$ over C, equipped with a structure of $\mathfrak{r}_a$-modules and with factorization of $\gamma_{a,b}$:

$$\mathfrak{r}_a \to \mathcal{V} \to \mathfrak{b}_a.$$

We can therefore identify $\mathcal{N}_{a,b}$ with the set of quotients Q of the finite $\mathfrak{r}_a$-module $\mathfrak{b}_a/\gamma_{a,b}(\mathfrak{r}_a)$. We note that the length of Q as $\mathfrak{r}_a$-module is

$$\lg(Q) = \deg(\mathfrak{b}_a) - \deg(\mathcal{V})$$

where $\deg(\mathfrak{b}_a)$ depends only on the degrees of the divisors D and E. It follows that $\mathcal{N}$ can be decomposed as disjoint union of open and closed subvarieties:

$$\mathcal{N} = \bigsqcup_s \mathcal{N}_s,$$

where $\mathcal{N}_s$ classifies maps $y : C \to [\mathcal{Y}/\mathbb{G}_m \times \mathbb{G}_m]$ corresponding to $(\mathcal{V}, x, v, v^\vee)$ such that $\deg(\mathcal{V}) = \deg(\mathfrak{b}_a) - s$.

The following statement is a theorem of Yun, [35, Prop. 3.5.2].

Theorem 4.4.1. *Assuming* $\deg(D)$ *and* $\deg(E)$ *are large with respect to* s. *Then the moduli space* $\mathcal{N}'_s$ *is smooth, and the morphism* $\mathcal{N}'_s \to \mathcal{B}'$ *is proper and small.*

We can now derive the perverse continuation principle for Theorem 2.5.1.

5. Adjoint action

5.1. Invariant theory Let G be a split reductive group over k acting on its Lie algebra $\mathfrak{g}$ by the adjoint action. The ring of G-invariant functions of $\mathfrak{g}$ is a polynomial ring

$$k[\mathfrak{g}]^G = k[\mathbf{a}_1, \dots, \mathbf{a}_n]$$

where $\mathbf{a}_1, \dots, \mathbf{a}_n$ are homogenous polynomials of degree $\nu_1, \dots, \nu_n$. Although there may be many choices of $\mathbf{a}_1, \dots, \mathbf{a}_n$, the integers $\nu_1 \leqslant \dots \leqslant \nu_n$, arranged in increasing order, are completely determined by $\mathfrak{g}$. For instant, when $G = GL_n$, we have $\nu_i = i$ and we may take

$$\mathbf{a}_i(x) = \mathrm{tr}(\wedge^i x)$$

but we may also take the invariant function $x \mapsto \mathrm{tr}(x^i)$ just as well.

Let $\mathfrak{g}^{\mathrm{reg}}$ denote the open subset of $\mathfrak{g}$ of regular elements i.e $x \in \mathfrak{g}$ such that the centralizer I_x is of dimension equal the rank n of $\mathfrak{g}$. By restricting the morphism

$$\mathbf{a} : \mathfrak{g} \to \mathfrak{a} = \mathrm{Spec}(k[\mathbf{a}_1, \dots, \mathbf{a}_n]).$$

to the open subset $\mathfrak{g}^{\mathrm{reg}}$ of regular elements, we obtain a smooth surjective morphism

$$\mathbf{a}^{\mathrm{reg}} : \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{a}$$

whose fibers are G-homogenous spaces. There exists an open subset $\mathfrak{a}^{\mathrm{rss}}$ of $\mathfrak{a}$ consisting of regular semisimple adjoint orbits. Over $\mathfrak{a}^{\mathrm{rss}}$, the subset $\mathfrak{g}^{\mathrm{reg}}$ coincide with $\mathfrak{g}$ i.e. $\mathbf{a}^{-1}(\mathfrak{g}^{\mathrm{rss}}) \subset \mathfrak{g}^{\mathrm{reg}}$. After Kostant [19] there exists a section

$$\mathbf{k} : \mathfrak{a} \to \mathfrak{g}^{\mathrm{reg}}. \tag{5.1.1}$$

of $\mathbf{a}^{\mathrm{reg}}$.

Just as for Kloosterman and Jacquet-Rallis integrals, the theory of adjoint orbital integrals in Lie algebra can be reinterpreted as the geometry of the morphism of formal arc spaces associated with the map

$$[\mathbf{a}] : [\mathfrak{g}/G] \to \mathfrak{a}. \tag{5.1.2}$$

There are more difficulties in this case as $[\mathbf{a}]$ is not generically an isomorphism. Even over the "nice" open subset $\mathfrak{a}^{\mathrm{rss}}$, $[\mathfrak{g}/G]$ is only a gerbe bounded by the centralizer group scheme in the following sense.

Let us denote I the centralizer group scheme over $\mathfrak{g}$, whose fiber over $x \in \mathfrak{g}$ is

$$I_x = \{g \in G | \mathrm{ad}(g)x = x\}.$$

The restriction I^{reg} of I to $\mathfrak{g}^{\mathrm{reg}}$ is a G-equivariant smooth commutative group scheme. After [27, 3.1], I^{reg} descends to $\mathfrak{a}$ i.e. there exists a unique smooth group scheme $J \to \mathfrak{a}$, up to unique isomorphism, equipped with a G-equivariant isomorphism

$$(\mathbf{a}^{\mathrm{reg}})^* J \to I^{\mathrm{reg}}. \tag{5.1.3}$$

This is equivalent to saying that $[\mathfrak{g}^{\mathrm{reg}}/G]$ is a gerbe over $\mathfrak{a}$ bounded by J, or in other words, $[\mathfrak{g}^{\mathrm{reg}}/G]$ is a torsor over $\mathfrak{a}$ under the action of the relative classifying stack $B_{\mathfrak{a}}J$. Moreover, one can trivialize this gerbe by means of the Kostant section.

A crucial observation to be made here is that the isomorphism can be extended uniquely to a homomorphism of group schemes,

$$h : \mathbf{a}^* J \to I. \tag{5.1.4}$$

see [27, 3.2]. Following a suggestion of Drinfeld, we will reformulate this homomorphism as an action of $B_{\mathfrak{a}}J$ on $[\mathfrak{g}/G]$, extending its simply transitive action on $[\mathfrak{g}^{\mathrm{reg}}/G]$. The quotient

$$Q = [[\mathfrak{g}/G]/B_{\mathfrak{a}}J]. \tag{5.1.5}$$

of $[\mathfrak{g}/G]$ by the action of $B_{\mathfrak{a}}J$ is naturally a 2-stack. Evaluated over an algebraically closed field, Q is the 2-category whose objects are elements $x \in \mathfrak{g}$; the automorphisms of each object x is the Picard groupoid $\mathrm{Aut}_Q(x)$ whose objects are elements of I_x and for $g \in I_x$, the 2-automorphisms of g are elements $j \in J_a$ such that $h(j)g = g$ where $a = \mathbf{a}(x)$.

The morphism $\mathbf{a} : [\mathfrak{g}/G] \to \mathfrak{a}$ of (5.1.2) can be factorized through Q

$$[\mathfrak{g}/G] \to Q \to \mathfrak{a}. \tag{5.1.6}$$

where $[\mathfrak{g}/G] \to Q$ is a gerbe bounded by J. The restriction of $Q \to \mathfrak{a}$ to the open 2-substack

$$Q^{\mathrm{reg}} = [[\mathfrak{g}^{\mathrm{reg}}/G]/B_{\mathfrak{a}}J]$$

is an isomorphism by (5.1). Since $\mathfrak{g}^{\mathrm{reg}}$ coincides with $\mathfrak{g}$ over $\mathfrak{a}^{\mathrm{rss}}$, the morphism $Q \to \mathfrak{a}$ is an isomorphism over $\mathfrak{a}^{\mathrm{rss}}$. The morphism

$$q : Q \to \mathfrak{a} \tag{5.1.7}$$

plays a similar role to the morphism (4.1.4) in the Jacquet-Rallis case.

5.2. Stable orbital integrals We consider the space $C_c^\infty(\mathfrak{g}(F))$ of locally constant functions with compact support in $\mathfrak{g}(F)$. We will attempt to relate the (stable) orbital integrals for the adjoint action of $G(F)$ on $\mathfrak{g}(F)$ with the geometry of the morphism of formal arc spaces associated with (5.1.7).

Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. For every function $\phi \in C_c^\infty(\mathfrak{g}(F))$, we define the orbital integral

$$O_\gamma(\phi) = \int_{G_\gamma(F)\backslash G(F)} \phi(\mathrm{ad}(g)^{-1}\gamma)\frac{dg}{dj} \tag{5.2.1}$$

for given Haar measures dg and dj of $G(F)$ and $G_\gamma(F)$ respectively. To put orbital integrals in family, we will need to choose Haar measures on centralizers $G_\gamma(F)$, as γ varies, in a consistent way.

A provisional solution is to restrict ourselves to functions with support in $\mathfrak{g}(\mathcal{O})$ so that we only have to consider regular semisimple $\gamma \in \mathfrak{g}(F)$ such that

$$a = \mathbf{a}(\gamma) \in \mathfrak{a}(\mathcal{O}) \cap \mathfrak{a}^{\mathrm{rss}}(F).$$

Let $J_a = a^*J$ the inverse image of J by $a : \mathbb{D} \to \mathfrak{a}$, and J_a^0 its subgroup scheme of neutral components of J_a. The locally compact group $J_a(F)$ is now equipped with the compact open subgroups $J_a(\mathcal{O})$ and $J_a^0(\mathcal{O})$. We will normalize the Haar measure on $J_a(F)$ such that $J_a^0(\mathcal{O})$ has volume one. For $I_\gamma(F)$ and $J_a(F)$ are canonically isomorphic, we can transport the chosen Haar measure from $J_a(F)$ to $I_\gamma(F)$.

The difference between conjugation and geometric conjugation creates another problem. Let $\gamma, \gamma' \in \mathfrak{g}(F)$ be regular semisimple elements such that $\mathbf{a}(\gamma) = \mathbf{a}(\gamma') = a$ then there exists $g \in G(\bar{F})$ such that $\mathrm{ad}(g)\gamma = \gamma'$. However γ and γ' may not be conjugate by an element in $G(F)$. Let $\mathbf{h}_a$ denote the set of $G(F)$-orbits in $\mathbf{a}^{-1}(a)(F)$. A cocycle calculation shows that $\mathbf{h}_a$ is a principal homogenous space under the finite abelian group

$$\ker[H^1(F, J_a) \to H^1(F, G)], \tag{5.2.2}$$

and in particular, it is a finite set. The sum of orbital integrals within the finite set $\mathbf{h}_a$ will be called the stable orbital integral:

$$\mathrm{SO}_a(\phi) = \sum_{\gamma \in \mathbf{h}_a} \mathrm{O}_\gamma(\phi). \tag{5.2.3}$$

The stable orbital integral $\mathrm{SO}_a(\phi)$ is the integration of ϕ along the fiber $\mathbf{a}^{-1}(a)$. For every $\phi \in C_c^\infty(\mathfrak{g}(\mathcal{O}))$, the stable orbital integral of ϕ can be regarded as a function on $\mathfrak{a}(\mathcal{O})$:

$$a \mapsto \mathrm{SO}_a(\phi). \tag{5.2.4}$$

We will now narrow down to the basic case $\phi = \mathbb{I}_{\mathfrak{g}(\mathcal{O})}$. We will express the stable orbital integral (5.2.3) as the mass of certain groupoid of formal arcs.

For every $a \in \mathfrak{a}(\mathcal{O}) \cap \mathfrak{a}^{\mathrm{rss}}(F)$, we consider the 2-category Q_a of maps $\mathbb{D} \to Q$ lying over a. Objects of Q_a are maps $x : \mathbb{D} \to \mathfrak{g}$ lying over a. Morphisms between $x_1, x_2 : \mathbb{D} \to \mathfrak{g}$ are $g \in G(\mathcal{O})$ such that $\mathrm{ad}(g)x_1 = x_2$; 2-morphisms between $g_1, g_2 : x_1 \to x_2$ are $j \in J_a(\mathcal{O})$ such that $g_1 h(j) = g_2$ where $h : J_a(\mathcal{O}) \to I_{x_1}(\mathcal{O})$ is defined in (5.1.4). The mass of Q_a is defined to be

$$\#Q_a = \sum_x \frac{1}{\#\mathrm{Aut}(x)} \tag{5.2.5}$$

where x ranges over the set of isomorphism classes of Q_a, and $\#\mathrm{Aut}(x)$ is the mass of the groupoid $\mathrm{Aut}(x)$. Under the assumption $a \in \mathfrak{a}(\mathcal{O}) \cap \mathfrak{a}^{\mathrm{rss}}(F)$, for every $x \in \mathfrak{g}(\mathcal{O})$ lying over $a \in \mathfrak{a}(\mathcal{O})$, the homomorphism $h : J_a(\mathcal{O}) \to I_x(\mathcal{O})$ is injective. For simplicity, assume that $J_a(\mathcal{O})$ is connected, then

$$\#\mathrm{Aut}(x) = \#(I_x(\mathcal{O})/J_a(\mathcal{O})). \tag{5.2.6}$$

and it follows that

$$\mathrm{SO}_a(I_{\mathfrak{g}(\mathcal{O})}) = \#Q_a \tag{5.2.7}$$

so that $\mathrm{SO}_a(\phi)$ is the mass of the 2-groupoid of maps $\mathbb{D} \to Q$ lying over $a \in \mathfrak{a}(\mathcal{O}) \cap \mathfrak{a}^{\mathrm{rss}}(F)$. When $J_a(\mathcal{O})$ is not connected, the mass calculation is more involved, see [30, 8.2], but the above formula also holds in that case. Therefore the stable orbital integral function (5.2.4) expresses the mass of the fibers of the formal arc spaces of Q over the formal arc space of $\mathfrak{a}$.

5.3. Waldspurger's nonstandard fundamental lemma If G_1, G_2 are reductive groups with isogenous root data, there is a canonical isomorphism $\mathfrak{a}_1 \xrightarrow{\sim} \mathfrak{a}_2$ [30, 1.12.6]. Waldspurger conjectured that for $a_1 \in \mathfrak{a}_1(\mathcal{O}) \cap \mathfrak{a}_1^{\mathrm{rss}}(F)$ and $a_2 \in \mathfrak{a}_2(\mathcal{O}) \cap \mathfrak{a}_2^{\mathrm{rss}}(F)$ corresponding one to each other via the isomorphism $\mathfrak{a}_1 \xrightarrow{\sim} \mathfrak{a}_2$, the equality of stable orbital integrals

$$\mathrm{SO}_{a_1}(\mathrm{I}_{\mathfrak{g}_1(\mathcal{O})}) = \mathrm{SO}_{a_2}(\mathrm{I}_{\mathfrak{g}_2(\mathcal{O})}) \tag{5.3.1}$$

holds. This identity is known as the non-standard fundamental lemma for Lie algebra. After (5.2.7), this is equivalent to an identity of masses

$$\#Q_{1,a_1} = \#Q_{2,a_2}. \tag{5.3.2}$$

where $Q_1 = [[\mathfrak{g}_1/G_1]/B_{\mathfrak{a}_1}J_1]$ and $Q_2 = [[\mathfrak{g}_2/G_2]/B_{\mathfrak{a}_2}J_2]$. This identity is non obvious to the extent that Q_1 and Q_2 bear no direct geometric relation.

5.4. Global model Let C be a smooth quasi-projective curve over k. For each map $a : C \to \mathfrak{a}$ that sends the generic point of C into the open subspace $\mathfrak{a}^{\mathrm{rss}}$, we consider the stack Q_a of maps $x : C \to Q$ lying over a. If $C' = a^{-1}(\mathfrak{a}^{\mathrm{rss}})$, then the restriction of x to C' is completely determined by $a' = a|_{C'}$ as $Q \to \mathfrak{a}$ is an isomorphism over $\mathfrak{a}^{\mathrm{rss}}$. It follows that a is completely determined by its restriction to the completion C_v of C at the places $v \in |C - C'|$. In other words, we have the product formula

$$Q_a = \prod_{v \in |C - C'|} Q_{a_v} \tag{5.4.1}$$

where Q_{a_v} is the stack of maps $x_v : C_v \to Q$ lying over $a_v : C_v \to \mathfrak{a}$ that is the restriction of a to C_v.

If C is a projective curve and $\mathfrak{a}$ is affine, all maps $a : C \to \mathfrak{a}$ are constant. There are thus not enough global maps $a : C \to \mathfrak{a}$ to approximate a given local map $a_v : C_v \to \mathfrak{a}$. The standard remedy to this failure is to allow $a : C \to \mathfrak{a}$ having poles of degree bounded by a large positive divisor, or in other words, to twist $\mathfrak{a}$ by an ample line bundle over C.

The homothety action of $\mathbb{G}_m$ on $\mathfrak{g}$ induces a compatible action on the invariant quotient $\mathfrak{a}$. This action can be lifted in an obvious way to the centralizer group scheme $I \to \mathfrak{g}$ and therefore induces an action of $\mathbb{G}_m$ on the regular centralizer group scheme $J \to \mathfrak{a}$. We deduce an action of $\mathbb{G}_m$ on the Drinfeld 2-stack Q defined in (5.1.5).

Let C be a smooth projective curve, and $\mathcal{L}$ a line bundle over C. For every $a : C \to [\mathfrak{a}/\mathbb{G}_m]$ over $\mathcal{L} : C \to B\mathbb{G}_m$ such that a maps the generic point of C into $[\mathfrak{a}^{\mathrm{rss}}/\mathbb{G}_m]$, we consider the stack Q_a of maps $x : C \to [Q/\mathbb{G}_m]$ lying over a. The product formula (5.4.1) holds with Q_{a_v} being the stack of maps $x_v : C_v \to [Q/\mathbb{G}_m]$ lying over $a_v : C_v \to [\mathfrak{a}/\mathbb{G}_m]$ that is the restriction of a to C_v.

If we denote $\mathcal{A}_{\mathcal{L}}$ the space of maps $a : C \to [\mathfrak{a}/\mathbb{G}_m]$ lying over $\mathcal{L} : C \to B\mathbb{G}_m$ and $\mathcal{Q}_{\mathcal{L}}$ the "space" of maps $x : C \to [Q/\mathbb{G}_m]$ also lying over $\mathcal{L}$, then for every $a \in \mathcal{A}_{\mathcal{L}}$, Q_a is the fiber of $\mathcal{Q}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$ over a.

Let G_1, G_2 be reductive groups with isogenous root data. It can be checked that for G_1, G_2 with isogenous root data, there is an isomorphism $\mathcal{A}_{1,\mathcal{L}} = \mathcal{A}_{2,\mathcal{L}}$. If the points $a_1 \in \mathcal{A}_{1,\mathcal{L}}$ and $a_2 \in \mathcal{A}_{2,\mathcal{L}}$ correspond via this isomorphism then the identity of local masses (5.3.2) implies the identity of masses

$$\#Q_{1,a_1} = \#Q_{2,a_2} \tag{5.4.2}$$

of fibers of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ over a_1 and a_2 respectively.

5.5. The Hitchin fibration Although the formulation of (5.4.2) involves only the morphism $Q \to \mathfrak{a}$, in order to prove it, it seems necessary to take into account the 2-stages morphism (5.1.6) $[\mathfrak{g}/G] \to Q \to \mathfrak{a}$. We consider the "space" $\mathcal{M}_{\mathcal{L}}$ of maps $m : C \to [\mathfrak{g}/G \times \mathbb{G}_m]$ lying over $\mathcal{L} : C \to B\mathbb{G}_m$. We then have morphisms of spaces of maps

$$\mathcal{M}_{\mathcal{L}} \to \mathcal{Q}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}. \tag{5.5.1}$$

For $\mathcal{L}$ being the canonical bundle, the morphism $f : \mathcal{M}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$ is essentially the Hitchin fibration [15], an algebraic completely integrable system. For an arbitrary line bundle of large degree, the generic fiber of $\mathcal{M}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$ is essentially an abelian variety i.e it is an abelian variety after deleting the component and automorphism groups.

For every $a \in \mathcal{A}_{\mathcal{L}}$, we denote $\mathcal{P}_a$ the Picard stack of J_a-torsors where $J_a = a^* J$ is the pullback of the regular centralizer group scheme $J \to \mathfrak{a}$ via $a : C \to [\mathfrak{a}/\mathbb{G}_m]$. The action of $B_{\mathfrak{a}} J$ on $[\mathfrak{g}/G]$ gives rise to an action of $\mathcal{P}_a$ on $\mathcal{M}_a$ where $\mathcal{M}_a$ is the fiber of $\mathcal{M}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$ over a. One can check that $Q_a = [\mathcal{M}_a/\mathcal{P}_a]$. If a is the generic point then $\mathcal{P}_a$ is an abelian variety, up to a component group and an automorphism group, and it acts on $\mathcal{M}_a$ simply transitively. In particular Q_a is reduced to a point at the generic point a.

If we assume that G is a semisimple group, then there exists an open subset $\mathcal{A}_{\mathcal{L}}^{\mathrm{ell}}$ of $\mathcal{A}_{\mathcal{L}}$ such that over $\mathcal{A}_{\mathcal{L}}^{\mathrm{ell}}$, the morphism $f^{\mathrm{ell}} : \mathcal{M}_{\mathcal{L}}^{\mathrm{ell}} \to \mathcal{A}_{\mathcal{L}}^{\mathrm{ell}}$ is a proper morphism and $\mathcal{M}_{\mathcal{L}}^{\mathrm{ell}}$ is a smooth Deligne-Mumford stack. It follows from Deligne's purity theorem that the perverse cohomology ${}^pH^i(f_*^{\mathrm{ell}}\mathbf{Q}_\ell)$ are pure perverse sheaves, and therefore geometrically semisimple.

The restriction $\mathcal{P}^{\mathrm{ell}}$ of $\mathcal{P}_{\mathcal{L}}$ is to $\mathcal{A}_{\mathcal{L}}$ is of finite type. It acts on ${}^pH^i(f_*^{\mathrm{ell}}\mathbf{Q}_\ell)$. We denote ${}^pH^i(f_*^{\mathrm{ell}}\mathbf{Q}_\ell)^{\mathrm{st}}$ the maximal direct factor where $\mathcal{P}^{\mathrm{ell}}$ acts trivially. We expect that ${}^pH^i(f_*^{\mathrm{ell}}\mathbf{Q}_\ell)^{\mathrm{st}}$ is completely determined by its generic fiber via the intermediate extension functor. This is the content of what we called the "support theorem" in the expository paper [31]. In that paper, we proved the support theorem under the assumption that the characteristic of the base field k is zero. In the case of positive characteristic, we were able to prove the support theorem only after restricting ${}^pH^i(f_*^{\mathrm{ell}}\mathbf{Q}_\ell)^{\mathrm{st}}$ to a smaller open subset of $\mathcal{A}^{\mathrm{ell}}$. This is not very satisfying but it is enough to derive the fundamental lemma [30]. Significant progress has been made toward extending the domain of validity of the support theorem, in particular beyond the elliptic locus by Chaudouard and Laumon [3], [2], by Migliorini, Shende and Viviani [23], and by de Cataldo [7].

If G_1 and G_2 are semisimple groups with isogenous root data, one can prove that the generic fibers of $\mathcal{M}_{1,\mathcal{L}} \to \mathcal{A}_{1,\mathcal{L}}$ and $\mathcal{M}_{2,\mathcal{L}} \to \mathcal{A}_{2,\mathcal{L}}$ are isogenous abelian varieties, up to the component groups and automorphism groups. It follows that ${}^pH^i(f^{ell}_{1,*}\mathbf{Q}_\ell)^{st}$ and ${}^pH^i(f^{ell}_{2,*}\mathbf{Q}_\ell)^{st}$ have isomorphic generic fiber. Assuming that the support theorem be valid over $\mathcal{A}^{ell}_{\mathcal{L}}$, we derive that ${}^pH^i(f^{ell}_{1,*}\mathbf{Q}_\ell)^{st}$ and ${}^pH^i(f^{ell}_{2,*}\mathbf{Q}_\ell)^{st}$ are isomorphic perverse sheaves. In particular, this is valid when $k = \mathbb{C}$. When k in finite field, we proved in [30] that ${}^pH^i(f^{ell}_{1,*}\mathbf{Q}_\ell)^{st}$ and ${}^pH^i(f^{ell}_{2,*}\mathbf{Q}_\ell)^{st}$ are isomorphic in some open subset of $\mathcal{A}^{ell}_{\mathcal{L}}$ that has enough points so that one can derive (5.3.2). It is desirable to find a proof the support theorem over $\mathcal{A}^{ell}_{\mathcal{L}}$ in positive characteristic in order to streamline the global to local argument.

5.6. The Langlands-Shelstad fundamental lemma The proof of the Langlands-Shelstad fundamental lemma follows essentially the same route as Waldspurger's nonstandard fundamental lemma. It is nonetheless considerably more complicated because of the presence of κ-orbital integrals.

The letter κ in κ-orbital integral refers to a character of the finite group (5.2.2):

$$\kappa : \ker[H^1(F, J_a) \to H^1(F, G)] \to \mathbf{C}^\times. \tag{5.6.1}$$

This finite group acts simply transitively on the finite set $\mathbf{h}_a$ of $G(F)$-conjugacy classes in the set of F-points on the fiber $\mathbf{a}^{-1}(a)$ of $\mathfrak{g} \to \mathfrak{a}$ over $a \in \mathfrak{a}(F)$. The Kostant section provides a convenient base point $\mathbf{k}(a) \in \mathbf{a}^{-1}(a)(F)$ and thus an identification of the finite set $\mathbf{h}_a$ and the finite group (5.2.2):

$$\mathrm{inv} : \mathbf{h}_a \to \ker[H^1(F, J_a) \to H^1(F, G)].$$

We define the κ-orbital integral attached to a as the linear combination of orbital integrals in $\mathbf{a}^{-1}(a)(F)$ weighted by the values of κ:

$$O^\kappa_a(\phi) = \sum_{\gamma \in \mathbf{h}_a} \kappa(\mathrm{inv}(\gamma)) O_\gamma(\phi). \tag{5.6.2}$$

One can attach to the pair (a, κ) an endoscopic group H and a stable conjugacy class a_H in the Lie algebra $\mathfrak{h}$ of H. The Langlands-Shelstad fundamental lemma asserts an equality between $O^\kappa_a(\phi)$ and $SO_{a_H}(\phi_H)$ where ϕ and ϕ_H are respectively the characteristic functions of $\mathfrak{g}(\mathcal{O})$ and $\mathfrak{h}(\mathcal{O})$ up to a power of q. We refer to the introduction of [30] for a precise statement of this equality.

We now introduce a new ingredient, the affine Springer fiber, which is necessary for a geometric interpretation of the κ-orbital integral similar to the stable orbital integral. For a more throughout, and more intuitive, discussion of the affine Springer fiber, we refer to the lecture notes of Yun in this volume.

For every $a \in \mathfrak{a}(\mathcal{O})$, we denote $\mathcal{M}^\bullet_a$ the space of maps $x : \mathbb{D} \to [\mathfrak{g}/G]$, lying over $a : \mathbb{D} \to \mathfrak{a}$, and equipped with an isomorphism $x \to \mathbf{k}(a)$ over the punctured disc $\mathbb{D}^\bullet$, $\mathbf{k}$ being the Kostant section. It can be proven that the reduced space associated with $\mathcal{M}_a^\bullet$ is an algebraic variety locally of finite type, usually known as the affine Springer fiber.

Our affine Springer fiber $\mathcal{M}_a^\bullet$ is acted on by the group $\mathcal{P}_a^\bullet$ classifying J_a-torsors over $\mathbb{D}$ equipped with a trivialization over the $\mathbb{D}^\bullet$. In other words $\mathcal{P}_a^\bullet$ is the space of maps $p : \mathbb{D} \to B_{\mathfrak{a}}J$ over $a : \mathbb{D} \to \mathfrak{a}$ equipped with an isomorphism over the punctured disc between p and the neutral map $p_0 : \mathbb{D} \to B_{\mathfrak{a}}J$ corresponding to the trivial J-torsor. We have an action of $\mathcal{P}_a^\bullet$ on $\mathcal{M}_a^\bullet$ derived from the "universal" action of $B_{\mathfrak{a}}J$ on $[\mathfrak{g}/G]$.

The stack Q_a of maps $x : \mathbb{D} \to Q$ lying over $a : \mathbb{D} \to \mathfrak{a}$ can be presented as the quotient of $\mathcal{M}_a^\bullet$ by $\mathcal{P}_a^\bullet$:

$$Q_a = [\mathcal{M}_a^\bullet/\mathcal{P}_a^\bullet]. \tag{5.6.3}$$

It can be shown that the character κ of (5.6.1) defines a homomorphism $\kappa : B\mathcal{P}_a^\bullet(k)/\sim \to \mathbf{C}^\times$. It follows that the κ-orbital integral for ϕ being the characteristic function of $\mathfrak{g}(\mathcal{O})$ can be expressed as κ-weighted mass

$$O_a^\kappa(\phi) = \#Q_a^\kappa(k) = \sum_{x \in Q_a/\sim} \frac{\kappa(\mathrm{cl}(x))}{\#\mathrm{Aut}(x)} \tag{5.6.4}$$

where x ranges over the set of isomorphism classes of Q_a, $\mathrm{cl}(x)$ is the image if x is the $\mathcal{P}_a^\bullet(k)/\sim$ in the group of isomorphism classes of $\mathcal{P}_a$-torsors over k.

The κ-weighted mass $\#Q_a^\kappa(k)$ appears as local factor of an endoscopic part of the relative cohomology of the Hitchin fibration. As in the case of the nonstandard fundamental lemma where the key geometric ingredient is the determination of the support of simple perverse sheaves occurring in the stable part of the cohomology of the Hitchin fibration, in the endoscopic case, we need to determine the support of simple perverse sheaves occurring in endoscopic parts of the cohomology of the Hitchin fibration. In [27] we proved that the support is contained in the image of the Hitchin base of the corresponding endoscopic group. In [30], we proved that there is equality after certain restrictions. We won't discuss this matter further as it has already been the subject of expository papers [28] and [31].

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