Arithmetic of certain integrable systems

Ngô Bao Châu

University of Chicago &
Vietnam Institute for Advanced Study in Mathematics

System of congruence equations

- Let us consider a system of congruence equations

$$\begin{cases} P_1(x_1, \ldots, x_n) = 0 \\ \ldots \\ P_m(x_1, \ldots, x_n) = 0 \end{cases}$$

System of congruence equations

- Let us consider a system of congruence equations

$$\begin{cases} P_1(x_1, \dots, x_n) = 0 \\ \dots \\ P_m(x_1, \dots, x_n) = 0 \end{cases}$$

- where $P_1, \dots, P_m \in \mathbb{F}_p[x_1, \dots, x_n]$ are polynomial with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

System of congruence equations

- Let us consider a system of congruence equations

$$\begin{cases} P_1(x_1, \ldots, x_n) = 0 \\ \ldots \\ P_m(x_1, \ldots, x_n) = 0 \end{cases}$$

- where $P_1, \ldots, P_m \in \mathbb{F}_p[x_1, \ldots, x_n]$ are polynomial with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
- We are interested in the number of solutions of this system with in $\mathbb{F}_p$, and more generally in $\mathbb{F}_{p^r}$ where $\mathbb{F}_{p^r}$ is the finite extension of degree r of $\mathbb{F}_p$.

Valued points of algebraic variety

- If we denote $X = \mathrm{Spec}\mathbb{F}_p[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$, the algebraic variety defined by the system of equations

$$P_1 = 0, \ldots, P_m = 0,$$

then $X(\mathbb{F}_p^r)$ is the set of solutions with values in $\mathbb{F}_{p^r}$.

Valued points of algebraic variety

- If we denote $X = \mathrm{Spec}\mathbb{F}_p[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$, the algebraic variety defined by the system of equations

$$P_1 = 0, \ldots, P_m = 0,$$

then $X(\mathbb{F}_p^r)$ is the set of solutions with values in $\mathbb{F}_{p^r}$.

- Let $X(\bar{F}_p) = \bigcup_{r \in \mathbb{N}} X(\mathbb{F}_{p^r})$ be the set of points with values in the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$

Valued points of algebraic variety

- If we denote $X = \mathrm{Spec}\mathbb{F}_p[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$, the algebraic variety defined by the system of equations
$$P_1 = 0, \ldots, P_m = 0,$$
then $X(\mathbb{F}_p^r)$ is the set of solutions with values in $\mathbb{F}_{p^r}$.
- Let $X(\bar{F}_p) = \bigcup_{r\in\mathbb{N}} X(\mathbb{F}_{p^r})$ be the set of points with values in the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$
- The Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $X(\bar{\mathbb{F}}_p)$. It is generated by the Frobenius element $\sigma(x) = x^p$, and
$$\mathrm{Fix}(\sigma^r, X(\bar{\mathbb{F}}_p)) = X(\mathbb{F}_{p^r}).$$

Grothendieck-Lefschetz formula

- For a prime number $\ell \neq p$, Grothendieck defined the groups of ℓ-adic cohomology of X

$$\mathrm{H}^i(X) = \mathrm{H}^i(X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell) \text{ and } \mathrm{H}^i_c = \mathrm{H}^i_c(X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$$

  for every algebraic variety X over $\mathbb{F}_p$

Grothendieck-Lefschetz formula

- For a prime number $\ell \neq p$, Grothendieck defined the groups of ℓ-adic cohomology of X

$$\mathrm{H}^i(X) = \mathrm{H}^i(X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell) \text{ and } \mathrm{H}^i_c = \mathrm{H}^i_c(X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$$

  for every algebraic variety X over $\mathbb{F}_p$
- and proved the Lefschetz fixed points formula

$$\#\mathrm{Fix}(\sigma_p^r, X(\bar{\mathbb{F}}_p)) = \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{tr}(\sigma_p^r, \mathrm{H}^i_c(X)).$$

Grothendieck-Lefschetz formula

- For a prime number $\ell \neq p$, Grothendieck defined the groups of ℓ-adic cohomology of X

$$\mathrm{H}^i(X) = \mathrm{H}^i(X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell) \text{ and } \mathrm{H}^i_c = \mathrm{H}^i_c(X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$$

  for every algebraic variety X over $\mathbb{F}_p$
- and proved the Lefschetz fixed points formula

$$\#\mathrm{Fix}(\sigma_p^r, X(\bar{\mathbb{F}}_p)) = \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{tr}(\sigma_p^r, \mathrm{H}^i_c(X)).$$

- Deligne proved that for every field isomorphism $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$, the inequality $|\iota(\alpha)| \leq p^{i/2}$ for all eigenvalues α of σ acting on $\mathrm{H}^i_c(X)$.

Equality of numbers of points

- We will be concerned with proving equality of type

$$\#X(\mathbb{F}_{p^r}) = \#X'(\mathbb{F}_{p^r})$$

for different algebraic varieties.

Equality of numbers of points

- We will be concerned with proving equality of type

$$\#X(\mathbb{F}_{p^r}) = \#X'(\mathbb{F}_{p^r})$$

  for different algebraic varieties.
- We would like to develop a principle of "analytic continuation of equalities": Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms of algebraic varieties. If the equality $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$ holds for every point y in a dense open subset U of Y, then it holds for every $y \in Y$.

Equality of numbers of points

- We will be concerned with proving equality of type

$$\#X(\mathbb{F}_{p^r}) = \#X'(\mathbb{F}_{p^r})$$

  for different algebraic varieties.
- We would like to develop a principle of "analytic continuation of equalities": Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms of algebraic varieties. If the equality $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$ holds for every point y in a dense open subset U of Y, then it holds for every $y \in Y$.
- This can't be true in general. The question is to find geometric assumptions on f and f' that guarantee this principle.

Equality of numbers of points

- We will be concerned with proving equality of type
$$\#X(\mathbb{F}_{p^r}) = \#X'(\mathbb{F}_{p^r})$$
for different algebraic varieties.
- We would like to develop a principle of "analytic continuation of equalities": Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms of algebraic varieties. If the equality $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$ holds for every point y in a dense open subset U of Y, then it holds for every $y \in Y$.
- This can't be true in general. The question is to find geometric assumptions on f and f' that guarantee this principle.
- The complex of ℓ-adic sheaves $f_!\mathbb{Q}_\ell$ interpolates all cohomology group with compact support $\mathrm{H}^i_c(X_y)$
$$\mathrm{H}^i(f_!\mathbb{Q}_\ell)_y = \mathrm{H}^i_c(X_y)$$
for all geometric points $y \in Y$. Geometric assumption on f give constraint on the copmlex $f_!\mathbb{Q}_\ell$.

The case of proper and smooth morphisms

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset U of Y, such that for all $y \in U(\mathbb{F}_{q^r})$, $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$.

The case of proper and smooth morphisms

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset U of Y, such that for all $y \in U(\mathbb{F}_{q^r})$, $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$.
- If $f : X \to Y$ and $f' : X' \to Y$ are proper and smooth morphisms then $\mathrm{H}^i(f_!\mathbb{Q}_\ell)$ and $\mathrm{H}^i(f'_!\mathbb{Q}_\ell)$ are ℓ-adic local systems for every $i \in \mathbb{Z}$.

The case of proper and smooth morphisms

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset U of Y, such that for all $y \in U(\mathbb{F}_{q^r})$, $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$.
- If $f : X \to Y$ and $f' : X' \to Y$ are proper and smooth morphisms then $\mathrm{H}^i(f_!\mathbb{Q}_\ell)$ and $\mathrm{H}^i(f'_!\mathbb{Q}_\ell)$ are ℓ-adic local systems for every $i \in \mathbb{Z}$.
- Deligne's theorem implies that

$$\mathrm{tr}(\sigma_y, \mathrm{H}^i_c(X_y)) = \mathrm{tr}(\sigma_y, \mathrm{H}^i_c(X'_y)).$$

The case of proper and smooth morphisms

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset U of Y, such that for all $y \in U(\mathbb{F}_{q^r})$, $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$.
- If $f : X \to Y$ and $f' : X' \to Y$ are proper and smooth morphisms then $\mathrm{H}^i(f_!\mathbb{Q}_\ell)$ and $\mathrm{H}^i(f'_!\mathbb{Q}_\ell)$ are ℓ-adic local systems for every $i \in \mathbb{Z}$.
- Deligne's theorem implies that

$$\mathrm{tr}(\sigma_y, \mathrm{H}^i_c(X_y)) = \mathrm{tr}(\sigma_y, \mathrm{H}^i_c(X'_y)).$$

- The Chebotarev density theorem implies that the ℓ-adic local systems $\mathrm{H}^i(f_!\mathbb{Q}_\ell)$ and $\mathrm{H}^i(f'_!\mathbb{Q}_\ell)$ are isomorphic.

The case of proper and smooth morphisms

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset U of Y, such that for all $y \in U(\mathbb{F}_{q^r})$, $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$.
- If $f : X \to Y$ and $f' : X' \to Y$ are proper and smooth morphisms then $\mathrm{H}^i(f_!\mathbb{Q}_\ell)$ and $\mathrm{H}^i(f'_!\mathbb{Q}_\ell)$ are ℓ-adic local systems for every $i \in \mathbb{Z}$.
- Deligne's theorem implies that

$$\mathrm{tr}(\sigma_y, \mathrm{H}^i_c(X_y)) = \mathrm{tr}(\sigma_y, \mathrm{H}^i_c(X'_y)).$$

- The Chebotarev density theorem implies that the ℓ-adic local systems $\mathrm{H}^i(f_!\mathbb{Q}_\ell)$ and $\mathrm{H}^i(f'_!\mathbb{Q}_\ell)$ are isomorphic.
- A local system is determined by its restriction to any dense open subset.

Singularities

- To obtain interesting cases, one has to drop the smoothness assumption.

Singularities

- To obtain interesting cases, one has to drop the smoothness assumption.
- Goresky-MacPherson's theory of perverse sheaves is very efficient in dealing with singularities of algebraic maps.

Singularities

- To obtain interesting cases, one has to drop the smoothness assumption.
- Goresky-MacPherson's theory of perverse sheaves is very efficient in dealing with singularities of algebraic maps.
- For every algebraic variety Y, the category $\mathcal{P}(Y)$ of perverse sheaves of Y is an abelian categories. For every morphism $f : X \to Y$, one can define perverse cohomology

$$^{p}\mathrm{H}^{i}(f_{!}\mathbb{Q}_{\ell}) \in \mathcal{P}(Y)$$

  in similar way as usual cohomology $\mathrm{H}^{i}(f_{!}\mathbb{Q}_{\ell})$ are usual ℓ-adic sheaves.

Purity and semi-simplicity

- Let $f : X \to Y$ be a proper morphism where X is a smooth variety. Then according to Deligne, $f_! \mathbb{Q}_\ell$ is a pure complex of sheaves.

Purity and semi-simplicity

- Let $f : X \to Y$ be a proper morphism where X is a smooth variety. Then according to Deligne, $f_! \mathbb{Q}_\ell$ is a pure complex of sheaves.
- As important consequence of Deligne's purity theorem, Beilinson, Bernstein, Deligne and Gabber proved that after base change to $Y \otimes \bar{\mathbb{F}}_p$, ${}^p\mathrm{H}^i(f_! \mathbb{Q}_\ell)$ is a direct sum of simple perverse sheaves.

Purity and semi-simplicity

- Let $f : X \to Y$ be a proper morphism where X is a smooth variety. Then according to Deligne, $f_! \mathbb{Q}_\ell$ is a pure complex of sheaves.
- As important consequence of Deligne's purity theorem, Beilinson, Bernstein, Deligne and Gabber proved that after base change to $Y \otimes \bar{\mathbb{F}}_p$, ${}^p\mathrm{H}^i(f_! \mathbb{Q}_\ell)$ is a direct sum of simple perverse sheaves.
- There exists over $Y \otimes \bar{\mathbb{F}}_p$ a decomposition in direct sum

$$f_! \mathbb{Q}_\ell = \bigoplus_{\alpha \in \mathfrak{A}} K_\alpha[n_\alpha]$$

where K_α are simple perverse sheaves and $n_\alpha \in \mathbb{Z}$.

Simple perverse sheaves

- Let $i : Z \to Y \otimes_{F_p} \bar{\mathbb{F}}_p$ be the immersion of an irreducible closed irreducible subscheme. Let $j : U \to Z$ be the immersion of a nonempty open subscheme. Let L be an irreducible local system on U, then

$$K = i_* j_{!*} L[\dim Z]$$

is a simple perverse sheaf on $Y \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$.

Simple perverse sheaves

- Let $i : Z \to Y \otimes_{F_p} \bar{\mathbb{F}}_p$ be the immersion of an irreducible closed irreducible subscheme. Let $j : U \to Z$ be the immersion of a nonempty open subscheme. Let L be an irreducible local system on U, then

$$K = i_* j_{!*} L[\dim Z]$$

  is a simple perverse sheaf on $Y \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$.
- According to Goresky and MacPherson, every simple perverse sheaf is of this form.

Simple perverse sheaves

- Let $i : Z \to Y \otimes_{F_p} \bar{\mathbb{F}}_p$ be the immersion of an irreducible closed irreducible subscheme. Let $j : U \to Z$ be the immersion of a nonempty open subscheme. Let L be an irreducible local system on U, then

$$K = i_* j_{!*} L[\dim Z]$$

  is a simple perverse sheaf on $Y \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$.
- According to Goresky and MacPherson, every simple perverse sheaf is of this form.
- The definition of the intermediate extension functot $j_{!*}$ is complicated. For us, what really matters is that the perverse sheaf is completely determined by the local system L, more generally, it is determined by the restriction of L to any nonempty open subscheme of U.

Support

- If K is a simple perverse sheaf on $Y \otimes \bar{\mathbb{F}}_p$, it is of the form $K = i_* j_{!*} L[\dim Z]$. In particular, $\mathrm{supp}(K) := Z$ is completely determined.

Support

- If K is a simple perverse sheaf on $Y \otimes \bar{\mathbb{F}}_p$, it is of the form $K = i_* j_{!*} L[\dim Z]$. In particular, $\mathrm{supp}(K) := Z$ is completely determined.
- Let $f : X \to Y$ be a proper morphism where X is a smooth variety. then, $f_! \mathbb{Q}_\ell$ can be decomposed into a direct sum

$$f_! \mathbb{Q}_\ell = \bigoplus_{\alpha \in \mathfrak{A}} K_\alpha[n_\alpha]$$

of simple perverse sheaves. The finite set

$$\mathrm{supp}(f) = \{Z_\alpha | Z_\alpha = \mathrm{supp}(K_\alpha)\}$$

is well determined. This is an important topological invariant of f.

Only full support

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper morphisms with X, X' smooth varieties. If

$$\mathrm{supp}(f) = \mathrm{supp}(f') = \{Y\},$$

then the analytic continuation principle applies as $f_! \mathbb{Q}_\ell$ and $f'_! \mathbb{Q}_\ell$ are determined by their restrictions to any nonempty open subset.

Only full support

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper morphisms with X, X' smooth varieties. If

$$\mathrm{supp}(f) = \mathrm{supp}(f') = \{Y\},$$

  then the analytic continuation principle applies as $f_! \mathbb{Q}_\ell$ and $f'_! \mathbb{Q}_\ell$ are determined by their restrictions to any nonempty open subset.
- This is true if f and f' are proper and smooth.

Only full support

- Let $f : X \to Y$ and $f' : X' \to Y$ be proper morphisms with X, X' smooth varieties. If

$$\mathrm{supp}(f) = \mathrm{supp}(f') = \{Y\},$$

  then the analytic continuation principle applies as $f_!\mathbb{Q}_\ell$ and $f'_!\mathbb{Q}_\ell$ are determined by their restrictions to any nonempty open subset.
- This is true if f and f' are proper and smooth.
- Are there more interesting cases?

Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$.

Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$.
- Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if X is smooth, then $f_! \mathbb{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.

Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$.
- Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if X is smooth, then $f_! \mathbb{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.
- In particular $\mathrm{supp}(f) = \{Y\}$

Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$.
- Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if X is smooth, then $f_! \mathbb{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.
- In particular $\mathrm{supp}(f) = \{Y\}$
- Argument: play the Poincaré duality against the cohomological amplitude.

Relative curve

- Let $f : X \to Y$ be a relative curve such that X is smooth, f is proper, for generic $y \in Y$, X_y is smooth and for every $y \in Y$, X_y is irreducible.

Relative curve

- Let $f : X \to Y$ be a relative curve such that X is smooth, f is proper, for generic $y \in Y$, X_y is smooth and for every $y \in Y$, X_y is irreducible.
- Then according to Goresky and MacPherson, $\mathrm{supp}(f) = \{Y\}$.

Relative curve

- Let $f : X \to Y$ be a relative curve such that X is smooth, f is proper, for generic $y \in Y$, X_y is smooth and for every $y \in Y$, X_y is irreducible.
- Then according to Goresky and MacPherson, $\mathrm{supp}(f) = \{Y\}$.
- Argument: play the Poincaré duality against the cohomological amplitude.

Poincaré duality versus cohomological amplitude

- Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.

Poincaré duality versus cohomological amplitude

- Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.
- The cohomological amplitude of a relative curve: $\mathrm{H}^i(f_!\mathbb{Q}_\ell[2]) = 0$ for $i \notin \{-2, -1, 0\}$

Poincaré duality versus cohomological amplitude

- Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.
- The cohomological amplitude of a relative curve: $\mathrm{H}^i(f_! \mathbb{Q}_\ell[2]) = 0$ for $i \notin \{-2, -1, 0\}$
- Assume there exists a simple perverse sheaf K_α such that $K_\alpha[n_\alpha]$ is a direct factor of $f_! \mathbb{Q}_\ell[-2]$ and $\dim(Z_\alpha) = 0$ where $Z_\alpha = \mathrm{supp}(K_\alpha)$.

Poincaré duality versus cohomological amplitude

- Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.
- The cohomological amplitude of a relative curve: $\mathrm{H}^i(f_! \mathbb{Q}_\ell[2]) = 0$ for $i \notin \{-2, -1, 0\}$
- Assume there exists a simple perverse sheaf K_α such that $K_\alpha[n_\alpha]$ is a direct factor of $f_! \mathbb{Q}_\ell[-2]$ and $\dim(Z_\alpha) = 0$ where $Z_\alpha = \mathrm{supp}(K_\alpha)$.
- $\mathrm{H}^0(K_\alpha) \neq 0$, the cohomological amplitude implies that $n_\alpha \geq 0$.

Poincaré duality versus cohomological amplitude

- Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.
- The cohomological amplitude of a relative curve: $\mathrm{H}^i(f_!\mathbb{Q}_\ell[2]) = 0$ for $i \notin \{-2, -1, 0\}$
- Assume there exists a simple perverse sheaf K_α such that $K_\alpha[n_\alpha]$ is a direct factor of $f_!\mathbb{Q}_\ell[-2]$ and $\dim(Z_\alpha) = 0$ where $Z_\alpha = \mathrm{supp}(K_\alpha)$.
- $\mathrm{H}^0(K_\alpha) \neq 0$, the cohomological amplitude implies that $n_\alpha \geq 0$.
- By Poincaré duality $K_\alpha^\vee[-n_\alpha]$ is also a direct factor of $f_!\mathbb{Q}_\ell[2]$ where $\mathrm{supp}(K_\alpha^\vee) = \mathrm{supp}(K_\alpha)$. It follows that $n_\alpha \leq 0$.

Poincaré duality versus cohomological amplitude

- Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.
- The cohomological amplitude of a relative curve: $\mathrm{H}^i(f_!\mathbb{Q}_\ell[2]) = 0$ for $i \notin \{-2, -1, 0\}$
- Assume there exists a simple perverse sheaf K_α such that $K_\alpha[n_\alpha]$ is a direct factor of $f_!\mathbb{Q}_\ell[-2]$ and $\dim(Z_\alpha) = 0$ where $Z_\alpha = \mathrm{supp}(K_\alpha)$.
- $\mathrm{H}^0(K_\alpha) \neq 0$, the cohomological amplitude implies that $n_\alpha \geq 0$.
- By Poincaré duality $K_\alpha^\vee[-n_\alpha]$ is also a direct factor of $f_!\mathbb{Q}_\ell[2]$ where $\mathrm{supp}(K_\alpha^\vee) = \mathrm{supp}(K_\alpha)$. It follows that $n_\alpha \leq 0$.
- It follows that $n_\alpha = 0$. But then $\mathrm{H}^0(K_\alpha)$ is a direct factor of $\mathrm{H}^2(f_!\mathbb{Q}_\ell) = \mathbb{Q}_\ell(-1)$. This is not possible.

Goresky-MacPherson's inequality

- Let $f : X \to Y$ be a proper morphism with fiber of dimension d. Assume X smooth. Let $Z \in \mathrm{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_\ell$.

Goresky-MacPherson's inequality

- Let $f : X \to Y$ be a proper morphism with fiber of dimension d. Assume X smooth. Let $Z \in \mathrm{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_\ell$.
- Then $\mathrm{codim}(Z) \leq d$.

Goresky-MacPherson's inequality

- Let $f : X \to Y$ be a proper morphism with fiber of dimension d. Assume X smooth. Let $Z \in \mathrm{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_\ell$.
- Then $\mathrm{codim}(Z) \leq d$.
- Moreover, if the geometric fibers of f are irreducible, then $\mathrm{codim}(Z) < d$.

Goresky-MacPherson's inequality

- Let $f : X \to Y$ be a proper morphism with fiber of dimension d. Assume X smooth. Let $Z \in \mathrm{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_\ell$.
- Then $\mathrm{codim}(Z) \leq d$.
- Moreover, if the geometric fibers of f are irreducible, then $\mathrm{codim}(Z) < d$.
- For abelian fibration, Goresky-MacPherson's inequality can be used to establish the full support theorem.

Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension d,

Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension d,
- P acts on M relatively over S.

Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension d,
- P acts on M relatively over S.
- We assume that the action has affine stabilizers: for every geometric point $s \in S$, for every $m \in M_s$, the stabilizer P_m is affine.

Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension d,
- P acts on M relatively over S.
- We assume that the action has affine stabilizers: for every geometric point $s \in S$, for every $m \in M_s$, the stabilizer P_m is affine.
- We assume that the Tate modules of P is polarizable.

Tate module in family

- Assume P has connected fibers, for every geometric point $s \in S$, there exists a canonical exact sequence

$$0 \to R_s \to P_s \to A_s \to 0$$

where A_s is an abelian variety and R_s is a connected affine group. This induces an exact sequence of Tate modules

$$0 \to T_{\mathbb{Q}_\ell}(R_s) \to T_{\mathbb{Q}_\ell}(P_s) \to T_{\mathbb{Q}_\ell}(A_s) \to 0.$$

Tate module in family

- Assume P has connected fibers, for every geometric point $s \in S$, there exists a canonical exact sequence

$$0 \to R_s \to P_s \to A_s \to 0$$

where A_s is an abelian variety and R_s is a connected affine group. This induces an exact sequence of Tate modules

$$0 \to T_{\mathbb{Q}_\ell}(R_s) \to T_{\mathbb{Q}_\ell}(P_s) \to T_{\mathbb{Q}_\ell}(A_s) \to 0.$$

- The Tate modules can be interpolated into a single ℓ-adic sheaf

$$\mathrm{H}_1(P/S) = \mathrm{H}^{2d-1}(g_!\mathbb{Q}_\ell)$$

with fiber $\mathrm{H}_1(P/S)_s = T_{\mathbb{Q}_\ell}(P_s)$. Polarization of the Tate module of P is an alternating form on $\mathrm{H}_1(P/S)$ vanishing on $T_{\mathbb{Q}_\ell}(R_s)$ and induces a perfect pairing on $T_{\mathbb{Q}_\ell}(A_s)$.

δ-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of P_s.

δ-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of P_s.
- For every $\delta \in \mathbb{N}$,

$$S_\delta = \{s \in S | \delta(s) = \delta\}$$

is locally closed.

δ-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of P_s.
- For every $\delta \in \mathbb{N}$,

$$S_\delta = \{s \in S | \delta(s) = \delta\}$$

  is locally closed.
- $P \to S$ is said to be δ-regular if $\mathrm{codim}(S_\delta) \geq \delta$ for every $\delta \in \mathbb{N}$.

δ-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of P_s.
- For every $\delta \in \mathbb{N}$,

$$S_\delta = \{s \in S | \delta(s) = \delta\}$$

  is locally closed.
- $P \to S$ is said to be δ-regular if $\mathrm{codim}(S_\delta) \geq \delta$ for every $\delta \in \mathbb{N}$.
- In particular, for $\delta = 1$, the δ-regularity means P is generically an abelian variety.

δ-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of P_s.
- For every $\delta \in \mathbb{N}$,

$$S_\delta = \{s \in S | \delta(s) = \delta\}$$

  is locally closed.
- $P \to S$ is said to be δ-regular if $\mathrm{codim}(S_\delta) \geq \delta$ for every $\delta \in \mathbb{N}$.
- In particular, for $\delta = 1$, the δ-regularity means P is generically an abelian variety.
- One can prove δ-regularity for all Hamiltonian completely integrable system.

δ-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of P_s.
- For every $\delta \in \mathbb{N}$,

$$S_\delta = \{s \in S | \delta(s) = \delta\}$$

  is locally closed.
- $P \to S$ is said to be δ-regular if $\mathrm{codim}(S_\delta) \geq \delta$ for every $\delta \in \mathbb{N}$.
- In particular, for $\delta = 1$, the δ-regularity means P is generically an abelian variety.
- One can prove δ-regularity for all Hamiltonian completely integrable system.
- δ-regularity is harder to prove in characteristic p.

Theorem of support for abelian fibration

- Theorem: Let $(f : M \to S, g : P \to S)$ be a δ-regular abelian fibration. Assume that M is smooth, the fibers of $f : M \to S$ are irreducible. Then

$$\mathrm{supp}(f) = \{S\}.$$

Theorem of support for abelian fibration

- Theorem: Let $(f : M \to S, g : P \to S)$ be a δ-regular abelian fibration. Assume that M is smooth, the fibers of $f : M \to S$ are irreducible. Then

$$\mathrm{supp}(f) = \{S\}.$$

- Corollary: Let (P, M, S) and (P', M', S) be δ-regular abelian fibrations as above (in particular, M_s and M'_s are irreducible). If the generic fibers of P and P' are isogenous abelian varieties, then for every $s \in S(\mathbb{F}_q)$, $\#M_s(\mathbb{F}_q) = \#M'_s(\mathbb{F}_q)$.

Theorem of support for abelian fibration

- Theorem: Let $(f : M \to S, g : P \to S)$ be a δ-regular abelian fibration. Assume that M is smooth, the fibers of $f : M \to S$ are irreducible. Then
$$\mathrm{supp}(f) = \{S\}.$$
- Corollary: Let (P, M, S) and (P', M', S) be δ-regular abelian fibrations as above (in particular, M_s and M'_s are irreducible). If the generic fibers of P and P' are isogenous abelian varieties, then for every $s \in S(\mathbb{F}_q)$, $\#M_s(\mathbb{F}_q) = \#M'_s(\mathbb{F}_q)$.
- Remark: In practice, one has to drop the condition M_s irreducible and P_s connected. In these cases, the formulations of the support theorem and the numerical equality are more complicated.

Theorem of support for abelian fibration

- Theorem: Let $(f : M \to S, g : P \to S)$ be a δ-regular abelian fibration. Assume that M is smooth, the fibers of $f : M \to S$ are irreducible. Then
$$\mathrm{supp}(f) = \{S\}.$$
- Corollary: Let (P, M, S) and (P', M', S) be δ-regular abelian fibrations as above (in particular, M_s and M'_s are irreducible). If the generic fibers of P and P' are isogenous abelian varieties, then for every $s \in S(\mathbb{F}_q)$, $\#M_s(\mathbb{F}_q) = \#M'_s(\mathbb{F}_q)$.
- Remark: In practice, one has to drop the condition M_s irreducible and P_s connected. In these cases, the formulations of the support theorem and the numerical equality are more complicated.
- This theorem is the key geometric ingredient in the proof of Langlands' fundamental lemma.

Upper bound on codimension

- For every closed irreducible subscheme Z of S, we set $\delta_Z = \min_{s \in S} \delta(s)$.

Upper bound on codimension

- For every closed irreducible subscheme Z of S, we set $\delta_Z = \min_{s \in S} \delta(s)$.
- We prove that if $Z \in \mathrm{supp}(f)$, then $\mathrm{codim}(Z) \leq \delta_Z$ and if the geometric fibers of $f : M \to S$ are irreducible then $\mathrm{codim}(Z) < \delta_Z$ unless $Z = S$.

Upper bound on codimension

- For every closed irreducible subscheme Z of S, we set $\delta_Z = \min_{s \in S} \delta(s)$.
- We prove that if $Z \in \mathrm{supp}(f)$, then $\mathrm{codim}(Z) \leq \delta_Z$ and if the geometric fibers of $f : M \to S$ are irreducible then $\mathrm{codim}(Z) < \delta_Z$ unless $Z = S$.
- By the δ-regularity, we have the inequality $\mathrm{codim}(Z) \geq \delta_Z$. The only possibility is $Z = S$.

Topological explanation

- Let $s \in Z$ such that $\delta(s) = \delta_Z$.

Topological explanation

- Let $s \in Z$ such that $\delta(s) = \delta_Z$.
- Assume there is a splitting $A_s \to P_S$ of the Chevalley exact sequence
$$0 \to R_s \to P_s \to A_s \to 0$$

Topological explanation

- Let $s \in Z$ such that $\delta(s) = \delta_Z$.
- Assume there is a splitting $A_s \to P_S$ of the Chevalley exact sequence
$$0 \to R_s \to P_s \to A_s \to 0$$
- Assume there exists an étale neighborhood S' of s, an abelian scheme $A' \to S'$ of special fiber A_s, and a homormorphism $A' \to P'$ extending the splitting $A_s \to P_s$.

Topological explanation

- Let $s \in Z$ such that $\delta(s) = \delta_Z$.
- Assume there is a splitting $A_s \to P_S$ of the Chevalley exact sequence
$$0 \to R_s \to P_s \to A_s \to 0$$
- Assume there exists an étale neighborhood S' of s, an abelian scheme $A' \to S'$ of special fiber A_s, and a homormorphism $A' \to P'$ extending the splitting $A_s \to P_s$.
- Then over S', A' acts almost freely on M' and on can factorize $M' \to S'$ as $M' \to [M'/A'] \to S'$ where the morphism $M' \to [M'/A']$ is proper and smooth, and the morphism $[M'/A'] \to S'$ is of relative dimension δ_s. Thus our inequality can be reduced to the Goresky-MacPherson inequality.

Implement the argument

- The assumptions are not satisfied in general. The generic fiber of P is usually an irreducible abelian variety and does not admit a factor of smaller dimension i.e the lifting $A' \to P'$ can't exist.

Implement the argument

- The assumptions are not satisfied in general. The generic fiber of P is usually an irreducible abelian variety and does not admit a factor of smaller dimension i.e the lifting $A' \to P'$ can't exist.
- To overcome this difficulty, one need to reformulate the above argument in terms of homological algebra instead of topology.