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Endoscopy Theory of Automorphic Forms

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Abstract

Historically, Langlands has introduced the theory of endoscopy in order to measure the failure of automorphic forms from being distinguished by their L-functions as well as the defect of stability in the Arthur-Selberg trace formula and ℓ-adic cohomology of Shimura varieties. However, the number of important achievements in the domain of automorphic forms based on the idea of endoscopy has been growing impressively recently. Among these, we will report on Arthur's classification of automorphic representations of classical groups and recent progress on the determination of ℓ-adic galois representations attached to Shimura varieties originating from Kottwitz's work. These results have now become unconditional; in particular, due to recent progress on local harmonic analysis. Among these developments, we will report on Waldspurger's work on the transfer conjecture and the proof of the fundamental lemma.

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1. Langlands' Functoriality Conjecture

This section contains an introduction of the functoriality principle conjectured by Langlands in [39].

1.1. *L*-functions of Dirichlet and Artin. The proof by Dirichlet for the infiniteness of prime numbers in an arithmetic progression of the form $m + Nx$ for some fixed integers m, N with $(m, N) = 1$, was a triumph of

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the analytic method in elementary number theory, *cf.* [13]. Instead of studying congruence classes modulo N which are prime to N, Dirichlet attached to each character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ of the group $(\mathbb{Z}/N\mathbb{Z})^\times$ of invertible elements in $\mathbb{Z}/N\mathbb{Z}$, the Euler product

$$L_N(s,\chi) = \prod_{p \nmid N} (1 - \chi(p)p^{-s})^{-1}. \tag{1}$$

This infinite product converges absolutely for all complex numbers s having real part $\Re(s) > 1$ and defines a holomorphic function on this domain of the complex plane. For $N = 1$ and trivial character χ, this function is the Riemann zeta function. As for the Riemann zeta function, general Dirichlet L-function has a meromorphic continuation to the whole complex plane. However, in contrast with the Riemann zeta function that has a simple pole at $s = 1$, the Dirichlet L-function associated with a non trivial character χ admits a holomorphic continuation. This property of holomorphicity was a key point in Dirichlet's proof for the infiniteness of prime numbers in an arithmetic progression. Another important property is the functional equation relating $L(s,\chi)$ and $L(1-s,\bar\chi)$.

Let $\sigma : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$ be a finite order character of the Galois group of the field of rational numbers $\mathbb{Q}$. For each prime number p, we choose an embedding of the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ into the algebraic closure $\bar{\mathbb{Q}}_p$ of the field of p-adic numbers $\mathbb{Q}_p$. This choice induces a homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ from the local Galois group at p to the global Galois group. The Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ of the finite field $\mathbb{F}_p$ is a canonical quotient of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. We have the exact sequence

$$1 \to I_p \to \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 1 \tag{2}$$

where I_p is the inertia group. Recall that $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is an infinite procyclic group generated by the substitution of Frobenius $x \mapsto x^p$ in $\bar{\mathbb{F}}_p$. Let the inverse of this substitution denote Fr_p.

Let $\sigma : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$ be a character of finite order. For all but finitely many primes p, say for all $p \nmid N$ for some integer N, the restriction of σ to the inertia group I_p is trivial. In that case $\sigma(\mathrm{Fr}_p) \in \mathbb{C}^\times$ is a well defined root of unity. Artin defines the L-function

$$L_N(s,\sigma) = \prod_{p \nmid N} (1 - \sigma(\mathrm{Fr}_p)p^{-s})^{-1}. \tag{3}$$

Artin's reciprocity law implies the existence of a Dirichlet character χ such that

$$L_N(s,\chi) = L_N(s,\sigma). \tag{4}$$

As a consequence, the $L_N(s,\sigma)$ satisfies all the properties of the Dirichlet L-functions. In particular, it is holomorphic for nontrivial σ and it satisfies a functional equation with respect to the change of variables $s \leftrightarrow 1-s$.

Finite abelian quotients of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ correspond to finite abelian extensions of $\mathbb{Q}$. According to Kronecker-Weber's theorem, abelian extensions are obtained by adding roots of unity to $\mathbb{Q}$. Since general extensions of $\mathbb{Q}$ are not abelian, it is natural to seek a non abelian generalization of Artin's reciprocity law.

Let $\sigma : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(n, \mathbb{C})$ be a continuous n-dimensional complex representation. Since Galois groups are profinite groups, the image of σ is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. There exists an integer N, such that for every prime $p \nmid N$, the restriction of σ to the inertia group I_p is trivial. In that case, $\sigma(\mathrm{Fr}_p)$ is well defined in $\mathrm{GL}(n, \mathbb{C})$, and its conjugacy class does not depend on the particular choice of embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. The Artin L-function attached to σ is the Euler product

$$L_N(s, \sigma) = \prod_{p \nmid N} \det(1 - \sigma(\mathrm{Fr}_p) p^{-s})^{-1}. \tag{5}$$

Again, this infinite product converges absolutely for a complex number s with real part $\Re(s) > 1$ and defines a holomorphic function on this domain of the complex plane. It follows from the Artin-Brauer theory of characters of finite groups that the Artin L-function has meromorphic continuation to the complex plane.

Conjecture 1 (Artin). *If σ is a nontrivial irreducible n-dimensional complex representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the L-function $L(s, \sigma)$ admits holomorphic continuation to the complex plane.*

The case $n = 1$ follows from Artin's reciprocity theorem and Dirichlet's theorem. The general case would follow from Langlands's conjectural nonabelian reciprocity law. According to this conjecture, it should be possible to attach to σ as above a cuspidal automorphic representation π of the group $\mathrm{GL}(n)$ with coefficients in the ring of the adeles $\mathbb{A}_{\mathbb{Q}}$ so that the Artin L-function of σ has the same Eulerian development as the principal L-function attached to σ. According to the Tamagawa-Godement-Jacquet theory *cf.* [62, 17], the latter extends to an entire function on complex plane that satisfies a functional equation. In the case $n = 2$, if the image of σ is solvable, the reciprocity law was established by Langlands and Tunnel by means of the solvable base change theory. The case where the image of σ in $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{SO}_3(\mathbb{C})$ is the the nonsolvable group of symmetries of the icosahedron is not known in general, though some progress on this question has been made [64].

1.2. Elliptic curves. Algebraic geometry is a generous supply of representations of Galois groups. However, most interesting representations have ℓ-adic coefficients instead of complex coefficients. Any system of polynomial equations with rational coefficients, homogeneous or not, defines an algebraic variety. The groups of ℓ-adic cohomology attached to it are equipped with a continuous action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In contrast with complex representations, ℓ-adic representations might not have finite image.

The study of the case of elliptic curves is the most successful so far. Let E be an elliptic curve defined over $\mathbb{Q}$. The first ℓ-adic cohomology group of E is a 2-dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In other words, we have a continuous 2-dimensional ℓ-adic representation

$$\sigma_{E,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2, \mathbb{Q}_\ell) \tag{6}$$

for every prime ℓ. The $\mathbb{Q}$-elliptic curve E can be extended to a $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$-elliptic curve E_N for some integer N, i.e. E can be defined by homogeneous equation with coefficients in $\mathbb{Z}[N^{-1}]$ such that for every prime $p \nmid N$, the reduction of E_N modulo p is an elliptic curve defined over the finite field $\mathbb{F}_p$. If $p \neq \ell$, this implies that the restriction of $\sigma_{E,\ell}$ to inertia I_p is trivial. It follows that the conjugacy class of $\sigma_{E,\ell}(\mathrm{Fr}_p)$ in $\mathrm{GL}(2, \mathbb{Q}_\ell)$ is well defined. The number of points on E_N with coefficients in $\mathbb{F}_p$ is given by the Grothendieck-Lefschetz fixed points formula

$$|E_N(\mathbb{F}_p)| = 1 - \mathrm{tr}(\sigma_{E,\ell}(\mathrm{Fr}_p)) + p. \tag{7}$$

It follows that $\mathrm{tr}(\sigma_{E,\ell}(\mathrm{Fr}_p))$ is an integer independent of the prime ℓ. Since it is also known that $\det(\sigma_{E,\ell}(\mathrm{Fr}_p)) = p$, the eigenvalues of $\sigma_E(\mathrm{Fr}_p)$ are conjugate algebraic integers of eigenvalue $p^{1/2}$, independent of ℓ. We can therefore drop the ℓ in the expressions $\mathrm{tr}(\sigma_{E,\ell}(\mathrm{Fr}_p))$ and $\det(\sigma_{E,\ell}(\mathrm{Fr}_p))$ as well as in the characteristic polynomial of $\sigma_{E,\ell}(\mathrm{Fr}_p)$.

The L-function attached to the elliptic curve E is defined by Euler product

$$L_N(s, E) = \prod_{p \nmid N} \det(1 - \sigma_E(\mathrm{Fr}_p) p^{-s})^{-1}. \tag{8}$$

Since the complex eigenvalues of $\sigma_E(\mathrm{Fr}_p)$ are of complex absolute value $p^{1/2}$, the above infinite product is absolute convergent for $\Re(s) > 3/2$ and converges to a homolomorphic function on this domain of the complex plane.

Shimura, Taniyama and Weil conjectured that the there exists a weight two holomorphic modular form f whose L-function $L(s, E)$ has the same Eulerian development as $L_N(s, E)$ at the places $p \nmid N$. It follows, in particular, that $L(s, E)$ has a meromorphic continuation to the complexe plane and it satisfies a functional equation. As it was shown by Frey and Ribet, a more spectacular consequence is the last Fermat's theorem is actually true. The Shimura-Taniyama-Weil conjecture is now a celebrated theorem of Wiles and Taylor [73, 63] in the semistable case. The general case is proved in [7].

The Shimura-Taniyama-Weil conjecture fits well with Langlands's reciprocity conjecture, *cf.* [39]. Though the main drive of Wiles's work consists of the theory of deformation of Galois representations, it needed as input the reciprocity law for solvable Artin representations $\sigma : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ that was proved by Langlands and Tunnell. The interplay between the p-adic theory of deformations of Galois representations and Langlands's functoriality principle should be a fruitful theme to reflect upon *cf.* [44].

1.3. The Langlands conjectures. Let G be a reductive group over a global field F which can be a finite extension of $\mathbb{Q}$ or the field of rational functions of a smooth projective curve over a finite field. For each absolute value v on F, F_v denotes the completion of F with respect to v, and if v is nonarchimedean, $\mathcal{O}_v$ denotes the ring of integers of F_v. Let $\mathbb{A}_F$ denote the ring of adeles attached to F, defined as the restricted product of the F_v with respect to $\mathcal{O}_v$.

By discrete automorphic representation, we mean an irreducible representation of the group $G(\mathbb{A}_F)$, the group of adeles points of G, that occurs as a subrepresentation of

$$L^2(G(F)\backslash G(\mathbb{A}_F))_\chi \tag{9}$$

where χ is an unitary character of the center of G [6]. Such a representation can develop as a completed tensor product $\pi = \hat{\bigotimes}_v \pi_v$ where π_v are irreducible admissible smooth representations of $G(F_v)$ for all nonarchimedean place v. For almost all nonarchimedean place v, π_v has a unique $G(\mathcal{O}_v)$-invariant line l_v. The Hecke algebra $\mathcal{H}_v$ of compactly supported complex valued functions on $G(F_v)$ that are bi-invariant under the action of $G(\mathcal{O}_v)$ acts on that line. Assume that G is unramified at v then $\mathcal{H}_v$ is a commutative algebra whose structure could be described in terms of a duality between reductive groups, [8].

Reductive groups over an algebraically closed field are classified by their root datum $(X^*, X_*, \Phi, \Phi^\vee)$, where X^* and X_* are the group of characters, respectively cocharacters of a maximal torus and $\Phi \subset X^*, \Phi^\vee \subset X_*$ are, respectively, the finite subset of roots and of coroots, *cf.* [61]. By the exchange of roots and coroots, we have the dual root datum which is the root datum of a complex reductive group $\hat{G}$. The reductive group G is defined over F and becomes split over a Galois extension E of F. The group $\mathrm{Gal}(E/F)$ acts on the root datum of G in fixing a basis. It thus defines an action of $\mathrm{Gal}(E/F)$ on the complex reductive group $\hat{G}$. The semi-direct product ${}^L G = \hat{G} \rtimes \mathrm{Gal}(E/F)$ was introduced by Langlands and is known as the L-group attached to G, *cf.* [39].

Suppose G unramified at a nonarchimedean place v; in other words, assume that the finite extension E is unramified over v. After a choice of embedding $E \to \bar{F}_v$, the Frobenius element $\mathrm{Fr}_v \in \mathrm{Gal}(\bar{\mathbb{F}}_v/\mathbb{F}_v)$, where $\mathbb{F}_v$ denotes the residue field of F_v, defines an element of $\mathrm{Fr}_v \in \mathrm{Gal}(E/F)$. There exists an isomorphism, known as the Satake isomorphism, between the Hecke algebra $\mathcal{H}_v$ and the algebra of $\hat{G}$-invariant polynomial functions on the connected component $\hat{G} \rtimes \{\mathrm{Fr}_v\}$ of ${}^L G = \hat{G} \rtimes \mathrm{Gal}(E/F)$. The line l_v acted on by the Hecke algebra $\mathcal{H}_v$ defines a semisimple element $s_v \in \hat{G} \rtimes \{\mathrm{Fr}_v\}$ up to $\hat{G}$-conjugacy in this component.

Unramified representations of $G(F_v)$ are classified by semisimple $\hat{G}$-conjugacy classes in $\hat{G} \rtimes \{\mathrm{Fr}_v\}$. In order to classify all irreducible admissible smooth representations of $G(F_v)$ for all non-archimedean v, Langlands introduced the group

$$L_{F_v} = W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$$

where W_{F_v} is the Weil group of F_v. The subgroup W_{F_v} of $\mathrm{Gal}(\bar{F}_v/F_v)$ consists of elements whose image in $\mathrm{Gal}(\bar{\mathbb{F}}_v/\mathbb{F}_v)$ is an integral power of Fr_v.

According to theorems of Laumon, Rapoport, and Stuhler in equal characteristic case, and Harris-Taylor and Henniart in unequal characteristic case, there is a natural bijection between the set of n-dimensional representations of L_{F_v} and the set of irreducible admissible smooth representations of $\mathrm{GL}_n(F_v)$ preserving L-factors and ϵ-factors of pairs, [51, 20, 22, 23].

According to Langlands, there should be also a group L_F attached to the global field F such that automorphic representations of $\mathrm{GL}_n(n, \mathbb{A}_F)$ are classified by n-dimensional complex representations of L_F. The hypothetical group L_F should be equipped with a surjective homomorphism to the Weil group W_F.

When F is the field of rational functions of a curve defined over a finite field $\mathbb{F}_q$, the situation is much better. Instead of complex representations of the hypothetical L-group L_F, one parametrizes automorphic representations by ℓ-adic representations of the Weil group W_F. Recall that in the function field case W_F is the subgroup of $\mathrm{Gal}(\bar{F}/F)$ consisting of elements whose image in $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is an integral power of Fr_q. In a tour de force, Lafforgue proved that there exists a natural bijection between irreducible n-dimensional ℓ-adic representation of the Weil group W_F and cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ following a strategy initiated by Drinfeld, who settled the case $n = 2$ [14, 46, 47]. In the number fields case, only a part of ℓ-adic representations of W_F coming from motives should correspond to a part of automorphic representations.

Let us come back to the general case where G is a reductive group over a global field that can be either a number field or a function field. According to Langlands, automorphic representations should be partitioned into packets parametrized by conjugacy classes of homomorphisms $L_F \to {}^L G$ compatible with the projections to W_F. At non-archimedean places, irreducible admissible smooth representations of $G(F_v)$ should also be partitioned into finite packets parametrized by conjugacy of homomorphism $L_{F_v} \to \hat{G} \rtimes W_{F_v}$ compatible with the projections to W_{F_v}. The parametrization of the local component of an automorphic representation should dervie from the global parametrization by the homomorphism $L_{F_v} \to L_F$ that is only well defined up to conjugation.

This reciprocity conjecture on global parametrization of automorphic representations seems for the moment out of reach, in particular because of the hypothetical nature of the group L_F. In constrast, Langlands' functoriality conjecture is not dependent on the existence of L_F.

Conjecture 2 (Langlands). *Let H and G be reductive groups over a global field F and let ϕ be a homomorphism between their L-groups ${}^L H \to {}^L G$ compatible with projection to W_F. Then for each automorphic representation π_H of $H(\mathbb{A}_F)$, there exists an automorphic representation π of $G(\mathbb{A}_F)$ such that at each unramified place v where π_H is parametrized by a conjugacy class $s_v(\pi_H)$ in $\hat{H} \rtimes \{\mathrm{Fr}_v\}$, the local component of π is also unramified and parametrized by $\phi(s_v(\pi_H))$.*

At least in the number field case, the existence of L_F seems to depend upon the validity of the functoriality principle. Some of the most important conjectures in number theory and in the theory of automorphic representations. As explained in [39], Artin conjecture follows from the case of functoriality when $\hat{H}$ is trivial. It is also explained in loc. cit how the generalized Ramanujan conjecture and the generalized Sato-Tate conjecture would also follow from the functoriality conjecture.

The approach based on a combination of the converse theorem of Cogdell and Piateski-Shairo, and the Langlangs-Shahidi method was succesful in establishing some startling cases of functoriality beyond endoscopy, *cf.* [26]. However, it suffers obvious limitation as Langlands-Shahidi method is based on the representation of a Levi component of a parabolic group on the Lie algebra of its unipotent radical.

Recently, the p-adic method was also successful in establishing a weak form of the functoriality conjecture. The most spectacular result is the proof of the Sato-Tate conjecture [21] deriving from this weak form. We will not discuss this topic in this survey.

So far, the most successful method in establishing special cases of functoriality is endoscopy. We will discuss this topic in more details in the next section.

2. Endoscopy Theory and Applications

The endoscopy theory is primarily focused in the structure of the packet of representations that have the same conjectural parametrization, either global $L_F \to {}^L G$ or local $L_{F_v} \to \hat{G} \rtimes W_{F_v}$. The existence of the packet is closely related to the lack of stability in the trace formula. As shown in [42], the answer to this question derives from the comparison of trace formulas. It is quite remarkable that the inconvenient unstability in the trace formula turned out to be a possibility. The quest for a stable trace formula bringing the necessity of comparing two trace formulas, turned out to be an efficient tool for establishing particular cases of functoriality.

A good number of known cases of functoriality fits into a general scheme that is nowadays known as the theory of endoscopy and twisted endoscopy: Jacquet-Langlands theory, solvable base change, automorphic induction and the Arthur lift from classical groups to linear groups.

Another source of endoscopic phenomenon was the study of continuous cohomology of Shimura varieties as first recognized by Langlands [40]. The work of Kottwitz has definitely shaped this theory by proposing precise conjecture on the ℓ-adic cohomology of Shimura variety as Galois module [34]. This description has been established in many important cases by means of comparison of the Grothendieck-Lefschetz fixed points formula and the Arthur-Selberg trace formula.

2.1. Packets of representations. First intuitions of endoscopy come from the theory of representations of $\mathrm{SL}(2,\mathbb{R})$. The restriction of discrete series representations of $\mathrm{GL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})$ is reducible. Their irreducible factors having the same Langlands parameter obtained by composition $W(\mathbb{R}) \to \mathrm{GL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C})$ and thus belong to the same packet. Packet of representations is understood to be dual stable conjugacy relation between conjugacy classes. For instance, the rotations of angle θ and $-\theta$ centered at the origin of the plane are not conjugate in $\mathrm{SL}(2,\mathbb{R})$, but become conjugate either in $\mathrm{GL}(2,\mathbb{R})$ or in $\mathrm{SL}(2,\mathbb{C})$.

In general, if G is a quasi-split reductive group over a local field F_v, and $\Pi_v(G)$ is the set of irreducible representations of G, Langlands conjectured that $\Pi(G)$ is a disjoint union of finite sets $\Pi_{v,\phi}(G)$ that are called L-packets and indexed by admissible homomorphisms $\phi_v : L_{F_v} \to {}^L G_v$. The work of Shelstad [59] in the real case suggested the following description of the set $\Pi_v(G)$ in general, *cf.* [42].

Let S_{ϕ_v} denote the centralizer of the image of ϕ_v in $\hat{G}$, and $S^0(\phi_v)$ its neutral component. Let $Z(\hat{G})$ denote the center of $Z(\hat{G})$ and $Z(\hat{G})^\Gamma$ denote the subgroup of invariants under the action of the Galois group Γ. The group $\mathcal{S}_{\phi_v} = S_{\phi_v}/S^0_{\phi_v} Z(\hat{G})^\Gamma$ should control completely the structure of the finite set Π_{ϕ_v} and also the characters of the representations belonging to Π_{ϕ_v}. If we further assume ϕ_v tempered, i.e its image is contained in a relatively compact subset of $\hat{G}$, then there should be a bijection $\pi \mapsto \langle s, \pi\rangle$ from Π_{ϕ_v} onto the set of irreducible characters of $\mathcal{S}_{\phi_v}$. In particular, the cardinal of the finite set Π_{ϕ_v} should equal the number of conjugacy classes of $\mathcal{S}_{\phi_v}$.

There is also a conjectural description of multiplicity in the automorphic spectrum of each member of a global L-packet. We can attach any admissible homomorphism $\phi : L_F \to {}^L G$ local parameter $\phi_v : L_{F_v} \to {}^L G_v$. By definition, the global L-packet Π_ϕ is the infinite product of local L-packets Π_{ϕ_v}. For a representation $\pi = \otimes_v \pi_v$ with $\pi_v \in \Pi_v$ to appear in the automorphic spectrum, all but finitely many local components must be unramified. For those representations, there is a conjectural description of its automorphic multiplicity $m(\pi,\phi)$ that was made precise by Kottwitz based on the case of SL_2 worked out by Labesse and Langlands *cf.* [38]. In [31], Kottwitz introduced a group $\mathcal{S}_\phi$ equipped with homomorphism $\mathcal{S}_\phi \to \mathcal{S}_{\phi_v}$. The conjectural formula for $m(\pi,\phi)$ is

$$m(\pi,\phi) = |\mathcal{S}_\phi|^{-1} \sum_{\epsilon \in \mathcal{S}_\phi} \prod_v \langle \epsilon_v, \pi_v \rangle.$$

For each v, ϵ_v denotes the image of ϵ in $\mathcal{S}_{\phi_v}$ and $\langle \epsilon_v, \pi_v\rangle$, the value of the character of $\mathcal{S}_{\phi_v}$ corresponding to π_v evaluated on ϵ_v.

If the above general description has an important advantage of putting the automorphic theory in perspective, it also suffers a considrable inconvenience of being dependent on the hypothetical Langlands group L_F.

For quite a long time, we have known only a few low rank cases including the case of inner forms of $\mathrm{SL}(2)$ due to Labesse and Langlands [38], the cyclic base change for GL(2) due to Saito, Shintani and Langlands [41] and the case of $U(3)$ and its base change due to Rogawski [58]. Later, the cyclic base change for GL(n) was established by Arthur and Clozel [3]. Recently, this field has been undergoing spectacular developments. For quasisplit classical groups, Arthur has been able to establish the existence and the description of local packets as well as an automorphic multiplicity formula for global packets [2]. For p-adic groups, the local description becomes unconditional based on the local Langlands conjecture for GL(n) proved by Harris-Taylor and Henniart. Arthur's description of global packet as well as his automorphic multiplicity formula is based on cuspidal automorphic representations of GL(n) instead of the hypothetical group L_F. This description relies on a little bit of intricate combinatorics that goes beyond the scope of this report. The unitary case was also settled by Moeglin [52], the case of inner forms of SL(n) by Hiraga and Saito [24]. The general case of Jacquet-Langlands correspondence has been also established by Badulescu [4].

Most of the above developments were made possible by the formidable machine that is the Arthur trace formula and its stabilization. The comparison of the trace formula for two different groups, one being endoscopic to the other, proved to be a quite fruitful method. Arthur's parametrization of automorphic forms on quasisplit classical groups derives from the possibility of realizing these groups as twisted endoscopic groups of GL(n) and the comparison between the twisted trace formula of GL(n) and the ordinary trace formula for the classical group. This procedure is known as the stabilization of the twisted trace formula. The structure of the L-packets derives from the stabilization of ordinary trace formula for classical groups. For both twisted and untwisted, Arthur needed to assume the validity of certain conjectures on orbital integrals: the transfer and the fundamental lemma.

2.2. Construction of Galois representations. Based on indications given in Shimura's work, Langlands proposed a general strategy to constructing Galois representations attached to automorphic representation incorporated in ℓ-adic cohomology of Shimura varieties. This domain also recorded important developments due to Kottwitz, Clozel, Harris, Taylor, Yoshida, Labesse, Morel, Shin and others.

In particular, a non negligible portion of the global Langlands correspondence for number fields is now known. A number field F is of complex multiplication if it is a totally imaginary quadratic extension of a totally real number field F^+. In particular, the complex conjugation induces an automorphism c of F that is independent of complex embedding of F. Let $\Pi = \bigotimes_v \Pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_F)$ such that $\Pi^\vee \simeq \Pi \circ c$, whose component at infinity Π_∞ has the same infinitesimal character as some irreducible algebraic representation satisfying certain regularity condition. Then for every prime number ℓ, there exists a continuous representation $\sigma : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(n, \bar{\mathbb{Q}}_\ell)$ so that for every prime p of F that does not lie above ℓ, the local component π_v of π corresponds to the ℓ-adic local representation of $\mathrm{Gal}(\bar{F}_v/F_v)$ via the local Langlands correspondence established by Harris-Taylor and Henniart. This important theorem is due to Clozel, Harris, and Labesse [11], Morel [53] and Shin [60] with some difference in the precision.

Under the above assumptions on the number field F and the automorphic representation Π, there exists a unitary group $U(F^+)$ with respect to the quadratic extension F/F^+ that gives rise to a Shimura variety and an automorphic representation π of U whose base change to $\mathrm{GL}(n, F)$ is Π. The base change from the unitary group U to the linear group $\mathrm{GL}(n, F)$ is a case of the theory of twisted endoscopy. It is based on a comparison of the twisted trace formula for $\mathrm{GL}(n, F)$ and the ordinary trace formula for $U(F^+)$. For more details, see [10, 37].

Following the work of Kottwitz on Shimura varieties, it is possible to attach Galois representation to automorphic forms. Algebraic cuspidal automorphic representations of unitary group appears in ℓ-adic cohomology of Shimura variety. In [35], Kottwitz proved a formula for the number of points on certain type of Shimura varieties with values in a finite field at a place of good reduction, and in [34], he showed how to stabilize this formula in a very similar manner to the stabilization of the trace formula. He also needed to assume the validity of the same conjectures on local orbital integrals as in the case of stabilization of the trace formula.

Kottwitz' formula for the number of points allow to show the compatibility with the local correspondence at the unramified places. More recently, Shin proved a formula for fixed points on Igusa varieties that looks formally similar to Kottwitz' formula that allows him to prove the compatibility with the local correspondence at a ramified place [60].

Morel was able to calculate the intersection cohomology of non-compact unitary Shimura varieties when the other authors confined themselves in the compact case [53]. The description of the intersection cohomology has been conjectured by Kottwitz.

We observe the remarkable similarity between Arthur's works on the classification of automorphic representations of classical groups and the construction of Galois representations attached to automorphic representations by Shimura varieties. Both need the stablization of a twisted trace formula and of an ordinary trace formula or similar formula thereof.

3. Stabilization of the Trace Formula

The main focus of the theory of endoscopy is the stabilization of the trace formula. The trace formula allows us to derive properties of automorphic representations from a careful study of orbital integrals. The orbital side of the trace formula is not stable but the defect of stability can be expressed by an endoscopic group. It follows the endoscopic case of the functoriality conjecture.

This section will give more details about the stabilization of the orbital side of the trace formula.

3.1. Trace formula and orbital integrals. In order to simplify the exposition, we will consider only semisimple groups G defined over a global field F. The Arthur-Selberg trace formula for G has the following form

$$\sum_{\gamma\in G(F)/\sim} \mathbf{O}_\gamma(f) + \cdots = \sum_{\pi} \mathrm{tr}_\pi(f) + \cdots \tag{10}$$

where γ runs over the set of anisotropic conjugacy classes of $G(F)$ and π over the set of discrete automorphic representations. The trace formula contains also more complicated terms related to hyperbolic conjugacy classes on one side and the continuous spectrum on the other side.

The test function f is of the form $f = \bigotimes_v f_v$ where for v, f_v is a smooth compactly supported function on $G(F_v)$ and for almost all nonarchimedean places v, f_v the unit function of the unramified Hecke algebra of $G(F_v)$. The global orbital integral

$$\mathbf{O}_\gamma(f) = \int_{I_\gamma(F)\backslash G(\mathbb{A})} f(g^{-1}\gamma g)dg \tag{11}$$

is convergent for anisotropic conjugacy classes $\gamma \in G(F)$. Here $I_\gamma(F)$ is the discrete group of F-points on the centralizer I_γ of γ. After choosing a Haar measure $dt = \bigotimes dt_v$ on $I_\gamma(\mathbb{A})$, we can express the above global integral as follows

$$\mathbf{O}_\gamma(f) = \mathrm{vol}(I_\gamma(F)\backslash I_\gamma(\mathbb{A}), dt) \prod_v \mathbf{O}_\gamma(f_v, dg_v/dt_v). \tag{12}$$

The torus I_γ has an integral form well defined up to finitely many places, and the measure dt is chosen so that $I_\gamma(\mathcal{O}_v)$ has volume one for almost all v. Over a nonarchimedean place, the local orbital integral

$$\mathbf{O}_\gamma(f_v, dg_v/dt_v) = \int_{I_\gamma(F_v)\backslash G(F_v)} f(g^{-1}\gamma g)\frac{dg_v}{dt_v} \tag{13}$$

is defined for every locally constant function $f_v \in C_c^\infty(G(F_v))$ with compact support. Local orbital integral $\mathbf{O}_\gamma(f_v, dg_v/dt_v)$ is convergent for every v and equals 1 for almost all v. The volume term is finite when the global conjugacy class γ is anisotropic.

Arthur introduced a truncation operator to deal with the continuous spectrum in the spectral expansion and hyperbolic conjugacy classes in the geometric expansion. In the geometric expansion, Arthur has more complicated local integrals that he calls weighted orbital integrals, see [2].

3.2. Stable orbital integrals. For $\mathrm{GL}(n)$, two regular semisimple elements in $\mathrm{GL}(n,F)$ are conjugate if and only if they are conjugate in $\mathrm{GL}(n,\bar{F})$, where $\bar{F}$ is an algebraic closure of F and this latter condition is tantamount to request that γ and γ' have the same characteristic polynomial. For a general reductive group G, we also have a characteristic polynomial map $\chi : G \to T/W$ where T is a maximal torus and W is its Weyl group. An element is said to be strongly regular semisimple if its centralizer is a torus. Strongly regular semisimple elements $\gamma, \gamma' \in G(\bar{F})$ have the same characteristic polynomial if and only if they are $G(\bar{F})$-conjugate. However, there are possibly more than one $G(F)$-conjugacy classes within the set of strongly regular semisimple elements having the same characteristic polynomial in $G(F)$. These conjugacy classes are said to be stably conjugate.

Let $\gamma, \gamma' \in G(F)$ be such that there exist $g \in G(\bar{F})$ with $\gamma' = g\gamma g^{-1}$. For all $\sigma \in \mathrm{Gal}(\bar{F}/F)$, since γ, γ' are defined over F, $\sigma(g)^{-1}g$ belongs to the centralizer of γ. The map

$$\sigma \mapsto \sigma(g)^{-1}g \tag{14}$$

defines a cocycle with values in $I_\gamma(\bar{F})$ whose image in $G(\bar{F})$ is a boundary. For a fixed $\gamma \in G(F)$, assumed strongly regular semisimple, the set of $G(F)$-conjugacy classes in the stable conjugacy class of γ can be identified with the subset A_γ of elements $\mathrm{H}^1(F, I_\gamma)$ whose image in $\mathrm{H}^1(F,G)$ is trivial. For local fields, the group $\mathrm{H}^1(F, I_\gamma)$ is finite but for global field, it can be infinite.

For a local non-archimedean field F, A_γ is a subgroup of the finite abelian group $\mathrm{H}^1(F, I_\gamma)$. One can form linear combinations of orbital integrals within a stable conjugacy class using characters of A_γ. In particular, the stable orbital integral

$$\mathbf{SO}_\gamma(f) = \sum_{\gamma'} \mathbf{O}_{\gamma'}(f)$$

is the sum over a set of representatives γ' of conjugacy classes within the stable conjugacy class of γ. One needs to choose in a consistent way Haar measures on different centralizers $I_{\gamma'}(F)$. For strongly regular semisimple γ, the tori $I_{\gamma'}$ for γ' in the stable conjugacy class of γ, are in fact canonically isomorphic, so that we can transfer a Haar measure from $I_\gamma(F)$ to $I_{\gamma'}(F)$. Obviously, the stable orbital integral $\mathbf{SO}_\gamma$ depends only on the characteristic polynomial of γ. If a is the characteristic polynomial of a strongly regular semisimple element γ, we set $\mathbf{SO}_a = \mathbf{SO}_\gamma$. A stable distribution is an element in the closure of the vector space generated by the distributions of the forms $\mathbf{SO}_a$ with respect to the weak topology.

In some sense, stable conjugacy classes are more natural than conjugacy classes. In order to express the difference between orbital integrals and stable orbital integrals, one needs to introduce other linear combinations of orbital integrals known as κ-orbital integrals. For each character $\kappa : A_\gamma \to \mathbb{C}^\times$, κ-orbital integral is a linear combination

$$\mathbf{O}^\kappa_\gamma(f) = \sum_{\gamma'} \kappa(\mathrm{cl}(\gamma'))\mathbf{O}_{\gamma'}(f)$$

over a set of representatives γ' of conjugacy classes within the stable conjugacy class of γ, $\mathrm{cl}(\gamma')$ being the class of γ' in A_γ. For any γ' in the stable conjugacy class of γ, A_γ and $A_{\gamma'}$ are canonical isomorphic so that the character κ on A_γ defines a character of A'_γ. Now, $\mathbf{O}^\kappa_\gamma$ and $\mathbf{O}^\kappa_{\gamma'}$ are not equal but differ by the scalar $\kappa(\mathrm{cl}(\gamma'))$ where $\mathrm{cl}(\gamma')$ is the class of γ' in A_γ. Even though this transformation rule is simple enough, we can't a priori define κ-orbital $\mathbf{O}^\kappa_a$ for a characteristic polynomial a as in the case of stable orbital integral. This is a source of an important technical difficulty in the theory of endoscopy: the transfer factor.

3.3. Stable distributions and the trace formula. Test functions for the trace formula are finite combination of functions f on $G(\mathbb{A})$ of the form $f = \bigotimes_{v\in|F|} f_v$ where for all v, f_v is a smooth function with compact support on $G(F_v)$ and for almost all finite place v, f_v is the characteristic function of $G(\mathcal{O}_v)$ with respect to an integral form of G which is well defined almost everywhere.

The trace formula defines a linear form in f. For each v, it induces an invariant linear form in f_v. There exists a Galois theoretical cohomological obstruction that prevents this linear form from being stably invariant. Let $\gamma \in G(F)$ be a strongly regular semisimple element. Let $(\gamma'_v) \in G(\mathbb{A})$ be an adelic element with γ'_v stably conjugate to γ for all v and conjugate for almost all v. There exists a cohomological obstruction that prevents the adelic conjugacy class (γ'_v) from being rational. In fact the map

$$\mathrm{H}^1(F, I_\gamma) \to \bigoplus_v \mathrm{H}^1(F_v, I_\gamma) \tag{15}$$

is not surjective in general. Let $\hat{I}_\gamma$ denote the dual complex torus of I_γ equipped with a finite action of the Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$. For each place v, the Galois group $\Gamma_v = \mathrm{Gal}(\bar{F}_v/F_v)$ of the local field also acts on $\hat{I}_\gamma$. By local Tate-Nakayama duality as reformulated by Kottwitz, $\mathrm{H}^1(F_v, I_\gamma)$ can be identified with the group of characters of $\pi_0(\hat{I}_\gamma^{\Gamma_v})$. By global Tate-Nakayama duality, an adelic class in $\bigoplus_v \mathrm{H}^1(F_v, I_\gamma)$ comes from a rational class in $\mathrm{H}^1(F, I_\gamma)$ if and only if the corresponding characters on $\pi_0(\hat{I}_\gamma^{\Gamma_v})$, after restriction to $\pi_0(\hat{I}_\gamma^{\Gamma})$, sum up to the trivial character. The original problem with conjugacy classes within a stable conjugacy class, complicated by the presence of the strict subset A_γ of $\mathrm{H}^1(F, I_\gamma)$, was solved in Langlands [42] and in a more general setting by Kottwitz [32].

In [42], Langlands outlined a program to derive from the usual trace formula a stable trace formula. The key point is to apply Fourier transform on the finite group $\pi_0(\hat{I}_\gamma^{\Gamma})$ and the part of the trace formula corresponding to the stable conjugacy class of γ becomes a sum over the group of characters of $\pi_0(\hat{I}_\gamma^{\Gamma})$.

By definition, the term corresponding to the trivial character of $\pi_0(\hat{I}_\gamma^\Gamma)$ is the stable trace formula. The other terms can be expressed as product of κ-orbital integrals.

Langlands conjectured that these κ-orbital integrals can also be expressed in terms of stable orbital integrals of endoscopic groups. The precise constant occuring in these conjectures were worked out in his joint work with Shelstad *cf.* [45]. There are in fact two conjectures: the transfer and the fundamental lemma that we will review in a similar but simpler context of Lie algebras. Admitting these conjectures, Langlands and Kottwitz proved that the correction terms in the elliptic part match with the stable trace formula for endoscopic groups. This equality is known under the name of the stabilization of the elliptic part of the trace formula.

The whole trace formula was eventually stabilized by Arthur under more local assumptions that are the weighted transfer and the weighted fundamental lemma *cf.* [1]. Arthur's classification of automorphic forms of quasisplit classical groups depends upon the stabilization of twisted trace formula. For this purpose, Arthur's local assumptions are more demanding: the twisted weighted transfer and the twisted weighted fundamental lemma.

3.4. The transfer and the fundamental lemma. We will state the two conjectures about local orbital integrals known as the transfer conjecture and the fundamental lemma in the case of Lie algebra. The statements in the case of Lie group are very similar but the constant known as the transfer factor more complicated.

Assume for simplicity that G is a split group over a local non-archimedean field F. Let $\hat{G}$ denote the connected complex reductive group whose root system is related to the root system of G by exchanging roots and coroots. Let γ be a regular semisimple F-point on the Lie algebra $\mathfrak{g}$ of G. Its centralizer I_γ is a torus defined over F. By the Tate-Nakayama duality, a character κ of $\mathrm{H}^1(F, I_\gamma)$ corresponds to a semisimple element of $\hat{G}$ that is well defined up to conjugacy. Let $\hat{H}$ be the neutral component of the centralizer of κ in $\hat{G}$. For a given torus I_γ, we can define an action of the Galois group of F on $\hat{H}$ that factors through the component group of the centralizer of κ in $\hat{G}$. By duality, we obtain a quasi-split reductive group H over F which is an endoscopic group of G.

The endoscopic group H is not a subgroup of G in general. Nevertheless, it is possible to transfer stable conjugacy classes from H to G, and from the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ to $\mathfrak{g}$. Assume for simplicity that H is also split. The inclusion $\hat{H} = \hat{G}_\kappa \subset \hat{G}$ induces an inclusion of Weyl groups $W_H \subset W$. It follows that there exists a canonical map $\mathfrak{t}/W_H \to \mathfrak{t}/W$ that realizes the transfer of stable conjugacy classes from $\mathfrak{h}$ to $\mathfrak{g}$. If $\gamma_H \in \mathfrak{h}(F)$ has characteristic polynomial $a_H \in \mathfrak{t}/W_H(F)$ mapping to the characteristic polynomial a of $\gamma \in G(F)$, we will say that the stable conjugacy class of γ_H transfers to the stable conjugacy class of γ.

Kostant has constructed a section $\mathfrak{t}/W \to \mathfrak{g}$ of the characteristic polynomial morphism $\mathfrak{g} \to \mathfrak{t}/W$ *cf.* [29]. For every $a \in (\mathfrak{t}/W)(F)$, the Kostant section defines a distinguished conjugacy class with the stable conjugacy class of a. As showed by Kottwitz *cf.* [36], the Kostant section provides us a rather simple definition of the Langlands-Shelstad transfer factor in the case of Lie algebra. Let $\Delta(\gamma_H, \gamma)$ be the unique complex function depending on regular semisimple conjugacy classes $\gamma_H \in \mathfrak{h}(F)$ and $\gamma \in \mathfrak{g}(F)$ with the characteristic polynomial $a_H \in (\mathfrak{t}/W_H)(F)$ of γ_H mapping to the characteristic polynomial $a \in (\mathfrak{t}/W)(F)$ of γ and satisfying the following property

- $\Delta(\gamma_H, \gamma)$ depends only on the stable conjugacy class of γ_H,
- if γ and γ' are stably conjugate then $\Delta(\gamma_H, \gamma') = \langle \mathrm{inv}(\gamma, \gamma'), \kappa \rangle \Delta(\gamma_H, \gamma)$ where $\mathrm{inv}(\gamma, \gamma')$ is the cohomological invariant lying in $\mathrm{H}^1(F, I_\gamma)$ defined by the coccyle (14),
- if γ is conjugate to the Kostant section at a, $\Delta(\gamma_H, \gamma) = |\Delta_G(\gamma)^{-1} \Delta_H(\gamma_H)|^{1/2}$ where Δ_G, Δ_H are the usual discriminant functions on $\mathfrak{g}$ and $\mathfrak{h}$ and $|.|$ denotes the standard absolute value of the non-archimedean field F.

Conjecture 3 (Transfer). *For every $f \in C_c^\infty(G(F))$ there exists $f^H \in C_c^\infty(H(F))$ such that*

$$\mathbf{SO}_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma) \mathbf{O}_\gamma^\kappa(f) \tag{16}$$

for all strongly regular semisimple elements γ_H and γ with the characteristic polynomial $a_H \in (\mathfrak{t}/W_H)(F)$ of γ_H mapping to the characteristic polynomial $a \in (\mathfrak{t}/W)(F)$ of γ.

Under the assumption γ_H and γ regular semisimple with the characteristic polynomial $a_H \in (\mathfrak{t}/W_H)(F)$ of γ_H mapping to the characteristic polynomial $a \in (\mathfrak{t}/W)(F)$ of γ, their centralizers in H and G are canonically isomorphic tori. We can therefore transfer Haar measures between those locally compact groups.

Assume that we are in unramified situation i.e. both G and H have reductive models over $\mathcal{O}_F$. Let $1_{\mathfrak{g}(\mathcal{O}_F)}$ be the characteristic function of $\mathfrak{g}(\mathcal{O}_F)$ and $1_{\mathfrak{h}(\mathcal{O}_F)}$ the characteristic function of $\mathfrak{h}(\mathcal{O}_F)$.

Conjecture 4 (Fundamental lemma). *The equality (16) holds for $f = 1_{\mathfrak{g}(\mathcal{O}_F)}$ and $f^H = 1_{\mathfrak{h}(\mathcal{O}_F)}$.*

In the case of Lie group instead of Lie algebra, there is a more general version of the fundamental lemma. Let $\mathcal{H}_G$ be the algebra of $G(\mathcal{O}_F)$-biinvariant functions with compact support on $G(F)$ and $\mathcal{H}_H$ the similar algebra for $H(F)$. Using Satake isomorphism, we have a canonical homomorphism $b : \mathcal{H}_G \to \mathcal{H}_H$.

Conjecture 5. *The equality (16) holds for any $f \in \mathcal{H}_G$ and for $f^H = b(f)$.*

In [68], Waldspurger also stated another beautiful conjecture in the same spirit. Let G_1 and G_2 be two semisimple groups with isogeneous root systems i.e. there exists an isomorphism between their maximal tori which maps a root of G_1 on a scalar multiple of a root of G_2 and conversely. In this case, there is an isomorphism $\mathfrak{t}_1/W_1 \simeq \mathfrak{t}_2/W_2$. We can therefore transfer regular semisimple stable conjugacy classes from $\mathfrak{g}_1(F)$ to $\mathfrak{g}_2(F)$ and back.

Conjecture 6 (Nonstandard fundamental lemma). *Let $\gamma_1 \in \mathfrak{g}_1(F)$ and $\gamma_2 \in \mathfrak{g}_2(F)$ be regular semisimple elements having the same characteristic polynomial. Then we have*

$$\mathbf{SO}_{\gamma_1}(1_{\mathfrak{g}_1(\mathcal{O}_F)}) = \mathbf{SO}_{\gamma_2}(1_{\mathfrak{g}_2(\mathcal{O}_F)}). \tag{17}$$

3.5. The long march.

Let us remember the long march to the conquest of the transfer conjecture and the fundamental lemma.

The theory of endoscopy for real groups is almost entirely due to Shelstad. She proved, in particular, the transfer conjecture for real groups. The fundamental lemma does not make sense for real groups.

Particular cases of the fundamental lemma were proved in low rank case by Labesse-Langlands for SL(2) [38], Kottwitz for SL(3) [30], Rogawski for U(3) [58], Hales, Schroder and Weissauer for Sp(4). The first case of twisted fundamental lemma was proved by Saito, Shintani and Langlands in the case of base change for GL(2). The conjecture 4 in the case of stable base change was proved by Kottwitz [33] for unit and then 5 by Clozel and Labesse independently for Hecke algebra. Kazhdan [27], and Waldspurger [66] proved 4 for SL(n). More recently, Laumon and myself proved the case U(n) [50] in equal characteristic.

The following result is to a large extent a collective work.

Theorem 7. *The conjectures 3, 4, 5 and 6 are true for p-adic fields.*

In the landmark paper [67], Waldspurger proved that the fundamental lemma implies the transfer conjectures. Due to his and Hales' works, the case of Lie group follows from the case of Lie algebra. Waldspurger also proved that the twisted fundamental lemma follows from the combination of the fundamental lemma with his nonstandard variant [68]. In [19], Hales proved that if we know the fundamental lemma for the unit for almost all places, we know it for the entire Hecke algebra for all places. In particular, if we know the fundamental lemma for the unit element at all but finitely many places, we also know it at the remaining places. More details on Hales' argument can be found in [53].

The problem is reduced to the fundamental lemma for Lie algebra. Following Waldspurger and, independently, Cluckers, Hales and Loeser, it is enough to prove the fundamental lemma for a local field in characteristic p, see [69] and [12].

For local fields of characteristic p, the approach using algebraic geometry was eventually successful. This approach originated in the work of Kazhdan and Lusztig who introduced the affine Springer fiber, *cf.* [28]. In [18], Goresky, Kottwitz and MacPherson gave an interpretation of the fundamental lemma in terms of the cohomology of the affine Springer. They also introduced the use of the equivariant cohomology and proved the fundamental lemma for unramified elements assuming the purity of cohomology of affine Springer fiber. Later in [49], Laumon proved the fundamental lemma for general element in the Lie algebra of unitary group also by using the equivariant cohomology and admitting the same purity assumption. The conjecture of purity of cohomology of affine Springer fiber is still unproved.

The Hitchin fibration was introduced in this context in [54]. Laumon and I used this approach, combined with [49], to prove the fundamental lemma for unitary group in [50]. The equivariant cohomology is no longer used for effective calculation of cohomology but to prove a qualitative property of the support of simple perverse sheaves occurring in the cohomology of Hitchin fibration. Later, I realized that the equivariant cohomology does not work in general simply due to the lack of toric action. The general case was proved in [56] with essentially the same strategy as in [50] but with a major difference. Since the equivariant cohomology does not provide a general argument for the determination of the support of simple perverse sheaves occurring in the cohomology of Hitchin fibration, an entirely different argument was needed. This new argument is a blend of an observation of Goresky and MacPherson on perverse sheaves and Poincaré duality with some particular geometric properties of algebraic integrable systems *cf.* [57].

4. Affine Springer Fibers and the Hitchin Fibration

In this section, we will describe the geometric approach to the fundamental lemma.

4.1. Affine Spriger fibers. Let $k = \mathbb{F}_q$ be a finite field with q elements. Let G be a reductive group over k and $\mathfrak{g}$ its Lie algebra. Let $F = k((\pi))$ and $\mathcal{O}_F = k[[\pi]]$. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. According to Kazhdan and Lusztig [28], there exists a k-scheme $\mathcal{M}_\gamma$ whose set of k points is

$$\mathcal{M}_\gamma(k) = \{g \in G(F)/G(\mathcal{O}_F) \mid \mathrm{ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_F)\}.$$

They proved that the affine Springer fiber $\mathcal{M}_\gamma$ is finite dimensional and locally of finite type.

The centralizer $I_\gamma(F)$ acts on $\mathcal{M}_\gamma(k)$. The group $I_\gamma(F)$ can be given a structure of infinite dimensional group $\tilde{\mathcal{P}}_\gamma$ over k, acting on $\mathcal{M}_\gamma$. There exists a unique quotient $\mathcal{P}_\gamma$ of $\tilde{\mathcal{P}}_\gamma$ such that the above action factors through $\mathcal{P}_\gamma$ and there exists an open subvariety of $\mathcal{M}_\gamma$ over which $\mathcal{P}_\gamma$ acts simply transitively.

Here is a simple but important example. Let $G = \mathrm{SL}_2$ and let γ be the diagonal matrix

$$\gamma = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}.$$

In this case $\mathcal{M}_\gamma$ is an infinite chain of projective lines with the point ∞ in each copy being identified with the point 0 of the next one. The group $\mathcal{P}_\gamma$ is $\mathbb{G}_m \times \mathbb{Z}$ with $\mathbb{G}_m$ acting on each copy of $\mathbb{P}^1$ by rescaling and the generator of $\mathbb{Z}$ acting by translation from each copy to the next one. The dense open orbit is obtained by removing from $\mathcal{M}_\gamma$ its double points.

We have a cohomological interpretation for stable κ-orbital integrals. Let us fix an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$ so that κ can be seen as taking values in $\bar{\mathbb{Q}}_\ell$. Then we have the formula

$$\mathbf{O}^\kappa_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sharp \mathcal{P}^0_\gamma(k)^{-1} \mathrm{tr}(\mathrm{Fr}_q, \mathrm{H}^*(\mathcal{M}_\gamma \otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell)_\kappa)$$

where Fr_q denotes the action of the geometric Frobenius on the ℓ-adic cohomology of the affine Springer fiber. In the case where the component group $\pi_0(\mathcal{P}_\gamma)$ is finite, $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa$ is the biggest direct summand of $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)$ on which $\mathcal{P}_\gamma$ acts through the character κ. By taking $\kappa = 1$, we obtained a cohomological interpretation of the stable orbital integral

$$\mathbf{SO}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sharp \mathcal{P}^0_\gamma(k)^{-1} \mathrm{tr}(\mathrm{Fr}_q, \mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_{st})$$

where the index st means the direct summand where $\mathcal{P}_\gamma$ acts trivially. When $\pi_0(\mathcal{P}_\gamma)$ is infinite, the definition of $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_{st}$ and $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa$ is a little bit more complicated.

Cohomological interptration of the fundamental lemma follows from the above cohomological interpretation of stable and κ-orbital integrals. In general, it does not seem possible to prove the cohomological fundamental lemma by a direct method because the ℓ-adic cohomology of the affine Springer fiber is as complicated as the orbital integrals. Nevertheless, in the case of unramified conjugacy classes, by using a large torus action of the affine Springer fiber and the Borel-Atiyah-Segal localization theorem for equivariant cohomology, Goresky, Kottwitz and MacPherson proved a formula for the ℓ-adic cohomology of unramified affine fibers in assuming the purity conjecture. It shuold be noticed however that there may be no torus action on the affine Springer fibers associated to most ramified conjugacy classes.

4.2. The Hitchin fibration. The Hitchin fibration appears in a quite remote area from the trace formula and the theory of endoscopy. It is fortunate that the geometry of the Hitchin fibration and the arithmetic of endoscopy happen to be just different smiling faces of Bayon Avalokiteshvara.

In [25], Hitchin constructed a large family of algebraic integrable systems. Let X be a smooth projective complex curve and Bun_G^{st} the moduli space of stable G-principal bundles on X. The cotangent bundle $T^*\mathrm{Bun}_G^{st}$ is natuturally a symplectic variety so that its algebra of analytic functions is equipped with a Poisson bracket $\{f, g\}$. It has dimension $2d$ where d is the dimension of Bun_G. Hitchin proves the existence of d Poisson commuting algebraic functions on $T^*\mathrm{Bun}_G$ that are algebraically independent

$$f = (f_1, \dots, f_d) : T^*\mathrm{Bun}_G^{st} \to \mathbb{C}^d. \tag{18}$$

The Hamiltonian vector fields associated to $f_1, \dots, f_d$ form d commuting vector fields along the fiber of f. Hitchin proved that generic fibers of f are open subsets of abelian varieties and Hamiltonian vector fields are linear.

To recall the construction of Hitchin, it is best to relax the stability condition and consider the algebraic stack Bun_G of all principal G-bundles instead of its open substack Bun_G^{st} of stable bundles. Following Hitchin, a Higgs bundle is a pair (E, ϕ), where $E \in \mathrm{Bun}_G$ is a principal G-bundle over X and ϕ is a global section of $\mathrm{ad}(E) \otimes K$, K being the canonical bundle of X. Over the stable locus, the moduli space $\mathcal{M}$ of all Higgs bundles coincide with $T^*\mathrm{Bun}_G^{st}$ by Serre's duality.

According to Chevalley and Kostant, the algebra $\mathbb{C}[\mathfrak{g}]^G$ of adjoint invariant function is a polynomial algebra generated by homogeneous functions $a_1, \dots, a_r$ of degree $e_1+1, \dots, e_r+1$ where $e_1, \dots, e_r$ are the exponents of the root system. If (E, ϕ) is a Higgs bundle then $a_i(\phi)$ is well defined as a global section of $K^{\otimes(e_i+1)}$. This defines a morphism $f : \mathcal{M} \to \mathcal{A}$ where $\mathcal{A}$ is the affine space

$$\mathcal{A} = \bigoplus_{i=1}^{r} \mathrm{H}^0(X, K^{\otimes(e_i+1)}).$$

whose dimension equals somewhat miraculously the dimension d of $\dim(\mathrm{Bun}_G)$. This construction applies also to a more general situation where K is replaced by an arbitrary line bundle, but of course the symplectic form as well as the equality of dimension are lost. It is not difficult to extend Hitchin's argument to prove that, after passing from the coarse moduli space to the moduli stack, the generic fiber of f is isomorphic to an extension of a finite group by an abelian variety. More canonically, the generic fiber of f is a principal homogeneous space under the action of the extension of a finite group by an abelian variety. On the infinitesimal level, this action is nothing but the action of the Hamiltonian vector fields along the fibers of f. We observe that Hamiltonian vector fields act also on singular fibers of f, and we would like to understand the the geometry of those fibers by this action.

In [54], we constructed a smooth Picard stack $g : \mathcal{P} \to \mathcal{A}$ that acts on $f : \mathcal{M} \to \mathcal{A}$. In particular, for every $a \in \mathcal{A}$, $\mathcal{P}_a$ acts on $\mathcal{M}_a$ in integrating the infinitesimal action of the Hamiltonian vector fields. For generic parameters a, the action of $\mathcal{P}_a$ on $\mathcal{M}_a$ is simply transitive but for degenerate parameters a, it is not. We observe the important product formula

$$[\mathcal{M}_a/\mathcal{P}_a] = \prod_{v \in X} [\mathcal{M}_{a,v}/\mathcal{P}_{a,v}] \tag{19}$$

that expresses the quotient $[\mathcal{M}_a/\mathcal{P}_a]$ as an algebraic stack as the product of affine Springer fibers $\mathcal{M}_{a,v}$ by its group of symmetry $\mathcal{P}_{a,v}$. For almost all v, $\mathcal{M}_{a,v}$ is a disrete set acted on simply transitively by $\mathcal{P}_{a,v}$.

In order to get an insight of the product formula, it is best to switch the base field from the field of complex numbers to a finite field k. In this case, it is instructive to count the number of k-points on the Hitchin fiber $\mathcal{M}_a$ as well as on the quotients $[\mathcal{M}_a/\mathcal{P}_a]$. In order to get actual numbers, we assume that the component group $\pi_0(\mathcal{P}_a)$ is finite. This is the case for a in an open subset $\mathcal{A}^{ell}$ of $\mathcal{A}$, called the elliptic part, to which we will restrict ourselves from now on.

More details about the following discussion can be found in [54, 55]. For $a \in \mathcal{A}^{ell}(k)$, the fiber $\mathcal{M}_a$ is a proper Deligne-Mumford stack and the number of its k-points can be expressed as a sum

$$|\mathcal{M}_a(k)| = \sum_{\gamma \in \mathfrak{g}(F)/\sim, \chi(f)=a} \mathbf{O}_\gamma(1_D) \tag{20}$$

over rational conjugacy classes $\gamma \in \mathfrak{g}(F)/\sim$, F denoting the function field of X within the stable conjugacy class defined by a, of global orbital integral (11) of certain adelic function 1_D, whose local expression $1_D = \prod_{v \in |X|} 1_{D_v}$ is given by the choice of a global section of the line bundle $K = \mathcal{O}_X(D)$. The number of k-points on the quotient $[\mathcal{M}_a/\mathcal{P}_a]$ can be expressed as a product of stable orbital integrals

$$|[\mathcal{M}_a/\mathcal{P}_a](k)| = \prod_{v \in |X|} \mathbf{SO}_a(1_{K_v}) \tag{21}$$

We will now look for an expression of the sum of global orbital integrals (20) in terms of stable orbital (21) plus correcting terms as in the stabilization of the trace formula. In our geometric terms, this expression becomes

$$|\mathcal{M}_a(k)| = |\mathcal{P}_a^0(k)| \sum_{\kappa} \mathbf{O}^\kappa_{\gamma_a}(1_D) \tag{22}$$

where $\mathbf{O}^\kappa_{\gamma_a}$ are κ-orbital integrals attached to the Kostant conjugacy class γ_a in the stable class a with respect to a Frobenius invariant character $\kappa : \pi_0(\mathcal{P}_a) \to \mathbb{Q}_\ell{}^\times$. The component group $\pi_0(\mathcal{P}_a)$ or the smile of Avalokiteshvara is the origin of endoscopic pain.

The cohomological interpretation of the formula (22) is the decomposition into direct sum of the cohomology of $\mathcal{M}_a$ with respect to the action of $\pi_0(\mathcal{P}_a)$

$$\mathrm{H}^*(\mathcal{M}_a \otimes_k \bar{k}, \mathbb{Q}_\ell) = \bigoplus_{\kappa:\pi_0(\mathcal{P}_a)\to \mathbb{Q}_\ell{}^\times} \mathrm{H}^*(\mathcal{M}_a \otimes_k \bar{k}, \mathbb{Q}_\ell)_\kappa. \tag{23}$$

It is not obvious to understand how this decomposition depends on a since the component group $\pi_0(\mathcal{P}_a)$ also depends on a. According to a theorem of Grothendieck, the component groups $\pi_0(\mathcal{P}_a)$ for varying a can be interpolated as fiber of a sheaf of abelian groups $\pi_0(\mathcal{P})$ for the étale topology of $\mathcal{A}$. Restricted to the elliptic part $\mathcal{A}^{ell}$, $\pi_0(\mathcal{P})$ is a sheaf of finite abelian groups. One of the difficulties to understand the decomposition (23) lies in the fact that $\pi_0(\mathcal{P})$ is not a constant sheaf. Nevertheless, the sheaf $\pi_0(\mathcal{P})$ acts on the perverse sheaves of cohomology

$$^p\mathrm{H}^n(f_*\mathbb{Q}_\ell|_{\mathcal{A}})$$

and decomposes it into a direct sum canonically indexed by a finite set of semisimple conjugacy classes of the dual group $\hat{G}$

$$^p\mathrm{H}^n(f_*\mathbb{Q}_\ell|_{\mathcal{A}^{ell}}) = \bigoplus_{[\kappa]\in\hat{G}/\sim} {}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell|_{\mathcal{A}^{ell}}).$$

This peculiar decomposition reflects the combinatorial complexity of the stabilization of the trace formula, see [54, 55]. Among the direct summand, the main term corresponding to $\kappa = 1$ is called the stable piece. For instance, the surprising appearance of semisimple conjugacy classes of the dual group reflects the presence of the equivalence classes of endoscopic groups in the stabilization of the trace formula.

The stabilization of the trace formula as envisionned by Langlands and Kottwitz suggests that the $[\kappa]$-part in the above decomposition should correspond to the stable part in the similar decomposition for an endoscopic group. This prediction can be realized in a clean geometric formulation after we pass to the étale scheme $\tilde{\mathcal{A}}$ over $\mathcal{A}$ *cf.* [56] which depends on the choice of a point $\infty \in X$. It was constructed in such a way that over $\tilde{\mathcal{A}}$, $\pi_0(\mathcal{P})$ becomes a quotient of the constant sheaf, whose sections over any connected test scheme are cocharacters of the maximal torus T. Over $\tilde{\mathcal{A}}^{ell}$, we obtain a finer decomposition

$$^p\mathrm{H}^n(f_*\mathbb{Q}_\ell|_{\tilde{\mathcal{A}}^{ell}}) = \bigoplus_{\kappa\in\hat{T}} {}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell|_{\tilde{\mathcal{A}}^{ell}})_\kappa$$

indexed by a finite subset of the maximal torus $\hat{T}$ in $\hat{G}$.

Let $\kappa \in \hat{T}$ correspond to a nontrivial piece in the above decomposition. The κ-component of the above direct sum is supported by the locus $\tilde{\mathcal{A}}^{ell}_\kappa$ in $\tilde{\mathcal{A}}^{ell}$ given by the elements $\tilde{a} \in \tilde{\mathcal{A}}^{ell}$ such that $\kappa : \mathbf{X}_*(T) \to \mathbb{Q}_\ell^\times$ factors through $\pi_0(\mathcal{P}_{\tilde{a}})$. This locus is not connected; its connected components are classified by homomorphism $\rho : \pi_1(X,\infty) \to \pi_0(\hat{G}_\kappa)$. Such a homomorphism defines a reductive group scheme H over X whose dual group is $\hat{H}_\rho$ by outer twisting. It can be checked that the connected component of $\tilde{\mathcal{A}}^{ell}_\kappa$ corresponding to ρ is just the Hitchin base $\mathcal{A}_{H_\rho}$ for the reductive group scheme H_ρ. Let $\iota_{\kappa,\rho} : \tilde{\mathcal{A}}_{H_\rho} \to \tilde{\mathcal{A}}$ denote this closed immersion.

Theorem 8. *Let G be a split semisimple group. There exists an isomorphism*

$$\bigoplus_n {}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell|_{\tilde{\mathcal{A}}^{ell}})_\kappa[2r](r) \sim \bigoplus_\rho (\iota_{\kappa,\rho})_* \bigoplus_n {}^p\mathrm{H}^n(f_{H_\rho,*}\mathbb{Q}_\ell|_{\tilde{\mathcal{A}}^{ell}_{H_\rho}})_{st}$$

where ρ are homomorphisms $\rho : \pi_1(X, \infty) \to \pi_0(\hat{G}_\kappa)$ and where r is some multiple of $\deg(K)$.

Here we stated our theorem in the case of split group, but it is valid for quasi-split group as well. In fact, the theorem was first proved for quasi-split unitary group by Laumon and myself in [50] before the general case was proved in [56]. To be more precise, the above theorem is proved under the assumption that the characteristic of the residue field is at least twice the Coxeter number of G.

The fundamental lemma for Lie algebra in equal characteristic case follows from the above theorem by a local-global argument. The unequal characteristic case follows from the equal characteristic case by theorem of Waldspurger [69] oand Cluckers, Hales, Loeser [12]. Waldspurger assumes that p does not divide the order of the Weyl group and Cluclers, Hales, Loeser needs a much stronger lower bound on p. In number field case, these assumptions do not matter as Hales proved that the validity of the fundamental lemma at almost all places implies its validity at the remaining places. Currently, the fundamental lemma for local fields of positive characteristic small with respect to G, is not known.

4.3. Support theorem. The main ingredient in the proof of theorem 8 is the determination of the support of simple perverse sheaves that appear as constituent of perverse cohomology of $f_* \mathbb{Q}_\ell$.

Let C be a pure ℓ-adic complex over a scheme S of finite type over a finite field k. Its perverse cohomology ${}^p\mathrm{H}^n(C)$ are then perverse sheaves and geometrically semisimple according to a theorem of Beilinson, Bernstein, Deligne and Gabber *cf.* [5]. According to Goresky and MacPherson, geometrically simple perverse sheaves are of the following form: let Z be a closed irreducible subscheme of $S \otimes_k \bar{k}$ with $i : Z \to S \otimes_k \bar{k}$ denoting the closed immersion, let U be a smooth open subscheme of Z with $j : U \to Z$ denoting the open immersion, let $\mathcal{K}$ be a local system on U, then $K = i_* j_{!*} \mathcal{K}[\dim(Z)]$ is a simple perverse sheaf, $j_{!*}$ being the functor of intermediate extension, and every simple perverse sheaf on $S \otimes_k \bar{k}$ is of this form. In particular, the support $Z = \mathrm{supp}(K)$ of a simple perverse sheaf is well defined. For a pure ℓ-adic complex C over a scheme S, we can ask the question what is the set of supports of simple perverse sheaves occurring as direct factors of the perverse sheaves of cohomology ${}^p\mathrm{H}^n(C)$.

The main topological ingredient in the proof of theorem 8 is the determination of this set of supports. We state only the result in characteristic zero. In characteristic p, we prove a weaker result, more complicated to state but enough for the purposes of the fundamental lemma.

Theorem 9. *Assume the base field k is the field of complex numbers. Then for any simple perverse sheaf K direct factor of ${}^p\mathrm{H}^n(f_* \mathbb{Q}_\ell|_{\tilde{\mathcal{A}}^{ell}})_{st}$, the support of K is $\tilde{\mathcal{A}}^{ell}$. Similarly, if K is a direct factor of ${}^p\mathrm{H}^n(f_* \mathbb{Q}_\ell|_{\tilde{\mathcal{A}}^{ell}})_\kappa$, then the support of K is of the form $\iota_\rho(\tilde{\mathcal{A}}_{H_\rho})$ for certain homomorphism $\rho : \pi_1(X, \infty) \to \pi_0(\hat{G}_\kappa)$.*

If we know two perverse sheaves having simple constituents of the same support, in order to construct an isomorphism between them, it is enough to construct an isomorphism over an open subset of the support. Over a small enough open subscheme, the isomorphism can be constructed directly.

Let us explain the proof of the nonstandard fundamental lemma conjectured by Waldspurger. Let G_1, G_2 be semisimple groups with isogeneous root systems. Their Hitchin moduli spaces $\mathcal{M}_1, \mathcal{M}_2$ map to the same base $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_2$. Let restrict to the elliptic locus and put $\mathcal{A} = \mathcal{A}^{ell}$. In order to prove $(f_{1*}\mathbb{Q}_\ell)_{st} \sim (f_{2*}\mathbb{Q}_\ell)_{st}$, it is enough to prove that they are isomorphic over an open subscheme of $\mathcal{A}$, as we know every simple perverse sheaf occurring in either one of these two complexes have support $\mathcal{A}^{ell}$. Over an open subscheme of $\mathcal{A}^{ell}$, $\mathcal{M}_1$ is acted on simply transitively by extension of a finite group by an abelian scheme and so is $\mathcal{M}_2$. The nonstandard fundamental lemma follows now from the fact that the above two abelian schemes are isogeneous and isogeneous abelian varieties have the same cohomology.

4.4. Weighted fundamental lemma. According to Waldspurger, the twisted fundamental lemma follows from the usual fundamental lemma and its nonstandard variant. Combining with his theorem that the fundamental lemma implies the transfer, the local results needed to stabilize the elliptic part of the trace formula and the twisted trace formula.

The classification of automorphic forms on quasisplit classical group requires the full power of the stabilization of the entire trace formula. For this purpose, Arthur needs more the twisted weighted fundamental lemma. This conjecture is an identity between twisted weighted orbital integrals.

The weighted fundamental lemma is now a theorem due to Chaudouard and Laumon *cf.* [9]. In the particular case of Sp(4), it was previously proved by Whitehouse *cf.* [72]. They introduced a condition of χ-stability in Higgs bundles such that the restriction of the Hitchin map $f : \mathcal{M} \to \mathcal{A}$ to the open subset $\mathcal{A}^\heartsuit$ of stable conjugacy classes that are generically regular semisimple and to moduli stack of χ-stable bundles $\mathcal{M}^\heartsuit_{\chi-st}$

$$f^\heartsuit_{\chi-st} : \mathcal{M}^\heartsuit_{\chi-st} \to \mathcal{A}^\heartsuit$$

is a proper morphism. This is an extension of the proper morphism $f^{ell} : \mathcal{M}^{ell} \to \mathcal{A}^{ell}$ that depends on a stability parameter χ. Chaudouard and Laumon extended the support theorem from f^{ell} to $f^\heartsuit_{\chi-st}$. They also showed that the number of points on a hyperbolic fiber of $\mathcal{A}^\heartsuit$ can be expressed in terms of weighted orbital integrals. The weighted fundamental lemma follows. It is quite remarkable that the moduli space depends on the stability parameter χ, though the number of points and the ℓ-adic complex of cohomology don't.

Finally, Waldspurger showed that the twisted weighted fundamental lemma follows from the weighted fundamental lemma and its nonstandard variant. He also showed that, if these statements are known for a local field of characteristic

p, tehy are also known for a p-adic local field with the same residue field, provided the residual characteristic does not divide the order of the Weyl group.

5. Functoriality Beyond Endoscopy

The unstability of the trace formula has been instrumental in establishing the first cases of the functoriality conjecture. The stable trace formula now fully established by Arthur should be the main tool in our quest for more general functoriality.

In [43], Langlands proposed new insights for the general case of functoriality principle. He observed that we are primarily concerned with the question how to distinguish automorphic representations π of G whose hypothetical parametrization $\sigma : L_F \to {}^L G$ has image contained in a smaller subgroup. Assume π of Ramanujan type (or tempered), the Zariski closure of the image of σ is not far from being determined by the order of the pole at 1 of the L-functions $L(s, \rho, \pi)$ for all representations ρ of ${}^L G$. Though we are not in position to work directly with these L-functions individually, the stable trace formula can be effective in dealing with the sum of L-functions attached to all automorphic representations π or the sum of their logarithmic derivative. Nontempered representations, especially the trivial representation, represent an obstacle to this strategy as they contribute to this sum the dominant term. The subsequent article [15], directly inspired from [43], might have proposed a method to subtract the dominant contribution. Other works [65, 48, 16], more or less inspired from [43], are the first encouraging steps on this new path that might lead us to the general case of functoriality.

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