On the Hitchin morphism for higher dimensional varieties

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Abstract

In this paper, we explore the structure of the Hitchin morphism for higher dimensional varieties. We show that the Hitchin morphism factors through a closed subscheme of the Hitchin base, which is in general a non-linear subspace of lower dimension. We conjecture that the resulting morphism, which we call the spectral data morphism, is surjective. In the course of the proof, we establish connections between the Hitchin morphisms for higher dimensional varieties, the invariant theory of the commuting schemes, and Weyl's polarization theorem. We use the factorization of the Hitchin morphism to construct the spectral and cameral covers. In the case of general linear groups and algebraic surfaces, we show that spectral surfaces admit canonical finite Cohen-Macaulayfications, which we call the Cohen-Macaulay spectral surfaces, and we use them to obtain a description of the generic fibers of the Hitchin morphism similar to the case of curves. Finally, we study the Hitchin morphism for some class of algebraic surfaces.

1 Introduction

For a smooth projective curve X over a field k, and a split reductive group G over k, a G-Higgs bundle over X is a pair (E,θ) consisting of a principal G-bundle E over X and an element $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes_{\mathscr{O}_X} \Omega^1_{X/k})$ called a Higgs field, where $\mathrm{ad}(E)$ is the adjoint vector bundle associated with E and $\Omega^1_{X/k}$ is the sheaf of 1-forms of X. In [12], Hitchin constructed a completely integrable system on the moduli space $\mathscr{M}_X$ of G-Higgs bundles over a curve X. This system can be presented as a morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ where $\mathscr{M}_X$ is the moduli space of Higgs bundles and $\mathscr{A}_X$ is a the affine space

$$\mathscr{A}_X = \bigoplus_{i=1}^{n} \mathrm{H}^0(X, S^{e_i}\Omega^1_{X/k}) \tag{1.1}$$

where $S^{e_i}\Omega^1_{X/k}$ being the e_ith are symmetric powers of the cotangent bundle $\Omega^1_{X/k}$. The morphism h_X is known as the Hitchin fibration. For curves X of genus $g>1$, h_X is surjective and its generic fiber is isomorphic to a disjoint union of abelian varieties if we discard automorphisms. This work intends to address these basic properties of the Hitchin morphism in the case of higher dimensional algebraic varieties.

Over a higher dimensional algebraic variety X, a G-Higgs bundle is a pair (E,θ) where E is a G-bundle over X equipped with a Higgs field

$$\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes_{\mathscr{O}_X} \Omega_X^1) \tag{1.2}$$

where $\mathrm{ad}(E)$ is the adjoint vector bundle of E, which is required to satisfy an integrability equation $\theta \wedge \theta = 0$. With given local coordinates $z_1, \ldots, z_d$ in a neighborhood U of $x \in X$ and given local trivialization of E, we can write $\theta = \sum_{i=1}^d \theta_i \mathrm{d}z_i$ where $\theta_i : U \to \mathfrak{g}$ are functions on U with values in the Lie algebra $\mathfrak{g}$ of G. The integrability equations satisfied by Higgs field is

$$[\theta_i, \theta_j] = 0$$

for all $1 \leq i, j \leq d$. Hitchin's construction, generalized to higher dimensional varieties by Simpson, provides a morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ where $\mathscr{A}_X$ is the affine space (1.1).

For higher dimensional algebraic varieties, the Hitchin morphism is very far from being surjective. We note that $h_X(E,\theta)$ could be defined for any $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes_{\mathscr{O}_X} \Omega_X^1)$ satisfying the integrability equation $\theta \wedge \theta = 0$ or not. We intent to understand the equations on $\mathscr{A}_X$ implied by the integrability condition $\theta \wedge \theta = 0$.

Our study of the Hitchin morphism for higher dimensional varieties follows the method of [18] in the one dimensional case. Namely, instead of studying the Hitchin morphism for a given variety X, we study certain universal morphisms independent of X. Those morphisms have to do with the construction of G-invariant functions on the scheme $\mathfrak{C}_G^d$ of commuting elements $x_1, \ldots, x_d$ in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The reductive group G acts diagonally on $\mathfrak{C}_G^d$ by adjoint action on $x_1, \ldots, x_d$.

Our study of G-invariant functions on $\mathfrak{C}_G^d$ can roughly divided into two parts. First, we investigate the generalization of the Chevalley restriction theorem to the commuting scheme. Second, we investigate the subring of G-invariant functions on $\mathfrak{C}_G^d$ derived from Weyl's polarization method. Both of these investigations are hindered by some notoriously difficult problems in commutative algebra, for instance the question whether $\mathfrak{C}_G^d$ is reduced. Although we offer no significant inroads in the solution of the problem on the reducedness of $\mathfrak{C}_G^d$, we will show a way to work around it to address our original problem of description of the Hitchin morphism. On the other hands, we attempt to state and hierarchize some problems which are related to and weaker than the reducedness of $\mathfrak{C}_G^d$, which seem to be worthy of further investigation.

Here is a summary of our results. For a higher dimensional smooth algebraic varieties X, the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ where $\mathscr{M}_X$ is the moduli stack of Higgs bundle and $\mathscr{A}_X$ is the affine space defined by the formula (4.1), is not surjective. We will define a closed subscheme $\mathscr{B}_X$ of $\mathscr{A}_X$, which is a non linear subspace of much lower dimension and prove that h_X factors through $\mathscr{B}_X$. We conjecture that the resulting morphism $\mathrm{sd}_X : \mathscr{M}_X \to \mathscr{B}_X$, which we call the spectral data morphism, is surjective. In the course of the proof, we establish connections between the Hitchin morphisms for higher dimensional varieties, the invariant theory of the commuting schemes, and Weyl's polarization theorem in classical invariant theory.

We use the factorization of the Hitchin morphism to construct the spectral and cameral covers and establish basic properties of them. In particular, we discover that, unlike the case of curves, the spectral and cameral covers are generally not flat in higher dimension. In the case $G = \mathrm{GL}_n$ and $\dim(X) = 2$, we construct an open subset $\mathscr{B}_X^{\heartsuit}$ of $\mathscr{B}_X$ such that for every $b \in \mathscr{B}_X^{\heartsuit}$, the corresponding spectral surface admits a canonical finite Cohen-Macaulayfication, called the Cohen-Macaulay spectral surface, and we use it to obtain a description of the Hitchin fiber $h_X^{-1}(b)$ similar to the case of curves. In particular, we show that $h_X^{-1}(b)$ is non empty for $b \in \mathscr{B}_X^{\heartsuit}$, and there is natural action of the Picard stack $\mathscr{P}_b$ of line bundles on the Cohen-Macaulay spectral surface on $h_X^{-1}(b)$. We also construct a subset $\mathscr{B}_X^{\diamondsuit}$ of $\mathscr{B}_X^{\heartsuit}$ such that for all $b \in \mathscr{B}_X^{\diamondsuit}$ the fiber $h_X^{-1}(b)$ is isomorphic to a disjoint union of abelian varieties after we discard automorphisms. For some class of algebraic surfaces (including elliptic surfaces), we can prove that $\mathscr{B}_X^{\diamondsuit}$ is an open dense subset of $\mathscr{B}_X^{\heartsuit}$, which is an open dense subset of $\mathscr{B}_X$.

Throughout this paper, we fix an algebraically closed field k of characteristic zero. To remove or weaken the assumption on the characteristic of k, we would have to refine many deep results in invariant theory. We hope to come back to this task in a future work.

2 Characteristics of Higgs bundles over curves

Hitchin's construction was revisited in [18] from the perspective of the theory of algebraic stacks. In loc. cit. the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ was derived from a natural morphism of algebraic stacks

$$h : \mathfrak{g}/G \to \mathfrak{g} /\!/ G \tag{2.1}$$

where $\mathfrak{g}/G$ and $\mathfrak{g} /\!/ G$ are the quotients of the Lie algebra of $\mathfrak{g}$ by the adjoint action of G in the framework of algebraic stacks and geometric invariant theory respectively. We recall that for every test scheme S, the groupoid of S-points of $\mathfrak{g}/G$ consist of all pairs (E, θ) where E is a principal G-bundles over S and $\theta \in \mathrm{H}^0(S, \mathrm{ad}(E))$ is a global section of the adjoint vector bundle $\mathrm{ad}(E)$ obtained from E by pushing out by the adjoint representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ of G. The categorical quotient $\mathfrak{g} /\!/ G$ is the affine scheme $\mathfrak{g} /\!/ G = \mathrm{Spec}(k[\mathfrak{g}]^G)$ whose ring of coordinates is the ring of all G-invariant polynomials on $\mathfrak{g}$. The concept of categorical quotient $\mathfrak{g} /\!/ G$ was devised by Mumford in [17] by which he means the initial object in the category of pairs (q, Q) where Q is a k-scheme and $q : \mathfrak{g} \to Q$ is a G-invariant morphism.

We also used the fundamental fact that the Chevalley restriction map is an isomorphism. Here, $\mathfrak{t}$ is a Cartan algebra, W its Weyl group. Since W-conjugate elements in $\mathfrak{t}$ are G-conjugate as elements of $\mathfrak{g}$, the restriction of a G-invariant function on $\mathfrak{g}$ is a W-invariant function on $\mathfrak{t}$ and therefore defines a homomorphism of algebras, the Chevalley restriction map

$$k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W \tag{2.2}$$

which is an isomorphism. We can also restate Chevalley's theorem by asserting that the morphism between the geometric invariant quotients

$$\mathfrak{t} /\!\!/ W \to \mathfrak{g} /\!\!/ G, \tag{2.3}$$

is an isomorphism.

Let us denote $\mathfrak{c} = \mathfrak{t} /\!\!/ W$. Since W acts on $\mathfrak{t}$ as a reflection group, according to another theorem of Chevalley, $\mathfrak{c}$ is also isomorphic to an affine space. The scalar action of $\mathbb{G}_m$ on $\mathfrak{t}$ induces an action of $\mathbb{G}_m$ on $\mathfrak{c}$. In fact, we can choose the coordinates $c_1, \ldots, c_n$ of the affine space $\mathfrak{c}$ such that as polynomial functions of $\mathfrak{t}$, they are homogenous of degree $e_1, \ldots, e_n$ i.e.

$$t(c_1, \ldots, c_n) = (t^{e_1} c_1, \ldots, t^{e_n} c_n). \tag{2.4}$$

The integers $d_1, \ldots, d_n$ are independent of the choice of $c_1, \ldots, c_n$.

Before proceeding further with the construction of the Hitchin map for curves, and as preparation for the higher dimensional case, let us state an elementary yet useful fact. Let V_d be a d-dimensional k-vector space which will also the space of morphisms $f : V_d \to \mathbb{A}^1$ satisfying $f(tv) = t^e f(v)$ can be canonically identified with the eth symmetric power $S^e V_d^*$ of the dual vector space V_d^*. This is just the restatement of the fact that the scalar action of $\mathbb{G}_m$ on V gives rise to the graduation of the algebra of polynomial functions on V i.e. $SV_d^* = \bigoplus_{e \in \mathbb{Z}_+} S^e V_d^*$. Although this seems completely obvious, this is an useful fact which shouldn't be overlooked. For instance, for $e = 1$, it says that all $\mathbb{G}_m$-equivariant algebraic map $f : V_d \to \mathbb{A}^1$ i.e. satisfying $f(tv) = tf(v)$ is automatically linear. For $d = 2$, all algebraic maps $f : V_d \to \mathbb{A}^1$ satisfying $f(tv) = t^2 f(v)$ is automatically quadratic and so on.

A Higgs field $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes \Omega_X^1)$ can also be seen as a $\mathscr{O}_X$-linear map $\mathscr{T}_X \to \mathrm{ad}(E)$ where $\mathscr{T}_X$ is the $\mathscr{O}_X$-module of local sections of the tangent bundles T_X of X, satisfying the integrability condition (3.1). As the integrability condition is void when X is a smooth algebraic curve, it be ignored in this section. We note that a $\mathscr{O}_X$-linear map $\mathscr{T}_X \to \mathrm{ad}(E)$ is equivalent to $\mathbb{G}_m$-equivariant morphism $\theta : T_X \to \mathfrak{g}/G$ lying over the map $X \to BG$ corresponding to the G-bundle E. By composing with the morphism $\mathfrak{g}/G \to \mathfrak{g} /\!\!/ G$ and the inverse of the isomorphism $\mathfrak{c} \to \mathfrak{g} /\!\!/ G$, we get a $\mathbb{G}_m$-equivariant morphism $a : T_X \to \mathfrak{c}$. For $i = 1, \ldots, n$, by composing with the functions $c_i : \mathfrak{c} \to \mathbb{A}^1$, we obtain $\mathbb{G}_m$-equivariant morphisms $a_i : T_X \to \mathbb{A}^1_{e_i}$ where $\mathbb{A}^1_{e_i}$ is a copy of the affine line which is acted on by $\mathbb{G}_m$ by the formula $t.x = t^{e_i} x$. Finally we note that the space of all $\mathbb{G}_m$-equivariant functions $a_i : T_X \to \mathbb{A}^1_{e_i}$ is the affine space of global section of the e_i symmetric power of the dual vector bundle $\Omega^1_{X/k}$ of 1-forms on X. Finally, we obtain the the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ where $\mathscr{A}_X$ is the affine space (4.1).

The main result of [12] asserts that, under the assumption $g_X \geq 2$, the generic fiber is isomorphic to a union of abelian varieties if we ignore isotropy groups. For instance, in the case $G = \mathrm{GL}_n$, Hitchin defines for every $a \in \mathscr{A}_X$ a spectral curve $X_a^\bullet$ which form a linear system of curves on the cotangent bundles of X. The assumption on the genus $g_X \geq 2$ implies that the system is ample and its generic fiber is a smooth projective curve. As $X_a^\bullet$ is smooth, the Hitchin fiber $\mathscr{M}_a = h_X^{-1}(a)$ is isomorphic to the Picard stack $\mathrm{Pic}(X_a^\bullet)$ which is isomorphic to a disjoint union of abelian varieties if we ignore automorphisms. For classical groups, Hitchin also constructs certain spectral curves using their standard representations. For abstract reductive groups, Donagi constructs a cameral cover $\tilde{X}_a$ of X for every $a \in \mathscr{A}_X$ and proves that the Hitchin fiber $\mathscr{M}_a$ is isomorphic to a union of abelian varieties if the cameral cover $\tilde{X}_a$ is a smooth curve.

For we will attempt to generalize the construction of cameral and spectral curves for Higgs bundles over higher dimensional varieties, let us recall their construction in the case of curves. The cameral construction, due to Donagi [8], derives the cameral covering $\pi_a : \tilde{X}_a \to X$ from the Cartesian diagram

$$\begin{array}{ccc} \tilde{X}_a & \xrightarrow{\tilde{a}} & \mathfrak{t}/\mathbb{G}_m \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[a]{} & \mathfrak{c}/\mathbb{G}_m \end{array} \tag{2.5}$$

in which the morphism $X \to \mathfrak{c}/\mathbb{G}_m$ at the bottom line derives from the $\mathbb{G}_m$-equivariant morphism $a : T_X \to \mathfrak{c}$. In other words, the cameral covering derives from the W-invariant morphism $\pi : \mathfrak{t} \to \mathfrak{t} /\!/ W = \mathfrak{c}$. The morphism $\pi : \mathfrak{t} \to \mathfrak{c}$ is finite flat and W-invariant, so is π_a which is derived from π by base change. Away from the discriminant locus of $\mathfrak{c}$, the morphism $\pi : \mathfrak{t} \to \mathfrak{c}$ is finite étale and Galois with Galois group W. In [18], we denote by $\mathscr{A}_X^\heartsuit$ the open subset of $\mathscr{A}_X$ consisting of maps $a : X \to \mathfrak{c}/\mathbb{G}_m$ whose image is not contained in the discriminant locus. By construction, for $a \in \mathscr{A}_X^\heartsuit$, $\tilde{X}_a \to X$ is generically a finite étale Galois morphism of Galois group W. We understand much better the geometry of the fiber $\mathscr{M}_a$ for $a \in \mathscr{A}_X^\heartsuit$. In particular, there is a natural Picard groupoid $\mathscr{P}_a$, constructed in [18], acting on $\mathscr{M}_a$ with a dense open orbit.

3 The Higgs stack and the spectral data morphism

Let X be a smooth projective variety of dimension d over k. A G-Higgs bundle over X is a G-bundle E over X equipped with a $\mathscr{O}_X$-linear map $\theta : \mathscr{T}_X \to \mathrm{ad}(E)$ from the tangent bundle $\mathscr{T}_X$ of X to the adjoint vector bundle $\mathrm{ad}(E)$ of E satisfying the "integrability" condition: for all vector fields v_1, v_2 we have

$$[\theta(v_1), \theta(v_2)] = 0. \tag{3.1}$$

Let $\mathfrak{C}_G^d$ is the closed subscheme of $\mathfrak{g}^d$ consisting of $(\theta_1, \ldots, \theta_d) \in \mathfrak{g}^d$ such that $[\theta_i, \theta_j] = 0$ for all indices i, j with $1 \leq i, j \leq d$. We note that the commuting relations are automatically satisfied in the case $d = 1$. Let V_d denote the typical d-dimensional k-vector space i.e. a k-vector space equipped with a basis $v_1, \ldots, v_d$. We will identify $\mathfrak{g}^d$ with the space of all linear maps $\theta : V_d \to \mathfrak{g}$ by attaching to $(\theta_1, \ldots, \theta_d) \in \mathfrak{g}^d$ the unique linear map $\theta : V_d \to \mathfrak{g}$ satisfying $\theta(v_i) = \theta_i$. The commuting variety $\mathfrak{C}_G^d$ can then be identified with closed subscheme of $\mathfrak{g}^d$ consisting of all k-linear maps $\alpha : V_d \to \mathfrak{g}$ such that $[\theta(v), \theta(v')] = 0$ for all $v, v' \in V_d$.

Granted with this description of $\mathfrak{C}_G^d$, we have an action of $\mathrm{GL}_d \times G$ on $\mathfrak{C}_G^d$ derived from the natural action of GL_d on V and the adjoint action of G on $\mathfrak{g}$. We will call the quotient

$$\mathfrak{C}_G^d/(\mathrm{GL}_d \times G) \tag{3.2}$$

in the sense of algebraic stack, the Higgs stack. It attaches to every test scheme S the groupoid of triples $(\mathscr{V}, \mathscr{E}, \theta)$ consisting of a vector bundle $\mathscr{V}$ of rank d over S, a principal G-bundle $\mathscr{E}$ over S, and a $\mathscr{O}_S$-linear map $\theta : \mathscr{V} \to \mathrm{ad}(\mathscr{E})$ satisfying $[\theta(v), \theta(v')] = 0$ for all local sections v, v' of $\mathscr{V}$. A Higgs field on d-dimensional smooth variety X can be represented by a map

$$\theta : X \to \mathfrak{C}_G^d/(\mathrm{GL}_d \times G) \tag{3.3}$$

lying over the map $X \to \mathbb{B}\mathrm{GL}_d$ representing to the cotangent bundle T_X^* where $\mathbb{B}\mathrm{GL}_d$ is the classifying stack of GL_d.

The construction of the Hitchin map derives roughly from G-invariant functions on $\mathfrak{C}_G^d$. Studying G-invariant maps on $\mathfrak{C}_G^d$ amounts to investigate the morphism

$$\mathfrak{C}_G^d/G \to \mathfrak{C}_G^d /\!/ G \tag{3.4}$$

between quotients, in the sense of algebraic stacks and geometric invariant theory respectively, of the commuting variety $\mathfrak{C}_G^d$ by the diagonal action of G. By definition, the geometric invariant quotient $\mathfrak{C}_G^d /\!/ G$ is the affine scheme whose ring of coordinates is the k-algebra

$$k[\mathfrak{C}_G^d /\!/ G] = k[\mathfrak{C}_G^d]^G$$

of G-invariant functions on $\mathfrak{C}_G^d$.

The commuting scheme $\mathfrak{C}_G^d$ has been studied intensively, especially in the case $d = 2$. It has a non-empty open locus $\mathfrak{C}_G^{d,\mathrm{rss}}$ consisting of commuting linear maps $\theta : V_d \to \mathfrak{g}$ such that the image $\theta(V_d)$ has non-empty intersection with the regular semi-simple locus $\mathfrak{g}^{\mathrm{rss}}$ of $\mathfrak{g}$. This open locus is smooth. In the case $d = 2$, Richardson proved that the underlying topological space of $\mathfrak{C}_G^2$ is irreducible, and in particular, the locus $\mathfrak{C}_G^{2,\mathrm{rss}}$ is dense in $\mathfrak{C}_G^2$. Results of Iarrobino on Hilbert punctual schemes on $\mathbb{A}^d$, with $d \geq 3$, implies that irreductibility is no longer true for $d \geq 3$.

There is a well known conjecture that the commuting scheme $\mathfrak{C}_G^2$ is reduced. The generalization of this conjecture to the cases $d \geq 3$ seems to be rather doubtful for we have very little understanding of other components of $\mathfrak{C}_G^d$ other than the component containing the open locus $\mathfrak{C}_G^{d,\mathrm{rss}}$. We have nothing new to offer in direction of the reducedness of $\mathfrak{C}_G^d$.

The geometric invariant quotient $\mathfrak{C}_G^d /\!/ G$ seems to behave better. In [14], Hunziker proved a weak version of the Chevalley restriction theorem for the commuting variety. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, the embedding $\mathfrak{t}^d \to \mathfrak{g}^d$ factors through $\mathfrak{C}_G^d$ for $\mathfrak{t}$ is commutative. Since orbits of the diagonal actions of W on $\mathfrak{t}^d$ are contained in orbits of the diagonal action of G of $\mathfrak{C}_G^d$, the restriction of a G-invariant function on $\mathfrak{C}_G^d$ to $\mathfrak{t}^d$ is W-invariant. In other words, we have a morphism

$$\mathfrak{t}^d /\!/ W \to \mathfrak{C}_G^d /\!/ G. \tag{3.5}$$

Based on fundamental result of Richardson [20], Hunziker proved that this morphism induces a universal homeomorphism, i.e., it is a finite morphism inducing a bijection on k-points. Since $\mathfrak{t}^d /\!/ W$ is clearly irreducible, the geometric invariant quotient $\mathfrak{C}_G^d /\!/ G$ of the commuting scheme is also irreducible.

Conjecture 3.1. *The morphism* (3.5) *is an isomorphism.*

We note that this conjecture is equivalent to asserting that the geometric invariant quotient $\mathfrak{C}_G^d /\!/ G$ is reduced and normal. Indeed, since $\mathfrak{t}^d /\!/ W$ is obviously reduced and normal, if (3.5) is an isomorphism then $\mathfrak{C}_G^d /\!/ G$ also is reduced and normal. Conversely, if $\mathfrak{C}_G^d /\!/ G$ is reduced and normal, then the map (3.5), known to be a normalization, has to be an isomorphism. Note also that Conjecture 3.1 together with (3.4) imply that there is a G-invariant morphism

$$\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!/ W \tag{3.6}$$

to be called the *spectral data morphism*, making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{t}^d & \longrightarrow & \mathfrak{C}_G^d \\ \downarrow & \swarrow & \downarrow \\ \mathfrak{t}^d /\!/ W & \longrightarrow & \mathfrak{C}_G^d /\!/ G \end{array} \tag{3.7}$$

For the existence of this morphism would be important to the study of the Hitchin fibration, we will state a conjecture, which is a weaker form of 3.1.

Conjecture 3.2. *There exists a G-invariant morphism* $\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!/ W$ *making the diagram* (3.7) *commutes.*

We note that Conjecture 3.2 implies that the geometric invariant quotient $\mathfrak{C}_G^d /\!/ G$ is reduced. Indeed, the right triangle of (3.7) gives rise to a commutative triangle of rings, which says that the composition of homomorphisms

$$k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^W \to k[\mathfrak{C}_G^d]$$

is the inclusion map. It follows that the homomorphism $k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^W$ is injective. Since $k[\mathfrak{t}^d]^W$ in an integral domain, $k[\mathfrak{C}_G^d]^G$ is also an integral domain, and in particular reduced.

In next section, Theorem 4.4, we will construct a canonical map $\mathfrak{C}_G^d(k) \to \mathfrak{t}^d /\!/ W(k)$ making the diagram (3.7) commute on the level of k-points. For the moment, let us construct this map in the case $G = \mathrm{GL}_n$. A k-point $\theta \in \mathfrak{C}_n^d(k)$ consists of commuting family of endomorphisms $x_1, \ldots, x_d$ on the typical n-dimensional k-vector space $E = k^n$. It equips with E a structure of module over the polynomial algebra $S(V_d) = k[v_1, \ldots, v_d]$. Let F denote the corresponding finite $S(V_d)$-module. We have a decomposition $F = \bigoplus_{\alpha \in V} F_\alpha$ where F_α is a $\mathrm{S}(V_d)$-module annihilated by some power of the maximal ideal $\mathfrak{m}_\alpha$ corresponding to the point $\alpha \in \mathbb{A}^d(k)$ where $\mathbb{A}^d = \mathrm{Spec}(S(V_d))$. This decomposition gives rise to a 0-cycle

$$z(\theta) = \sum_{\alpha \in \mathbb{A}^d(k)} \mathrm{lg}(F_\alpha)\alpha.$$

of length n in $\mathbb{A}^d$. This construction gives rise to a $\mathrm{GL}_n(k)$-invariant map $\mathfrak{C}_n^d(k) \to \mathrm{Chow}_n(\mathbb{A}^d)(k)$ where

$$\mathrm{Chow}_n(\mathbb{A}^d) = \mathrm{Spec}((S(V_d)^{\otimes n})^{\mathfrak{S}_n})$$

As $G = \mathrm{GL}_n$, one can identify $\mathrm{Chow}_n(\mathbb{A}^d)$ with $\mathfrak{t}^d /\!/ W$.

Theorem 3.3. *Conjecture 3.2 holds in the case of* GL_n. *In particular, for* GL_n, *the geometric invariant quotient* $\mathfrak{C}_G^d /\!/ G$ *is reduced.*

Proof. The construction of the canonical map $\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!/ W$ in the case $G = \mathrm{GL}_n$ is due to Deligne. For reader's convenience, we will recall his construction. First we note that for $G = \mathrm{GL}_n$, there is a canonical isomorphism $\mathfrak{t}^d /\!/ W = \mathrm{Chow}_n(\mathbb{A}^d)$. For any k-algebra R, we will construct a functorial map $\mathfrak{C}_G^d(R) \to \mathrm{Chow}_\nu(\mathbb{A}^d)(R)$ following Deligne [7, Section 6.3.1]. A collection of d matrices $\alpha_1, \ldots, \alpha_d \in \mathfrak{gl}_n(R)$ gives rise to a k-linear map $\alpha : V_d \to \mathfrak{gl}_n(R)$. If $\alpha_1, \ldots, \alpha_n$ commute with each others, α gives rise to a homomorphism of k-algebras

$$S(\alpha) : S(V_d) \to \mathfrak{gl}_n(R).$$

By composing with the determinant, we get a map $\det \circ S(\alpha) : S(V_d) \to R$ which is a multiplicative polynomial map which is homogenous of degree n. It must derives from a polynomial linear map

$$z(\alpha) : (S(V_d)^{\otimes n})^{\mathfrak{S}_n} \to R \tag{3.8}$$

characterized by the property that

$$z(\alpha)(f^{\otimes n}) = \det \circ S(\alpha)(f)$$

for $f \in S(V_d)$. Since $\det \circ S(\alpha)$ is multiplicative, $z(\alpha)$ is a homomorphism of k-algebras. In other words, $z(\alpha)$ defines a R-point of $\mathrm{Chow}_n(\mathbb{A}^d)$. This finishes the construction of the map $\mathrm{sd} : \mathfrak{C}_G^d \to \mathfrak{t}^d /\!/ W$.

We shall prove that the compostion $\mathfrak{C}_G^d \xrightarrow{\mathrm{sd}} \mathfrak{t}^d /\!/ \mathrm{W} \xrightarrow{(3.5)} \mathfrak{C}_G^d/G$ is the quotient map. Equivalently, the induced map

$$k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^{\mathrm{W}} \to k[\mathfrak{C}_G^d]$$

on rings of functions is the natural inclusion map. Consider the natural isomorphisms

$$k[\mathfrak{t}^d] \simeq k[T(i)_j]_{1\le i\le d, 1\le j\le n}, \quad k[\mathfrak{g}^d] \simeq k[X(i)_{a,b}]_{1\le i\le d, 1\le a,b\le n},$$

where $T(i)_j$ and $X(i)_{a,b}$ are the coordinate functions for the i-th copy of $\mathfrak{t}$ and $\mathfrak{g}$ in $\mathfrak{t}^d$ and $\mathfrak{g}^d$ respectively. Let $X(i) \in \mathfrak{g}(k[\mathfrak{g}^d])$ be the $n\times n$ matrix whose (a,b)-entry is given by $X(i)_{a,b}$ and let $T(i) \in k[\mathfrak{t}^d]$ be the restriction of $X(i)$ to $\mathfrak{t}^d$. We denote by $\bar{X}(i) \in \mathfrak{g}(k[\mathfrak{C}_G^d])$ (resp. $\bar{X}(i)_{a,b} \in k[\mathfrak{C}_G^d]$) for the image of $X(i)$ (resp. $X(i)_{a,b}$) under the natural map $\mathfrak{g}(k[\mathfrak{g}^d]) \to \mathfrak{g}(k[\mathfrak{C}_G^d])$ (resp. $k[\mathfrak{g}^d] \to k[\mathfrak{C}_G^d]$). It is known that (see, e.g., [19]) the ring of G-invariant functions $k[\mathfrak{g}^d]^G$ is generated by

$$\mathrm{Tr}(X(i_1)X(i_2)\cdots X(i_k))$$

where $k \in \mathbb{Z}_{\ge 0}$ and $1 \le i_1,...,i_k \le d$. As the restriction map $k[\mathfrak{g}^d]^G \to k[\mathfrak{C}_G^d]^G$ is surjective and $[\bar{X}(i),\bar{X}(j)] = 0 \in \mathfrak{g}(k[\mathfrak{C}_G^d])$, it follows that $k[\mathfrak{C}_G^d]^G$ is generated by the G-invariant functions

$$\mathrm{Tr}(\bar{X}(1)^{a_1}\cdots\bar{X}(d)^{a_d})$$

where $a_j \in \mathbb{Z}_{\ge 0}$. Note that the image of $\mathrm{Tr}(\bar{X}(1)^{a_1}\cdots\bar{X}(d)^{a_d})$ under the map $k[\mathfrak{C}_G^d]^G \to k[\mathfrak{t}^d]^{\mathrm{W}}$ is equal to

$$\mathrm{Tr}(T(1)^{a_1}\cdots T(d)^{a_d}).$$

Thus to prove the desired claim, it suffices to show that

$$z(\alpha)(\mathrm{Tr}(T(1)^{a_1}\cdots T(d)^{a_d}) = \mathrm{Tr}(\bar{X}(1)^{a_1}\cdots\bar{X}(d)^{a_d}) \tag{3.9}$$

where $z(\alpha) = \mathrm{sd}^* : k[\mathfrak{t}^d]^{\mathrm{W}} \to k[\mathfrak{C}_G^d]$ is the map in (3.8) in the universal case: $R = k[\mathfrak{C}_G^d]$ and $\alpha : V_d \to \mathfrak{gl}_d(R)$ corresponds to the identity map $\mathrm{id} \in \mathfrak{C}_G^d(R)$.

Let $V(1),...,V(d)$ be the coordinate vectors of V_d. For any $x \in k$ consider the element $f = x - V(1)^{a_1}\cdots V(d)^{a_d} \in S(V_d) \simeq k[V(1),...,V(d)]$. It follows from the definition of $z(\alpha)$ that

$$\begin{aligned} z(\alpha)(f^{\otimes n}) &= \det\circ S(\alpha)(f) = \det(x\mathrm{id} - S(\alpha)(V(1))^{a_1}\cdots S(\alpha)(V(d))^{a_d}) = \\ &= x^n - \mathrm{Tr}(S(\alpha)(V(1))^{a_1}\cdots S(\alpha)(V(d))^{a_d})x^{n-1} + \cdots. \end{aligned} \tag{3.10}$$

On the other hand, under the canonical isomorphism $(S(V_d)^{\otimes n})^{\mathfrak{S}_n} \simeq k[\mathfrak{t}^d]^{\mathrm{W}} \simeq k[T(i)_j]^{\mathfrak{S}_n}$, the element $f^{\otimes n}$ becomes

$$\prod_{j=1}^n (x - T(1)_j^{a_1}\cdots T(d)_j^{a_d}) = \det(x\mathrm{id} - T(1)^{a_1}\cdots T(d)^{a_d})$$

and it follows that

$$z(\alpha)(f^{\otimes n}) = z(\alpha)(\det(x\mathrm{id} - T(1)^{a_1}\cdots T(d)^{a_d})) =$$
$$= x^n - z(\alpha)(\mathrm{Tr}((T(1)^{a_1}\cdots T(d)^{a_d})))x^{n-1} + \cdots. \tag{3.11}$$

Comparing the coefficients of x^{n-1} in (3.10) and (3.11), we obtain

$$z(\alpha)(\mathrm{Tr}(T(1))^{a_1}\cdots(T(d))^{a_d}) = \mathrm{Tr}(S(\alpha)(V(1))^{a_1}\cdots S(\alpha)(V(d))^{a_d}). \tag{3.12}$$

Since $S(\alpha)(V(i)) = \bar{X}(i) \in \mathfrak{g}(k[\mathfrak{C}_G^d])$, it implies

$$z(\alpha)(\mathrm{Tr}(T(1))^{a_1}\cdots(T(d))^{a_d}) \overset{(3.12)}{=} \mathrm{Tr}(S(\alpha)(V(1))^{a_1}\cdots S(\alpha)(V(d))^{a_d}) = \mathrm{Tr}(\bar{X}(1)^{a_1}\cdots\bar{X}(d)^{a_d}).$$

Equation (3.9) follows. This completes the proof of the proposition. □

Although we don't know the validity of Conjectures 3.1 and 3.2 in general, we know they are true on the level of topological spaces. This will allow us to work around and predict the image of the Hitchin map.

Remark 3.4. In [11], Gan and Ginzburg proved the reducedness of $\mathfrak{C}_G^d /\!/ G$ in the case $G = \mathrm{GL}_n$, $d = 2$, by a different method.

4 Weyl's polarization and the Hitchin morphism

Let us recall Weyl's polarization for several matrices first, then its generalization to the case of general reductive groups. Weyl's polarization provides us a method to construct invariant functions on the space $(\mathfrak{gl}_n)^d$ of d arbitrary matrices $\theta_1,\ldots,\theta_d \in \mathfrak{gl}_n$. For all $x_1,\ldots,x_n \in k$, the coefficients of the characteristic polynomial of $x_1\theta_1 + \cdots + x_d\theta_d$ give rise to functions on $\mathfrak{gl}_n^d$ invariant under the diagonal action of GL_n. In fact, for every G-invariant function c on $\mathfrak{g}$, the function

$$(\theta_1,\ldots,\theta_d) \mapsto c(x_1\theta_1 + \cdots + x_d\theta_d)$$

is an G-invariant function on $\mathfrak{g}^d$. After [16], these functions don't form a set of generators of $k[\mathfrak{g}^d]^G$, but this is an impediment for us. As we will see, they are closed to form a complete set of generators for the quotient ring $k[\mathfrak{C}_G^d]^G$, which is more of our interest.

Let V_d denote the typical k-dimensional k-vector space given with a basis $v_1,\ldots,v_d$. For every affine variety Y equipped with an action of $\mathbb{G}_m$, the functor on the category of k-algebras which associates with each k-algebra R the set of $\mathbb{G}_m$-equivariant morphisms $y : V_d \otimes_k R \to Y$ is representable by an affine scheme, which will be denote $Y_{\mathbb{G}_m}^{V_d}$. For instance, if Y is the affine line $\mathbb{A}^1 = \mathrm{Spec}(k[x])$ equipped with an action of $\mathbb{G}_m$ in which x is homogenous of degree e in other words the action of $\mathbb{G}_m$ is $t.x = t^e x$. In this case, the set of $\mathbb{G}_m$-equivariant maps $V_d \otimes R \to Y$ is the component of degree e in $S(V_d^*) \otimes_k R$. It follows that in this case $Y_{\mathbb{G}_m}^{V_d}$ is representable by the affine space $S^e(\mathbb{A}^d)$ which is the eth symmetric power $\mathbb{A}^d = \mathrm{Spec}(S(V_d))$.

For $Y = \mathfrak{g}$, we can identify $\mathfrak{g}^d$ with the space $\mathfrak{g}_{\mathbb{G}_m}^{V_d}$ of $\mathbb{G}_m$-equivariant maps $\theta : V_d \to \mathfrak{g}$. In particular, a $\mathbb{G}_m$-equivariant morphism $\theta : V_d \to \mathfrak{g}$ is automatically linear. Let us consider the case $Y = \mathfrak{c}$ where $\mathfrak{c}$ is the geometric invariant quotient $\mathfrak{g} /\!/ G$ of $\mathfrak{g}$ by G acting by adjoint action. For $\mathfrak{c}$ is a isomorphic to an n-dimensional affine space with homogenous coordinates $c_1, \ldots, c_n$ of degree $e_1, \ldots, e_n$, the space $A = \mathfrak{c}_{\mathbb{G}_m}^{V_d}$ of $\mathbb{G}_m$-equivariant maps $V_d \to \mathfrak{c}$ is isomorphic to:

$$A \simeq \prod_{i=1}^{n} S^{e_i}(\mathbb{A}^d). \tag{4.1}$$

The isomorphism depends on the choice of homogenous coordinates $c_1, \ldots, c_n$.

Since the morphism $\mathfrak{g} \to \mathfrak{c}$ is G-invariant and $\mathbb{G}_m$-equivariant, it induces a G-invariant morphism

$$\mathrm{pol} : \mathfrak{g}^d \to A \tag{4.2}$$

which embodies Weyl's polarization for the diagonal action of G on $\mathfrak{g}^d$. With a choice of homogenous coordinates $c_1, \ldots, c_n$ of $\mathfrak{c}$, we have a morphism

$$\mathrm{pol} : \mathfrak{g}^d \to \prod_{i=1}^{n} S^{e_i}(\mathbb{A}^d). \tag{4.3}$$

For instance, in the case $G = \mathrm{GL}_n$, the ith coefficient c_i of the characteristic polynomial of $x_1\theta_1 + \cdots + x_d\theta_d$ is a dth symmetric form of variables $x_1, \ldots, x_d$ and defines a point on $S^d(\mathbb{A}^d)$. Instead of coefficients of the characteristic polynomial, we may also take the trace of the ith power of an endomorphism for $1 \le i \le n$ which also define a system of homogenous coordinates of $\mathfrak{c}$. The latter is what Simpson used to define the Hitchin map for GL_n for higher dimensional varieties. We have seen that the choice of coordinates of $\mathfrak{c}$ is unimportant as it just gives rise to different isomorphisms (4.1).

By restriction to the commuting scheme $\mathfrak{C}_G^d$, we obtain a G-invariant morphism

$$h : \mathfrak{C}_G^d \to A \tag{4.4}$$

To study the structure of the Hitchin morphism, and in particular the image thereof, we need to understand the image of the map (4.4) and its relation to the Chevalley restriction morphism (3.5).

For that purpose, we will also need to use Weyl's polarization construction for the diagonal action of W on $\mathfrak{t}^d$. The morphism $\mathfrak{t} \to \mathfrak{c} = \mathfrak{t} /\!/ W$ is W-invariant and $\mathbb{G}_m$-equivariant. As a result, we have a W-invariant morphism

$$\mathrm{pol}_W : \mathfrak{t}^d \to A. \tag{4.5}$$

Theorem 4.1. *The morphism* pol_W *of* (4.5) *is finite. It induces an injective map on the set of* k*-points. In other words, there exists a unique reduced closed subscheme* B *of* A *such that* pol_W *factors through a morphism*

$$b : \mathfrak{t}^d /\!\!/ W \to B, \tag{4.6}$$

which is a universal homeomorphism and normalization. For $G = \mathrm{GL}_n$, pol_W *is a closed embedding and* b *is an isomorphism.*

Proof. In the case $G = \mathrm{GL}_n$, b is a closed embedding, i.e., pol_W induces an isomorphism of $\mathfrak{t}^d /\!\!/ W$ on a closed subscheme of A, which is necessarily reduced since $\mathfrak{t}^d$ is reduced. This is the first fundamental theorem for symmetric groups, which is a classical theorem of Weyl [25, II.A.3]. After Hunziker [14], b is a closed embedding for type B,C. The general statement is the Theorem 2.15 of Losik-Michor-Popov, [16]. □

Remark 4.2. After Wallach [24], the map in (4.6) fails to be a closed embedding for type D.

Example 4.3. Let us describe the closed subscheme B of A in in the case $G = \mathrm{SL}_2$ and $d = 2$. In this case the Cartan algebra can be identified with $\mathfrak{t} \simeq \mathrm{Spec}(k[t])$. The Weyl group $W = \mathfrak{S}_2$ on $\mathfrak{t}$ by $w(t) = -t$ where w is the non-trivial element of W. The geometric invariant quotient $\mathfrak{c} = \mathrm{Spec}(k[u])$ with $u = t^2$ and the morphism $\mathfrak{g} \to \mathfrak{c}$ is given $u = \det(g)$. Since the exponent $e = 2$, we have $A = S^2(\mathbb{A}^2)$ is a 3-dimensional vector space. The map $\mathfrak{t}^2 = \mathbb{A}^2 \to A = S^2(\mathbb{A}^2)$ is given by $v \mapsto v^2$. In coordinates, this is the map $\mathbb{A}^2 \to \mathbb{A}^3$ given by $(x, y) \mapsto (x^2, 2xy, y^2)$. Thus B is the closed subscheme of $A = \mathbb{A}^3$ defined by the equation $b^2 - 4ac = 0$ which can be identified by the geometric invariant quotient of $\mathbb{A}^2$ by $\mathfrak{S}_2$ acting on $\mathbb{A}^2$ by $(x, y) \mapsto (-x, -y)$.

We have the following factorization of $h : \mathfrak{C}^d_G \to A$:

Theorem 4.4. *There exists a closed subscheme* B' *of* A*, which is a thickening of the closed subscheme* B *of* A*, as in Theorem 4.1, such that the morphism* $h : \mathfrak{C}^d_G \to A$ *of* (4.4) *factors through a morphism*

$$\mathrm{sd}' : \mathfrak{C}^d_G \to B'. \tag{4.7}$$

In particular, there is a canonical $G(k)$*-equivariant morphism* $\mathfrak{C}^d_G(k) \to \mathfrak{t}^d /\!\!/ \mathrm{W}(k)$*. For* $G = \mathrm{GL}_n$*, we have* $B' = B$ *and* (4.7) *is equal to the spectral data morphism* $\mathrm{sd} : \mathfrak{C}^d_G \mapsto \mathfrak{t}^d /\!\!/ /\mathrm{W} \simeq B$ *constructed in Theorem 3.3.*

Proof. We first prove this assertion on the level of k-points. This amounts to prove that a k-point of A lies in the image of the morphism h of (4.4) if and only if it lies in the image of the morphism pol_W of (4.5). Since the Cartan algebra $\mathfrak{t}$ is commutative, $\mathfrak{t}^d$ is contained in $\mathfrak{C}^d_G$ and therefore its image in A is contained in the image of $\mathfrak{C}^d_G(k)$. In the opposite direction, it follows from Richardson's theorem [20, 3.6] that the G-orbit passing by a point $(x_1, \ldots, x_d) \in \mathfrak{C}^d_G(k)$ is closed if and only if it has non-empty intersection with $\mathfrak{t}^d$. Indeed, Richardson proved that the G-orbit passing by $(x_1, \ldots, x_d) \in \mathfrak{g}^d$ is closed if and only if elements $x_1, \ldots, x_d$ generate a reductive Lie algebra. In the case where $x_1, \dots, x_d$ commute with each other, this means $x_1, \dots, x_d$ lie in a same Cartan subalgebra (or simultaneously diagonalizable in $\mathfrak{gl}_n$ case). Since every orbit has a closed orbit contained in its closure, for every $x \in \mathfrak{C}_G^d(k)$, there exists $y \in \mathfrak{t}^d(k)$ lying in the closure of the G-orbit of x. Since x is a G-invariant map, x and y have the same image in A. The second claim follows from Theorem 3.3. □

One may ask whether Theorem 4.4 holds for $B' = B$ for general G. This would follow from Conjecture 3.2.

5 Postulated image of the Hitchin morphism and cameral covers

Let X be a d-dimensional smooth and proper algebraic variety over k. A Higgs bundle over X is represented by a map $\theta : X \to \mathfrak{C}_G^d/(G \times \mathrm{GL}_d)$ lying over the map $\tau_X^* : X \to B\mathrm{GL}_d$ given by its cotangent bundle T_X^*. By composing it with the map $h : \mathfrak{C}_G^d/(G \times \mathrm{GL}_d) \to A/\mathrm{GL}_d$, we obtain the Hitchin morphism

$$h_X : \mathscr{M}_X \to \mathscr{A}_X$$

where $\mathscr{A}_X$ is the space of maps $X \to A/\mathrm{GL}_d$ lying over the τ_X^*. By choosing a system of homogenous coordinates $c_1, \dots, c_n$ of $\mathfrak{c}$ of degrees $e_1, \dots, e_n$, we can identify this intrinsic definition of $\mathscr{A}_X$ with the definition of $\mathscr{A}_X$ given in (4.1). In [22], Simpson constructed the Hitchin for $G = \mathrm{GL}_n$ by using the system of homogenous coordinates $c_1, \dots, c_n$ of $\mathfrak{c}$ with c_i being the invariant polynomial on $\mathfrak{gl}_n$ given by $c_i(x) = \mathrm{tr}(x^i)$.

Let $\mathscr{B}_X$ denote the space of maps $X \to B/\mathrm{GL}_d$, where B is the closed subscheme of A defined in Theorem 4.1, lying over τ_X^*. It is clear that $\mathscr{B}_X$ is a closed subscheme of $\mathscr{A}_X$. We call it the *postulated image* of the Hitchin map h_X. By replacing B by B' we have a thickening $\mathscr{B}'_X$ of $\mathscr{B}_X$. By Theorem 4.4, for every Higgs bundle $\theta : X \to \mathfrak{C}_G^d/(G \times \mathrm{GL}_d)$, its image $a = h_X(\theta) : X \to A/\mathrm{GL}_d$ factors through $b' : X \to B'/\mathrm{GL}_d$. There is no topological difference between $\mathscr{B}_X$ and $\mathscr{B}'_X$.

Theorem 5.1. *Let X be a d-dimensional smooth and proper algebraic variety over an algebraically closed field k of characteristic zero, $\mathscr{M}_X$ the moduli stack of Higgs bundles over X. Then the Hitchin morphism $h_X : \mathscr{M}_X \to \mathscr{A}_X$ factors through the spectral data morphism*

$$\mathrm{sd}'_X : \mathscr{M}_X \to \mathscr{B}'_X.$$

In particular, the image of every geometric point $\theta \in \mathscr{M}_X(k)$ belongs to $\mathscr{B}_X(k)$.

For every $b \in \mathscr{B}_X(k)$, the morphism $b : X \to B/\mathrm{GL}_d$ lifts to a morphism $X \to \mathfrak{t}^d /\!/ W$ and gives rise to a finite covering

$$\tilde{X}_b = X \times_{\mathfrak{t}^d /\!/ W} \mathfrak{t}^d \tag{5.1}$$

to be called the cameral covering.

Proof. The image of $\theta \in \mathscr{M}_X(k)$ by h_X is a map $a : X \to A/\mathrm{GL}_d$ lying over $\tau_X^* : X \to \mathbb{B}\mathrm{GL}_d$ representing the cotangent bundle of X. By Theorem 4.4, the map $a : X \to A/\mathrm{GL}_d$ factors through B'/GL_d. Since X is reduced, a factors through a morphism $b : X \to B/\mathrm{GL}_d$. Since the morphism $\mathfrak{t}^d /\!/ W \to B$ is a normalization, and X is normal, the morphism $b : X \to B/\mathrm{GL}_d$ lifts to a morphism $X \to \mathfrak{t}^d /\!/ W$, which gives rise the cameral covering $\tilde{X}_b$ as the Cartesian product (5.1). □

Example 5.2. Consider the case when X is a d-dimensional abelian variety. By choosing an isomorphism between the Lie algebra of X and the typical d-dimensional vector space V_d, we will have an isomorphism $\mathscr{A}_X = A$ and $\mathscr{B}_X = B$ which is a strict subset of A for $d \geq 2$. We can also prove that the spectral data map $\mathscr{M}_X(k) \to \mathscr{B}_X(k)$ is surjective by restricting ourselves to the subset of $\mathscr{M}_X(k)$ consisting of Higgs bundles (E, θ) where E is the trivial G-bundle.

Conjecture 5.3. *For every $b \in \mathscr{B}_X(k)$, the fiber $h_X^{-1}(b)$ is non-empty.*

Let B^{rss} denote the open dense locus of B where the morphism $\mathfrak{t}^d \to B$ is a finite étale and Galois morphism of Galois group W. This is a GL_d-equivariant open subset of B.

Definition 5.4. *We define $\mathscr{B}_X^{\heartsuit}(k)$ to be the open locus of $\mathscr{B}_X(k)$ consisting of maps $b : X \to B/\mathrm{GL}_d$ whose image has non-empty intersection with $B^{\mathrm{rss}}/\mathrm{GL}_d$.*

We will prove Conjecture 5.3 in the case $b \in \mathscr{B}_X^{\heartsuit}(k)$, $G = \mathrm{GL}_n$ and $d = 2$. In the case $G = \mathrm{GL}_n$ and $d = 1$, we can work work with spectral curves which are more convenient than cameral curves, in particular to construct Higgs bundles. Cameral and spectral covers are generally not flat in higher dimensions but, for $d = 2$, there is a canonical way to make them flat. This is why we will later restrict ourselves to the case of surfaces.

6 Spectral covers

Let us first recall the construction of the universal spectral cover for $d = 1$ and before discussing the case $d \geq 2$. For $G = \mathrm{GL}_n$, one can identify $\mathfrak{t}$ with the affine space $\mathbb{A}^n$ using the entries of diagonal matrices. In these coordinates $\pi : \mathbb{A}^n \to \mathfrak{c}$ is given by $(x_1, \ldots, x_n) \mapsto (c_1, \ldots, c_n)$ where

$$\begin{aligned} c_1 &= x_1 + \cdots + x_n, \\ c_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \\ &\cdots \\ c_n &= x_1 \ldots x_n \end{aligned}$$

are the elementary symmetric polynomials of variables $x_1, \ldots, x_n$ The Weyl group W is the symmetric group $\mathfrak{S}_n$ acting on $\mathbb{A}^n$ by permutation of coordinates; and the geometric invariant quotient $\mathfrak{c} = \mathbb{A}^n /\!/ \mathfrak{S}_n$ is isomorphic to $\operatorname{Spec}(k[c_1, \dots, c_n])$ by the fundamental theorem of symmetric polynomials.

To construct the universal spectral cover, we consider the action of the subgroup $\mathfrak{S}_{n-1}$ of $\mathfrak{S}_n$ on $\mathbb{A}^n$ by letting it permuting the coordinates $(x_1, \dots, x_{n-1})$ and leaving x_n fixed. The geometric invariant quotient $\mathfrak{c}^\bullet = \mathbb{A}^n /\!/ \mathfrak{S}_{n-1}$ is thus the affine space of coordinates $(c'_1, \dots, c'_{n-1}, x_n)$ with

$$c'_1 = x_1 + \dots + x_{n-1}, \dots, c'_{n-1} = x_1 \dots x_{n-1}$$

being elementary symmetric polynomials of variables $x_1, \dots, x_{n-1}$. The induced morphism $p : \mathfrak{c}^\bullet \to \mathfrak{c}$ is a finite flat morphism of degree n. One can represent the finite morphism $\mathfrak{c}^\bullet \to \mathfrak{c}$ in terms of equations by considering the morphism $\iota : \mathfrak{c}^\bullet \to \mathfrak{c} \times \mathbb{A}^1$ given $(c'_1, \dots, c'_{n-1}, x_n) \mapsto (c_1, \dots, c_n, t)$ with

$$t = x_n, c_1 = c'_1 + x_n, c_2 = c'_2 + c'_1 x_n, \dots, c_n = c'_{n-1} x_n. \tag{6.1}$$

This is a closed embedding which identifies $\mathfrak{c}^\bullet$ with the closed subscheme of $\mathfrak{c} \times \mathbb{A}^1$ defined by the equation $t^n - c_1 t^{n-1} + \dots + (-1)^n c_n = 0$.

We will now generalize this construction to the case $d \geq 2$. For $G = \mathrm{GL}_n$, we have $\mathfrak{t}^d = (\mathbb{A}^d)^n$. The geometric invariant quotient $\mathfrak{t}^d /\!/ W$ can be identified with the Chow scheme $\mathrm{Chow}_n(\mathbb{A}^d) = (\mathbb{A}^d)^n /\!/ \mathfrak{S}_n$ classifying zero-dimensional cycles of length n of $\mathbb{A}^d$. Weyl's first fundamental theorem of invariant theory for $\mathfrak{S}_n$ states that the morphism

$$\mathrm{Chow}_n(\mathbb{A}^d) \to \mathbb{A}^d \times S^2\mathbb{A}^d \times \cdots S^n\mathbb{A}^d \tag{6.2}$$

given by the same set of equations as (6.1) is a closed embedding. We will construct the universal spectral covering of $\mathrm{Chow}_n(\mathbb{A}^d)$ as follows. We will represent a point of $\mathrm{Chow}_n(\mathbb{A}^d)$ as an unordered collection of n points of $\mathbb{A}^d$

$$[x_1, \dots, x_n] \in \mathrm{Chow}_n(\mathbb{A}^d). \tag{6.3}$$

We consider the morphism

$$\chi_{\mathbb{A}^d} : \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d \to S^n(\mathbb{A}^d) \tag{6.4}$$

given by

$$\chi_{\mathbb{A}^d}([x_1, \dots, x_n], x) = (x - x_1) \dots (x - x_n) = x^n - c_1 x^{n-1} + \dots + (-1)^n c_n \tag{6.5}$$

where $c_, \dots, x_n$ are elementary symmetric polynomial (6.1) of variables $x_1, \dots, x_n \in \mathbb{A}^d$. We define the closed subscheme $\mathrm{Cayley}_n(\mathbb{A}^d)$ to be

$$\mathrm{Cayley}_n(\mathbb{A}^d) = \chi_{\mathbb{A}^d}^{-1}(\{0\}) \tag{6.6}$$

the fiber of $0 \in S^n(\mathbb{A}^d)$.

Proposition 6.1. *1. The projection of $p : \mathrm{Cayley}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^d)$ is a finite morphism which is étale over the open subset $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$ of $\mathrm{Chow}_n(X)$ consisting of multiplicity free 0-cycles.*

2. *For every point $a = [x_1^{n_1}, \dots, x_m^{n_m}] \in \mathrm{Chow}_n(\mathbb{A}^d)$ where $x_1, \dots, x_m$ are distinct points of A^d, and $n_1, \dots, n_m$ are positive integers such that $n_1 + \cdots + n_m = n$, the fiber of $\mathrm{Cayley}_n(\mathbb{A}^d)$ over a is the finite subscheme of $\mathbb{A}^d$*

$$\mathrm{Cayley}_n(a) = \bigsqcup_{i=1}^{m} \mathrm{Spec}(\mathscr{O}_{V,x_i}/\mathfrak{m}_{x_i}^{n_i}), \tag{6.7}$$

where $\mathscr{O}_{V,x_i}$ is the local ring of V at x_i, and $\mathfrak{m}_{x_i}$ its maximal ideal. In particular, as soon as $d \geq 2$ and $n \geq 2$, then the cover $\mathrm{Cayley}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^d)$ is not flat.

3. *Let F be a finite $\mathscr{O}_V$-module of length n and $a \in \mathrm{Chow}_n(\mathbb{A}^d)$ its spectral datum. Then F is supported by the finite subscheme $\mathrm{Cayley}_n(a)$ of $\mathbb{A}^d$. (This is a generalization of the Cayley-Hamilton theorem)*

Proof. We will first describe a set of the generators of the ideal defining the closed subscheme $\mathrm{Cayley}_n(\mathbb{A}^d)$ of $\mathrm{Chow}_n(\mathbb{A}_d) \times \mathbb{A}_d$. Every linear form $v : \mathbb{A}^d \to \mathbb{A}^1$ induces a map on Chow varieties $[v] : \mathrm{Chow}_n(\mathbb{A}^d) \to \mathrm{Chow}_n(\mathbb{A}^1)$ mapping $a = [x_1, \dots, x_n] \in \mathrm{Chow}_n(\mathbb{A}^d)$ on

$$v(a) = [v(x_1), \dots, v(x_n)] \in \mathrm{Chow}_n(\mathbb{A}^1).$$

As the diagram

$$\begin{array}{ccc} \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d & \xrightarrow{\chi_{\mathbb{A}^d}} & \mathrm{S}^n(\mathbb{A}^d) \\ {\scriptstyle [v]\times v}\big\downarrow & & \big\downarrow{\scriptstyle S^n(v)} \\ \mathrm{Chow}_n(\mathbb{A}^1) \times \mathbb{A}^1 & \xrightarrow[\chi_{\mathbb{A}^1}]{} & \mathrm{S}^n(\mathbb{A}^1) = \mathbb{A}^1 \end{array} \tag{6.8}$$

is commutative, the function $f_v = \chi_{\mathbb{A}^1} \circ ([v] \times v) : \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d \to \mathbb{A}^1$ vanishes on $\mathrm{Cayley}_n(\mathbb{A}^d)$. Explicitly for every $a = [x_1, \dots, x_n] \in \mathrm{Chow}_v(V)$, we have

$$f_v(a, x) = (v(x) - v(x_1)) \dots (v(x) - v(x_n)). \tag{6.9}$$

Moreover, for $\mathrm{S}^n(v)$ generates the ideal defining 0 in $S^n(\mathbb{A}^d)$ as v varies in V_d, the functions f_v generate the ideal defining $\mathrm{Cayley}_n(\mathbb{A}^d)$ inside $\mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$. This provides a convenient set of generators of this ideal albeit infinite and even innumerable as k may be.

1. Let $v_1, \dots, v_d$ be the standard basis of the typical d-dimensional vector space V_d whose symmetric algebra $S(V_d)$ is the ring of coordinates of $\mathbb{A}^d$. The functions $f_{v_1}, \dots, f_{v_d}$ cut out a closed subscheme Z of $\mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$ which is finite flat of degree n^d over $\mathrm{Chow}_n(\mathbb{A}^d)$. Since $\mathrm{Cayley}_n(\mathbb{A}^d)$ is a closed subscheme of Z, it is also finite over $\mathrm{Chow}_n(\mathbb{A}^d)$. This proves the first assertion of the proposition.

2. We will prove that for $a = [x_1^{n_1}, \dots, x_m^{n_m}] \in \mathrm{Chow}_n(\mathbb{A}^d)$ where $x_1, \dots, x_m$ are distinct points of $\mathbb{A}^d$, and $n_1, \dots, n_m$ are positive integers such that $n_1 + \dots + n_m = n$, $\mathrm{Cayley}_n(a)$ is the closed subscheme of $\mathbb{A}^d$ defined by the ideal $\mathfrak{m}_{x_1}^{n_1} \dots \mathfrak{m}_{x_m}^{n_m}$ of $S(V_d)$ where $\mathfrak{m}_{x_i}$ is the maximal ideal corresponding to the point $x_i \in \mathbb{A}^d$.

   Let us denote I_a the ideal of $S(V_d)$ defining the finite subscheme $\mathrm{Cayley}_n(a)$ in $\mathbb{A}^d$. We first prove that $I = I_{x_1} \dots I_{x_n}$ where A/I_{x_i} is supported by some finite thickening of the point x_i. For this we only need to prove that for every $x \notin \{x_1, \dots, x_m\}$, there exists a function $f \in I_a$ such that $f \notin \mathfrak{m}_x$. We recall that the ideal I_a is generated by the functions $f_v(a) : \mathbb{A}^d \to \mathbb{A}^1$ as v varies in V_d. Choose a linear form $v \in V_d$ a linear form on $\mathbb{A}^d$ such that $v(x) \neq v(x_i)$ for all $i \in \{1, \dots, m\}$, then we have $f_v(a)(x) \neq 0$ by (6.9).

   As $x_1, \dots, x_m$ play equivalent roles, we can focus our attention on x_1. It only remains to prove that the images of the functions $f_v(a)$ in the localization $S(V_d)_{x_1}$ of $S(V_d)$ at x_1, as v varies in V_d, generate the ideal $\mathfrak{m}_{x_1}^{n_1}$. From (6.9), we already know that $f_v(a) \in \mathfrak{m}_{x_1}^{n_1}$ for every $v \in V_d$. By the Nakayama lemma, we only need to prove that the images of $f_v(a)$ in $\mathfrak{m}_{x_1}^{n_1}/\mathfrak{m}_{x_1}^{n_1+1}$ generate this vector space as v varies in V_d. We observe that for $v \in V_d$ such that $v(x_1) \neq v(x_i)$ for $i \in \{2, \dots, m\}$, the factors $v(v) - v(x_2), \dots v(v) - v(v_m)$ are all invertible at x_1, it is enough to prove that for $T \in V^*$ satisfying the open condition $v(x_1) \neq v(x_i)$ for $i \in \{2, \dots, m\}$, the functions $(v(v) - v(x_1))^{n_1}$ generate $\mathfrak{m}_{x_1}^{n_1}/\mathfrak{m}_{x_1}^{n_1+1}$. Here we use again the fact the image of the n-th power map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ span $\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ and this conclusion doesn't change even after we remove from $\mathfrak{m}_x/\mathfrak{m}_x^2$ a closed subset of smaller dimension.

3. By the Chinese remainder theorem we are easily reduced to prove that if E is a finite $S(V_d)$-module of length n, supported by a finite thickening of $x \in \mathbb{A}^d$ then E is annihilated by $\mathfrak{m}_x^n$. Since E is supported by a finite thickening of $x \in V$ it has a structure of $S(V_d)_x$-module where $S(V_d)_x$ is the localization of $S(V_d)$ at x. We consider the decreasing filtration $E \supset \mathfrak{m}_x E \supset \mathfrak{m}_x^2 E \supset \cdots$. By the Nakayama lemma, we know that for $m \in \mathbb{N}$, $\mathfrak{m}_x^m E/\mathfrak{m}_x^{m+1} E = 0$ implies $\mathfrak{m}_x^m E = 0$. It follows that as long as $\mathfrak{m}_x^m E \neq 0$, we have $\dim_k(\mathfrak{m}_x^i E/\mathfrak{m}_x^{i+1} E) \geq 1$ for all $i \in \{0, \dots, m\}$ and it follows that $m + 1 \leq n$. We conclude that $\mathfrak{m}_x^n E = 0$.

This completes the proof of Proposition 6.1 □

There is another construction of a possibly slightly different spectral cover of $\mathrm{Chow}_n(\mathbb{A}^d)$. We consider the action of $\mathfrak{S}_{n-1}$ on $(\mathbb{A}^d)^n$ by letting it permuting the coordinates $(x_1, \dots, x_{n-1})$ and leaving x_n fixed. The geometric invariant quotient $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1}$ is a normal scheme equipped with a morphism $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to (\mathbb{A}^d)^n /\!/ \mathfrak{S}_n$ which is finite and generically finite étale of rank n. We also have a morphism $\iota : (\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$ given by $(c'_1, \dots, c'_{n-1}, x_n) \mapsto (c_1, \dots, c_n, x_n)$ as in (6.1). These equations imply that ι is a closed embedding which identifies $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1}$ with a reduced closed subscheme of $\mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$.

Proposition 6.2. *The morphism* $\iota : (\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$ *is a closed embedding. It factors through a universal homeomorphism*

$$(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to \mathrm{Cayley}_n(\mathbb{A}^d) \tag{6.10}$$

which is an isomorphism over $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$.

Proof. The morphism $\iota : (\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d$ is obtained by restriction from a morphism $\prod_{i=1}^{n-1} S^i\mathbb{A}^d \times \mathbb{A}^d \to \prod_{i=1}^{n} S^i\mathbb{A}^d \times \mathbb{A}^d$ given by the equation (6.1), which is clearly a closed embedding. It follows that ι is a closed embedding.

Let $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$ denote the open subscheme of $\mathrm{Chow}_n(\mathbb{A}^d)$ consisting of multiplicity free zero-cycles. Let us denote $(\mathbb{A}^d)^{n,\circ}$ the preimage of B° which is the complement in $(\mathbb{A}^d)^n$ of all diagonals. The morphism $(\mathbb{A}^d)^{n,\circ} \to \mathrm{Chow}_n^\circ(\mathbb{A}^d)$ is finite, étale and Galois of Galois group $\mathfrak{S}_n$. The morphism $(\mathbb{A}^d)^{n,\circ} \to (\mathbb{A}^d)^{n,\circ}/\mathfrak{S}_{n-1}$ is finite, étale and Galois of Galois group $\mathfrak{S}_{n-1}$. It follows that the morphism $(\mathbb{A}^d)^{n,\circ}/\mathfrak{S}_{n-1} \to \mathrm{Chow}_n^\circ(\mathbb{A}^d)$ is finite, étale of degree $|\mathfrak{S}_n|/|\mathfrak{S}_{n-1}| = n$.

Over $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$, the morphism $\iota : (\mathbb{A}^d)^{n,\circ} /\!/ \mathfrak{S}_{n-1} \to B^\circ \times \mathbb{A}^d$ clearly induces an isomorphism of $(\mathbb{A}^d)^{n,\circ} /\!/ \mathfrak{S}_{n-1}$ on $\mathrm{Cayley}_n^\circ(\mathbb{A}^d)$ the preimage of $\mathrm{Chow}_n^\circ(\mathbb{A}^d)$ in $\mathrm{Cayley}_n(\mathbb{A}^d)$. Since $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1}$ is an integral scheme, the function $x^n - c_1x^{n-1} + \cdots + (-1)^n c_n$ which vanishes over $(\mathbb{A}^d)^{n,\circ} /\!/ \mathfrak{S}_{n-1}$ has to vanish on all $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1}$. It follows that the morphism ι factors through a morphism $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to \mathrm{Cayley}_n(\mathbb{A}^d)$. This morphism is finite since $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1}$ is finite over $\mathrm{Chow}_n(\mathbb{A}^d)$. One can check directly that the morphism $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1} \to \mathrm{Cayley}_n(\mathbb{A}^d)$ induces a bijection over the k-points, which implies that it is a universal homeomorphism. □

Remark 6.3. Drinfeld asked the question whether the morphism (6.10) is an isomorphism, as in the case $d = 1$. Of course if $\mathrm{Cayley}_n(\mathbb{A}^d)$ is reduced, it would follow from the above proposition that the homeomorphism (6.10) is an isomorphism.

Recall that in the case $G = \mathrm{GL}_n$, the closed subscheme B of A constructed in Theorem 4.1, is $B = \mathrm{Chow}_n(\mathbb{A}^d)$. As the universal spectral cover on B, we will take

$$B^\bullet = \mathrm{Cayley}_n(\mathbb{A}^d)$$

instead of $(\mathbb{A}^d)^n /\!/ \mathfrak{S}_{n-1}$. The reason is that we dispose a nice description 6.1 of the fibers of $B^\bullet$ over B, and a generalization of the theorem of Cayley-Hamilton.

The morphism $p_n : B^\bullet \to B$ being GL_d-equivariant, it can be twisted by any rank d vector bundle. For every geometric point $b \in \mathscr{B}_X(k)$, we have a morphism $b : X \to B/\mathrm{GL}_d$ lying over $\tau_X^* : X \to \mathbb{B}\mathrm{GL}_d$. By forming the Cartesian product

$$\begin{array}{ccc} X_b^\bullet & \longrightarrow & B^\bullet/\mathrm{GL}_d \\ \downarrow & & \downarrow \\ X & \longrightarrow & B/\mathrm{GL}_d \end{array} \tag{6.11}$$

we obtain the spectral cover $X_b^\bullet$ of X corresponding to b. Since $B^\bullet \to B$ is a finite morphism, $X_B^\bullet \to X$ is a finite covering. If $b \in \mathscr{B}_X^\heartsuit$ i.e $b(X)$ has non-empty intersection with B°/GL_d, the covering $X_b^\bullet \to X$ is generically finite étale of degree n.

If X is a curve, and if the spectral curve $X_b^\bullet$ is integral, after Beauville-Narasimhan-Ramanan [4], there is an equivalence of categories between the category of Higgs bundles (E,θ) of spectral datum a and the category of torsion-free $\mathscr{O}_{X_b^\bullet}$ of generic rank 1. This equivalence can be generalized to the case $d \geq 1$ with aid of the concept of Cohen-Macaulay sheaves.

Let us recall some basic facts about Cohen-Macaulay sheaves. A coherent sheaf M on a purely n-dimensional scheme Y is said to be *maximal Cohen-Macaulay* if $\dim(\mathrm{Supp}(M)) = \dim(Y)$ and $\mathrm{H}^i(\mathbb{D}(M)) = 0$ for $i \neq n$. A family of maximal Cohen-Macaulay sheaves on Y parametrized by S is a coherent sheaf M on $Y \times S$ flat over S and satisfying the property: for every $s \in S$, the restriction M_s to $Y \times \{s\}$ is maximal Cohen-Macaulay. If S is Cohen-Macaulay, then the above conditions imply that M itself is maximal Cohen-Macaulay. If Y is proper, the functor that associates with every test scheme S the groupoid of all families of Cohen-Macaulay sheaves on Y parametrized by S is an algebraic stack, see [2, 2.1].

We also recall an important fact about Cohen-Macaulay module. Let M be a Cohen-Macaulay R-module of finite type. Suppose that R is a finite A-algebra with A being a regular ring. Then M is a locally free A-module of finite type. We refer to [5, section 2] for a nice discussion on Cohen-Macaulay modules and for further references therein, or the comprehensive treatment in [6].

Proposition 6.4. *For every $b \in \mathscr{B}_X^\heartsuit(k)$, the fiber $h_X^{-1}(b)$ of the Hitchin morphism is isomorphic to the algebraic stack of maximal Cohen-Macaulay sheaves of generic rank one on the spectral cover $X_b^\bullet$.*

Proof. Let $(E,\theta) \in h_X^{-1}(b)$ a Higgs bundle of rank n whose spectral datum is $b \in \mathscr{B}_X^\heartsuit$. Then $E = p_*F$ where F is a coherent sheaf over the cotangent T_X^*. By the Cayley-Hamilton theorem, F is supported by the spectral cover $X_b^\bullet \subset T_X^*$. We have then $E = p_{b*}F$ where F is a coherent sheaf on $X_b^\bullet$. Since $p_b : X_b^\bullet \to X$ is a finite morphism, and E is a vector bundle over X, F is a maximal Cohen-Macaulay sheaf. Moreover, since p_b is generically finite étale of degree n, F has generic rank one. Inversely, if F is a maximal Cohen-Macaulay sheaf of generic rank one over $X_b^\bullet$, then $E = p_{b*}F$ is a vector bundle of rank n over X. It is naturally equipped with a Higgs field $\theta : E \otimes_{\mathscr{O}_X} \mathscr{T}_X \to E$ as $X_b^\bullet$ is a closed subscheme of T_X^*. □

In spite of the simplicity of the above description of $h_X^{-1}(b)$, it is not of great use by itself alone. For instance, it doesn't imply that $h_X^{-1}(b)$ is non empty. The difficulty is that in general for the spectral cover $X_b^\bullet$ is not Cohen-Macaulay itself, we do not know any recipe to construct coherent Cohen-Macaulay sheaves on $X_b^\bullet$. At this point, we see that in order to obtain a useful description of $h_X^{-1}(b)$, one needs to construct a finite Cohen-Macaulayfication of $X_b^\bullet$. This can be done in the case of surfaces.

7 Cohen-Macaulay spectral surfaces

In the case of surfaces, for every $b \in \mathcal{B}_X^{\heartsuit}(k)$, the spectral surface $X_b^\bullet$ admits a canonical finite Cohen-Macaulayfication whose construction relies on the theory of Hilbert schemes of points on surfaces and Serre's theorem on extending vector bundles on smooth surfaces across closed subschemes of codimension two. We will first recall Serre's theorem on extending locally free sheaves across a closed subscheme of codimension 2, see [21, Prop. 7].

Theorem 7.1. *Let X be a smooth surface over k, Z a closed subscheme of codimension 2 of X and $j : U \to X$ the open immersion of the complement U of Z in X. Then the functor $V \to j_*V$ is an equivalence of categories between the category of locally free sheaves on U and locally free sheaves on X. Its inverse is the functor j^*.*

As we are now restricted to the case $G = \mathrm{GL}_n$ and $d = 2$, the subscheme B of $A = \mathbb{A}^2 \times S^2\mathbb{A}^2 \times \cdots \times S^n\mathbb{A}^2$ is canonically isomorphic to the Chow scheme $\mathrm{Chow}_n(\mathbb{A}^2)$ of zero-cycles of length n on $\mathbb{A}^2$. We recall that a point $b \in \mathcal{B}_X$ is a section $b : X \to \mathrm{Chow}_n(\mathbb{A}^2)/\mathrm{GL}_2$ lying over $\tau_X^* : X \to \mathbb{B}\mathrm{GL}_2$ representing the cotangent bundle T_X^*. In other words, b is a section of the relative Chow scheme

$$\mathrm{Chow}_n(T_X^*/X) \to X$$

obtained from $\mathrm{Chow}_n(\mathbb{A}^2)$ by twisting it by the GL_2-torsor attached to the cotangent bundle T_X^* of X.

Let $\mathrm{Chow}_n^\circ(\mathbb{A}^2)$ denote the open locus of $\mathrm{Chow}_n(\mathbb{A}^2)$ consisting of multiplicity free zero-cycles, and Q its complement. Let $\mathrm{Chow}_n^\circ(T_X^*/X)$ the corresponding open locus in $\mathrm{Chow}_n(T_X^*/X)$, and $Q(T_X^*/X)$ its complement. The $\mathcal{B}_X^{\heartsuit}$ is the open locus in $\mathcal{B}_X$ consisting of maps $b : X \to \mathrm{Chow}_n(\mathbb{A}^2)/\mathrm{GL}_2$ mapping the generic point of X to the open locus $\mathrm{Chow}_n^\circ(T_X^*/X)$. In other words

$$\mathcal{B}_X^{\heartsuit} = \{b \in \mathcal{B}_X \,|\, \dim b^{-1}(Q(T_X^*/X)) \leq 1\}. \tag{7.1}$$

We first recall some well known fact about the Hilbert schemes of 0-dimensional subschemes of a surface. Let $\mathrm{Hilb}_n(\mathbb{A}^2)$ denote the moduli space of zero-dimensional subschemes of length n of $\mathbb{A}^2$. A point of $\mathrm{Hilb}_n(\mathbb{A}^2)$ is a 0-dimensional subscheme Z of $\mathbb{A}^2$ of length n that will be of the form $Z = \bigsqcup_{\alpha \in \mathbb{A}^2} Z_\alpha$ where Z_α is a local 0-dimensional subscheme of $\mathbb{A}^2$ whose closed point is α. It is known that the Hilbert-Chow morphism

$$\mathrm{HC}_n : \mathrm{Hilb}_n(\mathbb{A}^2) \to \mathrm{Chow}_n(\mathbb{A}^2). \tag{7.2}$$

given by $Z \mapsto \sum_{\alpha \in \mathbb{A}^2} \mathrm{length}(Z_\alpha)\alpha$, where $\mathrm{length}(Z_\alpha)$ is the length of Z_α, is a resolution of singularities of Chow_n. It is clear that HC_n is an isomorphism over Chow_n°.

As the morphism (7.2) is GL_2-equivariant, we can twist it by any GL_2-bundle, and in particular by the GL_2-bundle associated to the cotangent bundle T_X^* over a smooth surface X

and by doing so we obtain

$$\mathrm{HC}_{T_X^*/X} : \mathrm{Hilb}_n(T_X^*/X) \to \mathrm{Chow}_n(T_X^*/X). \tag{7.3}$$

This morphism is a proper morphism and its restriction the open subset $\mathrm{Chow}_n^\circ(T_X^*/X)$ is an isomorphism. Here the open immersion $\mathrm{Chow}_n^\circ(T_X^*/X) \subset \mathrm{Chow}_n(T_X^*/X)$ is obtained from the open subscheme Chow_n° of Chow_n classifying multiplicity free 0-cycles by the process of GL_2-twisting by the cotangent bundle.

Proposition 7.2. *For every $b \in \mathscr{B}_X^\heartsuit(k)$, there exists a unique finite flat covering*

$$p_a^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X \tag{7.4}$$

of degree n, equipped with a X-morphism $\iota_b : X_b^{\mathrm{CM}} \to T_X^*$ satisfying the following property: there exists an open subset $U \subset X$, whose complement is a closed subset of codimension at least 2, such that ι_a is a closed embedding over U and for every $x \in U$, the fiber $X_b^{\mathrm{CM}}(x)$ is a point of $\mathrm{Hilb}_n(T_X^*/X)(x)$ lying over the point $b(x) \in \mathrm{Chow}_n(T_X^*/X)(x)$. Moreover, the morphism ι : $X_b^{\mathrm{CM}} \to T_X^*$ factors through the closed subscheme $X_b^\bullet$ of T_X^* and the implied morphism $X_b^{\mathrm{CM}} \to X_b^\bullet$ is a finite Cohen-Macaulayfication of $X_b^\bullet$.

Proof. Let U° be the preimage of $\mathrm{Chow}_n^\circ(T_X^*/X)$ by the section $a : X \to \mathrm{Chow}_n(T_X^*/X)(x)$. By assumption $b \in \mathscr{B}_X^\heartsuit$, U° is a non empty open subset of X. As the morphism $\mathrm{HC}_{T_X^*/X}$ of (7.3) is an isomorphism over $\mathrm{Chow}_n(T_X^*/X)$, we have a unique lifting

$$b_{\mathrm{Hilb}}^\circ : U^\circ \to \mathrm{Hilb}_n(T_X^*/X) \times_X U^\circ$$

laying over the restriction $b^\circ = b|U^\circ$.

Since the Hilbert-Chow morphism (7.3) is proper, there exists an open subset $U \subset X$, larger than U°, whose complement $X - U$ is a closed subscheme of codimension at least 2, such that $b' : U^\circ \to \mathrm{Hilb}_n'(T_X^*/X) \times_X U^\circ$ extends to

$$b_{\mathrm{Hilb}}^U : U \to \mathrm{Hilb}_n(T_X^*/X) \times_X U.$$

By pulling back from $\mathrm{Hilb}_n(T_X^*/X)$ the tautological family of subschemes of T_X^*, we get a finite flat morphism $U_b^+ \to U$ of degree n, equipped with a closed embedding $\iota_U : U_b^+ \to T_U^*$.

According to Serre's theorem on extending vector bundles over surfaces, there exists a unique the finite flat covering $X_b^{\mathrm{CM}} \to X$ of degree n extending the covering U_b^+ of U. The closed embedding $\iota_U : U_b^+ \to T_U^*$ extends to a finite morphism $X_b^{\mathrm{CM}} \to T_X^*$ which may no longer be a closed embedding.

By construction $p_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X$ is a finite flat morphism of degree n, it follows from smoothness of X that X_b^{CM} is a Cohen-Macaulay surface. Apply the generalized Cayley-Hamilton theorem to the vector bundle $p_{b*}^{\mathrm{CM}} \mathscr{O}_{X_b^{\mathrm{CM}}}$, as $\mathscr{O}$-module over T_X^*, it is supported by $X_b^\bullet$. It follows that the morphism $X_b^{\mathrm{CM}} \to T_X^*$ factors through $X_b^\bullet$. Since X_b^{CM} is finite over X, it is also finite over $X_b^\bullet$. As $X_b^{\mathrm{CM}} \to X_b^\bullet$ is an isomorphism over the nonempty open subset U°, it is a finite Cohen-Macaulayfication of $X_b^\bullet$. □

Instead of using the Hilbert scheme, we can construct X_b^{CM} over the height one points as follows. Let $U^\circ = b^{-1}(\mathrm{Chow}_n^\circ(\mathrm{T}_X^*/X))$ and Z' le complement of U'. Let z be the generic point of an irreducible component of Z'. The localization of X at x is $X_z = \mathrm{Spec}(\mathscr{O}_{X,z})$ where $\mathscr{O}_{X,z}$ is a discrete valuation ring. By restricting $p_{b*}\mathscr{O}_{X_b^\bullet}$ to $\mathscr{O}_{X,z}$ we get a module of finite type which may have torsion. By considering the quotient $\mathrm{Spec}(p_{b*}\mathscr{O}_{X_b^\bullet}/(p_{b*}\mathscr{O}_{X_b^\bullet}^{\mathrm{tors}}))$ we obtain a locally free $\mathscr{O}_{X,z}$-module and thus a section $X_z \to \mathrm{Hilb}_n(\mathrm{T}_X^*/X) \times_X X_z$ over $b|_{X_z}$. By uniqueness of such a section we have an isomorphism

$$\mathrm{Spec}(p_{b*}\mathscr{O}_{X_b^\bullet}/(p_{b*}\mathscr{O}_{X_b^\bullet})^{\mathrm{tors}}) \simeq \mathrm{Spec}(p_{b*}^{\mathrm{CM}}\mathscr{O}_{X_b^{\mathrm{CM}}}) \tag{7.5}$$

over the complement of a codimension two subscheme of X.

Remark 7.3. We don't know whether the construction of the Cohen-Macaulay spectral surface X_b^{CM} works well in families. The issue is that the construction makes use of the equivalence of categories from Theorem 7.1 which does not work well in families.

Theorem 7.4. *For every $b \in \mathscr{B}_X^\heartsuit(k)$, the fiber $h_X^{-1}(b)$ is the moduli stack of Cohen-Macaulay sheaves F of generic rank one over the Cohen-Macaulay spectral surface X_b^{CM}. It contains in particular the Picard stack $\mathscr{P}_b$ of line bundles on X_b^{CM}. The action of $\mathscr{P}_b$ on itself by translation extends to an action of $\mathscr{P}_b$ on $h_X^{-1}(b)$.*

In particular, $h_X^{-1}(b)$ is non empty.

Proof. Let $(V,\theta) \in \mathscr{M}_X$ be a Higgs bundle over X lying over $b \in \mathscr{B}_X^\heartsuit(k)$. The Higgs field $\theta : V \otimes \mathscr{T}_X \to V$ define a homomorphism $\mathrm{S}_X(\mathscr{T}_X) \to \mathrm{End}_X(V)$ which factors through $p_{a*}\mathscr{O}_{Y_a}$ by the generalized Cayley-Hamilton theorem.

Let $U^\circ = b^{-1}(\mathrm{Chow}_n^\circ(T_X^*/X))$ and Z' the complement of U°. Let z be the generic point of an irreducible component of Z'. The localization of X at x is $X_z = \mathrm{Spec}(\mathscr{O}_{X,z})$ where $\mathscr{O}_{X,z}$ is a discrete valuation ring. Over X_z we have a homomorphism

$$p_{b*}\mathscr{O}_{Y_b^\bullet} \otimes_{\mathscr{O}_X} \mathscr{O}_{X_z} \to \mathrm{End}_X(V) \otimes_{\mathscr{O}_X} \mathscr{O}_{X_z}.$$

Since the target is clearly torsion free, this homomorphism factors through (7.5). Thus over an open subset $U \subset X$ whose complement is of codimension two, the above morphism factors through a homomorphism of algebras

$$p_{b*}^{\mathrm{CM}}\mathscr{O}_{X_b^{\mathrm{CM}}} \otimes_{\mathscr{O}_X} \mathscr{O}_U \to \mathrm{End}_X(V) \otimes_{\mathscr{O}_X} \mathscr{O}_U.$$

By applying Serre's theorem again, we have a canonical homomorphism $p_{b*}^{\mathrm{CM}}\mathscr{O}_{X_b^{\mathrm{CM}}} \to \mathrm{End}_X(V)$ so that $V = \tilde{p}_{a*}F$ where F is a Cohen-Macaulay $\mathscr{O}_{X_b^{\mathrm{CM}}}$-module of generic rank one.

Since $p_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X$ is finite and flat, for every line bundle L on X_b^{CM}, $p_{b*}^{\mathrm{CM}}L$ is a vector bundle of rank n carrying a Higgs field. Thus $\mathscr{P}_b \subset h_X^{-1}(b)$. We have an action of $\mathscr{P}_b$ on $h_X^{-1}(b)$ given by $(L,F) \mapsto L \otimes_{\mathscr{O}_{X_b^{\mathrm{CM}}}} F$ where L is a line bundle on X_b^{CM} and F is a Cohen-Macaulay sheaf of generic rank one. □

Remark 7.5. Let $b \in \mathscr{B}_X^{\heartsuit}(k)$ such that the Cohen-Macaulay surface X_b^{CM} is integral. According to [1], the stack of Cohen-Macaulay sheaves on X_b^{CM} of generic rank one, which is the Hitchin fiber $h_X^{-1}(b)$ by the Theorem 7.4, admits a compactification $\mathrm{Pic}(X_b^{\mathrm{CM}})^{\square}$ given by the moduli stack of torsion free rank one sheaves on X_b^{CM}. On the other hand, by [23, Theorem 6.11], the (extended) Hitchin map $h_X^{\square} : \mathscr{M}_X^{\square} \to \mathscr{A}_X$ from $\mathscr{M}_X^{\square}$, the moduli stack of torsion free Higgs sheaves on X, to $\mathscr{A}_X$ is proper. One can show that the fiber $(h_X^{\square})^{-1}(b)$ is isomorphic to $\mathrm{Pic}(X_b^{\mathrm{CM}})^{\square}$.

Definition 7.6. *We define $\mathscr{B}_X^{\diamondsuit}(k)$ to be the subset of $\mathscr{B}_X^{\heartsuit}(k)$ consisting of those points b such that the corresponding Cohen-Macaulay spectral surface X_b^{CM} is normal.*

Lemma 7.7. *For $b \in \mathscr{B}_X^{\diamondsuit}(k)$, the neutral component $\mathscr{P}_b^0$ of $\mathscr{P}_b$ is a quotient of abelian variety by $\mathbb{G}_m$ acting trivially.*

Proof. This is a consequence of a theorem of Geisser [10, Theorem 1]. Geisser's theorem states that the multiplicative part the neutral part of the Picard variety P of an algebraic variety Y is trivial if and only if $\mathrm{H}^1_{\mathrm{et}}(Y, \mathbb{Z})$ is trivial whereas the unipotent part is trivial if and only if Y is semi-normal. If Y is normal, $\pi_1(Y)$ is a profinite group, being a quotient of the Galois group of the generic point, and therefore can't afford a nontrivial continuous homomorphism to $\mathbb{Z}$. It follows that $\mathrm{H}^1_{\mathrm{et}}(Y, \mathbb{Z})$ is trivial. On the other hand, a normal variety is certainly also semi-normal. Now after Geisser, the Picard variety P_b^0 associated to $\mathscr{P}_b^0$ is an abelian variety. We have $\mathscr{P}_b^0 = P_b^0/\mathbb{G}_m$. □

Proposition 7.8. *For $b \in \mathscr{B}_X^{\diamondsuit}(k)$, the action of $\mathscr{P}_b$ on the Hitchin fiber $h_X^{-1}(b)$ is free and $h_X^{-1}(b)$ is a disjoint union of $\mathscr{P}_b$-orbits.*

Proof. If a line bundle $L \in \mathscr{P}_b$ has a stabilizer $F \in h_X^{-1}(b)$ then, as any such F is locally free of rank one on the smooth locus U_b of X_b^{CM}, the line bundle L is trivial on U_b. Since X_b^{CM} is normal, the compliment $X_b^{\mathrm{CM}} \setminus U_b$ is zero dimensional, it implies L is trivial hence the action of $\mathscr{P}_b$ is free. We claim that the $\mathscr{P}_b$ orbits on $h_X^{-1}(b)$ are open and closed. The closedness follows from the lemma above. To show that $\mathscr{P}_b$-orbits are open, we observe that $h_X^{-1}(b)$ is isomorphic to the stack of reflexive sheaves of rank one on X_b^{CM} and, for any $F \in h_X^{-1}(b)$, the assignment sending $F' \in h_X^{-1}(b)$ to the reflexive hull of $F' \otimes_{X_b^{\mathrm{CM}}} F$ (that is, the double dual of $F' \otimes_{X_b^{\mathrm{CM}}} F$) defines an automorphism of $h_X^{-1}(b)$ mapping $\mathscr{P}_b$ isomorphically to the $\mathscr{P}_b$-orbit through F. Since $\mathscr{P}_b$ is open in $h_X^{-1}(b)$ (see [1]), it implies that $\mathscr{P}_b$-orbits are open in $h_X^{-1}(b)$. The proposition follows.

□

We expect that $\mathscr{B}_X^{\diamondsuit}(k)$ is a non-empty open subset of $\mathscr{B}_X(k)$ for most algebraic surfaces. The non-emptiness of $\mathscr{B}_X^{\diamondsuit}(k)$ is closely related to question on zero locus of symmetric differentials, which seems very little is known in higher dimension.

8 Surfaces fibered over a curve

In this section we investigate the spectral surfaces $X_b^\bullet$ and the Cohen-Macaulay spectral surface X_b^{CM} in the case when X is a fibration over a curve C and apply our findings to ruled and elliptic surfaces.

Let X be a smooth projective surface and let C be a smooth projective curve. Assume there is a proper flat surjective map $\pi : X \to C$ such that the generic fiber a smooth projective curve. We denote by $X^0 \subset X$ the largest open subset such that π is smooth. Consider the cotangent morphism $d\pi : T_C^* \times_C X \to T_X^*$. It induces a map

$$[d\pi] : \mathrm{Chow}_n(T_C^*/C) \times_C X \to \mathrm{Chow}_n(T_X^*/X)$$

on the relative Chow varieties.

For every section $b_C : C \to \mathrm{Chow}_n(T_C^*/C)$, the composition

$$b_X : X \simeq C \times_C X \overset{a_C \times \mathrm{id}_X}{\to} \mathrm{Chow}_n(T_C^*/C) \times_C X \overset{[d\pi]}{\to} \mathrm{Chow}_n(T_X^*/X)$$

is a section of $\mathrm{Chow}_n(T_X^*/X) \to X$ and the assignment $b_C \to b_X$ defines a map on the spaces of spectral data

$$\iota_\pi : \mathscr{B}_C \to \mathscr{B}_X. \tag{8.1}$$

We claim that the map above is a closed embedding. To see this we observe that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{B}_C & \xrightarrow{\iota_\pi} & \mathscr{B}_X \\ \downarrow{\scriptstyle \iota_C} & & \downarrow{\scriptstyle \iota_X} \\ \mathscr{A}_C & \xrightarrow{j_\pi} & \mathscr{A}_X \end{array} \tag{8.2}$$

where the vertical arrows are the embeddings in 4.1, and the bottom arrow is the embedding

$$j_\pi : \mathscr{A}_C = \bigoplus_{i=1}^{n} \mathrm{H}^0(C, S^i\Omega_C^1) \hookrightarrow \mathscr{A}_X = \bigoplus_{i=1}^{n} \mathrm{H}^0(X, S^i\Omega_X^1)$$

induced by the injection of vector spaces $\mathrm{H}^0(C, S^i\Omega_C^1) = \mathrm{H}^0(X, \pi^* S^i\Omega_C^1) \to \mathrm{H}^0(X, S^i\Omega_X^1)$. The claim follows. Note that, since $\dim C = 1$, the left vertical arrow in (8.2) is in fact an isomorphism. From now on we will view $\mathscr{B}_C$ as a subspace of $\mathscr{B}_X$. Since the cotangent map $d\pi : T_C^* \times_C X \to T_X^*$ is a closed imbedding over the open locus X^0, we have

$$\mathscr{B}_C^\heartsuit = \mathscr{B}_C \cap \mathscr{B}_X^\heartsuit.$$

For any $b \in \mathscr{B}_C$, we denote by $C_b^\bullet \to C$ the corresponding spectral curve and we define $X_b^+ = C^\bullet \times_C X$. The natural projection map $p_b^+ : X_b^+ \to X$ is finite flat of degree n. Since X is smooth, it follows that X_b^+ is a Cohen-Macaulay surface.

Lemma 8.1. *There exits a finite X-morphism $p_b^+ : X_b^+ \to X_b^\bullet$ which is a generic isomorphism if $b \in \mathscr{B}_C^\heartsuit$. If the fibration $\pi : X \to C$ has only reduced fibers, then for any $b \in \mathscr{B}_C^\heartsuit$, the map $q_b^+ : X_b^+ \to X_b^\bullet$ is isomorphic to the finite Cohen-Macaulayfication $q_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X_b^\bullet$ in (7.4) (which is well-defined since $b \in \mathscr{B}_X^\heartsuit$).*

Proof. Let $i_b^+ : X_b^+ \to T_X^*$ be the restriction of the cotangent morphism $d\pi : T_C^* \times_C X \to T_X^*$ to the closed sub-scheme $X_b^+ \subset T_C^* \times_C X$. By the Cayley-Hamilton theorem the map i_b^+ factors through the spectral surface $X_b^\bullet$. Let $q_b^+ : X_b^+ \to X_b^\bullet$ be the resulting map. As X_b^+ is finite over X, the map q_b^+ is finite. In addition, if $b \in \mathscr{B}_C^\heartsuit$, then both X_b^+ and $X_b^\bullet$ are generically étale over X of degree n and it implies that q_b^+ is a generic isomorphism.

Assume the fibers of π are reduced. Then the smooth locus X^0 of the map π is open and its complement $X - X^0$ is a closed subset of codimension 2. Since the map $i_b^+ : X_b^+ \to T_X^*$ is a closed embedding over X^0, Proposition 7.2 implies the finite flat covering $q_b^+ : X_b^+ \to X_b^\bullet$ is isomorphic to the finite Cohen-Macaulayfication $q_b^{\mathrm{CM}} : X_b^{\mathrm{CM}} \to X_b^\bullet$. □

Definition 8.2. *We define $\mathscr{B}_C^\diamondsuit$ to be the open subset of $\mathscr{B}_C^\heartsuit$ consisting of those points b such that the corresponding spectral curve C_b is smooth and irreducible.*

Corollary 8.3. *Assume the fibration $\pi : X \to C$ has only reduced fibers. Then we have $\mathscr{B}_C^\diamondsuit \subset \mathscr{B}_X^\diamondsuit$, that is, the surface X_b^{CM} is normal for $b \in \mathscr{B}_C^\diamondsuit$.*

Proof. Since X_b^{CM} is Cohen-Macaulay, by Serre's criterion for normality, it suffices to show that the $X_b^{\mathrm{CM}} \simeq X_b^+$ is smooth in codimension ≤ 1. The assumption implies the complement $X - X^0$ has codimension at least 2. Since C_b is smooth for $b \in \mathscr{B}_C^\diamondsuit$, the open subset $X_b^{+0} := \widetilde{C}_a \times_C X^0 \subset X_b^+$ is smooth (since the map $X_b^{+0} \to C_b$ and C_b are smooth) and the complement $X_b^+ - X_b^{+0}$ has codimension at least 2. The corollary follows. □

Example 8.4. Consider the case when $X = C \times \mathbb{P}^1$ and $n = 2$. We have

$$\mathscr{B}_X = \mathscr{B}_C = \mathrm{H}^0(C, \Omega_C^1) \oplus \mathrm{H}^0(C, \mathrm{S}^2\Omega_C^1).$$

Let $b = (b_1, b_2) \in \mathscr{B}_C^\heartsuit$ and $p_b : X_b^\bullet \to X$ be the corresponding spectral surface. Then étale locally over X, the surface $X_b^\bullet$ is isomorphic to the closed subscheme of $\mathrm{Spec}(k[x_1, x_2, t_1, t_2])$ defined by the equations

$$\begin{cases} t_1^2 + b_1 t_1 + b_2 = 0 \\ t_2(2t_1 + b_1) = 0 \\ t_2^2 = 0 \end{cases} \tag{8.3}$$

here x_1, x_2 are local coordinate of C and $\mathbb{P}_1$ and $b_i \in k[x_1]$. Let $Dis = (b_1^2 - 4b_2 = 0) \subset C$ be the discriminant divisor for b. From (8.3) we see that $X_b^\bullet$ is an étale cover of degree 2 away from the divisor $Dis \times \mathbb{P}^1 \subset X$. Note that the spectral surface $p_b : X_b^\bullet \to X$ is not flat over X as the push-forward $p_{b*}\mathcal{O}_{X_b^\bullet}$ has length three over $Dis \times \mathbb{P}^1$. The finite Cohen-Macaulayfication $X_b^{\mathrm{CM}} \to X_b^\bullet$ is given by the flat quotient $\mathrm{Spec}(p_{b*}\mathcal{O}_{X_b^\bullet}/(p_{b*}\mathcal{O}_{X_b^\bullet})^{\mathrm{tors}})$ which is isomorphic to $X_b^{\mathrm{CM}} \simeq C_b \times \mathbb{P}^1$. The Hitchin fiber $h_X^{-1}(b)$ is isomorphic to

$$h_X^{-1}(b) = h_C^{-1}(b) \times \mathscr{P}ic(\mathbb{P}^1) = h_C^{-1}(b) \times B\mathbb{G}_m.$$

Proposition 8.1. Let X be a smooth projective surface and $\pi : X \to C$ be either a ruled surface, or a non-isotrivial elliptic surface with reduced fibers. Then for every n, the pull-back map

$$\mathrm{H}^0(C, S^n\Omega_C^1) \to \mathrm{H}^0(X, S^n\Omega_X^1)$$

is an isomorphism.

It follows from the proposition above that in the case of ruled surfaces and non-isotrivial elliptic surface with reduced fibers, we have $\mathscr{A}_C = \mathscr{A}_X$. Since $\mathscr{B}_C = \mathscr{A}_C$, we have $\mathscr{B}_X = \mathscr{B}_C$ and $\mathscr{B}_X^\lozenge$ and $\mathscr{B}_X^\heartsuit$ are open dense in $\mathscr{B}_X$. For every $b \in \mathscr{B}_C$, we have a spectral curve $C_b^\bullet$ which is finite flat of degree n over C. We also have the spectral surface $X_b^\bullet$ which is a finite scheme over X embedded in its cotangent bundle T_X^*. The Cohen-Macaufication of $X_b^\bullet$ is $X_b^+ = C_b \times_C X$. In the case of elliptic surfaces, the morphism $X_b^{\mathrm{CM}} \to X_b^\bullet$ may not be an isomorphism, and X_b^{CM} may not be embedded in the cotangent bundle T_X^*. The existence of the Cohen-Macaulay spectral cover guarantees that $h_X^{-1}(b) \neq \emptyset$.

The Proposition 8.1 is obvious for ruled surfaces. Let us investigate it in the case of elliptic surfaces. We assume there is a proper flat map $\pi : X \to C$ from X to a smooth projective curve C with general fiber a smooth curve of genus one. We will focus on the case when $\pi : X \to C$ is not isotrivial, relatively minimal, and has reduced fibers (e.g., semi-stable non-isotrivial elliptic surfaces). Let X^0 denote the largest open subset of X such that the restriction of π to X^0 is a smooth morphism $\pi^0 : X^0 \to C$. Since the geometric fibers of π are all reduced, the complement of X^0 in X is a zero-dimensional subscheme. Over X^0, we have an exact sequence of tangent bundles

$$0 \to \mathscr{T}_{X^0/C} \to \mathscr{T}_{X^0} \to (\pi^0)^*\mathscr{T}_C \to 0 \tag{8.4}$$

For every $n \in \mathbb{N}$, we have the exact sequence of symmetric powers

$$0 \to S^{n-1}\mathscr{T}_{X^0} \otimes (\pi^0)^*\mathscr{T}_C \to S^n\mathscr{T}_{X^0} \to S^n(\pi^0)^*\mathscr{T}_C \to 0. \tag{8.5}$$

Let $\eta = \mathrm{Spec}(K)$ be the generic point of C and $E = X \times_C \eta$ which is an elliptic curve over η. The restriction of (8.4) to E is a short exact sequence making the rank two vector bundle $\mathscr{T}_X|_E$ a self-extension of the trivial line bundle of E. As we assume the elliptic fibration π is non isotrivial, the Kodaira-Spencer map is not zero and $\mathscr{T}_X|_E$ is a non-trivial self-extension of the trivial line bundle on E. After Atiyah, such a non-trivial extension is unique up to isomorphism

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E \to 0. \tag{8.6}$$

In other words, the restriction of (8.4) to the generic fiber X_η is isomorphic to (8.6).

Lemma 8.5. *The exact sequence of symmetric powers derived from* (8.6)

$$0 \to S^{n-1}\mathcal{E} \to S^n\mathcal{E} \to \mathcal{O}_E \to 0 \tag{8.7}$$

is not split.

Proof. Indeed if

$$0 \to L' \to V \to L \to 0 \tag{8.8}$$

is an extension of of a line bundle L by a line bundle L' then there is a canonical filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = S^n V$$

of $S^n V$ such that for every $i \in \{1,\dots,n\}$ we have $F_{i+1}/F_i \simeq L^i \otimes L'^{n-i}$ and more over the exact sequence

$$0 \to F_{n-1}/F_{n-2} \to F_n/F_{n-2} \to F_n/F_{n-1} \to 0 \tag{8.9}$$

is isomorphic to the sequence (8.8) tensored by $L^{\otimes(n-1)}$. In particular, if (8.8) is not split, then (8.9) is not split neither, and as a consequence, the exact sequence

$$0 \to F_{n-1} \to F_n \to L^{\otimes n} \to 0$$

is not split. □

Lemma 8.6. *For every $n \in \mathbb{N}$:*

1. $\dim_K \mathrm{Ext}^1(\mathcal{O}_E, S^n\mathcal{E}) = 1$
2. $\dim_K(\mathrm{Hom}(S^n\mathcal{E}, \mathcal{O}_E)) = 1$
3. *the restriction map* $\mathrm{Hom}(S^n\mathcal{E}, \mathcal{O}_E) \to \mathrm{Hom}(S^{n-1}\mathcal{E}, \mathcal{O}_E)$ *is zero.*

Proof. Induction on n using the Ext long exact sequences derived from (8.7). □

It follows from the above lemmas that, for every $n \in \mathbb{N}$, $S^n\mathcal{E}$ is the unique extension of $\mathcal{O}_E$ by $S^{n-1}\mathcal{E}$, up to isomorphism.

Now we prove that pulling back 1-forms defines an isomorphism

$$\mathrm{H}^0(C, S^n\Omega^1_C) \simeq \mathrm{H}^0(X, S^n\Omega^1_X).$$

This map is obviously injective, let us prove that it is also surjective. A symmetric form $\alpha \in \mathrm{H}^0(X, S^n\Omega^1_X)$ gives rise to a linear form $\alpha : S^n\mathscr{T}_X \to \mathscr{O}_X$. By restriction to the generic fiber E of the elliptic fibration, we obtain a map $\alpha_E : S^n\mathscr{E} \to \mathscr{O}_E$. By previous lemma, the restriction of α_E to $S^{n-1}\mathscr{E}$ is zero. It follows that in the exact sequence (8.5), the restriction of α to $S^{n-1}\mathscr{T}_{X^0} \otimes (\pi^0)^*\mathscr{T}_C$ is zero i.e. it factors through $(\pi^0)^*\mathscr{T}_C$. Since the complement of X^0 in X is zero dimensional, α factors through $(\pi^0)^*\mathscr{T}_C$ i.e. it comes from a symmetric form on C. This finishes the proof of Proposition 8.1.

These calculations show that the Hitchin fibration for ruled and elliptic surfaces are closely related to Hitchin fibration for the base curve. This is compatible with the fact that under the Simpson correspondence, stable Higgs bundles for a smooth projective surface X corresponds to irreducible representations of the fundamental group $\pi_1(X)$, and in the case of ruled and non-isotropic elliptic surfaces with reduced fibers we have $\pi_1(X) \simeq \pi_1(C)$ where C is the base curve (see, e.g., [9, Section 7]).

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