MATH GRE PREP: WEEK 1

UCHICAGO REU 2019

(1) Assume $A\colon \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation with $A(5,6) = 3$ and $A(2,1) = -1$. Compute $A(1,4)$.

(A) -2

(B) 1

(C) 2

(D) 5

(E) 6

(2) Suppose $\alpha, \beta > 0$. Compute:

$$\int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x}\, dx.$$

(A) $\log \beta\alpha$

(B) $2 \log \frac{\beta}{\alpha}$

(C) $2 \log \frac{\alpha}{\beta}$

(D) $\log \frac{\beta}{\alpha}$

(E) $\log \frac{\alpha}{\beta}$

(3) Integrate:

$$\int \frac{dx}{1+e^x}.$$

(A) $2x - \log(e^x + 1) + C$

(B) $x + \log(e^x + 1) + C$

(C) $x - \log(e^x + 1) + C$

(D) $x - \log(e^x + e^{-x}) + C$

(E) $x - \log(1 + e^{-x}) + C$

Date: July 8, 2019.

(4) At a banquet, n women and m men are to be seated in a row of $n+m$ chairs. If the entire seating arrangement is to be chosen at random, what is the probability that all of the men will be seated next to each other in m consecutive positions?

(A) $\frac{1}{\binom{n+m}{m}}$

(B) $\frac{m!}{\binom{n+m}{m}}$

(C) $\frac{n!}{(n+m)!}$

(D) $\frac{m!n!}{(n+m-1)!}$

(E) $\frac{m!(n+1)!}{(n+m)!}$

(5) Endow $\mathbb{R}$ with the right topology, generated by $\mathcal{T} = \{(a, \infty) \ : \ a \in \mathbb{R}\}$, and call this space X. Which of the following is false?

(A) X is σ-compact (it is the union of countably many compact subsets).

(B) X is sequentially compact (every sequence has a convergent subsequence).

(C) X is limit point compact (every infinite subset has a limit point in X).

(D) X is Lindelöf (every open cover of X has a countable subcover).

(E) X is pseudoocompact (every continuous function $f\colon X \to \mathbb{R}$ is bounded).

(6) Evaluate the sum:

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

(A) 3/4

(B) 1

(C) 3/2

(D) 2

(E) 3

(7) What is the remainder upon dividing 13^{2019} by 95?

(A) 1

(B) 13

(C) 74

(D) 12

(E) 61

(8) Evaluate the following limit:

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{k}{n}\right)^{1/n}.$$

(A) 1

(B) e^{-1}

(C) e^{-2}

(D) 0

(E) The limit does not exist.

(9) Let A be the annulus, $A = \{(x, y) \in \mathbb{R}^2 \ : \ 1/2 \leq \sqrt{x^2 + y^2} \leq 2\}$. Evaluate:

$$\iint_A 2x - 2ye^{x^2+y^2} dxdy.$$

(A) 1

(B) 0

(C) 2π

(D) -2π

(E) 4π

(10) Which of the following functions are holomorphic, with $x, y \in \mathbb{R}$?

I. $f(x+iy) = x^2 + iy^2$

II. $g(x+iy) = x + x^2 - y^2 + i(2xy + y)$

III. $h(x+iy) = y + e^x \cos y + i(x + e^x \sin y)$

(A) None of them are holomorphic.

(B) II only

(C) III only

(D) I and III only

(E) II and III only

(11) Let R be the group of the nonzero real numbers under multiplication, and define $a \star b = |a|b$.

I. $(R, \star)$ has a left identity.

II. $(R, \star)$ is left cancellative, i.e. $a \star b = a \star c$ implies $b = c$.

III. $(R, \star)$ forms a group.

Which of the above are true?

(A) All of them are true.

(B) I only

(C) II only

(D) I and II only

(E) None of them are true.

(12) Let $\phi(x)$ and $\psi(y)$ be two smooth functions defined on $\mathbb{R}$. Let S be a positively oriented circle of radius 1 around the origin. Which of the following is zero?

I. $\int_S \phi(y)\,dx + \psi(x)\,dy$

II. $\int_S \phi(xy)(ydx + xdy)$

III. $\int_S \phi(x)\psi(y)dx$

(A) None are zero.

(B) I only

(C) II only

(D) I and II only

(E) I, II, and III

(13) Evaluate the integral

$$\int_0^\pi \sin^3(x)dx.$$

(A) 1

(B) 4/3

(C) 7/2

(D) $\pi/2$

(E) π

(14) How many abelian groups are there of order 360, up to isomorphism?

(A) 3

(B) 6

(C) 10

(D) 15

(E) 30

(15) A man flips 10 coins. With H the number of heads, and T the number of tails, the man then flips $\max\{2H - T^2, 0\}$ coins. What is the expected number of heads of both groups?

(A) Between 0 and 8.

(B) Between 8 and 10.

(C) Between 10 and 12.

(D) Between 12 and 15.

(E) Between 15 and 20.

(16) A tank contains 150 L of salt water, with 0.7 kg of salt per liter. Salt water containing 0.5 kg of salt per liter is added at a rate of 7 liters per minutes. The tank is kept at a constant volume by draining water at the same rate. Assuming instantaneous mixing, at what time is there 90 kg of salt in the tank?

(A) $\log(2) \cdot 150/7$

(B) $\log(3) \cdot 150/7$

(C) $\log(4) \cdot 150/7$

(D) $\log(5) \cdot 150/7$

(E) $\log(6) \cdot 150/7$

(17) For which θ is $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ purely imaginary?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(D) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

(E) 0

(18) Which of the following conditions imply that two sets, A and B, have the same cardinality?

I. There exist $f\colon A \to B$ and $g\colon B \to A$ such that $g \circ f = Id_A$.

II. $A \subset B$ and there exists $f\colon A \to B$, and $g\colon B \to A$ such that $f \circ g = Id_B$.

III. $|A \setminus B| = |B \setminus A|$

Which of the above statements are true?

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) II and III only

(19) Let C be the circle of radius 2 about the origin in $\mathbb{C}$ traversed counter-clockwise. Compute the integral

$$\int_C \frac{1}{z^2+1} dz.$$

(A) 1

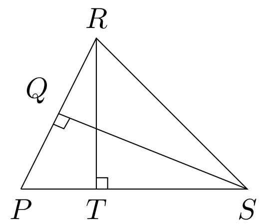
(B) 0

(C) i

(D) $-i/2$

(E) $-i$

(20)



In $\triangle PRS$, $RT = 7$, $PR = 8$, and $QS = 9$. Which of the following is closest to the length of side PS?

(A) 7.14

(B) 8.22

(C) 9.87

(D) 10.29

(E) 11.44

(21) Consider the following statements.

I. $(A \implies B) \implies C$

II. $A \implies (B \implies C)$

III. $(A \wedge B) \implies C$

IV. $B \implies (A \implies C)$

V. $(B \implies A) \implies C$

How many of the above (numbered) statements are logically distinct?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

(22) Consider the following attempted proof of the statement that if X is a compact subset of $\mathbb{R}$, then a continuous function $f\colon X \to \mathbb{R}$ is uniformly continuous. We use $B_\epsilon(x)$ to denote the open ball of radius ϵ about x.

I. Fix $\epsilon > 0$. As f is continuous for all $x \in X$ there exists δ_x such that if $y \in B_{\delta_x}(x)$, then $|f(x) - f(y)| < \epsilon/2$. Let $\mathcal{C} = \{B_{\delta_x} \mid x \in X\}$. Note $\mathcal{C}$ is an open cover of X.

II. By compactness of X there exists a finite subcover $\mathcal{C}'$ of $\mathcal{C}$, which we index by the set $X' \subset X$.

III. Set $\delta = \min_{x \in X'} \delta_x/2$. Then if $\delta/4 > |x - y|$, there exists $z \in X'$ such that $x, y \in B_{\delta_z}$.

IV. Thus as $|f(z) - f(x)|$ and $|f(z) - f(y)|$ are both less than $\epsilon/2$, by the triangle inequality $|f(x) - f(y)| < \epsilon$, so f is uniformly continuous.

In the above proof, at which step was the first error made? Or is there none at all?

(A) I

(B) II

(C) III

(D) IV

(E) The proof is correct.

Answers

(1) (D): Compute in the domain.

(2) (D): Rewrite integrand as integral of $\sin(xy)$, exchange order of integration.

(3) (C): Probably fastest to differentiate the answers.

(4) (E): Count them.

(5) (B): $\{-n\}_{n\in\mathbb{N}}$ does not converge.

(6) (C): Do the standard trick, or evaluate by differentiation of the geometric series.

(7) (D): $\varphi(95) = 72$, $2019 \equiv 3 \mod 72$, $13^3 \equiv 12 \mod 95$.

(8) (B): Use Stirling's approximation for $n!$.

(9) (B): Use symmetry of the integral. Or compute with Stokes theorem.

(10) (B): Use the Cauchy-Riemann equations.

(11) (D): The left identity is not a right identity.

(12) (C): Use Stokes to solve the integrals.

(13) (B): This can be evaluated with u-substitution after $\sin^2(x) = 1 - \cos^2(x)$.

(14) (B): Classification of finite abelian groups.

(15) (A): Compute (or estimate).

(16) (A): This is a first-order separable differential equation. Solve.

(17) (C): Compute.

(18) (E): In II, g is an injection.

(19) (B): Use the residue theorem.

(20) (D): Write the area of the triangle in two different ways.

(21) (C): The middle three statements are identical.

(22) (C): Consider if $\mathcal{C}' = \{(-1,1),(1,2)\}$.