(1) Assume $A: \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation with $A(5, 6) = 3$ and $A(2, 1) = -1$. Compute $A(1, 4)$.

(A) $-2$
(B) 1
(C) 2
(D) 5
(E) 6

(2) Suppose $\alpha, \beta > 0$. Compute:

$$\int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x} \, dx.$$

(A) $\log \frac{\beta}{\alpha}$
(B) $2 \log \frac{\beta}{\alpha}$
(C) $2 \log \frac{\alpha}{\beta}$
(D) $\log \frac{\beta}{\alpha}$
(E) $\log \frac{\alpha}{\beta}$

(3) Integrate:

$$\int \frac{dx}{1 + e^x}.$$

(A) $2x - \log(e^x + 1) + C$
(B) $x + \log(e^x + 1) + C$
(C) $x - \log(e^x + 1) + C$
(D) $x - \log(e^x + e^{-x}) + C$
(E) $x - \log(1 + e^{-x}) + C$

Date: July 8, 2019.
(4) At a banquet, \( n \) women and \( m \) men are to be seated in a row of \( n + m \) chairs. If the entire seating arrangement is to be chosen at random, what is the probability that all of the men will be seated next to each other in \( m \) consecutive positions?

(A) \( \frac{1}{\binom{n+m}{m}} \)

(B) \( \frac{m!}{\binom{n+m}{m}} \)

(C) \( \frac{n!}{(n+m)!} \)

(D) \( \frac{m!n!}{(n+m-1)!} \)

(E) \( \frac{m!(n+1)!}{(n+m)!} \)

(5) Endow \( \mathbb{R} \) with the right topology, generated by \( T = \{ (a, \infty) : a \in \mathbb{R} \} \), and call this space \( X \). Which of the following is false?

(A) \( X \) is \( \sigma \)-compact (it is the union of countably many compact subsets).

(B) \( X \) is sequentially compact (every sequence has a convergent subsequence).

(C) \( X \) is limit point compact (every infinite subset has a limit point in \( X \)).

(D) \( X \) is Lindelöf (every open cover of \( X \) has a countable subcover).

(E) \( X \) is pseudoocompact (every continuous function \( f : X \to \mathbb{R} \) is bounded).

(6) Evaluate the sum:

\[ \sum_{n=1}^{\infty} \frac{n^2}{3^n} \]

(A) \( \frac{3}{4} \)

(B) \( 1 \)

(C) \( \frac{3}{2} \)

(D) \( 2 \)

(E) \( 3 \)
(7) What is the remainder upon dividing $13^{2019}$ by 95?

(A) 1
(B) 13
(C) 74
(D) 12
(E) 61

(8) Evaluate the following limit:

$$\lim_{n \to \infty} \prod_{k=1}^{n} \left( \frac{k}{n} \right)^{1/n}.$$ 

(A) 1
(B) $e^{-1}$
(C) $e^{-2}$
(D) 0
(E) The limit does not exist.

(9) Let $A$ be the annulus, $A = \{(x, y) \in \mathbb{R}^2 : 1/2 \leq \sqrt{x^2 + y^2} \leq 2\}$. Evaluate:

$$\iint_A 2x - 2ye^{x^2+y^2} \, dx \, dy.$$ 

(A) 1
(B) 0
(C) $2\pi$
(D) $-2\pi$
(E) $4\pi$
(10) Which of the following functions are holomorphic, with \( x, y \in \mathbb{R} \)?

I. \( f(x + iy) = x^2 + iy^2 \)

II. \( g(x + iy) = x + x^2 - y^2 + i(2xy + y) \)

III. \( h(x + iy) = y + e^x \cos y + i(x + e^x \sin y) \)

(A) None of them are holomorphic.

(B) II only

(C) III only

(D) I and III only

(E) II and III only

(11) Let \( R \) be the group of the nonzero real numbers under multiplication, and define \( a \star b = |a|b \).

I. \((R, \star)\) has a left identity.

II. \((R, \star)\) is left cancellative, i.e. \( a \star b = a \star c \) implies \( b = c \).

III. \((R, \star)\) forms a group.

Which of the above are true?

(A) All of them are true.

(B) I only

(C) II only

(D) I and II only

(E) None of them are true.
(12) Let \( \phi(x) \) and \( \psi(y) \) be two smooth functions defined on \( \mathbb{R} \). Let \( S \) be a positively oriented circle of radius 1 around the origin. Which of the following is zero?

I. \( \int_S \phi(y) \, dx + \psi(x) \, dy \)

II. \( \int_S \phi(xy) (y \, dx + x \, dy) \)

III. \( \int_S \phi(x) \psi(y) \, dx \)

(A) None are zero.

(B) I only

(C) II only

(D) I and II only

(E) I, II, and III

(13) Evaluate the integral \( \int_0^\pi \sin^3(x) \, dx \).

(A) 1

(B) \( \frac{4}{3} \)

(C) \( \frac{7}{2} \)

(D) \( \frac{\pi}{2} \)

(E) \( \pi \)

(14) How many abelian groups are there of order 360, up to isomorphism?

(A) 3

(B) 6

(C) 10

(D) 15

(E) 30
(15) A man flips 10 coins. With $H$ the number of heads, and $T$ the number of tails, the man then flips $\max\{2H - T^2, 0\}$ coins. What is the expected number of heads of both groups?

(A) Between 0 and 8.

(B) Between 8 and 10.

(C) Between 10 and 12.

(D) Between 12 and 15.

(E) Between 15 and 20.

(16) A tank contains 150 L of salt water, with 0.7 kg of salt per liter. Salt water containing 0.5 kg of salt per liter is added at a rate of 7 liters per minutes. The tank is kept at a constant volume by draining water at the same rate. Assuming instantaneous mixing, at what time is there 90 kg of salt in the tank?

(A) $\log(2) \cdot \frac{150}{7}$

(B) $\log(3) \cdot \frac{150}{7}$

(C) $\log(4) \cdot \frac{150}{7}$

(D) $\log(5) \cdot \frac{150}{7}$

(E) $\log(6) \cdot \frac{150}{7}$

(17) For which $\theta$ is $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ purely imaginary?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(D) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

(E) 0
Which of the following conditions imply that two sets, \( A \) and \( B \), have the same cardinality?

I. There exist \( f: A \to B \) and \( g: B \to A \) such that \( g \circ f = Id_A \).

II. \( A \subset B \) and there exists \( f: A \to B \), and \( g: B \to A \) such that \( f \circ g = Id_B \).

III. \( |A \setminus B| = |B \setminus A| \)

Which of the above statements are true?

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) II and III only

Let \( C \) be the circle of radius 2 about the origin in \( \mathbb{C} \) traversed counter-clockwise. Compute the integral

\[
\int_C \frac{1}{z^2 + 1} \, dz.
\]

(A) 1
(B) 0
(C) \( i \)
(D) \( -i/2 \)
(E) \( -i \)
In $\triangle PRS$, $RT = 7$, $PR = 8$, and $QS = 9$. Which of the following is closest to the length of side $PS$?

(A) 7.14  
(B) 8.22  
(C) 9.87  
(D) 10.29  
(E) 11.44

(21) Consider the following statements.

I. $(A \implies B) \implies C$

II. $A \implies (B \implies C)$

III. $(A \land B) \implies C$

IV. $B \implies (A \implies C)$

V. $(B \implies A) \implies C$

How many of the above (numbered) statements are logically distinct?

(A) 1  
(B) 2  
(C) 3  
(D) 4  
(E) 5
Consider the following attempted proof of the statement that if $X$ is a compact subset of $\mathbb{R}$, then a continuous function $f : X \to \mathbb{R}$ is uniformly continuous. We use $B_\epsilon(x)$ to denote the open ball of radius $\epsilon$ about $x$.

I. Fix $\epsilon > 0$. As $f$ is continuous for all $x \in X$ there exists $\delta_x$ such that if $y \in B_{\delta_x}(x)$, then $|f(x) - f(y)| < \epsilon/2$. Let $\mathcal{C} = \{B_{\delta_x} \mid x \in X\}$. Note $\mathcal{C}$ is an open cover of $X$.

II. By compactness of $X$ there exists a finite subcover $\mathcal{C}'$ of $\mathcal{C}$, which we index by the set $X' \subset X$.

III. Set $\delta = \min_{x \in X'} \delta_x/2$. Then if $\delta/4 > |x - y|$, there exists $z \in X'$ such that $x, y \in B_{\delta_z}$.

IV. Thus as $|f(z) - f(x)|$ and $|f(z) - f(y)|$ are both less than $\epsilon/2$, by the triangle inequality $|f(x) - f(y)| < \epsilon$, so $f$ is uniformly continuous.

In the above proof, at which step was the first error made? Or is there none at all?

(A) I
(B) II
(C) III
(D) IV
(E) The proof is correct.
Answers

(1) (D): Compute in the domain.

(2) (D): Rewrite integrand as integral of $\sin(xy)$, exchange order of integration.

(3) (C): Probably fastest to differentiate the answers.

(4) (E): Count them.

(5) (B): $\{-n\}_{n \in \mathbb{N}}$ does not converge.

(6) (C): Do the standard trick, or evaluate by differentiating the geometric series.

(7) (D): $\varphi(95) = 72$, $2019 \equiv 3 \mod 72$, $13^3 \equiv 12 \mod 95$.

(8) (B): Use Stirling’s approximation for $n!$.

(9) (B): Use symmetry of the integral. Or compute with Stokes theorem.

(10) (B): Use the Cauchy-Riemann equations.

(11) (D): The left identity is not a right identity.

(12) (C): Use Stokes to solve the integrals.

(13) (B): This can be evaluated with $u$-substitution after $\sin^2(x) = 1 - \cos^2(x)$.

(14) (B): Classification of finite abelian groups.

(15) (A): Compute (or estimate).

(16) (A): This is a first-order separable differential equation. Solve.

(17) (C): Compute.

(18) (E): In $\Pi$, $g$ is an injection.
(19) (B): Use the residue theorem.

(20) (D): Write the area of the triangle in two different ways.

(21) (C): The middle three statements are identical.

(22) (C): Consider if $C' = \{(-1, 1), (1, 2)\}$. 