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Exceptional Lie groups and some related geometry

October 8, 2021

Outline of the talk

- Compact simple Lie groups – Maximal tori, Weyl groups and root systems
- Crash course in exceptional groups

— Applications —

- Exceptional groups occur in many contexts of mathematics and physics, and the main interest in this talk is
- Some instances of geometry arising from arrangements defined by Weyl groups of exceptional groups

Examples

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- Let V be a complex vector space of dimension n with an inner product; then the group of (complex) linear automorphisms of V which preserve the inner product is a compact Lie group denoted $U(n)$ (the *unitary* group). Then $M = (M^*)^{-1}$.

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In each case, the determinant gives rise to an exact sequence

$$1 \longrightarrow SG \longrightarrow G \xrightarrow{\det} K \longrightarrow 1$$

where K is the field $\mathbb{R}$ or $\mathbb{C}$ and G is the group above. Then: SG is a *simple compact Lie group* $SO(n)$ resp. $SU(n)$.

Maximal tori

For the two groups considered, the maximal tori are easily described.

Example

- For $U(n)$, it is the group $T = \{\mathrm{diag}(t_1, \ldots, t_n)\}$ of diagonal matrices with complex entries $t_i = e^{i\theta_i}$. For $SU(n)$ one has in addition $\prod_i^n t_i = 1$.

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- Let $B(\theta) := \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}$ be the 2×2 real matrix representing a rotation of θ degrees in $\mathbb{R}^2$. For the group $O(n)$ T is the group of diagonal *block* matrices $T = \{\mathrm{diag}(B_1(\theta_1), \ldots, B_{\frac{n}{2}}(\theta_{\frac{n}{2}}))\}$ when $n = 2m$ is even and $T = \{B_1(\theta_1), \ldots, B_{[\frac{n}{2}]}(\theta_{[\frac{n}{2}]}), 1)\}$ when $n = 2m+1$ is odd.

The Weyl group

Let $N(T) \subset G$ denote the normalizer of T; the group $W(G) = N(T)/T$ is called the *Weyl group* of G.

Example

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- For $SO(n)$, one has the permutations of the *blocks*; but in addition, in each block $\theta_i \mapsto -\theta_i$ preserves the torus. The Weyl group is $W(SO(2m+1)) = \Sigma_m \rtimes \mathbb{Z}_2^m$, for $SO(2m)$ the number of sign changes needs to be even, $W(SO(2m)) = \Sigma_m \rtimes \mathbb{Z}_2^{m-1}$.

Using quaternions

$\mathbb{H}$ is the $\mathbb{R}$-algebra generated by 1 and elements i, j, k such that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji, ik = -ki, jk = -kj.$$

$\mathbb{H}$ forms a *division algebra* over $\mathbb{R}$, meaning it is a multiplicative algebra over $\mathbb{R}$ without zero-divisors. $\mathbb{H}$ satisfies all the axioms of a field except commutativity, hence it can be used as the scalar field for a vector space V of dimension n over $\mathbb{H}$ (dimension 4 over $\mathbb{R}$).

Convention

One needs to fix the scalar multiplication; standard is: V is a *right* $\mathbb{H}$-vector space, then matrices operate from the left.

Using a $\mathbb{H}$-hermitian form, what was done above for the inner products can be extended to the case of *quaternionic matrices*. This leads to compact simple Lie groups called the *symplectic groups* $Sp(n)$. The Weyl group is $\Sigma_n \rtimes \mathbb{Z}_2^n$.

Root systems and classification

A representation of G is a group homomorphism of G in the automorphism group of a vector space. Conjugation in G, $x \mapsto c_s(x) := sxs^{-1}$, defines a map

$$\mathrm{Inn} : G \longrightarrow \mathrm{Aut}(G), s \mapsto c_s : G \longrightarrow G.$$

This leads to the *adjoint representation* of G, defined as follows:

$$\begin{array}{rcl} \mathrm{Ad} : G & \longrightarrow & GL(T_e(G)), \\ s & \mapsto & T(c_s) : T_e(G) \longrightarrow T_e(G). \end{array} \tag{1}$$

Representations are studied by restricting to T: any representation of T is multiplication by $(t_1^{a_1}, \dots, t_n^{a_n})$ with integer a_i. The torus itself is $T = \mathbb{R}^n/\mathbb{Z}^n$, and the exponents correspond to elements in a *lattice*. The *roots of G* are the eigenvalues of the adjoint representation. In this way a *root system* is defined, and leads to the classification

$$\boxed{A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8.}$$

Classical and exceptional groups

In the notation, the subscript denotes the *dimension of the maximal torus*

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Exceptional groups

$$G_2, F_4, E_6, E_7, E_8$$

The classification is called the Cartan-Killing classification and goes back to independent determinations by those two mathematicians. Any idea what these exceptional groups represent?

Octonions

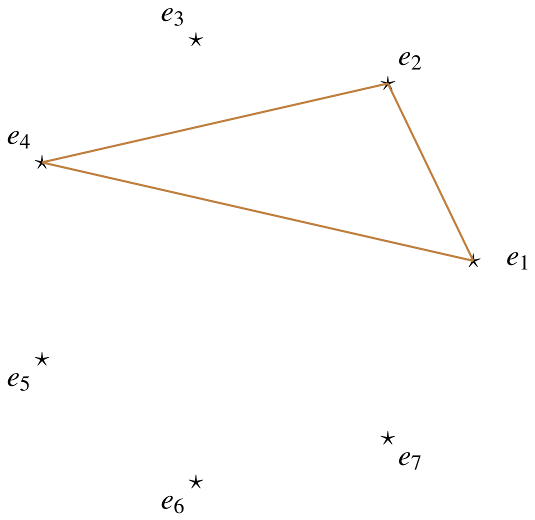
Note that $\mathbb{C} = \mathbb{R} + i\mathbb{R}, i^2 = -1, \mathbb{H} = \mathbb{C} + j\mathbb{C}, j^2 = -1$. Such algebras are called composisition algebras. The construction can be extended one more time, $\mathbb{O} = \mathbb{H} + e_3\mathbb{H}$ to give a division algebra. This algebra is given by the relations:

The division algebra of octonions (Cayley-Graves algebra)

$$\begin{aligned} e_j^2 = -e_0 = -1 \qquad & j = 1, \ldots, 7 \\ e_j e_k = -e_k e_j \qquad & j \neq k, \quad j,k = 1, \ldots, 7 \\ e_i \cdot e_{i+1} = e_{i+3}, \quad e_{i+1} \cdot e_{i+3} = e_i, \quad & e_{i+3} \cdot e_i = e_{i+1}, \quad \text{indices taken mod 7} \end{aligned} \tag{2}$$

in which $e_0, e_i, e_{i+1}, e_{i+3}$ form a subalgebra isomorphic to $\mathbb{H}$.

Multiplication in $\mathbb{O}$



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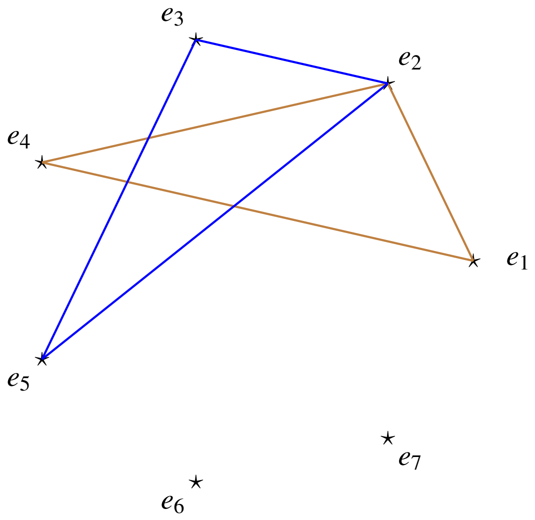

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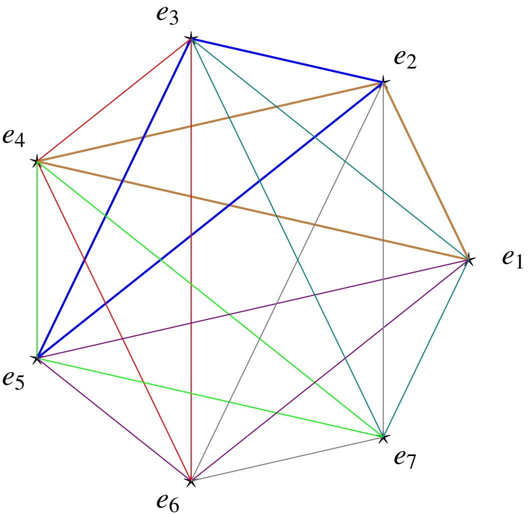

Each colored triangle is a subalgebra $\cong \mathbb{H}$. For example, this shows $e_3 \cdot e_5 = e_2$ amd $e_5 \cdot e_1 = -e_6$.

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- Jordan algebras were studied at the beginning of the twentieth century in a quest to find interesting algebras for quantum physics. Again there is a classification, with a *unique exceptional* one, a 27-dimensional algebra (the 3×3 algebra above).
- There is an exotic way to define, starting with a Jordan algebra and a $\mathbb{R}$-division algebra, a Lie algebra, the Tits-Vinberg construction.

The Tits-Vinberg construction

Let $\mathscr{A}$ be a composition algebra (one of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) and $\mathscr{J}$ a Jordan algebra. Define the algebra

$$\mathscr{T}(\mathscr{A}, \mathscr{J}) = \mathrm{Der}(\mathscr{A}) \oplus \mathrm{Der}(\mathscr{J}) + \mathscr{A}_0 \otimes \mathscr{J}_0.$$

Provided with a rather complicated bracket, this can be made to a Lie algebra. The algebra $\mathbf{H}_3(\mathscr{A})$ of "hermitian" 3×3-matrices for a composition algebra $\mathscr{A}$ is a Jordan algebra; taking for $\mathscr{A}$, $\mathscr{B}$ the various composition algebras with $\mathscr{T} = \mathscr{T}(\mathscr{A}, \mathbf{H}_3(\mathscr{B}))$ leads to the *magic square*:

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	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Appearances of exceptional groups in various contexts

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- The Weyl groups and root systems define arrangements which arise in various fascinating situations in algebraic geometry.

Main topic of this talk

Arrangements defined by Weyl groups

The Weyl group $W(G)$ can be defined as a reflection group, generated in $GL_n(\mathbb{R})$ (n the rank of G) by the *reflections on the roots*. Considering the corresponding *projective* arrangement one dimension lower defines a hyperplane arrangement in $\mathbf{P}^n(\mathbb{R})$ (or in complex projective space). A particular case in considered in "The geometry of some special arithmetic quotients". More details will be given in the upcoming monograph by the author.

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Weyl group actions and arrangements

The group G_2, being only of rank 2, defines only a set of points on $\mathbf{P}^1(\mathbb{C})$, but the others give rise to interesting arrangements.

- F_4 defines an arrangement of planes in $\mathbf{P}^3(\mathbb{C})$: the 24 planes form an arrangement which has two subsets of 12 roots, each corresponding to a subroot system of type D_4, called a *triple of desmic tetrahedra*, one arising from the long roots, one from the short ones.

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Weyl group actions and arrangements

- E_6 defines an arrangement of 36 hyperplanes ($\mathbf{P}^4$) in $\mathbf{P}^5(\mathbb{C})$. The group $W(E_6)$ is isomorphic to the group of incidence-preserving permutations of the 27 lines on a smooth cubic surface, and all the geometry has wonderful interpretations of geometric configurations. This geometry has been studied in detail in "The geometry of some special arithmetic quotients".

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- E_7 defines an arrangement of 63 hyperplanes ($\mathbf{P}^5$) in $\mathbf{P}^6(\mathbb{C})$. The group $W(E_7)$ is similarly related to the 28 bitangents of a quartic curve in $\mathbf{P}^2(\mathbb{C})$, and a similar correspondence to geometric properties is mirrored in the arrangement.

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- E_8 defines an arrangement of 120 hyperplanes ($\mathbf{P}^6$) in $\mathbf{P}^7(\mathbb{C})$. This arrangement is very "universal": all the above can be derived as subarrangements of this arrangement. The lattice in $\mathbb{R}^8$ defined by the roots can be identified with a very special order in $\mathbb{O}$, that is a sublattice which is also a subring. Many other connections exist.

The arrangement defined by the Weyl group $W(E_6)$

The Weyl group $W(E_6)$ is isomorphic to the group of permutations of the 27 lines on a cubic surface. This isomorphism has a beautiful geometric expression in the properties of the arrangment defined.

Properties of the 27 lines on a smooth cubic surface

- The are 45 special hyperplane sections, each of which intersects the cubic surface in 3 of the 27 lines; this plane is tangent at the 3 intersection points of the lines, hence the name *tritangent*.

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- There are 40 *triads of trihedral pairs*, such that the triple contains all 27 lines.

Subloci of the arrangement defined by $W(E_6)$

As a Weyl group, $W(E_6)$ is generated by *reflections on the roots*. Since the rank is 6, the projective arrangement is defined in $\mathbf{P}^5(\mathbb{C})$.

36 hyperplanes

- Since E_6 has 36 positive roots, this defines 36 hyperplanes (each a $\mathbf{P}^4(\mathbb{C})$), one corresponding to each double-six.
- Corresponding to the 120 triples of azygetic double-sixes, there are 120 $\mathbf{P}^3(\mathbb{C})$'s which are the intersection of 3 of the 36.
- Corresponding to the 270 pairs of syzegetic pairs of double sixes, there are 270 $\mathbf{P}^3(\mathbb{C})$'s which are the intersection of two of the 36.
- Corresponding to the 120 trihedral pairs, there are 120 $\mathbf{P}^1(\mathbb{C})$'s (lines) which are the intersection of 6 of the 36.

Where are the 27 lines?

The 27 weights

The group E_6 has two complex-conjugate 27-dimensional representations, in the exceptional Jordan algebra (the highest weight is the fundamental weight corresponding to the first (resp. last) root α_1 (resp. α_6)). This defines 27 weights, which are accordingly an orbit under the Weyl group. It is these weights which correspond to the 27 lines. (Note: everything taking place in the universal cover of the maximal torus identified with $H^1(T,\mathbb{Z})$).

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The arrangement defined by the 27 hyperplanes

This defines 27 hyperplanes ($\mathbf{P}^4$). Some of the interesting loci:

- Corresponding to the 45 tritangent planes, there are 45 $\mathbf{P}^3(\mathbb{C})$'s which are the intersection of 3 of the 27.
- Corresponding to the 216 pairs of skew lines there are 216 $\mathbf{P}^3(\mathbb{C})$'s which are the intersection of two of the 27.
- Recalling that each of the 120 trihedral pairs defines a set of 9 lines, there are 120 lines which are the intersection of 9 of the 27.
- etc...

Invariant polynomials of $W(E_6)$

The invariants of the classical groups described above are polynomials invariant under permutations of the coordinates, or under permutations of the squares of the coordinates. Here it is a more challenging matter. The degrees of the invariants are: 2, 5, 6, 8, 9, 12. Of these, only the quintic is *unique*. As a general definition, for invariants of *even* degree one can take the sum of the powers of the roots. For *odd* degrees, this sum is always 0 because with any root also its negative is a root.

The k^{th} powers of the 27 fundamental weights

Using the 27 weights (with equations $a_i = 0, b_i = 0, c_{ij} = 0$), one gets for any degree k an invariant of the corresponding degree.

$$I_k := \sum_{i,j} \left\{ a_i^k + b_i^k + c_{ij}^k \right\}.$$

The invariant quintic

A whole chapter in "The geometry of some special arithmetic quotients" is devoted to the quintic $I_5 = 0$, a 4-dimensional variety in $\mathbf{P}^5(\mathbb{C})$. Some of the beautiful properties are just listed here.

Loci on the quintic

- The *singular locus* of the quintic consists of 120 lines which intersect in 36 points.

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- The 36 intersections of I_5 with the root hyperplanes are copies of a quintic threefold known as the *Nieto quintic*.
- Both the Nieto quintic and the invariant quintic 4-fold are compactifications of *ball quotients*. This was conjectured in the lecture notes mentioned above and has been verified in the mean time. In fact, also the 45 $\mathbf{P}^3(\mathbb{C})$'s above are ball quotients.

This years discovery: the invariant nontic I_9

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  - The 120 lines are contained *simply*, leaving a curve of degree 1080 to explain.
  - The 120 lines are arranged in $\mathbf{P}^5(\mathbb{C})$ such that 4 of them lie in certain planes.

More geometry of I_9

- Restricted to these planes, 5 of the octics vanish and the remaining one splits off these 4 lines, leaving a quartic curve in each of the planes. These planes are the so-call c-planes, and the 4 lines intersect at 6 of the 36 singular points of intersection.

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- The intersection of I_9 with the *symmetric* hyperplanes $\mathbf{P}^4(\mathbb{C})$ is a nontic threefold $\mathscr{K}_9$. It intersects the Nieto quintic (the intersection of I_5 with the same hyperplane) in the union of 15 + 15 planes, the c-planes mentioned above, and the so-called Segre planes, each containing 3 of the nodes and 4 of the 10 isolated double points of $\mathscr{N}_5$, the c-planes being double in the intersection.

Part of a bigger story?

The proof that $\mathscr{N}_5$ is (birational to) a ball quotient requires manipulations of Chern numbers on $\mathscr{N}_5$ and a branched cover $W \longrightarrow \mathscr{N}_5$ of $\mathscr{N}_5$. The branch locus of that cover is contained in the union of the c-planes and the Segre planes, i.e., in the intersection $\mathscr{N}_5 \cap \mathscr{K}_9$. This means that the computation done already for $\mathscr{N}_5$ applies immediately to $\mathscr{K}_9$! Apart from the singular quartics of $\mathscr{K}_9$ in each of the c-planes, ...

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A conjecture on $\mathscr{K}_9$

There is a cover $Y \longrightarrow \mathbf{P}^4(\mathbb{C})$ which restricted to $\mathscr{N}_5$ is the cover mentioned above. Then the cover $Z \longrightarrow \mathscr{K}_9$, branched at the union of the c-planes and the Segre planes, is (birational to) a compactification of a ball quotient.

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A more daring conjecture

Let $\mathscr{V} \subset \mathbf{P}^4(\mathbb{C})$ be a symmetric variety which contains the Segre planes and/or the c-planes. Then the restriction of Y to $\mathscr{V}$ is (birational to) a ball quotient. Y is itself a 4-dimensional ball quotient.

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The compactification locus is the image of the 120 singular lines in the birational model.

... and beyond

Assuming the above conjectures are correct, we would have the following situation: in $\mathbf{P}^5(\mathbb{C})$ there are two hypersurfaces I_5 and I_9 which are (birational to) compactifications of ball quotients; these meet in 45 linear spaces $\mathbf{P}^3(\mathbb{C})$ which are 3-dimensional subball quotients.

Bigger and bigger...

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It is natural then to wonder whether $\mathbf{P}^5(\mathbb{C})$ is a ball quotient such that I_5 and I_9 are codimension 1 subballs?

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Move up to $\mathbf{P}^6(\mathbb{C})$, which is the ambient space for the $W(E_7)$-arrangement. Each of the 28 hyperplanes arising from the 28 weights contains exactly the configuration just considered. Is $\mathbf{P}^6(\mathbb{C})$ a ball quotient with 28 subball quotients $\mathbf{P}^5(\mathbb{C})$ as above?

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Currently our understanding of branched covers and birational geometry in dimension > 3 is not quite complete enough to describe explicitly what happens and what the birational models should precisely look like. Some progress here will be necessary.

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