Introduction to Braid Groups

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ABSTRACT. This paper is an introduction to the braid groups intended to familiarize the reader with the basic definitions of these mathematical objects. First, the concepts of the Fundamental Group of a topological space, configuration space, and exact sequences are briefly defined, after which geometric braids are discussed, followed by the definition of the classical (Artin) braid group and a few key results concerning it. Finally, the definition of the braid group of an arbitrary manifold is given.

Contents

1. Geometric and Topological Prerequisites

Any picture of the braid groups necessarily follows only after one has discussed the necessary prerequisites from geometry and topology. We begin by discussing what is meant by Configuration space.

Definition 1.1 Given any manifold M and any arbitrary set of m points in that manifold Q_m, the *configuration space* $F_{m,n}M$ is the space of ordered n-tuples $\{x_1, ..., x_n\} \in M - Q_m$ such that for all $i, j \in \{1, ..., n\}$, with $i \neq j$, $x_i \neq x_j$.

If M is simply connected, then the choice of the m points is irrelevant, as, for any two such sets Q_m and Q'_m, one has $M - Q_m \cong M - Q'_m$. Furthermore, $F_{m,0}$ is simply $M - Q_m$ for some set of M points Q_m. In this paper, we will be primarily concerned with the space $F_{0,n}M$.

Another space useful for our purposes is obtained by taking $B_{m,n}M = F_{m,n}M/\mathfrak{S}_n$, which just takes any $\{x_1, ..., x_n\} \in F_{m,n}M$ and sets all permutations of those coordinates equal. Furthermore, there is a canonical projection $\mathfrak{p} : F_{m,n}M \to B_{m,n}M$.

Definition 1.2 Given two continuous maps $f, g : X \to Y$, where X and Y are topological spaces, f and g are said to be *homotopic* if there exists a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. This map H is referred to as a *homotopy* from f to g.

A homotopy between maps is simply a method of continuously deforming one map into the other. For our purposes, it is most important in light of the following definitions.

Definition 1.3 A *path* in a topological space X is a continuous map $f : [0, 1] \to X$. If $f(0) = f(1)$, then f is referred to as a *loop*. A space is called *path connected* if any two points in the space are connected by a path.

Definition 1.4 Given a based, path connected topological space (X, x), the *fundamental group* of that space, $\pi_1(X)$, is the set of all loops starting and ending at x up to homotopy such that the endpoints are fixed (i.e. $h(0, t) = h(1, t) = x$ for all $t \in [0, 1]$).

There is a manner in which one may transform loops from one point into loops from the chosen basepoint. Given the space (X, x) and another point y, one may take a loop γ from y to y, and transform it into a loop from x to x. Simply take a path η from x to y, and analyze the composition $\eta\gamma\eta^{-1}$. Clearly, this corresponds to a loop from x to x. Analogously, one may transform loops from x to x into loops from y to y. Hence, the choice of basepoint is immaterial and is omitted from the notation.

Proposition 1.5 The fundamental group of a topological space X is, in fact, a group.

Proof. In order to show this, one needs to establish a multiplicative structure on $\pi_1(X)$. Take $f, g \in \pi_1(X)$. Then one may define $f \cdot g$ as follows:

$$(f \cdot g)(t) = \begin{cases} f(2t) & : 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

The associativity of this product may be easily checked. In addition, there exists an identity element $id(t) = x$, and inverses, which may be written as $f^{-1}(t) = f(1-t)$ for any $f \in \pi_1(X)$. One may easily verify that the group properties hold. □

This is one of the most important concepts in Algebraic Topology. It is particularly important to us for reasons which will be made apparent in later sections.

Definition 1.6 An *exact sequence* is a chain of groups and homomorphisms between them such that the image set of one homomorphism in the chain is equal to the kernel of the following one. In other words, it is a chain

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} G_n$$

such that for all $k \in \{1, ..., n-1\}$, one has $im(f_k) = ker(f_{k+1})$. A short exact sequence is one of the form

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1,$$

where 1 represents the trivial group. This forces f to be a monomorphism and g to be an epimorphism (in the nonabelian case, there is the additional requirement that f is the inclusion of a normal subgroup). Furthermore, this in turn induces an isomorphism $C \cong B/A$.

Now, there are two important lemmas about exact sequences which must be stated. Their proofs are a simple, but lengthy, application of diagram chasing.

Lemma 1.7 Given two exact sequences and a set of morphisms between the groups comprising them such that the following diagram commutes,

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 & \xrightarrow{f_3} & G_4 & \xrightarrow{f_4} & G_5 \\ \downarrow{g_1} & & \downarrow{g_2} & & \downarrow{g_3} & & \downarrow{g_4} & & \downarrow{g_5} \\ G_6 & \xrightarrow{f_5} & G_7 & \xrightarrow{f_6} & G_8 & \xrightarrow{f_7} & G_9 & \xrightarrow{f_8} & G_{10} \end{array}$$

the *five lemma* states that if g_1, g_2, g_4, and g_5 are isomorphisms, then g_3 must be one as well. A corollary to this regarding short exact sequences is as follows. Given two short exact sequences morphisms between the groups that compose them such that the following diagram commutes,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\
1 & \longrightarrow & G_4 & \xrightarrow{f_3} & G_5 & \xrightarrow{f_4} & G_6 & \longrightarrow & 1
\end{array}$$

the *short five lemma* states that if g_1 and g_3 are isomorphisms, then so is g_2. The *strong five lemma* replaces those isomorphisms with injective maps.

Definition 1.8 A *fibre bundle* is a quadruple (E, X, F, π), where E is called the *total space*, X is the *base space*, F is the *fibre*, and $\pi : E \to X$ is the projection of E onto X such that there exists an open neighborhood U of every point $x \in X$ which makes the following diagram commute:

$$\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\cong} & U \times F \\
{\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mathrm{proj}_1} & \\
U & &
\end{array}$$

where proj_1 is the standard projection on the first coordinate. A covering with such open neighborhoods with their respective homeomorphisms is referred to as a *local trivialization* of the bundle.

2. Geometric Braids

The theory of braids begins with a very intuitive geometrical description of the main objects of study.

Definition 2.1 A *geometric braid* on n strings is a subset $\beta \subset \mathbb{R}^2 \times [0, 1]$ such that β is composed of n disjoint topological intervals (maps from the unit interval into a space). Furthermore, β must satisfy the following conditions:

1.) $\beta \cap (\mathbb{R}^2 \times \{0\}) = \{(1, 0, 0), (2, 0, 0), ..., (n, 0, 0)\}$

2.) $\beta \cap (\mathbb{R}^2 \times \{1\}) = \{(1, 0, 1), (2, 0, 1), ..., (n, 0, 1)\}$

3.) $\beta \cap (\mathbb{R}^2 \times \{t\})$ consists of n points for all $t \in [0, 1]$

4.) For any string in β, there exists a projection $\text{proj}_i : \mathbb{R}^2 \times [0,1] \to [0,1]$ taking that string homeomorphically to the unit interval.

Two braids are considered isotopic if one may be deformed into the other in a manner such that each of the intermediate steps in this deformation yields a geometric braid.

Given an arbitrary braid β, tracing along the strands, one finds that the 0 endpoints are permuted relative to the 1 endpoints. This permutation is the called the *underlying permutation* of β. A braid is called *pure* if its underlying permutation trivial (i.e., its endpoints are unpermuted).

3. The Braid Group of a General Manifold

Definition 3.1 Taking $\mathbb{R}^2$ to be the euclidian plane, the *classical braid group* on n strings is simply $\pi_1 B_{0,n}\mathbb{R}^2$ and, analogously, $\pi_1 F_{0,n}\mathbb{R}^2$ is the corresponding pure braid group.

Clearly, the elements of each braid group thus defined are simply geometric braids, for one may think of these braids as the graph of a map from $[0,1]$ into the space $B_{0,n}E^2$ (starting and ending at the same point in the space). Thus, taking a base of n distinct points in $\mathbb{R}^2$, one may represent any element of the classical braid group as a geometric braid. Analogously for the pure braid group, but each string must start and end at the same point. Composition of braids is simply given by stacking one braid atop another.

Definition 3.2 Given any manifold M, and a basepoint in its configuration space, its *braid group* is given by $\pi_1 B_{0,n}M$. Its *pure (unpermuted) braid group* is given by $\pi_1 F_{0,n}M$.

Although this fact will not be proven (its proof is quite long and involved, and is not the focus of this paper), it is pretty clear that for all dimensions greater than 2, the braid group is trivial. A non-rigorous proof may be given as follows. For any manifold, braids comprise one dimensional objects living in $\dim M + 1$ dimensions. Hence, if $\dim M \geq 3$, then the dimension of the space occupied by the braids on that manifold is of dimension at least 4 hence, as the braids are composed of one-dimensional strings, there is enough room to unbraid them. It is just as clear that the braid group of any 1-manifold is trivial as well, as there is too little room to begin braiding in the first place. Henceforth, when we refer to a manifold, we mean a 2-manifold.

Given the fibre bundle map $\mathfrak{p}$ described above, it is apparent that $\pi_1 B_{0,n}M/\pi_1 F_{0,n}M \cong \mathfrak{S}_n$. This isomorphism may be thought of as follows. There is a natural surjective homomorphism from $B_{0,n}M$ to $\mathfrak{S}_n$ which maps a braid to its underlying permutation. Its kernel is simply the corresponding pure braid group.

This is an application of a more general principle of algebraic topology. Given a covering space E of a base space B, one finds that there exists an inclusion map $\pi_1(E) \xrightarrow{i} \pi_1(B)$ which induces the short exact sequence

$$1 \longrightarrow \pi_1(E) \xrightarrow{i} \pi_1(B) \longrightarrow \pi_1(B)/\pi_1(E) \longrightarrow 1$$

where $\pi_1(B)/\pi_1(E)$, which need not be a group in general, is identified with fibres over a point due to transitive group action on E.

For an arbitrary 2-manifold M, given an inclusion of $\mathbb{R}^2$ into M, take $i_{m,n} : F_{m,n}\mathbb{R}^2 \to F_{m,n}M$ to be the resulting inclusion, respecting subtraction of m points (since the choice of the points does not matter, one may simply choose them all to be in the neighborhood of the inclusion), and $i_{m,n*} : \pi_1 F_{m,n}\mathbb{R}^2 \to \pi_1 F_{m,n}M$ to be the associated group homomorphism.

Lemma 3.3 Given any connected, boundaryless q-manifold, with $q \in \{2, 3, ...\}$, and $n, r \in \mathbb{N}$ such that $n > r$, one has a map $\mu : F_{0,n}M \to F_{0,r}M$ such that $\mu(v_1, ..., v_n) = (v_1, ..., v_r)$. This map is a locally trivial fibre bundle, whose fibre is $F_{r,n-r}M$.

Proof. Take a point $p = (p_1, ..., p_r) \in F_{0,r}M$. Then $\mu^{-1}(p) \subset F_{0,n}M$ such that the first r coordinates are p. Given that $F_{r,n-r}M$ is independent of the choice of removed points, one may take $F_{r,n-r}M$ to be based on $M - p$. As the first r entries in $\mu^{-1}(p)$ are fixed as p, there exists a homeomorphism $\mu^{-1}(p) \cong F_{r,n-r}M$.

Now, for each p_i making up p, there exist open neigborhoods $U_i \subset M$ containing p_i such that $\overline{U}_i$ is a closed ball, and $U_j \cap U_k = \emptyset$ if

$j \neq k$. Given this, one may easily see that $U_1 \times \cdots \times U_r = U \subset F_{r,n-r}M$ is an open neighborhood of p, considering U with respect to the subspace topology of $F_{r,n-r}M$ relative to $(M - \{p_1, ..., p_r\})^{n-r}$. Now, as the construction of such functions is rather involved (see Kassel and Turaev for all the gory details), suppose without proof that there exist continuous maps $\eta_i : U_i \times \overline{U}_i \to \overline{U}_i$ with the following special properties. For any point $s \in U_i$, one has that the restriction $\eta_i|_s : \overline{U}_i \to \overline{U}_i$, with $s \in U_i$, mapping $v \mapsto \eta_i(s, v)$ is a homeomorphism fixing the boundary $\partial\overline{U}_i$ and sending p_i to s. Fixing a point $u = (u_1, ..., u_r) \in U$, these maps naturally induce an $\eta^u : M \to M$ such that for any $v \in M$,

$$\eta^u(v) = \begin{cases} \eta_i(u_i, v) & , v \in U_i \text{ for } i \in \{1, ..., r\} \\ v & , \text{ else} \end{cases}$$

This is pretty clearly a homeomorphism depending on u in a continuous fashion between $U \times F_{r,n-r}M$ and $\mu^{-1}(p)$ which commutes with the projections. Hence, $\mu|_U : \mu^{-1}(U) \to U$ is a trivial fibre bundle, thus proving the lemma. □

Clearly, this implies the analogous result for $F_{m,n}M \to F_{m,r}M$ with fibre $F_{m+r,n-r}$ by simply considering the result on the manifold minus m points.

Lemma 3.4 If for all $m \geq 0$, one has $\pi_2 F_{m,1}M = \pi_3 F_{m,1}M = 1$, then $\pi_2 F_{0,n}M = 1$ for all $n \in \mathbb{N}$.

Proof. Given the fibration $F_{m,n}M \to F_{m,1}$ due to theorem 4.3, one obtains the following homotopy exact sequence:

$$\cdots \longrightarrow \pi_3 F_{m,1}M \longrightarrow \pi_2 F_{m+1,n-1}M \longrightarrow \pi_2 F_{m,n}M \longrightarrow \pi_2 F_{m,1}M \longrightarrow \cdots$$

Given that $\pi_2 F_{m,1}M = \pi_3 F_{m,1}M = 1$, one finds that $\pi_2 F_{m+1,n-1}M \cong \pi_2 F_{m,n}M$. Thus, by inductive reasoning, one sees that $\pi_2 F_{n-1,1}M \cong \pi_2 F_{0,n}M = 1$. □

Theorem 3.5 Take $i : F_{n-1,1}M \to F_{0,n}M$ to be the inclusion of $F_{n-1,1}M$ into $F_{0,n}M$ by adding a distinct element to the set of $n-1$ missing points. If $\pi_2 F_{m,1}M = \pi_3 F_{m,1}M = \pi_0 F_{m,1}M = 1$ for all $m \geq 0$, then the following is exact:

$$1 \longrightarrow \pi_1 F_{n-1,1}M \xrightarrow{i_*} \pi_1 F_{0,n}M \xrightarrow{\pi_*} \pi_1 F_{0,n-1}M \longrightarrow 1$$

Proof. Given theorem 3.3, one obtains a locally trivial fibration mapping $F_{0,n}M \to F_{0,n-1}M$ whose fibre is $F_{n-1,1}M$. This induces the exact (by application of Lemma 3.4) sequence above. □

Theorem 3.6 Given a compact 2-manifold M not homeomorphic to the sphere or $\mathbb{R}P^2$, one has $ker(i_{0,n*}) = 1$.

Proof. Given the homomorphisms outlined as well as the following two exact sequences, one obtains the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1 F_{n-1,1}\mathbb{R}^2 & \xrightarrow{f_1} & \pi_1 F_{0,n}\mathbb{R}^2 & \xrightarrow{f_2} & \pi_1 F_{0,n-1}\mathbb{R}^2 & \longrightarrow & 1 \\ & & \downarrow i_{n-1,1*} & & \downarrow i_{0,n*} & & \downarrow i_{0,n-1*} & & \\ 1 & \longrightarrow & \pi_1 F_{n-1,1}M & \xrightarrow{f_1} & \pi_1 F_{0,n}M & \xrightarrow{f_2} & \pi_1 F_{0,n-1}M & \longrightarrow & 1 \end{array}$$

Now, $i_{n-1,1*}$ must be injective for all $n \in \mathbb{N}$. For the sake of brevity, the proof shall be omitted, however, for those readers interested in the proof, it follows from the fact that for any manifold N and set $Q_{n-1} \subset N$ of $n-1$ points, $F_{n-1,0}N = N - Q_{n-1}$ and the Seifert-van Kampen theorem. The proof now falls to a nifty use of induction. Given that $\pi_1 F_{0,1}\mathbb{R}^2 = \pi_1 \mathbb{R}^2 = 1$, as $F_{0,1}N = N$ for any manifold N, one finds that $i_{0,1*}$ must be injective as well. Now, suppose that $i_{0,n-1*}$ is injective. Then, by the strong five lemma, one finds that $i_{0,n*}$ is injective. Hence, $ker(i_{0,n*}) = 1$. □

This inclusion implies that classical braiding is possible on any connected 2-manifold other than the sphere and projective plane (which may also be shown to exhibit classical braiding in most cases).

4. The Artin Braid Group

Definition 4.1 The Artin Braid Group on n letters, B_n, is a finitely-generated group with generators $\sigma_1, \sigma_2, ...\sigma_{n-1}$ which satisfy the following relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ when } |i-j| \geq 2 \text{ for } i,j \in \{1,...,n-1\}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \in \{1,...,n-2\}$$

These relations are referred to as the braid relations. Any element of B_n is called a braid. B_1 is trivial by definition. For all other n, B_n is infinite. B_2 is isomorphic to $\mathbb{Z}$. Now, there exists an obvious surjective group homomorphism $\pi : B_n \to \mathfrak{S}_n$ simply defined by "following" the strands in a geometric sense and analyzing their underlying permutations. Alternatively, one may write this in terms of algebraic generators by letting $\pi(\sigma_i) = \pi(\sigma_i^{-1}) = (i, i+1)$ for all $i \in \{1, ..., n-1\}$. Hence, given that the symmetric group is not abelian for $n \geq 3$, neither is the braid group. The kernel of the homomorphism between the braid group and the symmetric group is referred to as the pure braid group P_n.

While the precise link between geometric braids and the classical braid group will be formalized below, it is still possible to describe in an intuitive sense what the link is. Each of the generators represents the twisting of two adjacent strands around one another in a specified direction (the inverse is defined analogously, just in the other direction). That one is able to obtain any geometric braid (a more difficult result) will be proved later on.

While the generators described above may be the simplest, they are not the only set. In fact, they are not even the smallest.

Theorem 4.2 B_n may be generated by two or fewer elements, for any $n \in \mathbb{N}$.

Proof. Consider the following two elements of B_n: σ_1 and $\alpha = \sigma_1 \cdots \sigma_{n-1}$. If one could find a method of representing any of the standard generators in terms of these two, then clearly the theorem would follow.

claim: Given any $i \in \{1, ..., n-1\}$ and any $k \in \{1, ..., i\}$, one finds that $\alpha^{i-1}\sigma_1\alpha^{1-i} = \alpha^{i-k}\sigma_k\alpha^{k-i}$

Proof. This lends itself to a nifty proof by induction. The first non-trivial case is that when $k = 2$. Making use of the braid relations, one finds that

$$\alpha^{i-1}\sigma_1\alpha^{1-i} = \alpha^{i-2}\sigma_1 \cdots \sigma_{n-1}\sigma_1\alpha^{1-i} = \alpha^{i-2}\sigma_1\sigma_2\sigma_1 \cdots \sigma_{n-1}\alpha^{1-i} = \alpha^{i-2}\sigma_2\alpha^{2-i}$$

Analogously, suppose that the same holds for $k-1$. Then, making use of the same relations, one finds that

$$\alpha^{i-1}\sigma_1\alpha^{1-i} = \alpha^{i-(k-1)}\sigma_{k-1}\alpha^{(k-1)-i} = \alpha^{i-k}\sigma_1\cdots\sigma_{n-1}\sigma_{k-1}\alpha^{(k-1)-i} =$$

$$\alpha^{i-k}\sigma_1\cdots\sigma_{k-1}\sigma_k\sigma_{k-1}\cdots\sigma_{n-1}\alpha^{(k-1)-i} =$$

$$\alpha^{i-k}\sigma_1\cdots\sigma_k\sigma_{k-1}\sigma_k\cdots\sigma_{n-1}\alpha^{(k-1)-i} = \alpha^{i-k}\sigma_k\alpha^{k-i}$$

Thus completing the proof of the claim. □

This, in turn, implies that for all $i \in \{1, ..., n-1\}$, one has $\sigma_i = \alpha^{i-1}\sigma_1\alpha^{1-i}$. Hence, $B_n =< \sigma_1, \alpha >$. Q.E.D. □

Now, the most important generator set (other than the primary one, of course) that we will discuss is that which generates the pure braid group. The pure braid group admits generators

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \text{ for } i < j$$

which satisfy the relations

$$A_{r,s}A_{i,j}A_{r,s}^{-1} = \begin{cases} A_{i,j} & , r < i < s < j \text{ or } i < r < s < j \\ A_{r,j}A_{i,j}A_{r,j}^{-1} & , s = j \\ (A_{i,j}A_{s,j})A_{i,j}(A_{i,j}A_{s,j})^{-1} & , r = i < j < s \\ (A_{r,j}A_{s,j}A_{r,j}^{-1}A_{s,j}^{-1})A_{i,j}(A_{r,j}A_{s,j}A_{r,j}^{-1}A_{s,j}^{-1})^{-1} & , r < i < s < j \end{cases}$$

It is clear that these are in the pure braid group from the "generators" definition of the symmetric braid group. As the proof of this is long, cumbersome, and not particularly interesting, it will be omitted. This being said, it is possible to informally prove these relatively easily using braid diagrams. These generators correspond to starting at the ith strand, wrapping around the jth one, and returning on the same side of the intermediate strands.

5. The Link Between the Geometric and Algebraic Pictures

Now, we come to the primary theorem of this paper—one of the most important theorems in the preliminary study of braids.

Theorem 5.1 The classical braid group is the classical (Artin) braid group.

Proof. Recall the projection $\mathfrak{p} : F_{0,n}\mathbb{R}^2 \to B_{0,n}\mathbb{R}^2$, taking $((1,0),...,(n,0))$ as the basepoint of $F_{0,n}\mathbb{R}^2$, there is a unique lift given any loop $l \in \pi_1 B_{0,n}\mathbb{R}^2$ based at $\mathfrak{p}(((1,0),...,(n,0)))$ into a function $l : [0,1] \to F_{0,n}\mathbb{R}^2$ permuting the different strands in $F_{0,n}\mathbb{R}^2$. Now, consider the following function $\widetilde{\pi} : \pi_1 B_{0,n}\mathbb{R}^2 \to \mathfrak{S}_n$, defined as follows. Given any braid $\beta \in \pi_1 B_{0,n}\mathbb{R}^2$, there exists a unique lift $\beta = (\widetilde{\beta}_1, ..., \widetilde{\beta}_n) : [0,1] \to F_{0,n}\mathbb{R}^2$. Using this lift, the function $\widetilde{\pi}$ is defined on an arbitrary braid as follows:

$$\widetilde{\pi}(\beta) = \begin{pmatrix} \widetilde{\beta}_1(0) & \cdots & \widetilde{\beta}_n(0) \\ \widetilde{\beta}_1(1) & \cdots & \widetilde{\beta}_n(1) \end{pmatrix} \in \mathfrak{S}_n,$$

which is simply a more technical way of saying that it encodes the braid's underlying permutation (this is a special case of path lifting in the theory of covering spaces). Clearly, $\pi_1 F_{0,n}\mathbb{R}^2$ is the kernel of this surjection. Now that we have established this function, it is time to reveal its utility. Let $\mathfrak{i}_n : B_n \to \pi_1 B_{0,n}\mathbb{R}^2$ be a homomorphism relating the two classical braid groups. Given this, one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_n & \longrightarrow & B_n & \xrightarrow{\pi} & \mathfrak{S}_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathfrak{i}_n|_{P_n}} & & \downarrow{\scriptstyle \mathfrak{i}_n} & & \downarrow{\scriptstyle id_{\mathfrak{S}_n}} & & \\ 1 & \longrightarrow & \pi_1 F_{0,n}\mathbb{R}^2 & \longrightarrow & \pi_1 B_{0,n}\mathbb{R}^2 & \xrightarrow{\widetilde{\pi}} & \mathfrak{S}_n & \longrightarrow & 1 \end{array}$$

Thus, if we can show that $\mathfrak{i}_n$ restricted to P_n is an isomorphism, then it is in general as well. Recalling the set of generators for the group P_n established in section 4, P_{n-1} may be thought of as the subgroup of P_n generated by all $A_{i,j}$ such that $1 \le i < j \le n-1$. Upon inspection, there is a useful homomorphism $\xi : P_n \to P_{n-1}$ defined as follows:

$$\xi(A_{i,j}) = \begin{cases} A_{i,j} & ,1 \le i < j \le n-1 \\ 1 & ,1 \le i < j = n \end{cases}$$

Geometrically, this simply corresponds to removing the nth strand. One finds that $ker(\xi)$ is simply the normal closure of $A_{1,n}, ..., A_{n-1,n}$ in P_n, however, by careful examination of the defining relations for this presentation, one finds that this is simply the subgroup $< A_{1,n}, ..., A_{n-1,n} >= U_n$. Given the manner in which it is defined, there exists an analogous homomorphism $\pi_* : \pi_1 F_{0,n}\mathbb{R}^2 \to \pi_1 F_{0,n-1}\mathbb{R}^2$ such that $ker(\pi_*) =$

$\pi_1 F_{n-1,1}\mathbb{R}^2$, which is simply the free group on $n-1$ generators. Clearly, the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_n & \longrightarrow & P_n & \xrightarrow{\xi} & P_{n-1} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathfrak{i}_n|_{U_n}} & & \downarrow{\scriptstyle \mathfrak{i}_n} & & \downarrow{\scriptstyle \mathfrak{i}_{n-1}} & & \\
1 & \longrightarrow & \pi_1 F_{n-1,1}\mathbb{R}^2 & \longrightarrow & \pi_1 F_{0,n}\mathbb{R}^2 & \xrightarrow{\tilde{\pi}} & \pi_1 F_{0,n-1}\mathbb{R}^2 & \longrightarrow & 1
\end{array}$$

Thinking graphically, one finds that one may represent the generators of U_n as separating one string from the others. If one thinks about this picture for a little while, it becomes apparent that these generate the free group on $n-1$ letters as well. This is due geometrically to the fact that one fixed strand wrapping around the other $n-1$ strands is homotopic to the plane punctured $n-1$ times. More rigorously, the image set of the generators forms a basis for the free group, so as both groups are finitely generated and $\mathfrak{i}_n|_{U_n}$ is surjective, it is an isomorphism. Now, we proceed by induction. Note that $\pi_1 F_{0,1}\mathbb{R}^2 = 1 = P_1$. Suppose that $\mathfrak{i}_{n-1}$ is an isomorphism. Then, by the five lemma, $\mathfrak{i}_n$ is an isomorphism. Hence, $B_n \cong \pi_1 B_{0,n}\mathbb{R}^2$, which concludes the proof. □

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6. Bibliography

1. Birman, J. S. (1974). *Braids, Links, and Mapping Class Groups.* Princeton, NJ: Princeton University Press.

2. Birman, Joan S., and Tara E. Brendle. *Braids: A Survey.* N.p.: n.p., n.d. arxiv.org. Web. 2011. ¡http://arxiv.org/abs/math/0409205¿.
3. Hatcher, A. (2002). *Algebraic Topology.* New York, NY: Cambridge University Press.
4. Kassel, C., & Turaev, V. (n.d.). *Braid Groups.* N.p.: Springer. Retrieved July, 2011, from Springerlink.
5. May, J. P. (1999). *A Concise Course in Algebraic Topology.* Chicago, IL: University of Chicago Press. Retrieved July, 2011, from math.uchicago.edu/ may