A 2-ADIC EXTENSION OF THE COLLATZ FUNCTION

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Abstract. We introduce the concept of p-adic integers before exploring the connections between the 2-adic integers and the Collatz Conjecture: an unsolved problem in mathematics whose iterative nature lends itself to p-adic analysis.

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1. Introduction to the Collatz Function

The Collatz Function was first described by Lothar Collatz in the 1950s[1], but it was not until 1963 that the function was presented in published form[1]. Although hundreds of papers have analyzed the Collatz Function, it continues to be pivotal to understanding several open problems in number theory. *The Collatz Function* is defined with $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

The Collatz Conjecture states that for each $n \in \mathbb{N}$, there exists a k such that $f^k(n) = 1$ (where f^k is the k^{th} iteration of f). As an example of a natural number which satisfies the Collatz Conjecture, examine 12, which when iterated through

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the Collatz Function ($k = 9$) leads to the following sequence: $12 \implies 6 \implies 3 \implies 10 \implies 5 \implies 16 \implies 8 \implies 4 \implies 2 \implies 1$. The Collatz Conjecture has been shown to be true for all $n \in \mathbb{N}$ such that $n < 2^{60}$ [4], but the general conjecture for all natural numbers remains unproven.

Before delving into the connection between The Collatz Conjecture and p-adic numbers, we must rigorously introduce the ring of p-adic integers.

2. Metric Spaces

Definition 2.1. A *Metric Space* (Z, d) is a set Z and a distance function $d : Z \times Z \to \mathbb{R}^{\geq 0}$ such that for all $x, y, z \in Z$ the following three statements[2] hold:

(1) $d(x, y) = 0$ if and only if $x = y$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) \leq d(x, z) + d(z, y)$.

The third property is known as the Archimedian Triangle Inequality and a stronger version of the Archimedian Triangle Inequality is defined below:

Definition 2.2. *The Ultrametric Triangle Inequality*: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

Definition 2.3. An *Ultrametric Space* is a metric space whose distance function satisfies the Ultrametric Triangle Inequality.

Note that it follows from properties (1) and (3) that for all $x, y \in Z$, $x \neq y$, $d(x, y) > 0$

Example 2.4. The Rational Numbers, $\mathbb{Q}$, form a metric space under the standard distance function

$$d(x, y) = \begin{cases} x - y & \text{if } x > y \\ y - x & \text{if x} \leq \text{y.} \end{cases} \tag{2.5}$$

Proof. **(Property 1)** Since by definition, $d(x, y) = x - y$ or $d(x, y) = y - x$, if $x = y$, then $d(x, y) = 0$. And similarly if $d(x, y) = 0$, then $x = y$.

(Property 2) Without loss of generality assume that $x > y$. So $d(x, y) = x - y$ and $d(y, x) = x - y$, so by substitution, $d(x, y) = d(y, x)$.

(Property 3) Without loss of generality assume that $x > y$. So $d(x, y) = x - y = x - y + z - z = x - z + z - y$. $x - z$ equals $d(x, z)$ or $-d(x, z)$, so $x - z \leq d(x, z)$. Similarly $z - y$ equals either $d(z, y)$ or $-d(z, y)$, so $z - y \leq d(z, y)$. So by substitution, $d(x, y) = x - z + z - y \leq d(x, z) + d(z, y)$. □

3. The p-adic Distance Function

For the remainder of this paper, p will refer to a fixed prime number.

Definition 3.1. *The p-adic valuation function* $V_p : \mathbb{Z} \to \mathbb{Z} \cup \{^+\infty\}$ is defined below:

$$V_p(x) = \begin{cases} ^+\infty & \text{if } x = 0 \\ \max\{z \mid p^z \text{ divides x}\} & \text{if x} \in \mathbb{Z} \text{ and } x \neq 0. \end{cases} \tag{3.2}$$

Note that by the Fundamental Theorem of Arithmetic, $V_p(xy) = V_p(x) + V_p(y)$. It is thus natural to extend V_p to the rational numbers a/b as follows: $V_p(a/b) = V_p(a) - V_p(b)$ for all $a, b \in \mathbb{Z}$ such that $a, b \neq 0$, We can now extend the fact that $V_p(xy) = V_p(x) + V_p(y)$ to our extended p-adic valuation function:

Lemma 3.3. *For all* $x, y \in \mathbb{Q}$, $V_p(xy) = V_p(x) + V_p(y)$.

Proof. If $xy \in \mathbb{Q}$ and $xy \notin \mathbb{Z}$, then let $x = q/r$ and $y = s/t$ so that $V_p(xy) = V_p(qs/rt) = V_p(qs) - V_p(rt)$ by the definition of the p-adic valuation function. So since this lemma is true for all integers, $V_p(qs) - V_p(rt) = V_p(q) + V_p(s) - V_p(r) - V_p(t) = V_p(q) - V_p(r) + V_p(s) - V_p(t) = V_p(q/r) + V_p(s/t)$, which by definition equals $V_p(x) + V_p(y)$. ☐

We can now present three properties of the p-adic valuation function:

(1) For all $x \in \mathbb{Z}$, $V_p(x) \geq 0$.

Proof. Let $y = V_p(x) = \max\{z \,|\, p^z \text{ divides } x\}$. Since x must be an integer, $p^0|x$. And since $0 \in \{z \in \mathbb{Z} \text{ such that } p^z|x\}$, $V_p(x) = \max\{0, z_1, z_2, z_3, ...\} \geq 0$. ☐

(2) For all $x \in \mathbb{Q}$, $V_p(x) = V_p(-x)$.

Proof. For $x \in \mathbb{Z}$, this property follows directly from the fact that if $p^k|z$, then $p^k| - z$. For $x \in \mathbb{Q}$, this property follows from Lemma (3.3) and the application of this property to integers as described in the previous case. ☐

(3) For all $x, y \in \mathbb{Q}$, $V_p(x + y) \geq \min\{V_p(x), V_p(y)\}$.

Proof. Without loss of generality, let $z = V_p(x) = \min\{V_p(x), V_p(y)\}$. So choose $a, b \in \mathbb{Z}$ such that $x = ap^z$ and $y = bp^z$ with a not divisible by p. Factoring out p^z results in $x + y = p^z(a + b)$. Furthermore, $V_p(x + y) = V_p(p^z(a + b)) = V_p(p^z) + V_p(a + b)$ by Lemma (3.3). This contracts to $V_p(x + y) = z + V_p(a + b)$. And since $V_p(a + b) \geq 0$ by property (1), $V_p(x + y) \geq z = V_p(x) = \min\{V_p(x), V_p(y)\}$. ☐

Definition 3.4. The *p-adic distance function* $d_p : \mathbb{Q} \text{x} \mathbb{Q} \to \mathbb{Q}^{\geq 0}$ is defined as follows:

$$d_p(x,y) = \begin{cases} \frac{1}{p^{V_p(x-y)}} & \text{if x} \neq \text{y} \\ 0 & \text{if x = y.} \end{cases} \tag{3.5}$$

Theorem 3.6. *The rational numbers* $\mathbb{Q}$ *together with the p-adic distance function* (3.5) *form an ultrametric space.*

Proof. **(Property 1)** This is an obvious consequence of the definition of $d_p(x, y)$.

(Property 2) If $x = y$, then by definition $d_p(x, y) = 0 = d_p(y, x)$. But if $x \neq y$, then it is apparent that $d_p(x, y)$ equals $d_p(y, x)$ if and only if $V_p(x - y)$ equals $V_p(y - x)$ which follows directly from Property (2) of the p-adic valuation function[2].

(Property 3) Note that $x-y = x-z+z-y$. So by property (3) of the p-adic valuation function, $V_p(x-y) = V_p(x-z+z-y) \geq \min\{V_p(x-z), V_p(z-y)\}$. Without loss of generality assume $d_p(x, z) = \max\{d_p(x, z), d_p(y, z)\}$. From the definition of the p-adic distance function, this occurs if and only if $V_p(x - z) = \min\{V_p(x - z), V_p(y - z)\}$. So $V_p(x - y) \geq V_p(x - z)$. And it follows that $p^{V_p(x-y)} \geq p^{V_p(x-z)}$. So $\frac{1}{p^{V_p(x-y)}} \leq \frac{1}{p^{V_p(x-z)}}$. Which results in the equation: $d_p(x, y) \leq d_p(x, z) = \max\{d_p(x, z), d_p(y, z)\}$. So $d_p(x, y)$ forms an ultrametric space over $\mathbb{Q}$. □

4. Completeness

Definition 4.1. A sequence $(a_0, a_1, ...)$ of elements from a metric space $M = \{X, d(x, y)\}$ is *Cauchy* if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(a_m, a_n) < \epsilon$ [5].

Definition 4.2. A metric space M is *complete* if every Cauchy sequence in M converges to an element of M.

Example 4.3. The metric space $M = \{\mathbb{Q}, ||\}$, defined in Example (2.4), is **not** a complete metric space.

Proof. Consider the sequence (a_i) where each a_i is the irrational number $e = 2.718...$ rounded to the i^{th} decimal place. It is obvious that the sequence is Cauchy because for any ϵ, choose N such that $10^{-N} < \epsilon$ so that $d(a_m, a_n) = |a_m - a_n|$ is less than ϵ. But since e is not a rational number, and since this Cauchy Sequence converges to e, M is not a complete metric space. □

5. The Field of P-adic Numbers

Definition 5.1. *The Field of p-adic Numbers* $\mathbb{Q}_p$ can be defined for each p prime as the completion of $\mathbb{Q}$ under the p-adic distance function $d_p(x, y)$. In other words, $\mathbb{Q}_p$ is the set of the limits of all Cauchy sequences in $\mathbb{Q}$.

The real numbers are a completion of the rational numbers under the standard absolute value metric, and each real number can be written as $x = \pm \sum_{i=n}^{\infty} a_i 10^{-i}$ where each a_i is a single digit (i.e. $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$). We have just defined the base-10 decimal representation of x. So, for example, $82.34 = 8 \cdot 10^1 + 2 \cdot 10^0 + 3 \cdot 10^{-1} + 4 \cdot 10^{-2} + 0 \cdot 10^{-3}$. This idea extends perfectly to p-adic numbers.

Theorem 5.2. *(Hensel's Representation of the p-adic numbers): There is a natural bijection between* $\mathbb{Q}_p$ *and* $\{\sum_{i=s}^{\infty} a_i p^i \,|\, n \in \mathbb{Z},\, a_i \in \{0, 1, ..., p-1\}, s \in \mathbb{Z}\}$.

A proof of this theorem would require making precise the definition of completion, which is beyond the scope of this paper. But the main idea is as follows: the mapping takes the sum $\sum_{i=s}^{\infty} a_i p^i$ to the sequence (b_n) where $b_n = \sum_{i=s}^{n} a_i p^i$. This is clearly a Cauchy Sequence. The remainder of the proof shows that every Cauchy Sequence "has the same limit as" one of these partial-sum sequences; where "has the same limit as" is an equivalence relation on Cauchy sequences that may be made precise.

Definition 5.3. *The p-adic Integers* are defined (using the Hensel Representation) as $\mathbb{Z}_p = \{\sum_{i=0}^{\infty} a_i p^i\}$

Theorem 5.4. *There is a natural embedding of* $\mathbb{Q}$ *in* $\mathbb{Q}_p$.

Proof. For each $q \in \mathbb{Q}$, examine the constant sequence $A = (q, q, q, q, q, q, ...)$ with $a_i = q$ for all $i \in \mathbb{N}$. A is Cauchy because for any a_m and a_n, $d_p(a_m, a_n) = d_p(q, q) = \frac{1}{p^{V_p(0)}} = 0$ which is by definition less than ϵ for all $\epsilon > 0$. It is equally evident that the sequence A converges to q. □

Theorem 5.5. *The embedding in Theorem* (5.4) *induces an embedding of* $\mathbb{Z}^{\geq 0}$ *in* $\mathbb{Z}_p$*; and the unique Hensel Representation of each* $z \in \mathbb{Z}^{\geq 0}$ *is* $\sum_{i=0}^{n} a_i p^i$ *for some* $n \in \mathbb{N}$.

Proof. **(Existence)** This proof uses induction. For $z = 0$, $n = 0$, because the p-adic Hensel Representation of z is $0 \cdot p^0 + 0 \cdot p^1 + ...$ Now assume that our theorem holds for all $z \in \mathbb{Z}^{\geq 0}$ such that z is less than or equal to a given k. Examine $k+1$. Let $q = \max\{s \,|\, k+1-p^s > 0\}$. Since $0 < k+1-p^q \leq k$, there exists an $n \in \mathbb{N}$ such that the p-adic Hensel Representation of $k+1-p^q$ is $\sum_{i=0}^{n} a_i p^i$. Also, $a_i = 0$ for all $i \geq q$ since $k+1-p^q < p^q$. So for the integer $k+1$, $n = q$.

(Uniqueness) Assume that there exist two distinct p-adic Hensel Representations of $z \in \mathbb{Z}^{\geq 0}$: $\sum_{i=0}^{n} a_i p^i$ and $\sum_{i=0}^{m} b_i p^i$. If $z \equiv r \pmod{p}$ for some $r \in \mathbb{Z}^{\geq 0}$, then a_0 must equal r and b_0 must equal r. Since this is true for all $r \in \mathbb{Z}^{\geq 0}$, $a_0 = b_0$. So substracting the equal terms a_0 and b_0 from their respective summations, we must still be left with distinct coefficients since we began with distinct representations of z. Now assume that $a_i = b_i$ for all i less than or equal to a given k. Examine a_{k+1} and b_{k+1}. If $z \equiv t \pmod{p^{k+2}}$ for some $t \in \mathbb{Z}^{\geq 0}$, then to preserve equality between each summation and z, a_{k+1} must equal q and b_{k+1} must equal q since each $a_i p^i$ equals $b_i p^i$ for $i \leq k$. So by induction, each coefficient a_i must equal b_i which contradicts our assumption that there exist two distinct p-adic Hensel Representations of z. □

Example 5.6. 2-adically, the Hensel Representation of 93 is: $93 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4 + 0 \cdot 2^5 + 1 \cdot 2^6 + 0 \cdot 2^7 + 0 \cdot 2^8 + 0 \cdot 2^9 + 0 \cdot 2^{10} + ...$

Example 5.7. 2-adically, $-1 = 1 + 1 \cdot 2 + 1 \cdot 2^2 + ...$, because adding 1 to each side, we arrive at the continuously contracting equation $0 = 2 + 1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 + ... = 0 + 2 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 + ... = 0 + 0 + 2 \cdot 2^2 + 1 \cdot 2^3 + ... = 0 + 0 + 0 + 2 \cdot 2^3 + ...$

6. Extending the Collatz Function to the 2-adic Integers $\mathbb{Z}_2$

In order to extend the Collatz Function to 2-adic integers, we must establish an idea of congruence between two different 2-adic numbers.

Definition 6.1. For $s, t \in Z_p$, we say s is *congruent to t (mod p^k)*, written as $s \equiv t \pmod{p^k}$, if the first k terms of the Hensel representations of s and t are equal [7].

Using this idea of congruence, we can extend the Collatz Function to $C : Z_p \to Z_p$ as follows: for $n = a_0 2^0 + a_1 2^1 + ...$,

$$C(n) = \begin{cases} n/2 & \text{if } a_0 = 0 \\ 3n+1 & \text{if } a_0 = 1. \end{cases} \tag{6.2}$$

Definition 6.3. A function $f : M \to M$ is *continuous* if for each a in the domain of f the following holds: for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, a) < \delta$ implies that $d(f(x), f(a)) < \epsilon$

Theorem 6.4. *$C(n)$ is continuous over the 2-adic integers with respect to the 2-adic distance function $d_2(x, y)$.*

Proof. Let $\epsilon > 0$ be arbitrary and $\delta = \min\{1, \epsilon/2\}$. For $a \in Z_2$, we will show that C is continuous at a. Let $x \in Z_2$ satisfy $d_2(a, x) < \delta$. We examine two possible cases:

(Case 1): $a \equiv 0 \pmod 2$. Assume $x \equiv 1 \pmod 2$ so that $x - a \equiv 1 \pmod 2$. Since $x - a$ is not divisible by 2, $V_2(x-a) = 0$. So $d_2(x, a) = \frac{1}{2^{V_2(x-a)}} = 1$. This is a contradiction since we would arrive at the inequality: $1 = d_2(x, a) < \delta \leq 1$. So $x \not\equiv 1 \pmod 2$. Since $x \equiv 0 \pmod 2$, the application of the Collatz Function to the even x and a terms results in: $d_2(C(x), C(a)) = \frac{1}{2^{V_2((x-a)/2)}} = \frac{1}{2^{V_2(x-a)-1}} = \frac{2}{2^{V_2(x-a)}} < 2\delta \leq \epsilon$.

(Case 2): $a \equiv 1 \pmod 2$. Assume $x \equiv 0 \pmod 2$ so that $x - a \equiv 1 \pmod 2$. Since $x - a$ is not divisible by 2, $V_2(x-a) = 0$. So $d_2(x, a) = \frac{1}{2^{V_2(x-a)}} = 1$. This is a contradiction since we would arrive at the inequality: $1 = d_2(x, a) < \delta \leq 1$. So $x \not\equiv 0 \pmod 2$. Since $x \equiv 1 \pmod 2$, the application of the Collatz Function to the odd terms x and a results in: $d_2(C(x), C(a)) = \frac{1}{2^{V_2(3x+1-3a-1)}} = \frac{1}{2^{V_2(3(x-a))}}$. And since 3 is not divisible by 2, by Lemma (3.3), our equation equals $\frac{1}{2^{V_2(x-a)}} < \delta \leq \epsilon/2 < \epsilon$. ☐

Examining the properties of the extended Collatz Function might seem like a roundabout way to attempt to prove the Collatz Conjecture, but, as described in the next section, simple mathematical properties can lead to meaningful results in number theory. Researchers have shown that the extended Collatz Function $C(n)$ is measure-preserving, strongly mixing, and ergodic [7], as well as not analytic, infinitely many times differentiable, surjective, and not injective [8]. The hope is that proving certain properties of $C(n)$ will lead to a proof of the Collatz Conjecture.

7. Examining the Collatz Conjecture Modulo 2

Assume that the Collatz Conjecture is false. This implies that there exists a non-empty set of natural numbers for which repeated applications of the Collatz Function never result in 1. Let S be that set of natural numbers. Let f be the Collatz Function (1.1). Note that if $z \in S$, then $f(z) \in S$.[1]

Since every element of S is greater than 1 (i.e. bounded below) and since S is non-empty, S has a smallest element. Let q be the smallest element of S.

Theorem 7.1. $q \equiv 1$ *(mod 2)*

[1]Assume $f(z) \notin S$. So $\exists k \in \mathbb{N}$ s.t. k applications of f to $f(z)$ will result in 1; this composition, when further composed with $f(z)$, yields the $k+1$ applications of the Collatz Function to z which result in 1 which is a contradiction because $z \in S$.

Proof. Assume $q \not\equiv 1 \pmod 2$. Then $q \equiv 0 \pmod 2$ since all integers are 1 or 0 (mod 2). So since q is even, $f(q) = q/2$ which is an element of S. But this contradicts the fact that q is the smallest element of S because for all $q \in \mathbb{N}$, $q/2 < q$. □

Note that since $q \equiv 1 \pmod 2$, $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$.

Theorem 7.2. $q \equiv 3$ *(mod 4)*

Proof. Assume $q \not\equiv 3 \pmod 4$. Then $q \equiv 1 \pmod 4$ by Theorem (7.1). So, q can be written as $q = 4n+1$ for some $n \in \mathbb{N}$. And since q is odd, $f(q) = 12n+4$. And since $12n+4$ is even, $f(12n+4) = 6n+2$. And since $6n+2$ is even, $f(6n+2) = 3n+1$. But since $3n+1 \in S$, and since $3n+1 < 4n+1$, we have a contradiction. So since $q \not\equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$. □

Note that since $q \equiv 3 \pmod 4$, $q \equiv 3 \pmod{16}$ or $q \equiv 7 \pmod{16}$ or $q \equiv 11 \pmod{16}$ or $q \equiv 15 \pmod{16}$.

Theorem 7.3. $q \not\equiv 3$ *(mod 16).*

Proof. Assume $q \equiv 3 \pmod{16}$. So q can be written as $q = 16n+3$ for some $n \in \mathbb{N}$. And since q is odd, $f(q) = 48n + 10$. And since $48n + 10$ is even, $f(48n + 10) = 24n+5$. And since $24n+5$ is odd, $f(24n+5) = 72n+16$. Three more applications of $f(x)$ to even integers results in $f(72n + 16) = 36n + 8$, $f(36n + 8) = 18n + 4$, and finally $f(18n + 4) = 9n + 2$. But by the same logic as is used in the proofs of the previous two theorems, since $9n+2 < 16n+3$ and since $9n+2 \in S$, we have a contradiction. So $q \not\equiv 3 \pmod{16}$. □

So to conclude this progression of theorems, $q = 7$, 11, or 15 (mod 16).

8. Conclusion

The conclusions of Section (7) might seem inconsequential, but further results of the same type could result in a proof of the Collatz Conjecture. Section (7) provides results of the form: if $q = a \pmod{2^k}$, then there exists an $n > k$ such that $q \neq a \pmod{2^n}$. If this result was proven for all $k \in \mathbb{N}$, then, by Theorem (5.5), q could not be an integer. And since q would not be an integer, the set S defined in section (7) would have to be empty. So the Collatz Conjecture would be true for all natural numbers.

Unfortunately the approach detailed in Section (7) does not lead to the above result for $k > 4$. But even so, the knowledge that the smallest element of the set of numbers which does not satisfy the Collatz Conjecture must be congruent to 7, 11, or 15 (mod 16) could accelerate the search for natural numbers which contradict the Collatz Conjecture. Instead of investigating every natural number up to a certain point for convergence to 1 (as many papers and a distributed computing project[4] have done in the past), a search can be conducted in a fraction of the time by ignoring all the integers which do not satisfy the properties of q which are proven above.

The p-adic numbers, although complex, offer an alternative way to examine the Collatz Conjecture. And perhaps using the extended Collatz Function or other functions like it, someone will eventually solve this seemingly simple problem which has resisted a solution for so many years.

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