Integer Partitions

University of Chicago REU 2026

Aditya Pahuja

performed on 2026-07-16; last edited 2026-07-22

Contents

1 Partition bijections

One of the fundamental counting problems is that of splitting a collection of n things into groups. There are several different flavors of this question: are the things distinct or identical? Do we care about the ordering of the groups? Are the groupings restricted in any way?

These notes address the instance of n identical things being split into groups whose order doesn't matter. This is more concretely phrased as expressing an integer as the sum of a nonincreasing sequence of nonnegative integers.

Definition 1.1 — A **partition** of an integer n is a decomposition of n into a sum of positive integers: namely, we say that a sequence $(a_1, a_2, \ldots, a_k)$ is a partition of n if $a_1 \geq a_2 \geq \cdots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = n$. The a_i are called the **parts** of this partition.

Where convenient, we will abbreviate $(a_1, a_2, \ldots, a_k)$ by just writing $a_1 + a_2 + \cdots + a_k$ for a given partition of n. For example, the seven partitions of 5 are

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

We generally refer to the number of partitions of n as $p(n)$, so $p(5) = 7$; by convention, $p(0) = 1$.

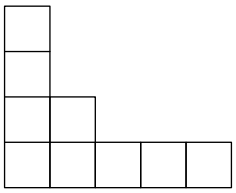
Counting partitions is generally hard, and as such, the main flavor of question that appears in the theory of integer partitions is showing that a certain class of partitions has the same size as some other (perhaps known/computable) quantity. For example:

Theorem 1.2. The number of partitions of n with k parts is the same as the number of partitions of n whose largest part is k.

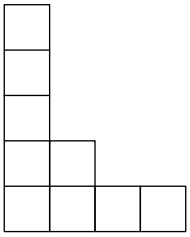
Statements like that of Theorem 1.2 often fall to bijective proofs. One technique for coming up with bijections is visually representing the partitions:

Definition 1.3 — The **Ferrers diagram** of a partition $(a_1, a_2, \ldots, a_k)$ of n is a set of k columns of boxes aligned at their bottom edge in which column j contains a_j boxes.

It is easier to describe these by just showing a picture; here is the Ferrers diagram of $(4, 2, 1, 1, 1)$.



With enough cleverness, we can manipulate these diagrams so as to generate new partitions. For example, we can form the **conjugate partition**, obtained by reflecting a Ferrers diagram so that columns become rows and rows become columns (i.e. read the diagram horizontally instead of vertically). For example, the conjugate of $(4, 2, 1, 1, 1)$ is $(5, 2, 1, 1)$, as shown below.



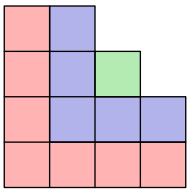
We also have the notion of a **self-conjugate** partition, which, as you may infer, is a partition that is invariant under conjugation. We shall use Ferrers diagrams to prove Theorem 1.2 very quickly, as well as a few other neat facts.

Proof of Theorem 1.2. Let $\mathcal{S}$ be the set of partitions of n with k parts, and let $\mathcal{T}$ be the set of partitions of n with largest part k. Then, we have a natural mapping $S \to T$ given by conjugation, and it is obvious that this map is bijective because it is inverted by just conjugating again. □

Some more partition facts:

Theorem 1.4. The number of partitions of n into distinct odd parts is the same as the number of self-conjugate partitions of n.

Proof. I'll just provide a picture for this one.



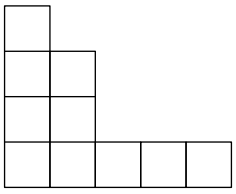
Do you see the bijection? □

Theorem 1.5. The number of partitions of n in which each part is at least two is $p(n) - p(n-1)$.

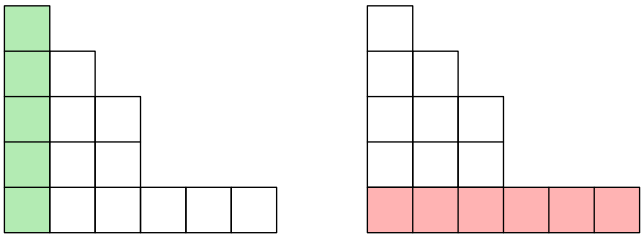
Proof. We just need to show that the number of partitions of n containing a 1 is $p(n-1)$. The corresponding bijection is easy: take a partition of n containing 1, and delete the 1! □

Theorem 1.6. The number of partitions of $a - c$ into exactly $b - 1$ parts, each of which is at most c, is the same as the number of partitions of $a - b$ into exactly $c - 1$ parts, each of which is at most b.

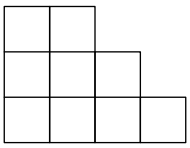
Proof. We will do a "proof by pictures", where the diagrams have $a = 15$, $b = 6$, and $c = 5$. One partition of $a - c = 10$ into $b - 1 = 5$ parts of size at most $c = 5$ is



Add a part (column) of size c and then subtract one from each part (delete the bottom row):



This represents a partition of $a-b$ because it was obtained by adding c and deleting b squares; it has largest part $c-1$ because the largest part prior to deleting the bottom row was the green column of size c; and it has at most b parts because we started with $b-1$ parts and added one, and deleting the bottom row doesn't change the number of parts. Taking the conjugate (depicted below) of this partition therefore gives a partition of $a-b$ into exactly $c-1$ parts, each of which is at most b.



Showing that this map is a bijection is left as an exercise. □

2 Interlude on generating functions

The principle behind a generating function is a common one in mathematics: when you are given a large collection of data, try to mash everything into one object; if this object is well-behaved enough, you can extract information about the constituent data.

In our case, we define the **(ordinary) generating function** of a sequence of real numbers a_0, a_1, ... to be the power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Unless we are choosing a particular value for x we do not really concern ourselves with issues of convergence; in this sense we consider $A(x)$ to be a "formal" power series.

Our goal is now to use algebraic operations on power series to try to recover whatever information we might need. For example, we can add, multiply, differentiate, integrate rewrite as rational functions, etc. Here is a simple application:

Theorem 2.1. The nth term of the sequence of integers given by $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = 3f_{n+1} - 2f_n$ for $n \geq 0$ is $f_n = 2^n - 1$.

Proof. Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$ be the generating function for f_n. Then $xF(x)$ and $x^2F(x)$ are given by shifting the indices of the coefficients by 1 and 2 respectively, so

$$F(x) - 3xF(x) + 2x^2F(x) = f_0 + f_1 x - 3f_0 x + \sum_{n=2}^{\infty}(f_n - 3f_{n-1} + 2f_{n-2})x^n = x,$$

implying that

$$F(x) = \frac{x}{1-3x+2x^2} = \frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{n=0}^{\infty}(2^n - 1)x^n$$

by geometric series. □

Of course, in principle you could have done e.g. induction to verify this, but this is a technique to *find* the answer!

> **Exercise.** Find a general formula for the nth Fibonacci number F_n, given by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. (Try to do this without using any prior knowledge of what the answer is!)

In combinatorics what tends to happen is that we have a sequence given by counting some configuration of n objects, so we form the generating function for the sequence; the generating function is then something we can handle with our myriad algebraic operations. We will show examples of this strategy in what follows.

3 Euler's pentagonal number theorem

Let us begin by working with the generating function for the sequence $p(n)$.

Theorem 3.1. The (ordinary) generating function for $p(n)$, which we will call $P(x)$, is given by

$$P(x) = \sum_{n\geq 0} p(n)x^n = \prod_{n\geq 1} \frac{1}{1-x^n}.$$

To be more explicit — unfurling the quotient into a geometric series — the product should be read as

$$\prod_{n\geq 1}(1 + x^n + x^{2n} + x^{3n} + \cdots).$$

Each term in the resulting product power series is produced by picking exactly one $(x^n)^j$ monomial for each n. The result is that each monomial obtained from expanding the product looks like

$$x^{a_1\cdot 1+a_2\cdot 2+a_3\cdot 3+\cdots},$$

so this term enumerates one of the partitions of $n = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots$. Summing across all such terms, we find that the product equals $\sum_{n=0}^{\infty} p(n)x^n$.

Remark 3.2. You may complain that you could've picked infinitely many non-one terms when expanding the product, as this would produce a nonsensical expression x^∞. To fix this, I will specify that the infinite product is actually the limit of finite products as you would expect, where "convergence" in this world means that the coefficient of x^n eventually stabilizes, for each n. If you do not trust me, you are free to go check for yourself that the product of geometric series that yields $P(x)$ does converge; perhaps this is instructive (same for other infinite products I write, e.g. $Q(x)$ below).

Since $P(x)$ is the product over all n of $(1-x^n)^{-1}$, it is a natural question to ask what the product of $1-x^n$ is equal to. In fact, if we understand what this product is, we can derive a recursive formula for $P(n)$. Specifically, letting

$$Q(x) = \prod_{n=1}^{\infty}(1-x^n) = \sum_{k=0}^{\infty} c_k x^k,$$

the identity

$$1 = P(x)Q(x) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} p(i)c_j x^k\right)$$

implies that the x^0 coefficient $p(i)c_j$ is equal to 1, and that the remaining coefficients are zero, which we express as

$$p(k) = p(k)c_0 = -\sum_{i=0}^{k-1} p(i)c_{k-i}.$$

As it turns out, we can compute the c_j explicitly. To do so we will start by interpreting what the coefficients of $Q(x)$ count. A monomial in the expansion of $Q(x)$ is obtained by making a choice of 1 or $-x^n$ for each $n = 1, 2, \ldots$. If we pick r non-one terms, say $-x^{n_i}$ for $i = 1, \ldots, r$ (where the n_i are distinct positive integers), the resulting monomial makes a contribution of $(-1)^r x^{n_1+\cdots+n_r}$ to the power series for $Q(x)$. Thus, the coefficient of x^n is

$$\sum_{\substack{n_1<\cdots<n_r\\ n_1+\cdots+n_r=n}} (-1)^r,$$

which is the number of partitions of n into an even number of distinct parts minus the number of partitions of n into an odd number of distinct parts. We now present one way to count this quantity.

Theorem 3.3 (Euler's pentagonal number theorem)**.** Let $p_e(n)$ be the number of partitions of n into an even number of distinct parts, and let $p_o(n)$ be the number of partitions of n into an odd number of distinct parts. Then,

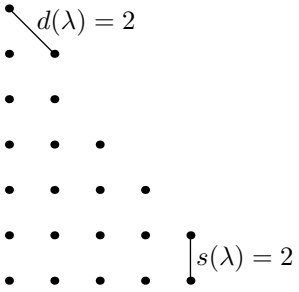
$$p_e(n) - p_o(n) = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m \pm 1) \\ 0 & \text{otherwise} \end{cases}$$

The numbers $\frac{1}{2}m(3m \pm 1)$, where m is an integer, are called **pentagonal numbers** (see Wikipedia for why).

Exercise 3.1. Show that I could equivalently have described the pentagonal numbers as the numbers obtainable as $\frac{1}{2}m(3m+1)$ for integer m, or alternatively the numbers $\frac{1}{2}m(3m-1)$ for integer m. That is, my sentence above is slightly redundant.

Proof. Let A and B respectively be the two sets of partitions in question. We will try to pair off elements of A and B, and our pairing will succeed in all but two cases.

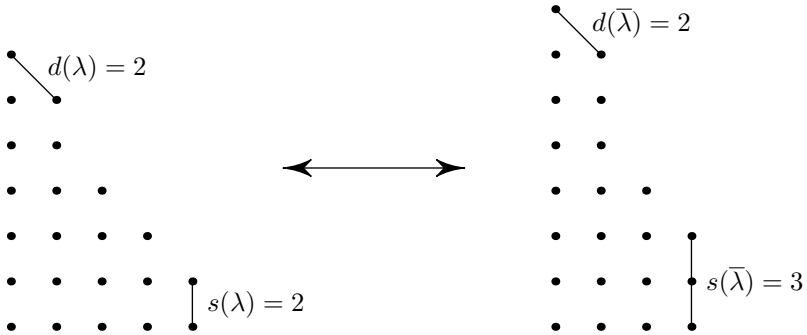
First, given a partition λ of n, we define the following two functions: $s(\lambda)$ is the smallest part of λ and $d(\lambda)$ is the greatest j with the property that the j largest parts of λ are consecutive. They are quite easy to read off of a given Ferrers diagram; here is an example with $\lambda = (7, 6, 4, 3, 2)$.



Given λ, we then define $\overline{\lambda}$ as follows:

- If $s(\lambda) > d(\lambda)$, then subtract one from each of the $d(\lambda)$ largest parts, and add in a new (smallest) part of size $d(\lambda)$.
- If $s(\lambda) \leq d(\lambda)$, then delete the smallest part and add one to each of the $s(\lambda)$ largest parts.

Here is what this looks like for $\lambda = (7, 6, 4, 3, 2)$ and its counterpart $\overline{\lambda} = (8, 7, 4, 3)$:



Most of the time, this will pair an element of A with an element of B. However, there are two points of failure.

- In the first case, our transformation tries to create a new smallest part. This can fail if $\lambda = (r+1, r+2, \ldots, 2r)$, in which case $\overline{\lambda} = (r, r, r+1, \ldots, 2r-1)$ no longer has distinct parts. Here, λ is a partition of $\frac{r(3r+1)}{2}$.

- In the second case, our transformation tries to get rid of the smallest part. This can fail if $\lambda = (r, r+1, \ldots, 2r-1)$, in which case $\overline{\lambda} = (1, r+2, r+3, \ldots, 2r)$ still has the same number of parts. Here, λ is a partition of $\frac{r(3r-1)}{2}$.

Thus, if $n = \frac{r(3r\pm1)}{2}$, then, based on the parity of n, $|A| - |B| = p_e(n) - p_o(n)$ is either $+1$ or -1. Indeed, the exact value is just $(-1)^r$ since the bad λs enumerated above both have r parts. Otherwise (if n isn't a pentagonal number) then the transformation works seamlessly, and $|A| = |B|$. □

Therefore, we have proven that

$$\prod_{k\geq 1}(1-x^k) = \sum_{m\in\mathbb{Z}}(-1)^m x^{m(3m-1)/2} = 1 + \sum_{m\geq 1}(-1)^m (x^{m(3m-1)/1} + x^{m(3m+1)/2}).$$

The promised recursive formula is

$$\boxed{p(n) = \sum_{m\geq 1}(-1)^m \left(p\left(n - \frac{m(3m-1)}{2}\right) + p\left(n - \frac{m(3m+1)}{2}\right)\right).}$$

Exercise (If you're bored)**.** Compute $p(1)$, $p(2)$, …, $p(15)$. Not so hard, right?

Remark 3.4. In the way of computation, the following asymptotic formula for $p(n)$ is also cool:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

where $\exp(x) = e^x$. That is, the ratio of the two expressions approaches 1 as n grows arbitrarily large.

The best known method for computing $p(n)$ for large n is the Hardy-Ramanujan-Rademacher formula:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{d}{dx} \frac{\sinh((\pi/k)(\frac{2}{3}(x-\frac{1}{24})^{1/2}))}{(x-\frac{1}{24})^{1/2}}\right]_{x=n}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ \gcd(h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k}$$

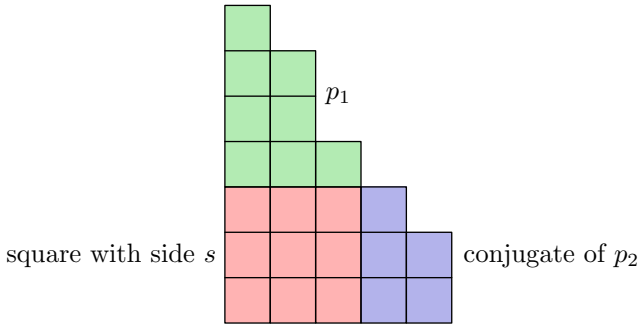
where $\omega_{h,k}$ is a particular 24th root of unity (I will spare you from the precise definition). It is a monstrosity of a formula and a truly breathtaking achievement, giving an exact formula for $p(n)$ that converges extremely rapidly (and actually gives the asymptotic formula above as a corollary).

4 Jacobi's triple product identity

4.1 Durfee squares

Definition 4.1 — The **Durfee square** of a partition is the largest square that can be drawn along the bottom and left edges of the partition's Ferrers diagram.

For example, the partition below has a Durfee square of side length 3 (drawn in red). The interesting thing is that drawing the Durfee square splits a partition into three pieces: the square itself (say of side length s), a partition with at most s parts, and a partition whose conjugate has at most s parts.



In fact, the set of triples (s, p_1, p_2), where $s \geq 0$ is an integer and p_1 and p_2 are partitions with at most s parts, is in bijection with the set of all partitions, with the bijective mapping given exactly by the above picture. This leads us very quickly to the following identity:

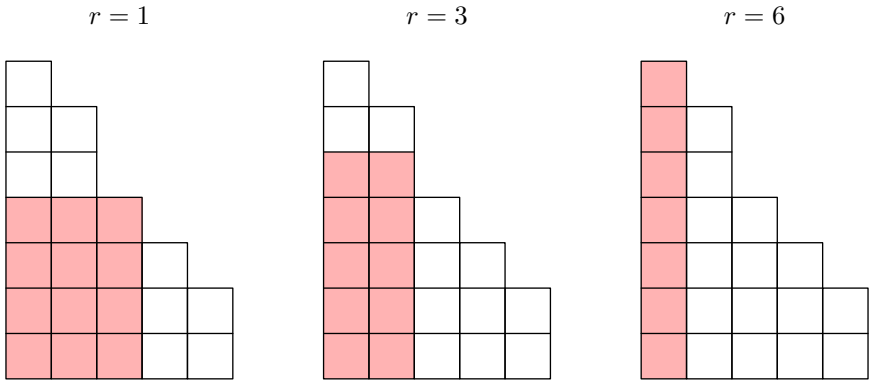
$$\sum_{s \geq 0} x^{s^2} \cdot \prod_{i=1}^{s} \frac{1}{(1-x^i)^2} = P(x).$$

Exercise. If the identity does not look obvious to you, convince yourself that it is true.

We can be more general with this construction.

Definition 4.2 — For an integer $r \geq 0$, the **r-Durfee rectangle** of a given partition is the largest bottom-edge-aligned and left-edge-aligned rectangle whose height is r more than its width that can be drawn within the Ferrers diagram. In particular, the 0-Durfee rectangle is the Durfee square.

Here are a couple of examples.



The previous identity generalizes seamlessly:

Theorem 4.3.

$$\sum_{s\geq 0} x^{s(s+r)} \prod_{i=1}^{s+r} \frac{1}{1-x^i} \cdot \prod_{i=1}^{s} \frac{1}{1-x^i} = P(x).$$

The proof goes in a very similar manner: fixing r, we get a bijection between arbitrary partitions and triples (s, p_1, p_2), where p_1 has at most s parts and p_2 has at most $s+r$ parts. The bijection is analogous to the square case above: a partition yields a triple in a unique way; and conversely, to recover a partition from a triple (s, p_1, p_2), take an $s \times (s+r)$ rectangle and glue p_1 and p_2 onto the sides with length s and $s+r$ respectively.

4.2 Jacobi's identity

We will now prove the following identity, which both implies the pentagonal number theorem and has much greater consequences (for example, we will use it to prove a famous theorem of Ramanujan, which says that $p(5n+4)$ is always a multiple of five).

Theorem 4.4 (Jacobi's triple product identity)**.**

$$\prod_{n\geq 1}(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = \prod_{n\geq 1}(1-x^{2n})^{-1}\sum_{r\in\mathbb{Z}} x^{r^2}z^r.$$

Proof. First, fixing r, let's describe what the $x^q z^r$ coefficient counts on the left-hand side. Since the LHS is symmetric in z^1 and z^{-1}, we can assume without loss of generality that r is positive. Such a term arises from picking r more $x^{2n-1}z$ terms than $x^{2n-1}z^{-1}$ terms when expanding the product — thus, the coefficient of $x^q z^r$ counts the number of ways to express q as

$$q = \sum_{i=1}^{s+r}(2a_i - 1) + \sum_{i=1}^{s}(2b_i - 1),$$

where $s \geq 0$ and the a_i and b_i are increasing sequences of positive integers.

Thus, if we write the LHS as

$$\sum_{r\in\mathbb{Z}} f_r(x) z^r,$$

we know that the coefficients of $f_r(x)$ count the number of ways to choose the a_i and b_i. Our goal then is to show that $f_r(x) = x^{r^2}\mathcal{P}(x^2)$.

Right now, we have expressed q as a sum of two partitions with odd, distinct parts. This should be vaguely reminiscent of Theorem 3.8, which dealt with sums of partitions, albeit unrestricted ones. To start with, we can write

$$\frac{q+2s+r}{2} = \sum_{i=1}^{s+r} a_i + \sum_{i=1}^{s} b_i.$$

This gets rid of the parity restriction, but it doesn't change the fact that the a_i are distinct and the b_i are distinct. The goal, then, is to convert an increasing sequence of positive integers to a nondecreasing sequence of nonnegative integers, which is the setting that Theorem 3.8 takes place in. The standard way that this is done is to consider $a_i - i$ and $b_i - i$ (you might see this pattern in a stars-and-bars context, too): after some algebra, this gives

$$\frac{q-r^2}{2} = s(s+r) + \sum_{i=1}^{s+r}(a_i - i) + \sum_{i=1}^{s}(b_i - i)$$

To be explicit, what we have done here is reparameterized $\frac{q-r^2}{2}$ as $s(s+r)$ plus a sum of unrestricted partitions with at most $s+r$ and s parts respectively

In other words, the number of ways to express q as a sum of two increasing sequences of $s+r$ and s odd numbers is the same as the number of ways to express $\frac{q-r^2}{2}$ as $s(s+r)$ plus two arbitrary partitions with at most $s+r$ and s parts, across all nonnegative integers s (where the number of partitions of a non-integer is zero, if you were particularly concerned about that). This latter object is exactly what Theorem 3.8 counts — indeed, the quantity we have been counting is in bijection with the partitions of $(q-r^2)/2$, so the quantity we have been counting is the $x^{(q-r^2)/2}$ coefficient of $P(x)$.

The rest is purely algebraic manipulation to extract

$$f_r(x) = \sum_{q\geq 0} p\left(\frac{q-r^2}{2}\right) x^q$$

in terms of $P(x)$. We can double all the exponents of $P(x)$ by substituting $x \mapsto x^2$, and then add r^2 to all the exponents by multiplying it with x^{r^2}:

$$\begin{aligned} P(x) &= p(0) + p(1)x + \cdots + p\left(\frac{q-r^2}{2}\right) x^{(q-r^2)/2} + \cdots \\ P(x^2) &= p(0) + p(1)x^2 + \cdots + p\left(\frac{q-r^2}{2}\right) x^{q-r^2} + \cdots \\ x^{r^2}P(x) &= p(0)x^{r^2} + p(1)x^{2+r^2} + \cdots + p\left(\frac{q-r^2}{2}\right) x^{q} + \cdots \end{aligned}$$

This implies that $f_r(x)$ agrees with $x^{r^2}P(x)$ on all its coefficients, so the power series are equal. □

From here, a proof of Euler's theorem takes one line: set $(x,z) = (q^{3/2}, -q^{-1/2})$ to get

$$\sum_{r\in\mathbb{Z}} (-1)^r q^{(3r^2-r)/2} = \prod_{n\geq 1} (1-q^{3n-2})(1-q^{3n-1})(1-q^{3n}) = \prod_{n\geq 1}(1-q^n).$$

4.3 Ramanujan's theorem

Our goal is to prove

Theorem 4.5. The number of partitions of $5n+4$ is a multiple of 5.

To do this we will use the following alternative form of Jacobi's identity:

$$\prod_{n\geq 1}(1-x^n)^3 = \sum_{r=0}^{\infty} x^{\frac{r^2+r}{2}} \cdot (-1)^r(2r+1).$$

To get this from the form we proved earlier, set $z = -xw$ (where w is just some other variable) to get

$$(1-w^{-1})\prod_{n\geq 1}(1-x^{2n}w)(1-x^{2n}w^{-1})(1-x^{2n}) = \sum_{r\in\mathbb{Z}} x^{r^2+r}w^r(-1)^r = \sum_{r\geq 0}(-1)^r x^{r^2+r}(w^r - w^{-r-1}).$$

Dividing out $w-1$ gives

$$w^{-1}\prod_{n\geq 1}(1-x^{2n}w)(1-x^{2n}w^{-1})(1-x^{2n}) = \sum_{r\geq 0}(-1)^r x^{r^2+r}(w^{r-1} + w^{r-2} + \cdots + w^{-r-1}),$$

and then plugging in $w=1$ and replacing x^2 with x gives the result (at this point we need to make that for each n, the coefficient of x^n doesn't do anything crazy after setting in $w=1$, but we already know explicitly that the coefficient of any given x^n is just a polynomial in w, so in particular a finite sum).

Proof of Theorem 4.5. We start with the observation that $p(5n-1)$ (writing $5n-1$ instead of $5n+4$ because 1 is smaller than 4) is the coefficient of x^{5n} in

$$x\prod_{i=1}^{\infty}(1-x^i)^{-1} = \left(x\prod_{i=1}^{\infty}(1-x^i)^4\right)\cdot\prod_{i=1}^{\infty}(1-x^i)^{-5} = F(x)G(x),$$

where $F(x)$ is defined as the product inside the parentheses in the middle term of the equality and $G(x) = P(x)^5$ is the other factor. If $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and $G(x) = \sum_{n=0}^{\infty} g_n x^n$, then the x^{5n} coefficient is

$$\sum_{i+j=5n} f_i g_j.$$

We will show that either f_i or g_j is always a multiple of five, so that the entire coefficient is a multiple of five.

Recall from number theory that if $f(x)$ is a polynomial then

$$f(x)^5 \equiv f(x^5) \pmod 5$$

(where the congruence mod 5 means that the corresponding coefficients agree mod 5); this generalizes to power series easily. Therefore, the coefficient g_j of x^j in $P(x)^5$ is a multiple of 5 if j isn't a multiple of 5.

In the other case, where j is a multiple of 5, then i is also a multiple of 5. Applying the alternative form of Jacobi's identity and Euler's pentagonal number theorem, we have

$$F(x) = x\cdot\prod_{m=1}^{\infty}(1-x^m)\cdot\prod_{k=1}^{\infty}(1-x^k)^3 = \sum_{m\in\mathbb{Z}}\sum_{k\geq 0}(-1)^{m+k}(2k+1)x^{1+m(3m-1)/2+k(k+1)/2}.$$

Now we figure out what m, k can be mod 5 given that the exponent $1+\frac{m(3m-1)}{2}+\frac{k(k+1)}{2} = i$ is 0 mod 5.

x	$\frac{x(3x-1)}{2}$ mod 5	$\frac{x(x+1)}{2}$ mod 5
0	0	0
1	1	1
2	0	3
3	2	1
4	2	0

From the table, we can see that the only way to get 1 plus an entry from each of the columns to be a multiple of 5 is if $\frac{m(3m-1)}{2} \equiv 1 \pmod 5$ and $\frac{k(k+1)}{2} \equiv 3 \pmod 5$, or equivalently $m \equiv 1 \pmod 5$ and $k \equiv 2 \pmod 5$. The contribution to the x^i coefficient by this pair (m,k) is $(-1)^{m+k}(2k+1)$, which is miraculously a multiple of five because $2k+1 \equiv 2\cdot 1+1 = 5 \pmod 5$, so we are done! □