PANTS DECOMPOSITION AND TEICHMÜLLER SPACE

PARIS YANG

Abstract. This exposition serves to introduce an application of pants decompositions of a hyperbolic surface. We first define some preliminary terms before going into what is meant by a pair of pants and pants decompositions. Then we will go through the concept of length and twist parameters in a pants decomposition. Finally, we build up to a result about Teichmüller space.

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1. Introduction

One may ask the question of how so-called "pairs of pants" are relevant in mathematics. The short answer is that it serves as foundational elements in parts of hyperbolic geometry and Topological Quantum Field Theory (TQFT). In the study of hyperbolic geometry, pairs of pants can be applied to Teichmüller spaces as well as in the Dehn-Nielsen Baer theorem. In the study of TQFTs, pairs of pants play a role in understanding Frobenius algebras in 2-dimensional TQFTs. For this paper, we will only discuss the relevance of pairs of pants in studying Teichmüller spaces. In particular, we will apply pants decomposition to show that the Teichmüller space of a surface with genus g, where $g \geq 2$, has dimension $6g - g$. We will mostly follow [12] and [13].

In section 2, we give some preliminary definitions that will be used later on throughout the paper.

In section 3, we introduce what pairs of pants are and prove a few of their properties.

In section 4, we define some terms regarding geodesics and mention a proposition that relates (disjoint) simple closed curves to (disjoint) simple closed geodesics

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in hyperbolic space.

In section 5, we introduce a few results from hyperbolic trigonometry and then go through the construction of right-angled, convex, hyperbolic hexagon.

In section 6, we discuss pant seams and a few results regarding pant seams. More specifically, we also see the purpose of mentioning hyperbolic hexagons from the previous section.

In section 7, we provide some intuition for the next section and also define what pant cuffs are.

Finally in section 8, we will introduce what is a Teichmüller space and construct length and twist parameters necessary for Fenchel-Nielsen coordinates. We will ultimately prove the theorem that the Teichmüller space has dimension $6g-6$.

2. Some Definitions

For the following definitions, we will consider X to be a metric space.

Definition 2.1. A *curve* in X is a continuous function $\gamma : [a, b] \to X$, where $[a, b]$ is a closed interval in $\mathbb{R}$ with $a < b$. If a curve encloses an area and satisfies the property that $\gamma(a) = \gamma(b)$, then it is called a *closed curve*. A closed curve $\gamma : [a, b] \to X$ is called a *simple closed curve* if it is injective on the interval $[a, b)$ (i.e. It does not intersect itself anywhere).

Definition 2.2. A *manifold* M is a topological space that is Hausdorff, second countable and locally Euclidean. A manifold is said to be of dimension n if it is locally Euclidean of dimension n.

Definition 2.3. A *surface* is a connected 2-dimensional manifold. A *closed surface* is a surface that is compact.

Definition 2.4. A *triangulation* of a closed surface M is a finite family of families $\{\phi_i : \Delta^2 \to M\}_{i=1}^m$ such that the following properties hold: (1) ϕ_i maps a standard 2-simplex homeomorphically onto a triangle $T_i \subseteq M$, (2) The union of all the triangles T_i for $i = 1, \ldots, m$ is M and (3) For $i \neq j$, the intersection of T_i and T_j is either a common edge or vertex of each triangle or empty.

Definition 2.5. Consider a triangulation $\{\phi_i : \Delta^2 \to M\}$, where Δ^2 denotes a 2-simplex (triangle) with positive orientation, for a surface M. The triangulation is *oriented* if each edge of the triangulation is of opposite orientations from the two adjacent triangles that it is an edge of. (See Figure 1)

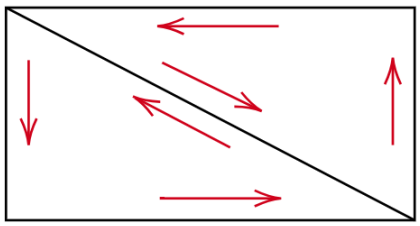

Figure 1. Diagonal as the edge with opposite orientations and its corresponding adjacent oriented triangles

Definition 2.6. A triangulation $\{\phi_i : \Delta^2 \to M\}$ for a surface M is *orientable* if replacing ϕ_i by ϕ_i or $\phi_i p$, where p is the reflection of Δ^2 in the line $y = x$, results in an oriented triangulation of M.

Definition 2.7. A surface M is called an *orientable surface* if it has an orientable triangulation.

Definition 2.8. Let M be a closed surface with v vertices, e edges and t triangles in its triangulation. Then the *Euler Characteristic* of M (denoted $\chi(M)$) is given by the equation $\chi(M) = v - e + t$.

Definition 2.9. The *genus* (denoted g) of a surface M is the largest number of simple closed curves that can be drawn on the surface without separating the surface. Intuitively speaking, it is the "number of handles" on a surface. It is given by the equation $g = \frac{2-\chi(M)}{2}$.

Definition 2.10. Let X and Y be topological spaces. Let f and g be continuous maps from X to Y. Then f and g are said to be *homotopic* (denoted $f \simeq g$) if there is a continuous map $h : X \times [0, 1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for each x in X. We call the map h to be a *homotopy* between f and g.

Note. Throughout this paper, we will consider closed, connected, orientable surfaces when we say surfaces.

3. Pairs of Pants

In this section, we will introduce what is meant by so-called pairs of pants and pants decompositions. Note that pants decompositions of hyperbolic surfaces always exist under the assumptions mentioned at the end of the previous section (see [9]). We will also use properties of the Euler Characteristic of a pair of pants to prove two statements about pants decompositions of surfaces with genus $g \geq 2$.

Definition 3.1. A *pair of pants* (see Figure 2) is a surface that is homeomorphic to a three-holded sphere (or also a disk with two holes). It is a genus zero surface and has three boundary components.

Proposition 3.2. *The Euler Characteristic of a pair of pants P is -1.*

Proof. We will prove this by applying the Euler Characteristic formula to one of the possible triangulations of a pair of pants. For simplicity, we will show this in 2D space using triangles instead of regular 2-simplicies. First note that since a three-holded sphere is homeomorphic to a disk with two holes, by transitivity we have that a pair of pants is also homeomorphic to a disk with two holes. This means that we can use a triangulation of a disk with two holes and one such triangulation is depicted in Figure 3. This particular triangulation consists of 39 edges, 18 vertices and 20 triangles. By applying the formula, we get that $\chi(P) = 18 - 39 + 20 = -1$ as desired. □

Definition 3.3. A *pants decomposition* of a surface is a set of disjoint, simple closed curves that cut the surface into pairs of pants or three-holed spheres (see e.g. Figure 4).

Lemma 3.4. *Given a surface M with genus $g \geq 2$, M can be decomposed into $2g - 2$ pairs of pants.*

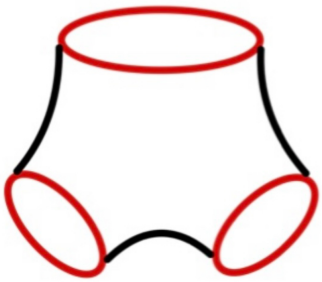

FIGURE 2. A pair of pants

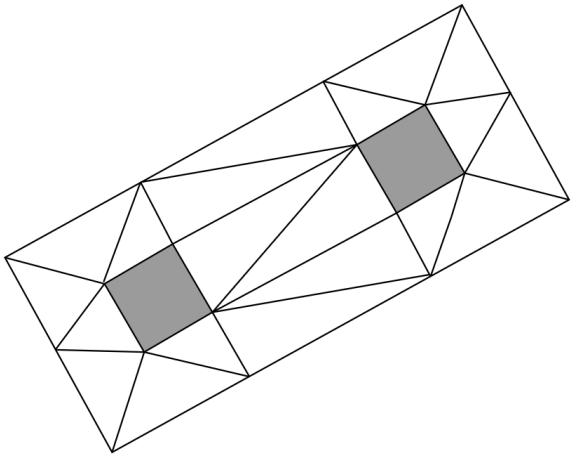

FIGURE 3. A possible triangulation of disk with two holes

Proof. Suppose we have a pants decomposition of surface M (i.e. cut M into pairs of pants P). From Proposition 3.2, by using a valid triangulation representation of a pair of pants that is homeomorphic to Figure 3, we have that each pair of pants contributes -1 to the Euler Characteristic. Let the number of pants, denoted $n(P)$, be represented by $-\chi(M)$. Since the genus of a surface and the Euler Characteristic have the relationship that $g = \frac{2-\chi(M)}{2}$, we can apply this formula to count the number of pants we have in our pants decomposition of M. By substituting $n(P) = -\chi(M)$, we get that $g = \frac{2+n(P)}{2}$ and rewriting this gives us $n(P) = 2g - 2$. Hence, we have that M can be decomposed into $2g - 2$ pairs of pants. □

The following result follows from applying Lemma 3.4.

Proposition 3.5. *Given a surface M with genus $g \geq 2$, M can be cut up into pairs of pants using $3g - 3$ simple closed curves.*

Proof. From Lemma 3.4, we have that M can be decomposed into $2g - 2$ pairs of pants. By definition, a pair of pants has three boundary components. This gives us $3(2g - 2) = 6g - 6$ simple closed curves. When we decompose a surface into

pairs of pants, we cut the surface along simple closed curves and each simple closed curve corresponds to a boundary component of a pair of pants. When we cut along a simple closed curve, we are creating two edges, so each boundary component is being shared by two pairs of pants. This means that we are overcounting by a factor of two, so by getting rid of the overcounts, we have that the total number of simple closed curves in M is exactly $3g - 3$. □

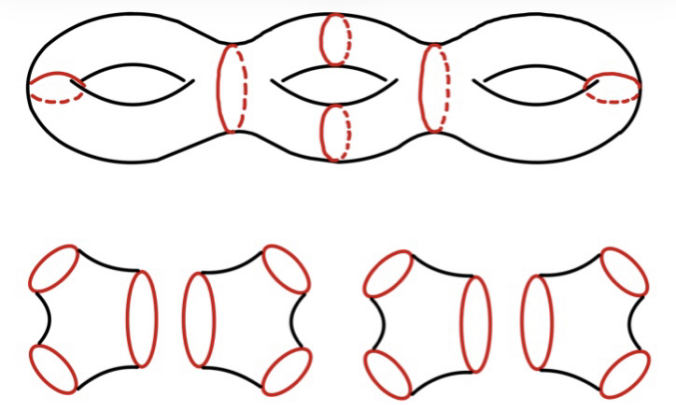

FIGURE 4. Pants decomposition of a surface with genus three into four pairs of pants using six simple closed curves

4. Closed Geodesics

For the following definitions, we will consider X to be a metric space.

Definition 4.1. A *geodesic line* in X is a locally distance preserving function $\gamma : \mathbb{R} \to X$. A *geodesic* is the image of a geodesic line.

Definition 4.2. Let $\gamma : \mathbb{R} \to X$ be a geodesic line. The *period* of γ is a positive real number p satisfying $\gamma(t + p) = \gamma(p)$ for all t in $\mathbb{R}$. We say that a geodesic line is *periodic* if it has a period.

Definition 4.3. A *closed geodesic line* in X is the image of a periodic geodesic line $\gamma : X \to \mathbb{R}$.

Definition 4.4. A closed geodesic $\gamma : X \to \mathbb{R}$, having smallest period p, is a *simple, closed geodesic* if γ restricted to the closed interval $[0, p]$ is a simple closed curve.

Definition 4.5. Let $\alpha, \beta : [0, 1] \to X$ be two closed curves. If there is a homotopy $H : [0, 1]^2 \to X$ from α to β satisfying $H(0, t) = H(1, t)$ for every t, then α and β are *freely homotopic*. If α is not freely homotopic to either of β or β^{-1}, then α and β are *homotopically distinct*.

The following proposition will be used throughout later parts of the paper, but will not be proven. For in depth proofs of this result, see [4][Ch.1], [8][pg.92] and [12][Section 9.6].

Proposition 4.6. *Let M be a hyperbolic surface. Every homotopically distinct, hyperbolic, simple closed curve is represented by simple closed geodesics of M. Moreover, every disjoint, homotopically distinct, hyperbolic, simple closed curve is represented by a disjoint simple closed geodesics of M.*

5. Hyperbolic Hexagons

In hyperbolic geometry, it is possible to construct right-angled hexagons (i.e. hexagons in the hyperbolic plane where interior angles are all right angles). In this section, we will show the existence and uniqueness of right-angled, convex, hyperbolic hexagons through construction. We will assume some background on hyperbolic trigonometry, but we will first list a few results from hyperbolic trigonometry without proof. We will later see an application of right-angled hexagons in pants decompositions.

Proposition 5.1. *Let a, b, c be the angles of a hyperbolic triangle and a', b', c' be the lengths of the respective opposite sides. Then:*

$$\frac{\sinh a'}{\sin a} = \frac{\sinh b'}{\sin b} = \frac{\sinh c'}{\sin c}.$$

This is known as the *Law of Sines*.

Proposition 5.2. *Let a, b, c be the angles of a hyperbolic triangle and a', b', c' be the lengths of the respective opposite sides. Then:*

$$\cosh c' = \frac{\cosh a \cosh b + \cosh c}{\sinh a \sinh b}.$$

This is known as the *Law of Cosines*.

Proposition 5.3. *Let Q be a hyperbolic, convex, quadrilateral with three right angles and fourth angle γ, and let a, b be the lengths of the sides opposite of the angle γ. Then:*

$$\cos\gamma = \sinh a \sinh b.$$

Proposition 5.4. *Let P be a hyperbolic, convex, pentagon with four right angles and fifth angle γ, let c' be the length of the side of P opposite γ, and let a, b be the lengths of the sides of P adjacent to the side opposite γ. Then:*

$$\cosh c' = \frac{\cosh a \cosh b + \cosh\gamma}{\sinh a \sinh b}.$$

Further details on hyperbolic trigonometry as well as proofs of the above propositions can be found in section 3.5 of [12].

Proposition 5.5. *Let a, b, c be positive real numbers. Then there is a right-angled, convex, hyperbolic hexagon, unique up to congruence, with alternate sides of lengths a, b, c, respectively.*

Proof. We first show the existence of right-angled, convex, hyperbolic hexagon. Let $c' \in \mathbb{R}^+$ satisfy the equation

$$\cosh c' = \frac{\cosh a \cosh b + \cosh c}{\sinh a \sinh b}.$$

Let $S_{c'}$ be the geodesic segment that has length c'. We will construct two more geodesic segments on the same side of $S_{c'}$. On one of the endpoints of $S_{c'}$, construct the geodesic segment S_a of length a such that S_a is perpendicular to $S_{c'}$. Similarly, on the other endpoint of $S_{c'}$, construct the geodesic segment S_b of length b such that S_b is perpendicular to $S_{c'}$. Then we choose hyperbolic lines $l_{a'}$ and $l_{b'}$ orthogonal to the sides of length b and a at the endpoint of S_b and S_a that are opposite the endpoints of $S_{c'}$, respectively.

Consider the side of length c. Without loss of generality, assume that $c \geq a, b$. We want to show that the following situations cannot happen: (1) $l_{b'}$ intersects S_b, (2) $l_{a'}$ intersects S_a, and (3) $l_{a'}$ intersects $l_{b'}$. Suppose by contradiction that $l_{b'}$ intersects S_b at some point. Then this would create a quadrilateral having three right angles, fourth angle γ, and opposide sides of length a and c', so by Proposition 5.3, we would have

$$\cos\gamma = \sinh a \sinh c'.$$

By using the identity that

$$\cosh^2(c') - \sinh^2(c') = 1,$$

we get that

$$\sinh^2(a) \sinh^2(c') = \sinh^2(a)(\cosh^2(c') - 1).$$

However, after some trigonometric manipulations, we get that

$$\sinh^2 a \sinh^2 c' > \frac{\cosh^2 c}{\sinh^2 b} > 1,$$

which is a contradiction. The argument to show that $l_{a'}$ and S_a do not intersect is entirely analogous. For the third situation, suppose by contradiction that $l_{a'}$ and $l_{b'}$ intersect at some point. Then this would create a pentagon having four right angles and fifth angle γ, so by Proposition 5.4, we would have

$$\cosh c' = \frac{\cosh a \cosh b + \cosh\gamma}{\sinh a \sinh b}.$$

However, this is also a contradiction because $\cosh c > \cosh\gamma$. Hence, all three of the above situations cannot happen.

Next, we construct the hyperbolic line l_c to be the common perpendicular to lines $l_{a'}$ and $l_{b'}$, such a line exists by [12][pg.90]. Let l_a and l_b be two hyperbolic lines containing the geodesic segments S_a and S_b, respectively. Then since l_c intersects $l_{a'}$ at some point and $l_{a'}$ and $S_{c'}$ are on the same side of l_a, we have that l_c and $l_{c'}$ are on the same side of l_a. Similarly, since l_c intersects $l_{b'}$ at some point and $l_{b'}$ and $S_{c'}$ are on the same side of l_b, we have that l_c and $l_{c'}$ are also on the same side of l_b. Call the geodesic segment of l_c that connects $l_{a'}$ to $l_{b'}$ to be S_c. Then this forms a right-angled, convex, hexagon with alternate sides S_a, S_b and S_c. Let the geodesic segment S_c be of length d. By Proposition 5.2, we have that

$$\cosh c' = \frac{\cosh a \cosh b + \cosh d}{\sinh a \sinh b}.$$

Since we also chose c' at the beginning uniquely to satisfy

$$\text{cosh}c' = \frac{\text{cosh}a\ \text{cosh}b + \text{cosh}c}{\text{sinh}a\ \text{sinh}b},$$

this implies that $c = d$. Hence, we have a right-angled, convex, hexagon with alternate sides of length a, b, c.

The uniqueness of right-angled, convex, hexagons determined by the lengths of their three alternate sides, up to congruence, follows from the identity that

$$\text{cosh}c = -\text{cosh}a\ \text{cosh}b + \text{sinh}a\ \text{sinh}b\ \text{cosh}c'$$

and details can be found in [6][pg.85] and [5][pg.40]. □

6. Pant Seams

As seen in the last section, we have the existence and uniqueness of right-angled, convex hexagons in hyperbolic space. First, we will give two definitions of a seam, one more descriptive than the other, but they are essentially the same. Then, we will discuss how these right-angled, convex hexagons are related to the structure of a hyperbolic pair of pants. It turns out that if we cut a pair of pants along its seams, we get two right-angled, convex, hyperbolic hexagons that are congruent.

Definition 6.1. A *seam* is a geodesic arc connecting two distinct boundary components and perpendicular to both of them at their endpoints. More precisely, a *seam* of a pair of pants P is the image S of an injective geodesic curve $\sigma : [a, b] \to M$, where M is a surface, such that the point $\sigma(a)$ is on a boundary component A of P, the point $\sigma(t)$ is in the interior of P for $a < t < b$, the point $\sigma(b)$ is on another boundary component B of P and S is perpendicular to A and B.

Definition 6.2. For a surface M with boundary, the *double* of M is obtained by gluing two copies of M together along their common boundary.

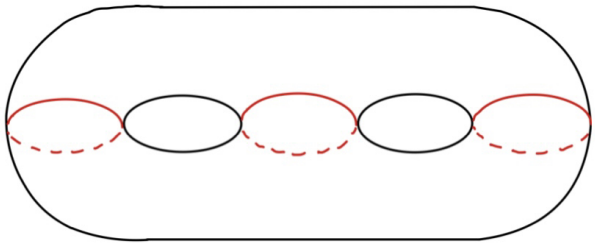

Figure 5. The double of a pair of pants

Proposition 6.3. *Let P be a pair of pants in a hyperbolic surface M such that each boundary component of P is a simple closed geodesic of M. Then any two boundary components of P are joined by a unique seam of P.*

Proof. Suppose a pair of pants P has boundary components C_1, C_2, and C_3. Let P' be a copy of P and let D be the double of the pair of pants P. Note that D is a closed, orientable surface, so in particular D is compact. If we extend the hyperbolic structure of the interiors of P and P' to D such that C_1, C_2, C_3 are now closed geodesics of D (see [12][pg.413]), we have that D is in fact complete.

We will first construct a seam of P using simple closed geodesics of D. Let $\alpha : [0,1] \to P$ be the starting simple curve, where $\alpha(t)$ is in the interior of P for $0 < t < 1$ and $\alpha(0), \alpha(1)$ are on the boundary components of C_1 and C_2, respectively. Let α' be the corresponding simple curve in the copy P'. Since $\alpha\alpha'$ is a non-trivial closed curve (i.e. not homotopic to a point, puncture or boundary component) in D, it is freely homotopic to a simple closed curve δ with image being a unique simple closed geodesic G in D by [10][pg.3]. By [12][theorem 9.6.7], geodesic G meets the closed geodesics C_1 and C_2 transversely at a single point. Then $S = G \cap P$ is the image of the geodesic curve G connecting C_1 and C_2 in P.

To show that $S = G \cap P$ is a seam of P, let $\rho : D \to D$ be the map such that $\rho(x) = x'$ and $\rho(x') = x$, where x' is the corresponding point of P' for each point x of P. Then ρ is an isometry of D. Since $\rho(\alpha\alpha'^{-1}) = \alpha'\alpha^{-1}$, we have that $\alpha'\alpha{-}1$ is freely homotopic to $\rho\delta$ and is represented by the simple closed geodesic ρG of D. By [10][pg.3], $\rho G = G$ and G is perpendicular to C_1 and C_2, so it follows that $S = G \cap P$ is perpendicular to C_1 and C_2. Hence, we have that S is a seam of P that connects C_1 and C_2.

To show uniqueness of seam S, suppose by contradiction that we have another seam T of P that connects C_1 to C_2 and is perpendicular to both C_1 and C_2. Then the union of T with its corresponding seam in P', say $E = T \cup T'$, is a simple closed geodesic of D. Let $\pi_1, \pi_2 : [0,1] \to P$ be simple curves starting in C_1 with images being S and T respectively. Then π_1 and π_2 are freely homotopic to each other by constructing a homotopy preserving the endpoints of C_1 and C_2. This implies that $\pi_1\pi_1^{-1}$ and $\pi_2\pi_2^{-1}$ are freely homotopic to each other, so by [10][pg.3], we can conclude that $G = E$, so the seam S is indeed unique. □

Proposition 6.4. *Under the same assumptions as in Proposition 6.3, the three seams of P are mutually disjoint.*

Proof. see [12][pg.414] □

Proposition 6.5. *Let P be a pair of pants in a hyperbolic surface M, where the boundary components of P are simple closed curves of M. Let a, b, c be the lengths of the boundary components of P. Then the structure of P is uniquely determined by a, b, c up to isometry.*

Proof. Let H_1 and H_2 be the right-angled, convex, hyperbolic hexagons that we get after cutting P along its seams (see Figure 6). Let a, b, c be the lengths of the boundary components of P. Since H_1 and H_2 have matching lengths for their three seam alternate sides, by Proposition 5.5, we have that H_1 and H_2 are congruent. This implies that their nonseam alternate sides also have the same lengths. We know that the entire boundary component of P has lengths a, b, c, so this means that the nonseam alternate sides of H_1 and H_2 have lengths $a/2, b/2, c/2$, respectively. Hence, it follows from the fact that we can cut a pair of pants into two congruent right-angled, convex, hyperbolic hexagons that the structure of P is determined by the lengths of a, b and c up to isometry. □

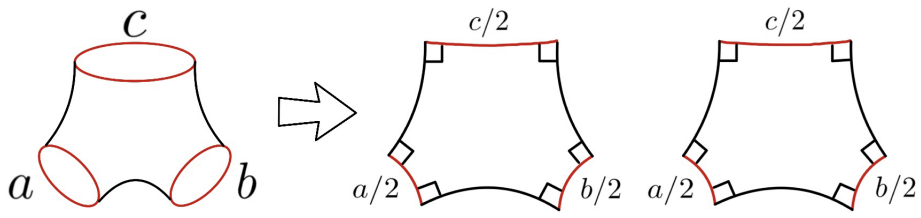

FIGURE 6. Result after cutting a pair of pants along its seams

7. Pants Decompositions

In this section, we will briefly discuss the general idea of gluings, markings and curve systems. It will serve as some motivation for the next section. For more details, see e.g. [3] and [1].

We have already seen that we can cut along seams of a hyperbolic pair of pants to obtain two congruent right-angled, convex, hyperbolic hexagons. It would make sense that we should be able to glue the two hexagons in a way that we get back a hyperbolic pair of pants. The general idea is given two congruent right-angled, convex, hyperbolic hexagons, we can glue along the sides of the hexagons, which are geodesic arcs of the same length. The geodesic arcs that we are gluing along form a surface with a boundary, which is our pair of pants. The interior of the pair of pants will be equipped with a hyperbolic metric and each boundary component is a closed geodesic (see [2]). The *cuffs* are the simple closed curves that cut a surface M into pairs of pants, so by gluing together pairs of pants along their cuffs with the same lengths, we can obtain a hyperbolic surface M.

In general, we refer to a marked hyperbolic surface as a hyperbolic surface with some additional structure that is specified. For our purpose, in the context of hyperbolic pairs of pants, we will refer to markings in terms of lengths and twists to specify the geodesic arcs on the surface, which we will see later. A curve system on a hyperbolic pair of pants typically refers to a set of non-intersecting simple closed curves, so for our purpose, it will consist of the three boundary components and the seams of our pair of pants.

8. Fenchel-Nielsen Coordinates

In this section, we will first introduce what a Teichmüller space is and then discuss length and twist parameters involved in the gluing of any two cuffs. We will construct the length and twist parameters and then introduce explicitly the coordinate system that is formed on a Teichmüller space by these length and twist parameters in a pants decomposition. Finally, we will show that under this coordinate system, the Teichmüller space has dimension $6g-6$. The proofs in this section will follow from [12].

Definition 8.1. The *Teichmüller space* of M, denoted $\mathcal{T}(M)$, is the set of all hyperbolic structures on M such that two hyperbolic structures are equivalent if there is an isometry homotopic to the identity between them.

Now we will begin constructing the length and twist parameters that will be used in the Fenchel-Nielson coordinates.

Let M be an oriented hyperbolic surface with genus $g \geq 2$ and equipped with hyperbolic structure Φ. First, we pick $3g-3$ disjoint, homotopically distinct, non-trivial simple closed curves on M. Let these curves be denoted by $\alpha_i : [0,1] \to M$ for $i = 1, \ldots, 3g-3$. Next, we need to choose $g+1$ disjoint, homotopically distinct, non-trivial simple closed curves on M to be the set of curves that represent the seams of pairs of pants. Let these curves be denoted by $\beta_j : [0,1] \to M$ for $j = 1, \ldots, g+1$. We will use these two sets of curves to construct our length and twist parameters.

We will start with the length parameter, By Proposition 4.6, the disjoint, simple closed curves $\alpha_1, \ldots, \alpha_{3g-3}$ can be represented by corresponding disjoint, simple closed geodesics, say $A_1, \ldots, A_{3g-3}$ of our hyperbolic surface (M, Φ). By Lemma 3.4, we know that $A_1, \ldots, A_{3g-3}$ cuts M into $2g-2$ pairs of pants. Recall that each pair of pants can be cut into two congruent right-angled, convex, hexagons and so Proposition 6.5 tells us that the lengths of the boundary components of a pair of pants determines its structure up to isometry. From these results, it makes sense to use the lengths of the boundary components to be our natural choice for length parameters. Hence, we let l_i be the lengths of A_i for $i = 1, \ldots, 3g-3$ and more precisely, we will use $\log l_i$ for $i = 1, \ldots, 3g-3$ as our length parameters.

For the twist parameters, we need to measure the amount of twist that is associated with the boundary components and seams in the pairs of pants. By Proposition 4.6, the disjoint, simple closed curves $\beta_1, \ldots, \beta_{g+1}$ can be represented by corresponding disjoint, simple closed geodesics $B_1, \ldots, B_{g+1}$. By [12][theorem 9.6.7], we restrict these geodesics $B_1, \ldots, B_{g+1}$ to their respective images connecting boundary components in the pairs of pants. We can construct a homotopy that preserves the endpoints of $A_1, \ldots, A_{3g-3}$ so that the image of the geodesics $B_1, \ldots, B_{g+1}$ are homotopic to pant seams (see [13][pg.90]). Suppose that P_i and Q_i are two pairs of pants in M that share the boundary component A_i. Fix the orientation of the pair of pants P_i to match the orientation of the boundary component A_i. Let $2d_i$ denote the total number of degrees which the homotopies move the endpoints of P_i lying in A_i. Here the number d_i represents how much the image of the geodesic of the pair of pants P_i is wrapping around the two seams of P_i with endpoints lying in A_i (see Figure 7[12][pg.416]). Since every boundary component is shared by two pairs of pants, let $d_i^{'}$ be the number that represents how much the pair of pants Q_i is wrapping around the two seams of Q_i with endpoints lying in A_i as well. Let $t_i = d_i - d_i^{'}$ be the twist coefficient of A_i. This twist coefficient t_i measures how much twist is associated with gluing of P_i and Q_i at A_i with respect to the specified marking and orientation of M.

We note that it is not possible for two copies of A_i for any i to have independent twists using any homotopy after performing gluings, so in fact the twisting coefficient t_i is a homotopy invariant (see [13][pg.91]). Note also that t_i is a well-defined number. If we cut M along its seam, then glue it back together with a twisting coefficient of $2\pi n$ degrees for any integer n, we will obtain back a surface that is isometric to what we have started with. Moreover, the twisting coefficient t_i is congruent modulo 2π to whichever angle that we have to rotate Q_i around A_i by in order to ensure that the corresponding seams of P_i and Q_i match.

Let $\mathcal{H}(M)$ be the set of all hyperbolic structures for M. Define a function $F : \mathcal{H}(M) \to \mathbb{R}^{6g-6}$ by setting $F(\Phi) = (\log\, l_1, t_1, \log\, l_2, t_2, \ldots, \log\, l_{3g-3}, t_{3g-3})$. The length-twist parameters in pants decomposition give a coordinate system on (M, Φ) (the surface M equipped with hyperbolic structure Φ). The components, which are dependent on length and twist coefficients, are known as the *Fenchel-Nielsen coordinates* of M in the Teichmüller space $\mathcal{T}(M)$.

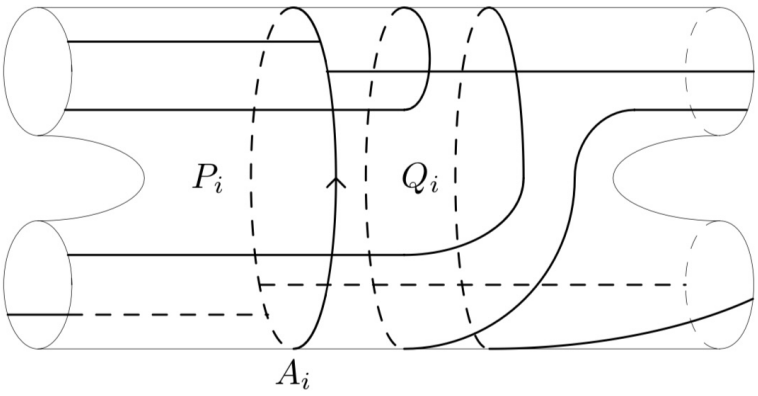

FIGURE 7. Four geodesic sections and seams ending in the geodesic A_i (Credits: [12][pg.416])

Remark. It is also common to see the version of the Fenchel-Nielsen coordinates without the use of logarithm, namely $F(\Phi) = (l_1, t_1, l_2, t_2, \ldots, l_{3g-3}, t_{3g-3})$, where (l_i, t_i) for $i = 1, \ldots, 3g-3$ are the length and twist coefficients for the set of $3g-3$ simple closed geodesics. Different sources may use a different version as convention depending on the context. For further details, see e.g. [4][pg.297].

Lemma 8.2. *Let M be an oriented surface with genus $g \geq 2$. Then the function $F : \mathcal{H}(M) \to \mathbb{R}^{6g-6}$ induces a function $\overline{F} : \mathcal{T}(M) \to \mathbb{R}^{6g-6}$.*

Proof. Let $h : M \to M$ be a homeomorphism homotopic to the identity map of M. Consider the hyperbolic structures Φ and Φh on M. Then from the choice of h, h is an orientation preserving isometry from $(M, \Phi h)$ to (M, Φ). From this, we can observe that for each i, $h^{-1}A_i$ is a simple closed geodesic of $(M, \Phi h)$. We also have that $h^{-1}A_i$ is freely homotopic to A_i for each i because h^{-1} is homotopic to the identity map as well. The disjoint, simple closed curves $\alpha_1, \ldots, \alpha_{3g-3}$ can be represented by the corresponding disjoint, simple closed geodesics $h^{-1}A_i, \ldots, h^{-1}A_{3g-3}$ and the disjoint, simple closed curves $\beta_1, \ldots, \beta_{g+1}$ can be represented by the corresponding disjoint, simple closed geodesics $h^{-1}B_1, \ldots, h^{-1}B_{g+1}$. Since h^{-1} is an orientation preserving isometry, $h^{-1}A_i$ has the same length and twist parameter as A_i for each i. Hence, $F(\Phi h) = F(\Phi)$ and so F induces a function $\overline{F} : \mathcal{T}(M) \to \mathbb{R}^{6g-6}$. □

Theorem 8.3. *Let M be an oriented surface with genus $g \geq 2$. Then the function $\overline{F} : \mathcal{T}(M) \to \mathbb{R}^{6g-6}$ is a homeomorphism.*

Proof. To show that $\overline{F}$ is a bijection, we will first show that $\overline{F}$ is injective. Suppose Φ and Φ' are two hyperbolic structures of M such that $F(\Phi) = F(\Phi')$. We need to show that the hyperbolic structures $\Phi = \Phi'$. Consider the simple closed curves $\alpha_1, \ldots, \alpha_{3g-3}$. Let $A_1, \ldots, A_{3g-3}$ be the simple closed geodesic representation in (M, Φ) and let $A_1^{'}, \ldots, A_{3g-3}^{'}$ be the simple closed geodesic representation in (M, Φ'). Then the length and twist coefficients of A_i and $A_i^{'}$ are the same for each i. By Proposition 6.5, $h : (M, \Phi') \to (M, \Phi)$ is an orientation preserving isometry that maps each geodesic $A_i^{'}$ onto the geodesic A_i. Similarly, for the seams, we can consider the simple closed curves $\beta_1, \ldots, \beta_{g+1}$. Let $B_1, \ldots, B_{g+1}$ be the simple closed geodesic representation on (M, Φ) and let $B_1^{'}, \ldots, B_{g+1}^{'}$ be the simple closed geodesic representation in (M, Φ'). Then $h(B_1^{'}), \ldots, h(B_{g+1}^{'})$ are the simple closed geodesics in (M, Φ) that form a continuous set of seams of (M, Φ) with same amount of twist about $A_1, \ldots, A_{3g-3}$ as the set of seams $B_1, \ldots, B_{g+1}$. From this together with Proposition 4.6, we have that $h(B_j)^{'}$ is freely homotopic to B_j for each j so $h(B_j^{'}) = B_j$. By using an argument involving the construction of a homotopy from considering the geodesics $A_1^{'}, \ldots, A_{3g-3}^{'}$ and $B_1^{'}, \ldots, B_{g+1}^{'}$ as forming a 1-skeleton of a cell structure for M, it can be deduced that $\Phi^{'} = \Phi h$ and $\Phi = \Phi^{'}$ in $\mathcal{T})(M)$ (for the details, [12][pg.417]).

Next, we will show that $\overline{F}$ is surjective. Let $(s_1, t_1, \ldots, s_{3g-3}, t_{3g-3})$ be a point of $\mathbb{R}^{6g-6}$ and set $l_i = e^{s_i}$ for $i = 1, \ldots, 3g-3$. By Propositions 5.5 and 6.5, since each pair of pants is the gluing of two congruent right-angled, convex, hyperbolic hexagons, there is a total of $4g-4$ right-angled, convex, hyperbolic hexagons that make up the $2g-2$ pairs of pants. Since each pair of pants has three boundary components, there is a total of $6g-6$ boundary components. These boundary components have lengths $l_1, l_1, l_2, l_2, \ldots, l_{3g-3}, l_{3g-3}$ and they can be arranged in one-to-one correspondence with the $2g-2$ pairs of pants of M. This can be done in a way that ensures the indexing of the lengths of the boundary components of each hyperbolic pair of pants is in correspondence with the indexing of the boundary components of the respective pair of pants of M. By choice, we can choose seam preserving homeomorphisms from pairs of pants of M to corresponding hyperbolic pairs of pants and then assign an orientation on the hyperbolic pairs of pants such that orientation remains preserved under these homeomorphisms.

Our goal is to show that $F(\Phi h) = (s_1, t_1, \ldots, s_{3g-3}, t_{3g-3})$. Let the twisting coefficient t_i be expressed as $\theta_i + 2\pi k_i$, where $0 \leq \theta_i < 2\pi$ and k_i is some integer. Let M' be the surface that we get from gluing the hyperbolic pairs of pants that we previously got. We perform this gluing by gluing them along the two boundary components of length l_i using an orientation reversing isometry with θ_i twist in the direction that is consistent with that of $\alpha_i([0,1])$ for each i. By [12][pg.389], M' has a hyperbolic structure such that the circle C_i that we get after gluing two boundary components of length l_i is a simple closed geodesic of length l_i for each i. From this, we can extend the homeomorphisms between pairs of pants of M and M' to an orientation preserving homeomorphism $h : M \to M'$, where $\alpha_i[0,1]$ is mapped onto the circle C_i for each i.

Let $\Phi = \phi_i : U_i \to H^2$ be the hyperbolic structure of M'. Then it follows that $\Phi h = \phi_i h : h^{-1}(U_i) \to H^2$ is a hyperbolic structure for M, with h being an orientation preserving isometry from $(M, \Phi h)$ to (M', Φ). If we represent $\alpha_i[0,1]$ by A_i for each i, then A_i is a simple closed geodesic of $(M, \Phi h)$ with length l_i and

A_i has twist coefficient congruent to θ_i modulo 2π. Suppose we replace h with h composed with the correct number of Dehn twists (see e.g. Martelli Hyp Geo) about C_i for each i. Then we can assume that the twist coefficient of A_i for each i is exactly t_i. Hence, this gives us that $F(\Phi h) = (s_1, t_1, \ldots, s_{3g-3}, t_{3g-3})$ and so $\overline{F}$ is surjective. Since we have shown both injectivity and surjectivity, we have that $\overline{F} : \mathcal{T}(M) \to \mathbb{R}^{6g-6}$ is a bijection by definition.

For proof of the Fenchel-Nielsen coordinates being continuous and having continuous inverse, see e.g. [7][pg.8], [11][pg.83], and [14][pg.8]. It then follows that $\overline{F} : \mathcal{T}(M) \to \mathbb{R}^{6g-6}$ is indeed a homeomorphism and so the Teichmüller space has dimension $6g - 6$. □

Remark. Some textbooks may refer to a version of Teichmüller space being homeomorphic to $\mathbb{R}^{3g-3} \times \mathbb{R}^{3g-3}$. It is exactly the same, but this version shows more explicitly that the space is decomposed into $3g - 3$ lengths and $3g - 3$ twists.

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