CONSTRUCTION OF THE ALEXANDER INVARIANT

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Abstract. In this paper, we motivate and detail the construction of the Alexander invariant, a powerful invariant in knot theory. This paper assumes familiarity with algebraic topology at the level of the fundamental group and homology theory.

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1. Introduction

Knots need no introduction. They sit on top of the shoes you're wearing, decorate the gifts you wrap, and secure your grocery bags when you walk home from Trader Joe's. Moreover, if you are anything like the author of this paper, all of the knots just listed are exactly the same knot. But how could you verify this? Perhaps you could try untying and retying them in different ways to see if they "look the same." While intuitively, this may make sense, this approach could lead you astray. Consider the following two knots:

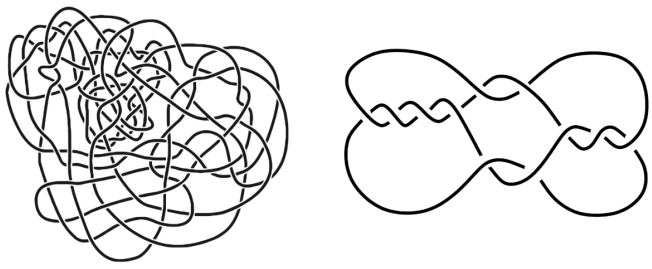


Figure 1. Haken's Gordian Knot[1]and Goeritz's Knot [Prevos21]

You may be surprised to learn that they are, in fact, the same exact knot. You will be even more surprised, or perhaps even slightly disturbed, to learn that neither of them are knotted at all; they are "unknots." However, what does this mean for us, as mathematicians? Well, of course, it means that we must stop talking about the real world and start talking about some definitions.

Definition 1.1. A *knot* is an embedding of the circle S^1 into $\mathbb{R}^3$. For example, the *unknot* is the identity embedding $S^1 \to S^1$.

Definition 1.2. Two embeddings $K_1, K_2 : S^1 \to \mathbb{R}^3$ are *ambient isotopic* if there exists a smooth family of homeomorphisms $H_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0, 1]$, such that H_0 is the identity map, and $H_1(K_1(S^1)) = K_2(S^1)$.

We say two knots are equivalent if they are ambient isotopic. Intuitively, this means that the first knot K_1 can be smoothly deformed into the other knot K_2 without cutting or gluing, and vice versa.

The central goal of knot theory is to create and study *knot invariants*, functors that assign to each knot K an object $f(K)$ in such a way that knots of the same type are assigned equivalent objects. The primary goal of this paper is to offer a glimpse into the delicate balancing act between "computational difficulty" and "amount of data encoded" that goes behind the creation and application of topological invariants. To this end, we follow Rolfsen's beloved text *Knots and Links* for the majority of this exposition. We will introduce various invariants and properties of knots, before ultimately constructing the ubiquitous Alexander invariant.

2. The Wirtinger Presentation

Our goal in this section is two-fold. First, we hope to introduce a basic invariant that is straightforward and effective distinguishing knot types. Second, we hope to use this invariant to inform and validate alternative invariants later in this paper.

With this in mind, we turn to the knot group, which relies on a low-dimensional topologist's favorite tool: the fundamental group. We begin by observing that the complement of a knot, $\mathbb{R}^3 - K$, is a topological invariant. Consequently, distinguishing loops within the knot complement that wrap around parts of K can help us classify knots.

Definition 2.1. Given a knot K in $\mathbb{R}^3$, the fundamental group $\pi_1(\mathbb{R}^3 - K)$ of the complement is the *knot group* of K. Note that this group is isomorphic to $\pi_1(S^3 - K)$ via the inclusion isomorphism.

This helps us quickly calculate the knot group of the unknot.

Lemma 2.2. *The unknot knot group* $\pi_1(\mathbb{R}^3 - S^1) \cong \mathbb{Z}$

Proof. First, we simplify the knot complement space into a more recognizable form. Identify S^3 with the boundary of $D^2 \times D^2$, yielding a decomposition

$$S^3 \;=\; (S^1 \times D^2) \cup (D^2 \times S^1).$$

[1]The name of this knot was inspired by the famous legend of the Gordian knot. According to the story, Alexander the Great faced the challenge of untying the intricate knot, with a prophecy declaring that whoever succeeded would rule Asia. Instead of painstakingly untying it or proving that it was ambient isotopic to the unknot, Alexander sliced through the knot with his sword.

Inside $S^1 \times D^2$, let $\gamma = S^1 \times \{0\}$. Removing γ gives

$$S^3 \setminus \gamma \;=\; \big((S^1 \times D^2) \setminus \gamma\big) \cup (D^2 \times S^1).$$

Because these two pieces are glued along their common boundary torus, their union deformation retracts onto a single solid torus. Hence,

$$S^3 \setminus \gamma \;\cong\; S^1 \times D^2.$$

Then:

$$\pi_1(\mathbb{R}^3 - S^1) \cong \pi_1(S^3 - S^1) \cong \pi_1(S^1 \times D^2) \cong \pi_1(S^1) \cong \mathbb{Z}. \qquad \square$$

In fact, the unknot is the *only* knot group isomorphic to $\mathbb{Z}$, but this is interestingly not as trivial to verify, with polynomial-time unknotting algorithms being the focus of many mathematicians today[2]. However, we now focus our attention on a procedure for writing down the *Wirtinger Presentation* of a knot group, given a projection of the knot. While the proof that this procedure actually works is omitted for brevity, details can be found in [Rolfsen76] and/or [Stillwell90].

Our first step is to draw a "projection" of the knot (in this case, the trefoil knot), which is a finite number of arcs $\alpha_1, \ldots, \alpha_n$ lying on a plane. Next, we orient the arcs so that each arc α_i leads naturally into $\alpha_{(i+1) \bmod n}$. One should be able to trace the arcs of the knot by their assigned orientation continuously.

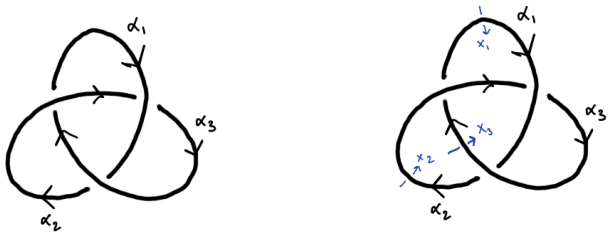
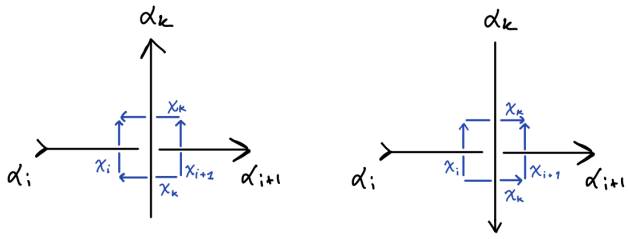


FIGURE 2. First and second step of the Wirtinger Presentation's procedure

For the second step, draw a short arrow labeled x_i passing under each arc α_i in a right-to-left direction. Each arrow represents a loop in $\mathbb{R}^3 - K$, and is described as follows: Take the basepoint $*$ to be your nose. The loop consists of the oriented triangle starting at $*$, traveling to the tail of x_i, following x_i to its head, and then returning to $*$. The generators of the knot group are associated with these loops $x_1, x_2, \ldots, x_n$, and relations are derived from the crossings in the knot projection.

At each crossing, identify the three arcs involved: the overcrossing arc and the two undercrossing arcs. There are two possibilities:



[2]For more information, see [Burton14].

Identify all crossings in the projection from Figure 2 to generate a set of relations.

With the generators and relations identified, the group presentation of the knot group can be formulated. The generators of the group are the loops, $\{x_1, x_2, \ldots, x_n\}$, and the relations are those derived from the crossings. The knot group is then expressed as:

$$\pi_1(S^3 - K) = \langle x_1, x_2, \ldots, x_n \mid (\text{crossing relations}) \rangle.$$

Finally, the presentation can be simplified by identifying and removing redundant generators or relations. In the example of the trefoil knot above, we have the relations $x_1x_3 = x_3x_2$, and $x_2x_1 = x_1x_3$, which can be condensed into the presentation $\langle x_1, x_2 \mid x_1^2 = x_2^3 \rangle$. Now, using the machinery of group theory, we can show that the trefoil isn't secretly the unknot.

For notational ease, we now denote the fundamental group of the knot complement $\pi_1(S^3 - K)$ as simply $\pi_1(K)$.

Theorem 2.3. *The knot group of the trefoil knot, $\pi_1(\text{trefoil})$, is not $\mathbb{Z}$*

Proof. Consider the group of permutations S_3 and the map from the free group $F(x_1, x_2)$ to S_3 given by $x_1 \mapsto (12)$ and $x_2 \mapsto (123)$.

Since $(12)^2 = (123)^3 = 1$, this defines a homomorphism $h : \pi_1(\text{trefoil}) \to S_3$. Observe that $(12)(123) = (13)$ and $(123)(12) = (23)$. Thus, the image of h includes all of S_3, which is a nonabelian group. Therefore, $\pi_1(\text{trefoil})$ is nonabelian, and cannot be $\mathbb{Z}$. □

Moreover, we can now prove the following:

Theorem 2.4. $\pi_1(K) \cong \mathbb{Z} \iff K$ *is the unknot.*

Proof. We have already shown the backwards direction in Lemma 2.2. The details of the forward direction can be found on page 103 of [Rolfsen76]. □

Corollary 2.5. *For a knot $K \subset S^3$, $H_1(S^3 - K) \cong \mathbb{Z}$.*

Proof. This fact follows from Theorem 2.4 and the fact that $\pi_1(X)^{\text{ab}} \cong H_1(X)$. □

Finally, we conclude this section by highlighting an example in which the knot group is not effective in delineating between non-equivalent knots. We prove this by showing that the two non-equivalent knots below have the same knot group.

Example 2.6. The square knot and the granny knot, respectively [Wolfram].

At first glance, these may look the same, but notice that the square knot is symmetrical while the granny knot's strand crossings are inverted when comparing between the left and right hand side. This is because the square knot is the connected sum of a trefoil and its mirror image, the left-handed trefoil, while the granny knot is the sum of two regular (right-handed) trefoils.

Formally, the *connected sum* of two oriented knots K_1 and K_2, denoted $K_1 \# K_2$, is the knot obtained by removing a small arc from each and joining the pairs of endpoints with oriented, non-intersecting arcs to form a single oriented knot.

Now, we compute the knot group of the square knot via the Seifert-van Kampen theorem. The square knot is the connected sum of a right-handed trefoil T_+ and a left-handed trefoil T_-, which we can write formally as $K_{\text{square}} = T_+ \# T_-$. The complements of the trefoils are $E_+ = S^3 - T_+$ and $E_- = S^3 - T_-$, and the complement of K_{square} decomposes as $E_{\text{square}} = E_+ \cup E_-$, with an annulus $E_+ \cap E_- \cong S^1 \times [0,1]$ forming the intersection.

Using Seifert–van Kampen, the knot group of E_{square} satisfies:

$$\pi_1(E_{\text{square}}) = \pi_1(E_+) *_{\pi_1(E_+ \cap E_-)} \pi_1(E_-).$$

Since $\pi_1(E_+ \cap E_-) \cong \mathbb{Z}$ (generated by the core of the annulus), this subgroup maps to the subgroup generated by the meridian loops $\langle x_+ \rangle \subset \pi_1(E_+)$ and $\langle x_- \rangle \subset \pi_1(E_-)$. The connected sum identifies $x_+ = x_-$.

Each trefoil $T_\pm$ had its knot group calculated earlier in this section via the Wirtinger presentation[3].

$$\pi_1(T_+) = \langle x_+, y_+ \mid x_+^2 = y_+^3 \rangle, \quad \pi_1(T_-) = \langle x_-, y_- \mid x_-^2 = y_-^3 \rangle.$$

The connected sum identifies a meridian generator from $\pi_1(T_+)$ with one from $\pi_1(T_-)$, producing the final amalgamated group:

$$\begin{aligned}
\pi_1(K_{\text{square}}) &= \pi_1(T_+) *_{\mathbb{Z}} \pi_1(T_-) \\
&= \langle x_+, y_+, x_-, y_- \mid x_+ y_+ x_+ = y_+ x_+ y_+,\ x_- y_- x_- = y_- x_- y_-,\ x_+ = x_- \rangle, \\
&= \langle x_+, y_+, y_- \mid x_+ y_+ x_+ = y_+ x_+ y_+,\ x_+ y_- x_+ = y_- x_+ y_- \rangle.
\end{aligned}$$

Analogously, because the granny knot can be seen as the connected sum of two right-handed trefoils, we have:

$$\begin{aligned}
\pi_1(K_{\text{granny}}) &= \pi_1(T_+) *_{\mathbb{Z}} \pi_1(T_+) \\
&= \langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1 = y_1 x_1 y_1,\ x_2 y_2 x_2 = y_2 x_2 y_2,\ x_1 = x_2 \rangle, \\
&= \langle x_1, y_1, y_2 \mid x_1 y_1 x_1 = y_1 x_1 y_1,\ x_1 y_2 x_1 = y_2 x_1 y_2 \rangle.
\end{aligned}$$

Thus, $\pi_1(K_{\text{granny}}) \cong \pi_1(K_{\text{square}})$.

Unfortunately, we don't currently have the tools to prove that these knots aren't equivalent. However, in Theorem 6.5, we will close this problem and see that squares are not grannies.

3. Seifert Surfaces

While the Wirtinger Presentation gives an effective method for distinguishing between knots, the fundamental group of a knot complement is oftentimes too unwieldy and difficult to compute for our uses. For example, given the presentations of the trefoil and figure eight knot groups, it is not immediately obvious that the two knots are distinct.

In mathematics, we often observe a direct correlation between how difficult a particular identity or object is to compute and how much information it yields. We see this correlation in the case of homology groups; homotopy groups are much

[3]In fact, Dehn gave an argument in 1914 that the mirror image trefoils are not ambient isotopic to each other, despite having the same knot group. This is another example of where the knot group fails to distinguish non-equivalent knots. Details can be found in [Dehn87].

harder to calculate, and homology groups pay the price by containing less precise information than them. Analogously, this leads us to the theory of Seifert surfaces, a knot invariant that records less information than the knot group, but is much easier to compute and see the variance between two distinct knots. Later, we will see that it also plays a role in defining a much more powerful invariant.

To define a Seifert surface, we must first define an important characteristic.

Definition 3.1. A *collar* for a boundary component B of a manifold M with boundary is an embedding

$$f : B \times [0, \varepsilon) \to M$$

such that $f(x, 0) = x$ for all $x \in B$ and $f(B \times [0, \varepsilon))$ is an open neighborhood of B in M.

Now, we can establish the following:

Definition 3.2. The *Seifert surface* of a knot K is a compact, connected, bicollared manifold $M \subset S^3$ such that $\partial M = K$. Here, "bicollared" generalizes the definition of a collar where the embedding maps in both directions. Note that this implies orientability of the manifold, as it gives M a "top" and a "bottom."

Informally, we can think of a Seifert surface as "filling in" the spaces between the components of a knot. The author likes to imagine a bubble wand in the shape of a knot, where the Seifert surface is represented as the soapy film spanning some of the wires of the wand.

Example 3.3. A non-example and example Seifert surface for the trefoil knot, respectively. Quick check for the reader: why is the Möbius strip *not* a Seifert surface?

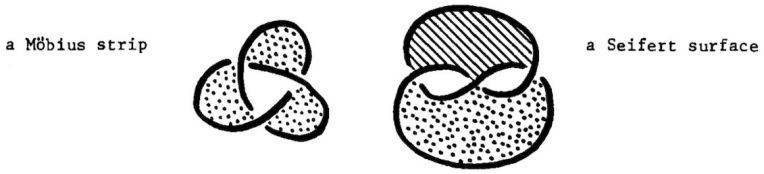


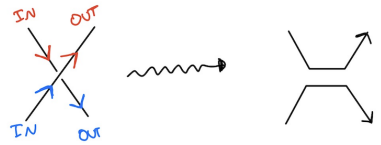
FIGURE 3. From [Rolfsen76]

Naturally, two questions concerning Seifert surfaces arise: do they always exist, and are they always unique? First, we will see that existence holds.

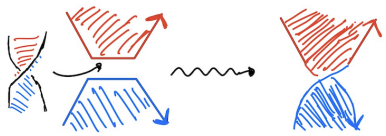
Theorem 3.4 (Seifert's Algorithm)**.** *For every knot $K \subset S^3$, there exists a Seifert surface M such that $\partial M = K$.*

Proof. We aim to demonstrate the effectiveness of an algorithm that takes in a knot K as an input and returns a Seifert surface M as an output. We detail the procedure in steps, and encourage the reader to follow along with their favorite knot. After presenting the proof, the author will give an example doing the same.

(1) Assign an orientation to K and consider its projection.
(2) Notice that for any crossing in the projection, there there are two strands of the knot that point towards the intersection of the crossing and two strands that point away from the intersection of the crossing.

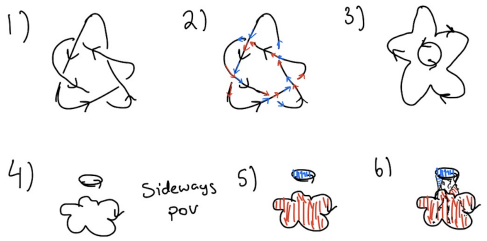


(3) Connect each of the strands going into the crossing with the adjacent strand going out of the crossing. We now have a disjoint collection of closed curves.
(4) Each of these closed curves can be made into disjoint disks. If a curve is nested inside another curve, simply push it off the plane (since we are actually working in S^3). These disks are called "Seifert circles."
(5) If the Seifert circle's boundary is clockwise oriented facing upwards, paint the upwards face red and the downwards face blue, and vice versa for a Seifert circle counter-clockwise oriented facing upwards.
(6) Connect disks by twisted bands at the points that were previously crossings of the knot to preserve orientation.



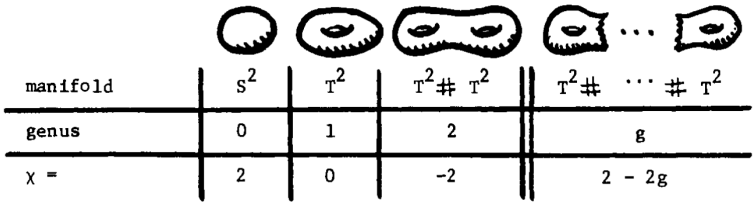
Now, you have an orientable 2-manifold M where $\partial M = K$. □

Example 3.5. Seifert's algorithm for the cinque-foil knot.



To answer the question of uniqueness, it should be clear to the reader by now that there are multiple different ways one could create a Seifert surface from the same knot and that two different knots could have the same Seifert surface. However, the power of creating Seifert surfaces out of knots is that we get access to 2-manifold theory, which comes with a powerful classification theorem.

Theorem 3.6 (Classification of 2-Manifolds)**.** *Every closed, orientable, connected 2-manifold is homeomorphic with one that appears in the table below, and is classified by its genus $g \geq 0$ and Euler characteristic χ.*



manifold	S^2	T^2	$T^2 \# T^2$	$T^2 \# \cdots \# T^2$
genus	0	1	2	g
χ =	2	0	-2	2 - 2g

Proof. Details can be found in [Rolfsen76], including more information about the Euler characteristic χ and the genus of 2-manifolds. □

We define the genus of a knot as follows.

Definition 3.7. The *genus of a knot or link* K in $\mathbb{R}^3$ or S^3 is defined as the minimum genus among all Seifert surfaces for K. That is, we write:

$$g(K) = g(M),$$

where M is a Seifert surface of minimal genus, often referred to as a *minimal Seifert surface*. The genus $g(K)$ is an invariant of the knot K

Although we don't get uniqueness for all knots, we do have uniqueness for one special knot.

Proposition 3.8. *A Seifert surface M for a knot K in S^3 has genus $g(M) = 0$ if and only if K is the unknot.*

Proof. $\implies$: We want to show that if $g(M) = 0$, then K is the unknot. A Seifert surface M with genus $g(M) = 0$ is a disk. If M is a disk, then $\partial M = K$ is a simple closed curve in S^3 that bounds a disk in S^3. By definition, any knot that bounds a disk in S^3 is the unknot. Hence, K must be the unknot.

$\impliedby$: Now, we want to show that if K is the unknot, then $g(M) = 0$. If K is the unknot, it bounds an embedded disk M in S^3. A disk has genus $g = 0$, and since the genus of a Seifert surface is the minimum genus among all orientable surfaces bounded by K, it follows that $g(M) = 0$. □

Naturally, one wonders how to calculate the genus of a Seifert surface. There are multiple strategies, and we introduce two helpful formulas.

Lemma 3.9. *Let M be a compact, orientable surface with b boundary components. Then*

$$g(M) \;=\; 1 \;-\; \frac{\chi(M) + b}{2},$$

where $\chi(M)$ is the Euler characteristic of M, and $g(M)$ is the genus.

Proof. We will prove this by filling in the boundary components of M with disks.

Suppose M has boundary components $\partial M = C_1 \cup C_2 \cup \cdots \cup C_b$. Attach a disk D_i to each boundary component C_i via a homeomorphism of their boundaries. Denote the resulting closed surface by $\widetilde{M}$. Since each disk D_i is orientable (in fact, simply connected), attaching disks to the boundary of an orientable surface preserves orientability, so $\widetilde{M}$ is an orientable closed surface.

Each disk D_i has Euler characteristic $\chi(D_i) = 1$. When attaching b disks (one along each boundary component), the total change to the Euler characteristic is b.

Hence,

$$\chi(\widetilde{M}) \;=\; \chi(M) + b.$$

Because $\widetilde{M}$ is now *closed* and orientable, we can use the genus-Euler-characteristic identity from Theorem 3.6 for a closed orientable surface:

$$\chi(\widetilde{M}) \;=\; 2 \;-\; 2\,g(\widetilde{M}).$$

But note that filling in boundary components with disks does not change the number of "handles" of the surface, so $\widetilde{M}$ and M have the same genus, i.e., $g(\widetilde{M}) = g(M)$.

Thus,

$$\chi(\widetilde{M}) = 2 - 2\,g(M).$$

Equating our two formulas for $\chi(\widetilde{M})$,

$$\chi(M) + b \;=\; 2 \;-\; 2g(M).$$

Solve for $g(M)$:

$$g(M) \;=\; 1 \;-\; \frac{\chi(M) + b}{2}.$$ □

Corollary 3.10. *Let L be an n-component link with a diagram having c crossings and s Seifert circles. The orientable Seifert surface M constructed via Seifert's algorithm has genus*

$$g(M) \;=\; 1 \;-\; \frac{s + n - c}{2}.$$

Proof. The surface M has boundary exactly L, so the number of boundary components is $b = n$. Starting with s disjoint disks (one per Seifert circle), each crossing contributes a band (Euler characteristic 0) that merges two disks. Thus, we have: $\chi(M) \;=\; s \;-\; c$.

By the previous Lemma 3.9, we have:

$$g(M) \;=\; 1 \;-\; \frac{\chi(M) + b}{2}.$$

Substituting $\chi(M) = s - c$ and $b = n$ gives

$$g(M) = 1 - \frac{s + n - c}{2}.$$

□

Although there does not exist a "magic bullet" to tell if a given Seifert surface is minimal, we have the following proposition to help out.

Proposition 3.11. *Suppose M is a Seifert surface for $K = \partial M$ in S^3, and $D \subset S^3$ is a disk such that:*

(1) $\partial D \cap M = \emptyset$,
(2) $\partial D \subset \mathrm{int}(M)$,
(3) $M - \partial D$ is connected.

Then M is not a minimal surface for K.

Proof. We construct a new Seifert surface M' for K with strictly smaller genus than M. First, take a small tubular neighborhood $A \cong \partial D \times I$ of ∂D in M, where $A \subset \text{int}(M)$. Define $M' := (M - A) \cup D$, replacing the annular neighborhood A with the disk D.

Since $\partial D \cap \partial M = \emptyset$, the boundary of M' is unchanged: $\partial M' = K$. Furthermore, M' remains connected because $M - \partial D$ is connected by assumption. The genus of M' is strictly less than the genus of M because the loop ∂D represents a "handle" in M, and replacing A with D effectively removes this handle.

Thus, M' is a Seifert surface for K of smaller genus, so M is not a minimal genus surface for K. □

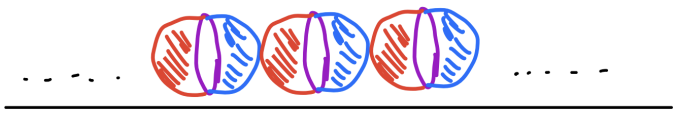
4. Infinite Cyclic Covering

We now leverage the Seifert surfaces in the previous section to construct a class of covering spaces for the knot complement. We opt for a more "geometric" construction at the cost of precision, but the interested reader can find the details in pages 9-14 of [Scherich18].

Construction 4.1. The *infinite cyclic cover*, $\widetilde{X}$.

Take a Seifert surface M for our knot K. Recall that M is a 2-manifold living in S^3 (or $\mathbb{R}^3$) whose boundary is exactly K. First, imagine slicing the Seifert surface "in half" along a core curve so that we effectively have two copies of M. Then, bend these two surfaces away from each other. Next, replicate and glue these surfaces to each other, so that each new copy is glued to the boundary knot of the previous one. In this manner, we are effectively "unrolling" the knot complement infinitely along the meridian direction.

Example 4.2. Construction of the infinite cyclic covering for the unknot.



The knot K itself remains in the base space, unchanged, but from the covering space viewpoint, we see infinitely many lifts of its complement. Each new copy of the Seifert surface sits in a new "layer" of the complement, with the boundary (a lift of the knot) meeting the boundary of the previous layer. To see this as a covering space, observe that the ith iteration of the manifold complement space $Y_i = \mathbb{R}^3 - M_i$ projects onto the complement of the Seifert, and N_i, the "gluing space" of the i and $i+1$ iteration of the cyclic covering, projects onto the interior of the ith Seifert surface. Together, Y_i and N_i cover the knot complement $\mathbb{R}^3 - K$.

Each time you go around the meridian loop of K in one copy, you pass into the next copy in this infinite chain. Thus, we are able to arrive at the following facts, whose proofs are omitted but are easy to believe and can be found in [Rolfsen76] or [Scherich18].

Proposition 4.3. *$\widetilde{X}$ is a path-connected open 3-manifold. There is a map $p : \widetilde{X} \to X$ which is a regular covering space. There is a covering automorphism $\tau : \widetilde{X} \to \widetilde{X}$, which takes Y_i to Y_{i+1} and N_i to N_{i+1}, and τ generates the group $\text{Aut}(\widetilde{X})$, which is infinite cyclic.*

To understand the importance of the infinite cyclic cover, we recall a definition from covering space theory.

Definition 4.4. Let X be a topological space with fundamental group $\pi_1(X)$. The *universal abelian cover* of X is the covering space $\widetilde{X}$ corresponding to the commutator subgroup $C = [\pi_1(X), \pi_1(X)]$, which is the kernel of the abelianization homomorphism:

$$\pi_1(X) \to \pi_1(X)^{\mathrm{ab}} \cong H_1(X).$$

Theorem 4.5. *The infinite cyclic cover of the knot complement, $\widetilde{X}$, is the universal abelian cover of X.*

Proof. We only need to use the fact that the group of deck transformations $\mathrm{Aut}(\widetilde{X})$ is isomorphic to $\mathbb{Z}$. Consider the standard exact sequence associated to a covering space:

$$1 \longrightarrow \pi_1(\widetilde{X}) \xrightarrow{p_*} \pi_1(X) \longrightarrow \mathrm{Aut}(\widetilde{X}) \longrightarrow 1.$$

Since $\mathrm{Aut}(\widetilde{X}) \cong \mathbb{Z}$, it follows that the image $p_*\big(\pi_1(\widetilde{X})\big)$ is a normal subgroup of $\pi_1(X)$ of index 1 in the abelianization. In other words,

$$p_*\big(\pi_1(\widetilde{X})\big) \supseteq \big[\pi_1(X), \pi_1(X)\big],$$

where $[\pi_1(X), \pi_1(X)]$ is the commutator subgroup of $\pi_1(X)$.

Now look at the induced map on the quotient:

$$\frac{\pi_1(X)}{[\pi_1(X),\, \pi_1(X)]} \longrightarrow \mathrm{Aut}(\widetilde{X}) \cong \mathbb{Z}.$$

Its kernel is exactly $p_*\big(\pi_1(\widetilde{X})\big) / [\pi_1(X), \pi_1(X)]$. Because the map is surjective onto $\mathbb{Z}$, it must be an isomorphism onto its image. This forces

$$p_*\big(\pi_1(\widetilde{X})\big) = [\pi_1(X), \pi_1(X)].$$

In other words, the subgroup of $\pi_1(X)$ corresponding to the covering $\widetilde{X}$ *is* the commutator subgroup. Hence, $\widetilde{X}$ is the universal abelian cover of X. □

5. The Alexander Invariant

Finally, we are ready to introduce and construct the Alexander invariant. It is important to preface, however, that the Alexander invariant constructed in this section is the algebraic and theoretical underpinning behind the *Alexander polynomial*, a polynomial that can be computed combinatorially. While elementary expositions of computing Alexander polynomials can be found in [Adams04], we hope to find a much deeper understanding of why and how the invariant works, and focus on it.

Definition 5.1. Let Λ denote the ring of *finite Laurent polynomials* with integer coefficients. Concretely, we define $\Lambda = \mathbb{Z}[t, t^{-1}]$, so a typical element is of the form

$$p(t) = c_{-r}\, t^{-r} + \cdots + c_{-1}\, t^{-1} + c_0 + c_1\, t + \cdots + c_s\, t^s$$

where each $c_i \in \mathbb{Z}$. Addition and multiplication are as in ordinary polynomial rings, extended naturally to allow powers of t that are negative. The units in Λ are exactly the monomials $\pm t^i$.

Now, consider a knot $K \subset S^3$ and let $X = S^3 - K$ denote its complement. The space X admits an infinite cyclic covering, which we denote by $\widetilde{X}$. By Theorem 4.5, this covering corresponds to the kernel of the abelianization map $\pi_1(X) \to \mathbb{Z}$. The covering transformations (i.e. the group of deck transformations) of $\widetilde{X}$ form an infinite cyclic group, generated by a homeomorphism $\tau : \widetilde{X} \longrightarrow \widetilde{X}$.

We are interested in the integral homology groups $H_i(\widetilde{X})$, which become naturally endowed with a module structure over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$. This arises by letting the formal variable t act on homology via the deck transformation τ. More precisely, given a Laurent polynomial

$$p(t) \;=\; c_{-r}\, t^{-r} \;+\; \cdots \;+\; c_0 \;+\; c_1\, t \;+\; \cdots \;+\; c_s\, t^s \quad \in \quad \Lambda$$

and a homology class $\alpha \in H_i(\widetilde{X})$, we define

$$p(t) \cdot \alpha \;=\; c_{-r}\, \tau_*^{-r}(\alpha) \;+\; \cdots \;+\; c_0\, \alpha \;+\; c_1\, \tau_*^{1}(\alpha) \;+\; \cdots \;+\; c_s\, \tau_*^{s}(\alpha).$$

Here, $\tau_*^k : H_i(\widetilde{X}) \to H_i(\widetilde{X})$ is the isomorphism on homology induced by applying the deck transformation τ exactly k times (if k is positive) or its inverse $-k$ times if k is negative.

This construction makes each $H_i(\widetilde{X})$ into a Λ–module. In particular, we are interested in $H_1(\widetilde{X})$.

Remark 5.2. One typically fixes the choice of the generator τ by requiring that it lifts a meridian loop in X. For classical knots in S^3, any choice with linking number 1 (a concept that we will define more precisely later) essentially yields the same Alexander module structure.

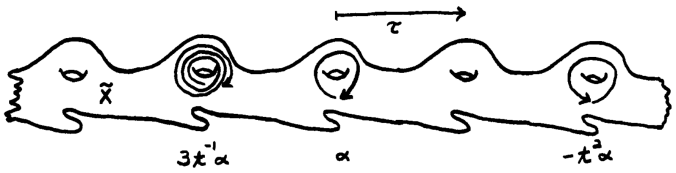


FIGURE 4. The one-dimensional homology element $3t^{-1} + t - t^2$ [Rolfsen76]

Now, we aim to verify that the algebraic construction we have defined works the way we want it to.

Proposition 5.3. *The Λ–multiplication defined by*

$$p(t) \cdot \alpha \;=\; \sum_{i=-r}^{s} c_i\, \tau_*^i(\alpha) \quad \textit{for} \quad p(t) = \sum_{i=-r}^{s} c_i\, t^i$$

makes $H_1(\widetilde{X})$ into a (left unitary) module over $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Proof. We must check that the assignment

$$\Lambda \times H_1(\widetilde{X}) \;\longrightarrow\; H_1(\widetilde{X}), \quad (p(t), \alpha) \;\mapsto\; p(t) \cdot \alpha$$

satisfies the axioms for a left module:

(1) *Distributivity over addition in* Λ*:* For $p(t), q(t) \in \Lambda$ and $\alpha \in H_1(\widetilde{X})$,
$$\big(p(t) + q(t)\big) \cdot \alpha \;=\; p(t) \cdot \alpha \;+\; q(t) \cdot \alpha.$$
This follows immediately from the definition, since $p(t)+q(t)$ just combines the coefficients c_i of $p(t)$ and $q(t)$ additively.

(2) *Distributivity over addition in* $H_1(\widetilde{X})$*:* For $p(t) \in \Lambda$ and $\alpha, \beta \in H_1(\widetilde{X})$,
$$p(t) \cdot (\alpha + \beta) \;=\; p(t) \cdot \alpha + p(t) \cdot \beta.$$
This again uses the fact that homology is an abelian group, and each τ_*^i is a group homomorphism on $H_1(\widetilde{X})$.

(3) *Associativity:* We must check $\big(p(t)\, q(t)\big) \cdot \alpha = p(t) \cdot \big(q(t) \cdot \alpha\big)$. By definition, $p(t)\, q(t)$ is the product in Λ, so expanding yields $\sum_j \sum_i c_i d_j\, t^{i+j}$. On the other hand, $q(t) \cdot \alpha$ is $\sum_i d_i\, \tau_*^i(\alpha)$, and then applying $p(t)$ amounts to multiplying by t^j in homology, i.e. τ_*^j, and summing over j. The net result is the same sum $\sum_{i,j} c_i d_j\, \tau_*^{i+j}(\alpha)$, proving associativity.

(4) *Unity:* Let $1(t) = 1$ be the multiplicative identity in Λ. Then $1 \cdot \alpha = \alpha$ by definition, since no powers of τ_* are applied.

Thus, all axioms of a left unitary module hold. □

Proposition 5.4. *If two knots $K, K' \subset S^3$ are equivalent, then their Alexander invariants $H_i(\widetilde{X})$ and $H_i(\widetilde{X}')$ (as Λ–modules) are isomorphic in each dimension, after possibly adjusting the choice of the generator τ.*

Proof. Suppose two knots $K, K' \subset S^3$ are equivalent, meaning there exists an ambient isotopy $H\colon S^3 \times [0,1] \to S^3$ such that $H_0 = \mathrm{id}$ and $H_1(K) = K'$, where $H_t(-) = H(-,t)$ is a homeomorphism for each $t \in [0,1]$. This ambient isotopy induces a homeomorphism $f = H_1\colon (S^3, K) \to (S^3, K')$ taking K onto K'. The homeomorphism f restricts to a homeomorphism $X \to X'$, where $X = S^3 - K$ and $X' = S^3 - K'$.

Because the covering spaces $\widetilde{X}$ and $\widetilde{X}'$ are each defined by taking the kernel of the abelianization map in π_1, the induced map on fundamental groups (and covering spaces) lifts to a homeomorphism
$$\widetilde{f} : \widetilde{X} \;\longrightarrow\; \widetilde{X}'.$$
Moreover, f takes a chosen meridian loop in X to a meridian loop in X', so $\widetilde{f}$ intertwines the deck transformations τ and τ' (up to possibly inverting one if the orientation is reversed).

Hence $\widetilde{f}_*$ induces an isomorphism on homology $H_i(\widetilde{X}) \to H_i(\widetilde{X}')$ that respects the action by τ (again, possibly replacing t by t^{-1}). Thus $H_i(\widetilde{X})$ and $H_i(\widetilde{X}')$ become isomorphic Λ–modules, up to the choice of identifying t with $t^{\pm 1}$. □

We can now happily conclude that our construction is in fact a knot invariant and calculate the Alexander invariant of a very special knot.

Theorem 5.5. *Let $X = S^3 - S^1$ be the complement of the unknot in S^3.*

Then its infinite cyclic cover $\widetilde{X}$ has
$$H_1(\widetilde{X}) \;=\; 0,$$
i.e. the first Alexander module of the trivial knot is trivial.

Proof. Recall from Lemma 2.2 that $\pi_1(S^3 - S^1) \cong \pi_1(S^1 \times D^2) \cong \mathbb{Z}$. Thus, because the universal cover of S^1 is $\mathbb{R}$, we have:

$$\widetilde{X} \;\cong\; \mathbb{R} \times D^2,$$

which is simply-connected. This lines up with the visual image from Example 4.2

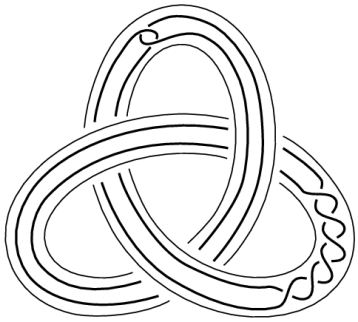
Since $\widetilde{X}$ is simply-connected, it has no nontrivial 1-dimensional homology. Therefore

$$H_1(\widetilde{X}) \;=\; 0.$$

Thus the first Alexander invariant of the unknot, which is defined to be $H_1(\widetilde{X})$ as a module over $\mathbb{Z}[t, t^{-1}]$, is indeed trivial. □

Remember, knot invariants are a balancing act; simpler computations can come at the expense of more precise data. Consequently, while the unknot uniquely determines its knot group, it does not own exclusive rights to its Alexander invariant.

Example 5.6. A Whitehead double of a trefoil knot, which has a trivial Alexander invariant.



In fact, Example 5.6 is a part of a whole class of knots that we can identify the Alexander invariant of.

Definition 5.7. Let $K \subset S^3$ be a nontrivial knot, and let $N(K)$ denote a tubular neighborhood of K. A *doubled knot* with twisting number m, denoted by $D_m(K)$, is constructed as follows. First, take two parallel copies K_1 and K_2 of K lying on $\partial N(K)$. Then, attach a band B between K_1 and K_2 that introduces exactly m full twists. The union $D_m(K) = K_1 \cup B \cup K_2$ forms a single-component knot in S^3.

Note that Example 5.6 is a doubled knot with twisting number $m = 0$. The following proposition shows that its Alexander invariant is trivial.

Proposition 5.8. *Let K be a doubled knot with twisting number m, and let $X = S^3 - N(K)$ denote its complement, where $N(K)$ is an open tubular neighborhood of K. Then:*

$$H_1(\tilde{X}) \cong \Lambda/(mt^2 + (1-2m)t + m),$$

and that all higher homology groups $H_i(\tilde{X})$ for $i \geq 2$ vanish.

Proof. First, we construct the infinite cyclic cover of X. Consider a simple cellular decomposition of X consisting of single 0-cell v, a single 1-cell e corresponding to the meridian μ of K, and a single 2-cell attached via a map representing the longitude λ, which incorporates the twisting number m.

In the infinite cyclic cover $\tilde{X}$, this cellular decomposition lifts to an infinite sequence of cells:

- 0-cells $\{v_n\}_{n\in\mathbb{Z}}$.
- 1-cells $\{e_n\}_{n\in\mathbb{Z}}$ connecting v_n to v_{n+1}.
- 2-cells $\{f_n\}_{n\in\mathbb{Z}}$ attached according to the twisting number m.

The cellular chain complex C_* of $\tilde{X}$ with coefficients in $\mathbb{Z}$ is given by

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

Due to the periodicity introduced by the twisting number m, it suffices to analyze the chain complex within a single period.

The boundary maps are determined as follows:

$$\partial_1(e_n) = v_{n+1} - v_n.$$

This corresponds to the module map ∂_1 acting as $t-1$, where t is the generator of the Laurent polynomial ring Λ acting by $t(v_n) = v_{n+1}$.

For the 2-cells, the attachment reflects the relation induced by the twisting number m:

$$\partial_2(f_n) = me_n + (1-2m)e_{n+1} + me_{n+2}.$$

Algebraically, this relation can be expressed as

$$\partial_2 = m + (1-2m)t + mt^2.$$

The first homology group is given by

$$H_1(\tilde{X}) = \ker(\partial_1)/\mathrm{im}(\partial_2).$$

From the boundary maps, we have the relation

$$m + (1-2m)t + mt^2 = 0$$

in the module Λ. Therefore, the module structure of $H_1(\tilde{X})$ is

$$H_1(\tilde{X}) \cong \Lambda/(mt^2 + (1-2m)t + m).$$

Since the cellular decomposition of $\tilde{X}$ involves only 0-cells, 1-cells, and 2-cells, the chain complex C_* satisfies $C_i = 0$ for $i \geq 3$. This implies that $H_i(\tilde{X}) = 0$ for $i \geq 3$. For $i = 2$, the exactness of the chain complex in degree 2 ensures that $H_2(\tilde{X}) = 0$. Thus:

$$H_i(\tilde{X}) \cong 0 \quad \text{for all } i \geq 2. \qquad \square$$

Ultimately, we see that the Alexander invariant's reliance on a coarser tool than the knot group (homology rather than homotopy) leads it to be much less effective for unknot detection than the knot group. However, the balancing act of invariants that has been the recurring theme of this paper is not so straightforward. Although the Alexander module takes shortcuts in some aspects, its construction allows it to distinguish certain knots the knot group cannot.

6. Squares aren't Grannies

In the previous section, we demonstrated that the Alexander invariant is not a complete invariant for the unknot. This stands in contrast to Example 2.6, where we discussed the square knot and the granny knot, which cannot be distinguished by the knot group, but, as we will see in this section, can be distinguished by the Alexander module.

Intuitively, the reason the knot group fails to distinguish these two knots is that it depends only on the free product of the groups of their constituent trefoils, and not on the specific way the trefoils are connected. Since both the square knot and the granny knot are composed of trefoil knots, their knot groups are isomorphic.

On the other hand, the Alexander module encodes information about the orientation and linking of a knot through the infinite cyclic cover of the knot complement, rather than getting data from just the knot complement. We leverage the structure of the infinite cyclic cover through the action of the deck transformation group $\mathbb{Z}$, which reflects how loops in the cover are shifted, capturing linking behavior and orientation behavior.

Thus, in this section, we will finally be able to see that squares are not grannies. First, we must define and recall the necessary algebraic constructions to extract the orientation and linking data from the Alexander invariant.

Definition 6.1. Let K_1 and K_2 be two oriented knots in S^3. The *linking number* $\mathrm{lk}(K_1, K_2)$ is defined as:

$$\mathrm{lk}(K_1, K_2) = \frac{1}{2} \sum_{p \in K_1 \cap K_2} \epsilon(p),$$

where the sum is taken over all crossings p between K_1 and K_2 in a diagram, and $\epsilon(p) \in \{+1, -1\}$ is the sign of the crossing p, determined by the relative orientation of the strands of K_1 and K_2 at p. While this is the most intuitive definition, [Rolfsen76] gives 7 other equivalent definitions.

Definition 6.2. Let S be a compact, oriented surface. We can consider $H_1(S)$ as a $\mathbb{Z}$-module (denoted $H_1(S;\mathbb{Z})$) by defining a *basis* for the first homology group as a set of oriented closed curves $\{a_1, a_2, \ldots, a_n\}$ on S such that:

(1) The curves are linearly independent over $\mathbb{Z}$: no nontrivial linear combination $\sum_{i=1}^{n} c_i a_i$ with $c_i \in \mathbb{Z}$ bounds a 2-chain in S.
(2) The curves generate $H_1(S;\mathbb{Z})$: every 1-cycle in S is homologous to a linear combination $\sum_{i=1}^{n} c_i a_i$ for some $c_i \in \mathbb{Z}$.

The number of basis elements n is equal to the rank of $H_1(S;\mathbb{Z})$, which is $2g$ for a surface of genus g.

Definition 6.3. Let S be a Seifert surface for an oriented knot K in S^3, and let $\{a_1, a_2, \ldots, a_n\}$ be a basis for $H_1(S;\mathbb{Z})$. The *Seifert matrix* V associated with S is an $n \times n$ matrix whose entries V_{ij} are defined as:

$$V_{ij} = \mathrm{lk}(a_i, a_j^+),$$

where a_j^+ denotes the curve obtained by pushing a_j slightly off S in the positive normal direction , and $\mathrm{lk}(a_i, a_j^+)$ is the linking number of a_i with a_j^+.

The Seifert matrix can be used to calculate the Alexander polynomial and to streamline the computations in the previous section.

Theorem 6.4. *Let K be an oriented knot in S^3, and let S be a Seifert surface for K. Let V be the Seifert matrix associated with S, and let t be the variable over which $\mathbb{Z}[t, t^{-1}]$ is defined. Then, the Alexander module is the cokernel of the matrix $tV - V^T$.*

Proof. Let $\widetilde{X}$ denote the infinite cyclic covering of the knot complement $X = S^3 - K$. Choose a cellular decomposition of X compatible with the Seifert surface S. This decomposition consists of a single 0-cell, $2g$ 1-cells corresponding to a basis $\{a_i\}$ of $H_1(S;\mathbb{Z})$, and a single 2-cell attached along the boundary of S to encode the relations in $H_1(S;\mathbb{Z})$. Lifting this decomposition to the infinite cyclic cover $\tilde{X}$, we obtain an infinite sequence of 0-cells $\{v_n\}_{n\in\mathbb{Z}}$, $2g$ 1-cells $\{e_{n,i}\}_{i=1}^{2g}$ connecting v_n to v_{n+1}, and 2-cells $\{f_n\}_{n\in\mathbb{Z}}$ attached according to the relations imposed by S.

The cellular chain complex C_* of $\tilde{X}$ is given by

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where C_0, C_1, and C_2 are free Λ-modules generated by the 0-cells, 1-cells, and 2-cells, respectively. The boundary maps are as follows:

- The map $\partial_1 : C_1 \to C_0$ sends each 1-cell to $(t-1)$ times the generator of C_0, reflecting the fact that the 1-cells connect 0-cells in adjacent copies of X.
- The map $\partial_2 : C_2 \to C_1$ encodes the relations imposed by the Seifert surface. Specifically, for each 2-cell f_n, the attaching map corresponds to the relation
$$\partial_2(f_n) = tVe_n - V^T e_n,$$
where V is the Seifert matrix and V^T is its transpose.

The first homology group $H_1(\tilde{X};\mathbb{Z})$ is computed as the quotient

$$H_1(\tilde{X};\mathbb{Z}) = \ker(\partial_1)/\mathrm{im}(\partial_2).$$

Since $\ker(\partial_1)$ consists of elements $x \in C_1$ such that $(t-1)x = 0$, and $\mathrm{im}(\partial_2)$ is generated by the columns of $tV - V^T$, it follows that

$$H_1(\tilde{X};\mathbb{Z}) \cong \Lambda^{2g}/(tV - V^T)\Lambda^{2g}.$$

Thus, the Alexander module is isomorphic to the cokernel of $tV - V^T$. □

We are now ready to prove the following theorem promised in the first section of this paper:

Theorem 6.5. *Squares aren't grannies*

Proof. In this theorem, we prove that the square knot and granny knot are not equivalent. Recall that the square knot and the granny knot are both composite knots formed by taking the connected sum of two trefoil knots. The square knot is the connected sum of a left-handed trefoil T_- and a right-handed trefoil T_+, while the granny knot is the connected sum of two left-handed trefoils $T_- \# T_-$.

To distinguish these knots, we compute their Alexander invariants explicitly using Theorem 6.4.

The Seifert matrix for T_- is

$$V_- = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

and for T_+ it is

$$V_+ = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The Seifert matrix for the square knot is

$$V_{\text{Square}} = \begin{pmatrix} V_- & 0 \\ 0 & V_+ \end{pmatrix},$$

and for the granny knot it is

$$V_{\text{Granny}} = \begin{pmatrix} V_- & 0 \\ 0 & V_- \end{pmatrix}.$$

For the square knot,

$$tV_{\text{Square}} - V_{\text{Square}}^T = \begin{pmatrix} tV_- - V_-^T & 0 \\ 0 & tV_+ - V_+^T \end{pmatrix},$$

where

$$tV_- - V_-^T = \begin{pmatrix} -t+1 & -1 \\ t & -t+1 \end{pmatrix}, \quad tV_+ - V_+^T = \begin{pmatrix} t-1 & 1 \\ -t & t-1 \end{pmatrix}.$$

Thus,

$$tV_{\text{Square}} - V_{\text{Square}}^T = \begin{pmatrix} -t+1 & -1 & 0 & 0 \\ t & -t+1 & 0 & 0 \\ 0 & 0 & t-1 & 1 \\ 0 & 0 & -t & t-1 \end{pmatrix}.$$

For the granny knot,

$$tV_{\text{Granny}} - V_{\text{Granny}}^T = \begin{pmatrix} tV_- - V_-^T & 0 \\ 0 & tV_- - V_-^T \end{pmatrix} = \begin{pmatrix} -t+1 & -1 & 0 & 0 \\ t & -t+1 & 0 & 0 \\ 0 & 0 & -t+1 & -1 \\ 0 & 0 & t & -t+1 \end{pmatrix}.$$

The Alexander module of the square knot is

$$\mathbb{Z}[t, t^{-1}]^4 / \text{im}(tV_{\text{Square}} - V_{\text{Square}}^T),$$

with relations

$$\begin{cases} (-t+1)x_1 - x_2 = 0, \\ tx_1 + (-t+1)x_2 = 0, \\ (t-1)x_3 + x_4 = 0, \\ -tx_3 + (t-1)x_4 = 0. \end{cases}$$

The Alexander invariant of the granny knot is

$$\mathbb{Z}[t, t^{-1}]^4 / \text{im}(tV_{\text{Granny}} - V_{\text{Granny}}^T),$$

with relations

$$\begin{cases} (-t+1)x_1 - x_2 = 0, \\ tx_1 + (-t+1)x_2 = 0, \\ (-t+1)x_3 - x_4 = 0, \\ tx_3 + (-t+1)x_4 = 0. \end{cases}$$

Note that in the square knot, the first and third relations involve $t-1$ and $-t+1$, reflecting the opposite orientation of the trefoils. In the granny knot, the first and third relations are identical, as both trefoils have the same orientation.

This difference in presentation does not immediately imply that the two Alexander modules are not isomorphic. However, we can calculate the *signature* of the respective matrices (# of positive eigenvalues − # of negative eigenvalues), evaluating at $t = -1$ to yield the following computations.

Substituting $t = -1$ into $tV_{\text{Square}} - V_{\text{Square}}^T$:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Since this is a block diagonal matrix, we can easily compute the eigenvalues: $3, 1, -1, -3$. Thus, the signature of the matrix is $2 - 2 = 0$.

Similarly, for the granny knot, substituting $t = -1$ into $tV_{\text{Granny}} - V_{\text{Granny}}^T$:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Our eigenvalues are $3, 3, 1, 1$. Thus, the signature is $4 - 0 = 4$.

The signatures differ, so the matrices are not secretly equal. Thus, the square knot and the granny knot are distinct. □

Remark 6.6. The above theorem cannot be proven with the Alexander polynomial, as the Alexander polynomial of both square and granny knots are $(t^2 - t + 1)^2$. The balancing act of knot invariants remains as delicate as ever.

7. Acknowledgments

I came into this summer without knowing any algebra or topology, so I owe the entirety of this paper to both of my wonderfully patient and kind mentors. In Chicago, I had the privilege of learning much of the machinery that support this paper from Professor Peter May. In Austin (and on Zoom), Professor Irving Dai was incredibly gracious to share his time and intuition with some random kid that was just from the same city he taught in. I would also like to thank Brian Streit for conversations on the material and support.

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