PARTIAL REGULARITY THEORY OF THE NAVIER STOKES EQUATIONS

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Abstract. This expository paper concerns the partial regularity theory of the Navier-Stokes equations in dimension three. In 1982, Caffarelli, Kohn, and Nirenberg showed that the singular set of a suitable weak solution has Hausdorff dimension at most one. In this paper, we present an alternative proof of partial regularity by Vasseur in [Vas07] inspired by the De Giorgi method for the regularity of elliptic equations. Partial regularity implies a local epsilon result for the exponent $p = 2$. Finally, we mention an improvement by Gustafson, Kang, and Tsai in [GKT07] to reduce the exponent from 2 to $\frac{5}{3}$ with further interpolation of the cubic velocity term.

Contents

1. Introduction 1
2. Setup for De Giorgi iteration 3
2.1. Pressure decomposition 4
3. Partial regularity 5
4. Local epsilon-regularity criterion 9
4.1. Control of velocity 10
4.2. Control of pressure variation 11
5. Hausdorff dimension of singular set 14
6. Reduction in exponent to $\frac{5}{3}$ 15
6.1. Control of the quadratic velocity terms 15
6.2. Control of the cubic velocity term 16
6.3. Control of the pressure term 17
Acknowledgments 19
References 19

1. Introduction

The Navier-Stokes equations (NSE) in dimension $d = 3$ are given by

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla P - \Delta u = 0, \quad t \in (0, \infty), x \in \Omega, \tag{1}$$

$$\operatorname{div} u = 0, \tag{2}$$

where Ω is a regular subset of $\mathbb{R}^3$. Here, $u : \mathbb{R}^3 \to \mathbb{R}^3$ represents the velocity while $P : \mathbb{R}^3 \to \mathbb{R}$ represents the pressure. They present a form of conservation momentum for a fluid, where the material time derivative of the velocity is equal to the

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sum of the effects of the pressure gradient and viscosity (with coefficients normalized to one).

The initial boundary value problem is endowed with the conditions

$$u(0,\cdot) = u^0 \in L^2(\Omega),$$
$$u(t,x) = 0, \quad t \in (0,\infty), x \in \Omega.$$

We also rely on the notion of "suitable weak solutions" which verify the generalized energy inequality in the sense of distribution

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(u\frac{|u|^2}{2}) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} \leq 0, \quad t \in (0,\infty), x \in \Omega. \tag{3}$$

In 1982, Caffarelli, Kohn and Nirenberg published the stunning result [CKN82] that such suitable weak solutions exist, and that the Hausdorff dimension of the singular set is bounded above by one. The main goal of this expository paper, mainly based on [Vas07], is to provide a proof of this partial regularity result.

The first part of the proof states that control of $\|u\|_{L_t^\infty L_x^2}$, $\|u\|_{L_t^2 H_x^1}$, and $\|P\|_{L_t^{1+} L_x^1}$ yields boundedness and regularity on a smaller set.

Theorem 1.1. *Let $p > 1$. There exists a universal constant C^* such that any solution u of (1), (2), (3) in $[-1,1] \times B(1)$ which satisfies*

$$\sup_{t\in[-1,1]} \left(\int_{B(1)} |u|^2 \,\mathrm{d}x \right) + \int_{-1}^1 \int_{B(1)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t + \left[\int_{-1}^1 \left(\int_{B(1)} |P| \,\mathrm{d}x \right)^p \mathrm{d}t \right]^{2/p} \leq C^* \tag{4}$$

must be bounded by 1 on $[-\frac{1}{2}, 1] \times B(1/2)$. In particular, u is regular in this region.

In Section 3, we present a sketch of the proof of Theorem 1.1 that relies on a technique devised by De Giorgi's solution to resolve Hilbert's 19th problem. The De Giorgi method involves a superlinear iteration with increasing cutoffs over shrinking domains that approximate the smaller set. Precisely, we make use of the following recursive inequality

Lemma 1.2. *Suppose there exists a constant C such that the sequence U_k satisfies*

$$U_k \leq C^k U_{k-1}^\delta,$$

where $\delta > 1$. Then $U_k \to 0$.

Subsequently, Theorem 1.1 leads to a local epsilon-regularity result depending on the scaled L^2 norm of ∇u. The proof relies instead on a linear iteration in Section 4 on the rescaled solutions of the NSE, which do not need to be small initially. The key insight lies in how the $\|u\|_{L_t^2 H_x^1}$ norm dominates the $\|u\|_{L_t^\infty L_x^2}$ and $\|P\|_{L_t^p L_x^1}$ norms at progressively smaller scales.

Theorem 1.3. *Let u be a suitable weak solution. For every $p \geq 2$, there exists a universal constant $\varepsilon_p > 0$, such that if near $(t_0, x_0) \in (0,\infty) \times \Omega$ the following holds*

$$\limsup_{r\to 0} \frac{1}{r^{5-2p}} \int_{Q_r(t_0,x_0)} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t < \varepsilon_p, \tag{5}$$

then (t_0, x_0) is a regular point.

Remark 1.4. A smaller choice of p represents a weaker assumption and stronger result, because we work on a bounded domain of finite measure.

A standard covering argument allows us to then show that Hausdorff dimension of the singular set is smaller than one.

We finish the paper by showing that Theorem 1.3 holds for $p \geq \frac{5}{3}$ in Section 6. This stronger result is possible with the help of another iteration that incorporates $\|u\|_3$ and $\|P\|_{\frac{3}{2}}$.

2. Setup for De Giorgi iteration

We discuss the preliminaries required to apply the De Giorgi method in order to prove Theorem 1.1.

Definition 2.1. We take the following sets that dyadically approach $B_{\frac{1}{2}}$

$$\begin{aligned}\beta_k &:= B_{\frac{1}{2}(1+2^{-3k})},\\ \beta_{k-\frac{1}{3}} &:= B_{\frac{1}{2}(1+2\cdot 2^{-3k})},\\ \beta_{k-\frac{2}{3}} &:= B_{\frac{1}{2}(1+4\cdot 2^{-3k})},\end{aligned}$$

as well as cylinders of the form

$$T_k := \frac{1}{2}(-1-2^{-k}), \quad \Gamma_k := [T_k, 1] \times \beta_k.$$

We always have

$$[-\frac{1}{2}, 1] \times B_{\frac{1}{2}} \subset \cdots \subset \Gamma_{k+1} \subset \Gamma_k \subset \cdots \subset [-1,1] \times B_1.$$

Definition 2.2. We introduce the function

$$v_k := [|u| - (1-2^{-k})]_+.$$

The quantities v_k^2 act as level sets of energy since they vanish for $|u| < 1-2^{-k}$ and have order $|u|^2$ for $|u| \gg 1-2^{-k}$.

Definition 2.3. We define the following quantities:

$$\begin{aligned}d_k^2 &:= \frac{(1-2^{-k})\mathbf{1}_{\{|u|\geq(1-2^{-k})\}}}{|u|}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2,\\ U_k &:= \sup_{t\in[T_k,1]}\left(\int_{\beta_k}|v_k(t,x)|^2\,\mathrm{d}x\right) + \int_{\Gamma_k}|d_k(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The quantity d_k is approximately equal to $|\nabla|u||$ when $|u| \simeq 1-2^{-k}$, and $|\nabla u|$ when $|u| \gg 1-2^{-k}$. We also verify that U_0 corresponds to the first two terms on the left hand side of (4).

We finish this subsection with a link between d_k and the gradient of v_k.

Lemma 2.4. *There exists a constant C such that for every k, and every $F \in L_t^\infty L_x^2(\Gamma_k)$ and $\nabla F \in L_{t,x}^2(\Gamma_k)$,*

$$\|F\|_{L_{t,x}^{\frac{10}{3}}(\Gamma_k)} \leq C\big(\|F\|_{L_t^\infty L_x^2(\Gamma_k)} + \|F\|_{L_t^\infty L_t^2(\Gamma_k)}^{\frac{2}{5}}\|\nabla F\|_{L_{t,x}^2(\Gamma_k)}^{\frac{3}{5}}\big).$$

Proof. Sobolev embedding gives

$$\|F\|_{L^2_t L^6_x(\Gamma_k)} \leq C\big(\|F\|_{L^\infty_t L^2_x(\Gamma_k)} + \|\nabla F\|_{L^2_{t,x}(\Gamma_k)} \big).$$

Then we derive

$$\begin{aligned}\|F\|_{L^{\frac{10}{3}}_{t,x}(\Gamma_k)} &\leq \|F\|^{\frac{2}{5}}_{L^\infty_t L^2_x(\Gamma_k)} \|F\|^{\frac{3}{5}}_{L^2_t L^6_x(\Gamma_k)} \\ &\leq C\big(\|F\|_{L^\infty_t L^2_x(\Gamma_k)} + \|F\|^{\frac{2}{5}}_{L^\infty_t L^x(\Gamma_k)} \|\nabla F\|^{\frac{3}{5}}_{L^2_{t,x}(\Gamma_k)} \big).\end{aligned}$$

from Hölder's inequality. □

We introduce cutoff functions $\phi_k \in C^\infty(\mathbb{R}^3)$ which verify

$$\phi_k(x) = 1 \text{ in } \beta_{k-\frac{2}{3}} \text{ and } \phi_k(x) = 0 \text{ in } \beta^c_{k-1}$$

with $|\nabla \phi_k| \leq C 2^{3k}$ and $|\nabla^2 \phi_k| \leq C 2^{6k}$.

Lemma 2.5. *We can split the function u into*

$$u = u\frac{v_k}{|u|} + u\left(1 - \frac{v_k}{|u|}\right), \text{ where } \left|u\left(1 - \frac{v_k}{|u|}\right)\right| \leq 1 - 2^{-k}.$$

Moreover, the following gradients are bounded with respect to d_k

$$\frac{v_k}{|u|}|\nabla u|, \mathbf{1}_{\{|u| \geq 1-2^{-k}\}}|\nabla |u||, |\nabla v_k| \leq d_k \text{ and } \left|\nabla \frac{u v_k}{|u|}\right| \leq 3 d_k.$$

Corollary 2.6. *We have $\|v_{k-1}\|_{L^{\frac{10}{3}}(\Gamma_{k-1})} \leq C U_{k-1}^{\frac{1}{2}}$.*

Proof. By Lemma 2.4, we have

$$\begin{aligned}\|v_{k-1}\|_{L^{\frac{10}{3}}(\Gamma_k)} &\leq C\big(\|v_{k-1}\|_{L^\infty_t L^2_x(\Gamma_k)} + \|v_{k-1}\|^{\frac{2}{5}}_{L^\infty_t L^2_x(\Gamma_k)} \|\nabla v_{k-1}\|^{\frac{3}{5}}_{L^2_{t,x}(\Gamma_k)} \big) \\ &\leq C\big(\|v_{k-1}\|_{L^\infty_t L^2_x(\Gamma_k)} + \|v_{k-1}\|^{\frac{2}{5}}_{L^\infty_t L^2_x(\Gamma_k)} \|d_k\|^{\frac{3}{5}}_{L^2_{t,x}(\Gamma_k)} \big).\end{aligned}$$

□

2.1. **Pressure decomposition.** The pressure is handled with a decomposition of the following form. Taking the divergence of the Navier–Stokes equation yields

$$\operatorname{div}\operatorname{div}(u \otimes u) + \Delta P = 0.$$

As u is divergence-free, we can subtract $\tilde{u}$ from it for

$$\operatorname{div}\operatorname{div}((u - \tilde{u}) \otimes (u - \tilde{u})) + \Delta P = 0.$$

For a cutoff η, we define the local pressure by computing the Newtonian potential of the localized advection divergence

$$P^{\mathrm{loc}} = (-\Delta)^{-1} \operatorname{div}\operatorname{div}\left(\eta (u - \tilde{u}) \otimes (u - \tilde{u})\right), \tag{6}$$

and the non-local pressure $P^{\mathrm{har}} = P - P^{\mathrm{loc}}$. This notation reflects how P^{har} is harmonic in the set $\{\eta = 1\}$:

$$\Delta P^{\mathrm{har}} = \operatorname{div}\operatorname{div}((1 - \eta)(u - \tilde{u}) \otimes (u - \tilde{u})).$$

Proposition 2.7. *Let (u,P) be a solution to the NSE in Γ_{k-1}. Then $P|_{\beta_{k-\frac{2}{3}}}$ can be decomposed into two parts*

$$P|_{\beta_{k-\frac{2}{3}}} = P_k^{\mathrm{loc}}|_{\beta_{k-\frac{2}{3}}} + P_k^{\mathrm{har}}|_{\beta_{k-\frac{2}{3}}},$$

where

$$\begin{aligned}\left\|\nabla P_k^{\mathrm{har}}\right\|_{L_t^p L_x^\infty([T_{k-1},1]\times\beta_{k-\frac{1}{3}})} + \left\|P_k^{\mathrm{har}}\right\|_{L_t^p L_x^\infty([T_{k-1},1]\times\beta_{k-\frac{1}{3}})}\\ \leq C2^{12k}\left(\|P\|_{L_t^p L_x^1(\Gamma_{k-1})} + \|u\|^2_{L_t^\infty L_x^2(\Gamma_{k-1})}\right).\end{aligned}$$

Moreover, P_k^{loc} is the solution on $[T_{k-1},1]\times\mathbb{R}^3$ to

$$-\Delta P_k^{\mathrm{loc}} = \sum_{ij}\partial_i\partial_j[\phi_k u_j u_i].$$

3. Partial regularity

We highlight the main decompositions for the proof of partial regularity and omit certain estimates. The details of the proof can be found in [Vas07].

Proposition 3.1. *Let $p>1$. There exist universal constants $C_p,\delta_p>1$ such that for any solution to the NSE in $[1,1]\times B_1$, the condition $U_0\leq 1$ implies*

$$U_k \leq C_p^k(1+\|P\|_{L_t^pL_x^1([0,1]\times B_0)})U_{k-1}^{\delta_p}.$$

We claim that Proposition 3.1 implies Theorem 1.1 with the help of Lemma 1.2. For sufficiently small U_0, we see that $U_k\to 0$ as $k\to\infty$. Then $(u-1)_+=0$ on $B_{\frac{1}{2}}$ implies $u\leq 1$ on $B_{\frac{1}{2}}$.

Proof. Adding the relation

$$\partial_t\frac{v_k^2-|u|^2}{2} + \operatorname{div}\left(u\frac{v_k^2-|u|^2}{2}\right) + u\left(\frac{v_k}{|u|}-1\right)\nabla P - u\left(\frac{v_k}{|u|}-1\right)\Delta u = 0$$

to the generalized energy inequality (3) yields

$$\partial_t\frac{v_k^2}{2} + \operatorname{div}\left(u\frac{v_k^2}{2}\right) + d_k^2 - \Delta\frac{v_k^2}{2} + \operatorname{div}(uP) + u\left(\frac{v_k}{|u|}-1\right)\nabla P \leq 0.$$

Let $\eta_k\in C^\infty(\mathbb{R}^3)$ be cutoff functions such that

$$\eta_k(x)=1 \text{ in } \beta_k \text{ and } \eta_k(x)=0 \text{ in } \beta^c_{k-\frac{1}{3}}$$

with $|\nabla\eta_k|\leq C2^{3k}$ and $|\nabla^2\eta_k|\leq C2^{6k}$. Then

$$\begin{aligned}&\int_{\mathbb{R}^3}\eta_k\frac{|v_k(t,x)|^2}{2}\,\mathrm{d}x + \iint_{[\sigma,t]\times\mathbb{R}^3}\eta_k d_k^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &\leq \int_{\mathbb{R}^3}\eta_k\frac{|v_k(\sigma,x)|^2}{2}\,\mathrm{d}x + \iint_{[\sigma,t]\times\mathbb{R}^3}\nabla\eta_k u\frac{|v_k(s,x)|^2}{2}\,\mathrm{d}x\,\mathrm{d}s\\ &+ \iint_{[\sigma,t]\times\mathbb{R}^3}\Delta\eta_k\frac{|v_k(s,x)|^2}{2}\,\mathrm{d}x\,\mathrm{d}s - \iint_{[\sigma,t]\times\mathbb{R}^3}\eta_k\left(\operatorname{div}(uP) + \left(\frac{v_k}{|u|}-1\right)u\nabla P\right)\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By integrating in σ between T_{k-1} and T_k, we obtain

$$U_k \le 2 \sup_{t\in[T_k,1]} \left(\int \eta_k(x) \frac{|v_k(t,x)|^2}{2} \,\mathrm{d}x + \int_{T_k}^t \int \eta_k(x) d_k^2(s,x) \,\mathrm{d}x \,\mathrm{d}s \right)$$

$$\le C2^{6k} \iint_{Q_{k-1}} |v_k(s,x)|^2 \,\mathrm{d}x \,\mathrm{d}s + C2^{3k} \iint_{Q_{k-1}} |v_k(s,x)|^3 \,\mathrm{d}x \,\mathrm{d}s \tag{7}$$

$$+ 2\int_{T_{k-1}}^1 \left| \int \eta_k(x) \left(\operatorname{div}(uP) + \left(\frac{v_k}{|u|} - 1 \right) u \nabla P \right) \mathrm{d}x \right| \mathrm{d}t. \tag{8}$$

Since, $v_k > 0$ implies $v_{k-1} > 2^{-k}$, we can apply Tchebychev's inequality and Corollary 2.6 such that for all $k, q > 1$,

$$\left\| \mathbf{1}_{\{v_k>0\}} \right\|_{L^q_{t,x}(\Gamma_{k-1})} \le C2^{\frac{10k}{3q}} U_{k-1}^{\frac{5}{3q}}, \tag{9}$$

$$\left\| \mathbf{1}_{\{v_k>0\}} \right\|_{L^\infty_t L^q_x(\Gamma_{k-1})} \le C2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}. \tag{10}$$

We can now control the velocity terms (7) with

$$\begin{aligned} &2^{6k} \left\| v_k^2 \right\|_{L^{\frac{5}{3}}(\Gamma_{k-1})} \left\| \mathbf{1}_{\{v_k>0\}} \right\|_{L^{\frac{5}{3}}(\Gamma_{k-1})} + 2^{3k} \left\| v_k^3 \right\|_{L^{\frac{10}{9}}(\Gamma_{k-1})} \left\| \mathbf{1}_{\{v_k>0\}} \right\|_{L^{10}(\Gamma_{k-1})} \\ &\le C2^{6k} U_{k-1} 2^{\frac{4k}{3}} U_{k-1}^{\frac{2}{3}} + C2^{3k} U_{k-1}^{\frac{3}{2}} 2^{\frac{k}{3}} U_{k-1}^{\frac{1}{6}} \\ &\le C2^{6k+\frac{4k}{3}} U_{k-1}^{\frac{5}{3}}. \end{aligned}$$

□

Meanwhile, the pressure terms (8) are bounded by

$$\int_{T_{k-1}}^1 \left| \int_{\beta_{k-\frac{1}{3}}} \eta_k \frac{v_k u}{|u|} \nabla P_k^{\mathrm{har}} \,\mathrm{d}x \right| \mathrm{d}t \tag{11}$$

$$+ \int_{T_{k-1}}^1 \left| \int_{\beta_{k-\frac{1}{3}}} \eta_k \left(\operatorname{div}(uP_k^{\mathrm{loc}}) + \left(\frac{v_k}{|u|} - 1 \right) u \nabla P_k^{\mathrm{loc}} \right) \mathrm{d}x \right| \mathrm{d}t. \tag{12}$$

Step 4: Bound of the non-local pressure term (11) involving P_k^{har}. We interpolate $v_k, \nabla P_k^{\mathrm{har}}$, and $\mathbf{1}_{\{v_k>0\}}$. We can bound $\left\| \nabla P_k^{\mathrm{har}} \right\|$ with Proposition 2.7 and use Corollary 2.6 on $\left\| \mathbf{1}_{\{v_k>0\}} \right\|$.

We then consider different values of p separately.

- For large values of $p \ge 2$ with $p' \le 2$, we control (11) by

$$\begin{aligned} &C \left\| v_k \right\|_{L^\infty_t L^2_x(\Gamma_{k-1})} \left\| \nabla P_k^{\mathrm{har}} \right\|_{L^p_t L^\infty_x([T_{k-1},1]\times\beta_{k-\frac{1}{3}})} \left\| \mathbf{1}_{\{v_k>0\}} \right\|_{L^2_{t,x}(\Gamma_{k-1})} \\ &\le C U_{k-1}^{\frac{1}{2}} \left\| \nabla P_k^{\mathrm{har}} \right\|_{L^p_t L^\infty_x([T_{k-1},1]\times\beta_{k-\frac{1}{3}})} 2^{\frac{5k}{3}} U_{k-1}^{\frac{5}{6}} \\ &\le C2^{\frac{5k}{3}} \left\| \nabla P_k^{\mathrm{har}} \right\|_{L^p_t L^\infty_x([T_{k-1},1]\times\beta_{k-\frac{1}{3}})} U_{k-1}^{\frac{4}{3}}. \end{aligned}$$

- For $p<2$, we get instead the weaker bound of

$$\begin{aligned}
&C\|v_k\|_{L^\infty_t L^2_x}\left\|\nabla P^{\mathrm{har}}_k\right\|_{L^p_t L^\infty_x}\left\|\mathbf{1}_{\{v_k>0\}}\right\|_{L^{p'}_{t,x}}\left\|\mathbf{1}_{\{v_k>0\}}\right\|_{L^\infty_t L^{\frac{2p}{2-p}}_x}\\
&\le CU_{k-1}^{\frac12}\bigg(\|P\|_{L^p_t L^1_x}+\|u\|^2_{L^\infty_t L^2_x}\bigg)2^{\frac{10k}{3p'}}U_{k-1}^{\frac{5}{3p'}}2^{\frac{k(2-p)}{p}}U_{k-1}^{\frac{2-p}{2p}}\\
&\le C2^{\frac{7k}{3}-\frac{4k}{3p}}\bigg(\|P\|_{L^p_t L^1_x}+\|u\|^2_{L^\infty_t L^2_x}\bigg)U_{k-1}^{\frac53-\frac{2}{3p}}.
\end{aligned}$$

Thus, for any $p>1$ and $U_0\le 1$, there exists $\alpha_p>0$ and $\delta_p>1$ such that (11) is bounded by

$$C2^{k\alpha_p}U_{k-1}^{\delta_p}(\|P\|_{L^p_t L^1_x}+\|u\|^2_{L^\infty_t L^2_x})\le C2^{k\alpha_p}U_{k-1}^{\delta_p}(\|P\|_{L^p_t L^1_x}+1). \tag{13}$$

Step 5: Bound of the local pressure term (12) involving P^{loc}_k.
We split the term involving P^{loc}_k into three $P^{\mathrm{loc}}_{k1}+P^{\mathrm{loc}}_{k2}+P^{\mathrm{loc}}_{k3}$, defined by

$$\begin{aligned}
-\Delta P^{\mathrm{loc}}_{k1}&=\sum_{ij}\partial_i\partial_j\bigg\{\phi_k u_j\bigg(1-\frac{v_k}{|u|}\bigg)u_i\bigg(1-\frac{v_k}{|u|}\bigg)\bigg\}\\
-\Delta P^{\mathrm{loc}}_{k2}&=\sum_{ij}\partial_i\partial_j\bigg\{2\phi_k u_j\bigg(1-\frac{v_k}{|u|}\bigg)u_i\frac{v_k}{|u|}\bigg\}\\
-\Delta P^{\mathrm{loc}}_{k3}&=\sum_{ij}\partial_i\partial_j\bigg\{\phi_k u_j\frac{v_k}{|u|}u_i\frac{v_k}{|u|}\bigg\}.
\end{aligned}$$

For all $1<q<\infty$, the Riesz Theorem gives $\left\|P^{\mathrm{loc}}_{k1}\right\|_{L^q(\Gamma_{k-1})}\le C_q$. We also have

$$\mathrm{div}(uP^{\mathrm{loc}}_{k1})+u\bigg(\frac{v_k}{|u|}-1\bigg)\nabla P^{\mathrm{loc}}_{k1}=\mathrm{div}\left(v_k\frac{u}{|u|}P^{\mathrm{loc}}_{k1}\right)-P^{\mathrm{loc}}_{k1}\,\mathrm{div}\left(\frac{uv_k}{|u|}\right). \tag{14}$$

For arbitrary $\frac53<q<\infty$, we check that

$$\begin{aligned}
&\int_{T_{k-1}}^1\bigg|\int\eta_k(x)\bigg\{\mathrm{div}(uP^{\mathrm{loc}}_{k1})+\bigg(\frac{v_k}{|u|}-1\bigg)u\nabla P^{\mathrm{loc}}_{k1}\bigg\}\,\mathrm{d}x\bigg|\,\mathrm{d}t\\
&\le 2^{3k}C_q\|v_k\|_{L^{\frac{10}{3}}}\left\|P^{\mathrm{loc}}_{k1}\right\|_{L^q}\left\|\mathbf{1}_{\{|u|\ge 1-2^{-k}\}}\right\|_{L^{\frac{10q}{7q-10}}}\\
&\quad+C_q\left\|P^{\mathrm{loc}}_{k1}\right\|_{L^q}\|d_k\|_{L^2}\left\|\mathbf{1}_{\{|u|\ge 1-2^{-k}\}}\right\|_{L^{\frac{2q}{q-2}}}\\
&\le C_q2^{k\alpha_q}(U_{k-1}^{\frac{5(q-1)}{3q}}+U_{k-1}^{\frac{4q-5}{3q}}).
\end{aligned}$$

We now turn to the terms involving P^{loc}_{k2} and P^{loc}_{k3}. We apply the Riesz Theorem and Corollary 2.6 again for

$$\begin{aligned}
\left\|P^{\mathrm{loc}}_{k2}\right\|_{L^{\frac{10}{3}}}&\le C\sum_{ij}\left\|u_j(1-\frac{v_k}{|u|})\right\|_{L^\infty}\left\|\frac{v_ku_i}{|u|}\right\|_{L^{\frac{10}{3}}}\le C\|v_k\|_{L^{\frac{10}{3}}}\le CU_{k-1}^{\frac12},\\
\left\|P^{\mathrm{loc}}_{k3}\right\|_{L^{\frac53}}&\le C\sum_{ij}\left\|\frac{u_jv_k}{|u|}\right\|_{L^{\frac{10}{3}}}\left\|\frac{v_ku_i}{|u|}\right\|_{L^{\frac{10}{3}}}\le C\|v_k\|^2_{L^{\frac{10}{3}}}\le CU_{k-1}.
\end{aligned}$$

With a further decomposition of the pressure gradients, we can now bound (12) by

$$\begin{aligned}
&\int_{T_{k-1}}^{1}\left|\int \eta_k(x)\Big\{\operatorname{div}(u(P_{k2}^{\mathrm{loc}}+P_{k3}^{\mathrm{loc}}))+\left(\frac{v_k}{|u|}-1\right)u\nabla(P_{k2}^{\mathrm{loc}}+P_{k3}^{\mathrm{loc}})\Big\}\,\mathrm{d}x\right|\mathrm{d}t\\
&\le\int_{T_{k-1}}^{1}\int_{\beta_{k-\frac{1}{3}}}|\nabla\eta_k||u||P_{k2}^{\mathrm{loc}}+P_{k3}^{\mathrm{loc}}|+\eta_k(|\nabla P_{k2}^{\mathrm{loc}}|+|\nabla P_{k3}^{\mathrm{loc}}|)\,\mathrm{d}x\,\mathrm{d}t\\
&\le C2^{3k}\int_{T_{k-1}}^{1}\int_{\beta_{k-\frac{1}{3}}}(1+v_k)(|P_{k2}^{\mathrm{loc}}|+|P_{k3}^{\mathrm{loc}}|)(|\nabla P_{k2}^{\mathrm{loc}}|+|\nabla P_{k3}^{\mathrm{loc}}|)\,\mathrm{d}x\,\mathrm{d}t\\
&\le C2^{3k}\Big(\left\|\mathbf{1}_{\{|u|\ge 1-2^{-k}\}}\right\|_{L^{\frac{10}{7}}(\beta_{k-1})}\left\|P_{k2}^{\mathrm{loc}}\right\|_{L^{\frac{10}{3}}(\beta_{k-\frac{1}{3}})}\\
&\quad+\left\|\mathbf{1}_{\{|u|\ge 1-2^{-k}\}}\right\|_{L^{\frac{5}{2}}(\beta_{k-1})}\left\|P_{k3}^{\mathrm{loc}}\right\|_{L^{\frac{5}{3}}(\beta_{k-\frac{1}{3}})}\Big)\\
&\le C2^{\alpha_{1k}}U_{k-1}^{\frac{5}{3}}+C2^{\alpha_{2k}}U_{k-1}^{\frac{4}{3}},
\end{aligned}$$

so we have desired inequality for all of the terms.

4. Local epsilon-regularity criterion

For $r > 0$, let B_r represent a ball in $\mathbb{R}^3$ of radius r centered at the origin, and denote $Q_r = (-r^2, r^2) \times B_r$.

Definition 4.1. We define the rescaled solutions (u_r, P_r) by

$$u_r(t,x) = ru(r^2t, rx), \qquad P_r(t,x) = r^2P(r^2t, rx).$$

If (u, P) is a solution to the Navier–Stokes equation in Q_r, then (u_r, P_r) is a solution to the Navier–Stokes equation in Q_1.

We fix a smooth radial cutoff function $\eta \in C_c^\infty(B_1)$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\frac{1}{2}}$, and $\int_{B_1} \eta \,\mathrm{d}x = 1$. Take $\eta_\lambda(x) = \eta(\frac{x}{\lambda})$. We define the weighted average velocity and pressure by

$$\tilde{u}(t) = \int_{B_1} u(t,x)\eta(x)\,\mathrm{d}x, \qquad \tilde{P}(t) = \int_{B_1} P(t,x)\eta(x)\,\mathrm{d}x.$$

$\tilde{u}_r$ and $\tilde{P}_r$ denotes the weighted average of u_r and P_r. For instance, $\tilde{u}_r$ means first rescale, then average:

$$\tilde{u}_r(t) = \int_{B_1} u_r(t,x)\eta(x)\,\mathrm{d}x = \int_{B_1} ru(r^2t, rx)\eta(x)\,\mathrm{d}x.$$

We will also use the heat kernel

$$\psi_\lambda(t,x) = \frac{1}{(2\lambda^2 - t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4(2\lambda^2 - t)}}.$$

This function ψ_λ solves the backwards heat equation in $(-\infty, 2\lambda^2) \times \mathbb{R}^3$

$$\partial_t \psi_\lambda + \Delta \psi_\lambda = 0, \qquad \psi_\lambda(2\lambda^2, x) = \delta_0(x)$$

where δ_0 is the Dirac delta at the origin. Check that ψ_λ has the following properties:

$$|\psi_\lambda(-1,x)| \le 1 \qquad \text{for } x \in \mathbb{R}^3, \tag{15}$$

$$|\psi_\lambda(t,x)| \ge \frac{1}{C\lambda^3} \qquad \text{for } x \in B_\lambda, -\lambda^2 \le t \le \lambda^2, \tag{16}$$

$$|\psi_\lambda| + |\nabla \psi_\lambda| \le C \qquad \text{for } x \in \mathbb{R}^3 \setminus B_{\frac{1}{2}}, -1 \le t \le \lambda^2, \tag{17}$$

$$|\psi_\lambda| \le \frac{C}{\lambda^3}, |\nabla \psi_\lambda| \le \frac{C}{\lambda^4} \qquad \text{for } x \in B_1, -1 \le t \le \lambda^2. \tag{18}$$

Proposition 4.2. *For $p \in (1, \frac{4}{3})$, there exists $\lambda \in (0, \frac{1}{4})$ and $C_\lambda > 1$, such that any suitable weak solution (u, P) in Q_1 verifies*

$$\|u_\lambda\|^2_{L_t^\infty L_x^2(Q_1)} + \frac{1}{\lambda^8}\left\|P_\lambda - \tilde{P}_\lambda\right\|^2_{L_t^p L_x^2(Q_1)} \tag{19}$$

$$\le \frac{1}{8}\left(1 + \frac{1}{\lambda^4}\|\nabla u\|^2_{L^2_{t,x}(Q_1)} + \frac{1}{\lambda^{13}}\|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)}\right)\|u\|^2_{L_t^\infty L_x^2(Q_1)} \tag{20}$$

$$+ \frac{1}{8\lambda^8}\left\|P - \tilde{P}\right\|^2_{L_t^p L_x^2(Q_1)} + \frac{C}{\lambda^2}\|\nabla u\|^2_{L^2_{t,x}(Q_1)} + \frac{C}{\lambda^{11}}\|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)} \tag{21}$$

Proof. To control the velocity, we use the backwards heat kernel ψ_λ in the local energy inequality.

4.1. **Control of velocity.** We multiply (3) by $\eta(x)\psi_\lambda(t,x)$ and integrate on $[-1,s]\times\mathbb{R}^3$ for some $s\in[-1,\lambda^2]$ to find

$$\int_{\mathbb{R}^3}\psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x+\int_{-1}^{\lambda^2}\int_{\mathbb{R}^3}\psi_\lambda(s,x)\eta(x)|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t \tag{22}$$

$$\leq\int_{\mathbb{R}^3}\psi_\lambda(-1,x)\eta(x)\frac{|u(-1,x)|^2}{2}\,\mathrm{d}x \tag{23}$$

$$+\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u\left(\frac{|u|^2}{2}-\frac{|\tilde{u}|^2}{2}\right)\mathrm{d}x\right|\mathrm{d}t \tag{24}$$

$$+\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}(\psi_\lambda\Delta\eta+2\nabla\eta\cdot\nabla\psi_\lambda)\frac{|u|^2(t,x)}{2}\,\mathrm{d}x\right|\mathrm{d}t \tag{25}$$

$$+\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u(P-\tilde{P})\,\mathrm{d}x\right|\mathrm{d}t, \tag{26}$$

where we use the fact that $\operatorname{div}u=0$ and $\tilde{u},\tilde{P}$ are independent of x.

For $-\lambda^2\leq s\leq\lambda^2$, we have

$$\int_{\mathbb{R}^3}\psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x\geq\frac{1}{C\lambda^3}\int_{B_\lambda}\frac{|u(s,x)|^2}{2}\,\mathrm{d}x=\frac{1}{C\lambda^2}\left\|u_\lambda\left(\frac{s}{\lambda^2},\cdot\right)\right\|^2_{L^2(B_1)}.$$

Next, we bound the four terms (23)-(26) on the right hand side. (23) can be controlled using bound (15):

$$\int_{\mathbb{R}^3}\psi_\lambda(-1,x)\eta(x)\frac{|u(-1,x)|^2}{2}\,\mathrm{d}x\leq\frac{1}{2}\|u\|^2_{L^\infty_tL^2_x(Q_1)}.$$

For (24), we apply Sobolev embedding and Hölder's inequality for

$$\begin{aligned}\left\|\frac{|u|^2}{2}-\frac{|\tilde{u}|^2}{2}\right\|_{L^1_tL^2_x(Q_1)}&\leq C\left\|\nabla\frac{|u|^2}{2}\right\|_{L^1_tL^{\frac{3}{2}}_x(Q_1)}\\&\leq C\|u\|_{L^2_tL^6_x(Q_1)}\|\nabla u\|_{L^2_{t,x}(Q_1)}\\&\leq C(\|u\|_{L^\infty_tL^2_x(Q_1)}+\|\nabla u\|_{L^2_{t,x}(Q_1)})\|\nabla u\|_{L^2_{t,x}(Q_1)}\end{aligned}$$

so we have

$$\begin{aligned}&\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u\left(\frac{|u|^2}{2}-\frac{|\tilde{u}|^2}{2}\right)\mathrm{d}x\right|\mathrm{d}t\\&\quad\leq\frac{C}{\lambda^4}\|u\|_{L^\infty_tL^2_x(Q_1)}\left\|\frac{|u|^2}{2}-\frac{|\tilde{u}|^2}{2}\right\|_{L^1_tL^2_x(Q_1)}\\&\quad\leq\frac{C}{\lambda^4}\left(\|u\|^2_{L^\infty_tL^2_x(Q_1)}\|\nabla u\|_{L^2_{t,x}(Q_1)}+\|u\|_{L^\infty_tL^2_x(Q_1)}\|\nabla u\|^2_{L^2_{t,x}(Q_1)}\right)\\&\quad\leq\frac{C}{\lambda^4}\|u\|^2_{L^\infty_tL^2_x(Q_1)}\|\nabla u\|_{L^2_{t,x}(Q_1)}+\frac{C}{\lambda^4}(1+\|u\|^2_{L^\infty_tL^2_x(Q_1)})\|\nabla u\|^2_{L^2_{t,x}(Q_1)},\end{aligned}$$

using the identity that $x\leq 1+x^2$.

For (25), we obtain with (17)

$$\int_{-1}^{\lambda^2}\int_{\mathbb{R}^3}(|\nabla\eta||\nabla\psi_\lambda|+\psi_\lambda|\Delta\eta|)\frac{|u|^2}{2}\,\mathrm{d}x\,\mathrm{d}t\leq C\|u\|^2_{L^\infty_tL^2_x(Q_1)}.$$

The last term (26) is bounded by

$$\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u(P-\tilde{P})\,\mathrm{d}x\right|\mathrm{d}t \le \frac{C}{\lambda^4}\|u\|_{L_t^\infty L_x^2(Q_1)}\left\|P-\tilde{P}\right\|_{L_t^1 L_x^2(Q_1)} \le C\|u\|^2_{L_t^\infty L_x^2(Q_1)} + \frac{C}{\lambda^8}\left\|P-\tilde{P}\right\|^2_{L_t^1 L_x^2(Q_1)}.$$

When combining the above estimates for (23)-(26), we always take the smallest power of λ. We conclude the bounds for the energy dissipation of u_λ by the right-hand side of (28):

$$\|u_\lambda\|^2_{L_t^\infty L_x^2(Q_1)} \le C\lambda^2\left(1+\frac{1}{\lambda^4}\|\nabla u\|^2_{L^2_{t,x}(Q_1)}\right)\|u\|^2_{L_t^\infty L_x^2(Q_1)} + \frac{C}{\lambda^6}\left\|P-\tilde{P}\right\|^2_{L_t^1 L_x^2(Q_1)} + \frac{C}{\lambda^2}\|\nabla u\|^2_{L^2_{t,x}(Q_1)}$$

4.2. **Control of pressure variation.** We use the pressure decomposition introduced in (6). For the local pressure term P^{loc}, recall that η is supported in B_1. We apply the Riesz Theorem for the singular integral operator $(-\Delta)^{-1}\operatorname{div}\operatorname{div}$ and conclude

$$\begin{aligned}\left\|P^{\mathrm{loc}}\right\|_{L_t^p L_x^2(Q_1)} &\le \|\eta(u-\tilde{u})\otimes(u-\tilde{u})\|_{L_t^p L_x^2(Q_1)}\\ &\le C\left\||u-\tilde{u}|^{2-\frac{2}{p}}\right\|_{L_t^\infty L_x^{\frac{p}{p-1}}(Q_1)}\left\||u-\tilde{u}|^{\frac{2}{p}}\right\|_{L_t^p L_x^{\frac{2p}{2-p}}(Q_1)}\\ &\le C\|u-\tilde{u}\|^{2-\frac{2}{p}}_{L_t^\infty L_x^2(Q_1)}\|u-\tilde{u}\|^{\frac{2}{p}}_{L_t^2 L_x^{\frac{4}{2-p}}(Q_1)}\\ &\le C\|u\|^{2-\frac{2}{p}}_{L_t^\infty L_x^2(Q_1)}\|\nabla u\|^{\frac{2}{p}}_{L^2_{t,x}(Q_1)},\end{aligned}$$

by the Sobolev inequality for $p<\frac{4}{3}$.

As for the non-local term P^{har}, the Poincaré inequality in B_λ asserts

$$\int_{B_\lambda}|P^{\mathrm{har}}(x)-(P^{\mathrm{har}})_{B_\lambda}(x)|^2\,\mathrm{d}x \le C\lambda^2\int_{B_\lambda}|\nabla P^{\mathrm{har}}|^2\,\mathrm{d}x.$$

Next, since P^{har} is harmonic in $B_{\frac{1}{2}}$, ∇P^{har} is also harmonic, so we can use the mean-value property to conclude

$$\begin{aligned}\int_{B_\lambda}|\nabla P^{\mathrm{har}}(x)|^2\,\mathrm{d}x &= \int_{B_\lambda}\left|\frac{1}{|B_{\frac{1}{8}}|}\int_{B_{\frac{1}{8}}}\nabla P^{\mathrm{har}}(x+y)\,\mathrm{d}y\right|^2\mathrm{d}x\\ &\le \int_{B_\lambda}\frac{1}{|B_{\frac{1}{8}}|}\int_{B_{\frac{1}{8}}}|\nabla P^{\mathrm{har}}(x+y)|^2\,\mathrm{d}y\,\mathrm{d}x\\ &\le \frac{1}{|B_{\frac{1}{8}}|}\int_{B_\lambda}\int_{B_{\frac{3}{8}}}|\nabla P^{\mathrm{har}}(z)|^2\,\mathrm{d}z\,\mathrm{d}x\\ &\le (8\lambda)^3\int_{B_{\frac{3}{8}}}|\nabla P^{\mathrm{har}}(z)|^2\,\mathrm{d}z.\end{aligned}$$

Finally, since P^{har} is harmonic in $B_{\frac{1}{2}}$, by Caccioppoli inequality we have

$$\int_{B_{\frac{3}{8}}} |\nabla P^{\mathrm{har}}(z)|^2 \,\mathrm{d}z \leq C \int_{B_{\frac{1}{2}}} |P^{\mathrm{har}} - \tilde{P}^{\mathrm{har}}|^2 \,\mathrm{d}x.$$

By concatenating the above three inequalities, we establish a λ^5 decay rate for P^{har}:

$$\begin{aligned}\left\|P^{\mathrm{har}} - (P^{\mathrm{har}})_{B_\lambda}\right\|^2_{L^p_t L^2_x(Q_\lambda)} &\leq C\lambda^5 \left\|P^{\mathrm{har}} - \tilde{P}^{\mathrm{har}}\right\|^2_{L^p_t L^2_x(Q_1)} \\ &\leq C\lambda^5 \left(\left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)} + \left\|P^{\mathrm{loc}} - \tilde{P}^{\mathrm{loc}}\right\|^2_{L^p_t L^2_x(Q_1)}\right).\end{aligned}$$

Together with the estimate on P^{loc} we conclude

$$\begin{aligned}\|P - (P)_{B_\lambda}\|^2_{L^p_t L^2_x(Q_\lambda)} &\leq C\lambda^5 \left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)} + C \left\|P^{\mathrm{loc}} - \tilde{P}^{\mathrm{loc}}\right\|^2_{L^p_t L^2_x(Q_1)} \\ &\leq C\lambda^5 \left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)} + C \|u\|^{2\left(2-\frac{2}{p}\right)}_{L^\infty_t L^2_x(Q_1)} \|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)} \\ &\leq C\lambda^5 \left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)} + C(1 + \|u\|^2_{L^\infty_t L^2_x(Q_1)}) \|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)},\end{aligned}$$

since $p < \frac{4}{3}$ guarantees $2 - \frac{2}{p} \leq \frac{1}{2} < 1$ with $x^{2-\frac{2}{p}} < 1 + x$.

The pressure variation scales in the following way

$$\begin{aligned}\left\|P_\lambda - \tilde{P}_\lambda\right\|^2_{L^p_t L^2_x(Q_1)} &= \left[\int_{-1}^1 \left(\int_{B_1} \left|P_\lambda(t,x) - \int_{B_1} P_\lambda(t,y)\eta(y)\,\mathrm{d}y\right|^2 \mathrm{d}x\right)^{\frac{p}{2}} \mathrm{d}t\right]^{\frac{2}{p}} \\ &= \lambda^4 \left[\int_{-1}^1 \left(\int_{B_1} \left|P(\lambda^2 t, \lambda x) - \int_{B_1} P(\lambda^2 t, \lambda y)\eta(y)\,\mathrm{d}y\right|^2 \mathrm{d}x\right)^{\frac{p}{2}} \mathrm{d}t\right]^{\frac{2}{p}} \\ &= \lambda^{4-\frac{4}{p}-3} \left[\int_{-\lambda^2}^{\lambda^2} \left(\int_{B_\lambda} |P(s,z) - (P)_{B_\lambda}(s)|^2 \,\mathrm{d}z\right)^{\frac{p}{2}} \mathrm{d}s\right]^{\frac{2}{p}} \\ &= \lambda^{1-\frac{4}{p}} \|P - (P)_{B_\lambda}\|^2_{L^p_t L^2_x(Q_\lambda)} \\ &\leq C\lambda^{6-\frac{4}{p}} \left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)} + C\lambda^{1-\frac{4}{p}}(1 + \|u\|^2_{L^\infty_t L^2_x(Q_1)}) \|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)}\end{aligned}$$

thus

$$\begin{aligned}&\frac{1}{\lambda^8} \left\|P_\lambda - \tilde{P}_\lambda\right\|^2_{L^p_t L^2_x(Q_1)} \\ &\leq C\lambda^{6-\frac{4}{p}} \left(\frac{1}{\lambda^8} \left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)}\right) + \frac{C}{\lambda^{7+\frac{4}{p}}}(1 + \|u\|^2_{L^\infty_t L^2_x(Q_1)}) \|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)}\end{aligned}$$

Together with the control of the velocity terms, we have shown that

$$\begin{aligned}
&\|u_\lambda\|^2_{L^\infty_t L^2_x(Q_1)} + \frac{1}{\lambda^8}\left\|P_\lambda - \tilde{P}_\lambda\right\|^2_{L^p_t L^2_x(Q_1)} \\
&\quad \le C\lambda^2\left(1 + \frac{1}{\lambda^4}\|\nabla u\|^2_{L^2_t L^2_x(Q_1)} + \frac{1}{\lambda^{13}}\|\nabla u\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)}\right)\|u\|^2_{L^\infty_t L^2_x(Q_1)} \\
&\quad + \frac{C}{\lambda^6}\left\|P - \tilde{P}\right\|^2_{L^p_t L^2_x(Q_1)} + \frac{C}{\lambda^2}\|\nabla u\|^2_{L^2_t L^2_x(Q_1)} + \frac{C}{\lambda^{11}}\|\nabla u\|^{\frac{4}{p}}_{L^2(Q_1)},
\end{aligned}$$

where we pick the worst i.e. smallest exponents of λ.

The proof is completed by choosing λ small such that $C\lambda^2 \le \frac{1}{8}$. □

Corollary 4.3. *There exists a universal constant $\varepsilon_2 > 0$, such that if near $(t_0, x_0) \in (0,\infty)\times\Omega$ the following holds*

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r(t_0,x_0)}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t < \varepsilon_2, \tag{27}$$

then (t_0, x_0) is a regular point.

Proof. We have fixed λ in the proof of Proposition 4.2. Define

$$U(r) := \|u_r\|^2_{L^\infty_t L^2_x(Q_1)} + \frac{1}{\lambda^8}\left\|P_r - \tilde{P}_r\right\|^2_{L^p_t L^2_x(Q_1)}.$$

By our assumption, for all sufficiently small r, we control $\|\nabla u_r\|_{L^2(Q_1)}$ such that

$$\left(1 + \frac{1}{\lambda^4}\|\nabla u_r\|^2_{L^2_{t,x}(Q_1)} + \frac{1}{\lambda^{13}}\|\nabla u_r\|^{\frac{4}{p}}_{L^2_{t,x}(Q_1)}\right) \le 2.$$

Without loss of generality, we take u such that

$$\frac{C}{\lambda^2}\|\nabla u\|^2_{L^2_{t,x}(Q_1)} + \frac{C}{\lambda^{11}}\|\nabla u\|^4_{L^2(Q_1)} \le \frac{C^*}{8}.$$

Then by applying Proposition 4.2 to u_r, we can find some $\lambda \in (0, \frac{1}{4})$ and obtain the recursive relation

$$U(\lambda r) \le \frac{1}{4}U(r) + \frac{C^*}{8}.$$

Therefore, we get

$$\limsup_{k\to+\infty} U(\lambda^k r) \le \frac{C^*}{2},$$

so there exists some r_0 such that $U(r_0) \le C^*$. But we can now apply Theorem 1.1 to u_{r_0}. □

5. Hausdorff dimension of singular set

Definition 5.1 (Hausdorff measure)**.** Recall that for a Borel set E, the 1-dimensional Hausdorff measure of E is given by

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E),$$

where

$$\mathcal{H}^\alpha_\delta(E) = \inf\left\{ \sum_k (\mathrm{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^\infty F_k, \quad \mathrm{diam} F_K \leq \delta \right\}.$$

Lemma 5.2 (Vitali Covering)**.** *Let F be an arbitrary collection of balls B such that $\sup \mathrm{diam}(B) < \infty$. Then there exists a countable subcollection $G \subset F$ such that the balls are pairwise disjoint and*

$$\bigcup_{B \in F} B \subset \bigcup_{C \in G} 5C.$$

Proposition 5.3. *Let u be a suitable weak solution of the NSE and $\mathrm{Sing}(u)$ be its set of singular points. Then Corollary 4.3 implies $\mathcal{H}^1(\mathrm{Sing}(u)) = 0$.*

Proof. Let arbitrarily small $\delta > 0$ be given. For any $(t, x) \in \mathrm{Sing}(u)$, we have

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q_r(t,x)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \geq \epsilon_2.$$

So we can pick $0 < 5r < \delta$ such that

$$\frac{1}{r} \int_{Q_{r(t,x)}} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \geq \frac{\epsilon_2}{2},$$

where $\{Q_r(t, x)\}$ is a cover of $\mathrm{Sing}(u)$.

By Lemma 5.2, we can pick a countable subcover at the cost of expanding the cylinders by a factor of five. Then

$$\begin{aligned} \mathcal{H}^1_\delta(\mathrm{Sing}(u)) &\leq \sum_i 5 r_i \\ &\leq \frac{5}{\epsilon_2} \sum_i \int_{Q_{r_i}(t_i, x_i)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

But suitable weak solutions satisfy

$$\int_{(0,\infty) \times \mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t < \infty,$$

so

$$\begin{aligned} \mathcal{H}^1_\delta(\mathrm{Sing}(u)) &\leq C \sum_i |Q_{r_i}(t_i, x_i)| \\ &\leq C \sum_i r_i^5 \leq \delta^4 \sum_i r_i \to 0 \end{aligned}$$

by our choice of δ. □

6. Reduction in exponent to $\frac{5}{3}$

This iteration is similar to that in Section 4, but we include the norms $\|u\|_{L^3}$ and $\|P\|_{L^{\frac{3}{2}}}$ as well.

Proposition 6.1. *There exists $\lambda \in (0, \frac{1}{4})$ and $C_\lambda > 1$, such that any suitable weak solution (u, P) in Q_1 verifies*

$$\begin{aligned} &\|u_\lambda\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u_\lambda\|^2_{L^2(Q_1)} + \frac{1}{\lambda^2}\left(\|u_\lambda\|^3_{L^3(Q_1)} + \|P_\lambda\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right) \\ &\quad \le \frac{1}{4}\left(\|u\|^2_{L^\infty_t L^2_x(Q_1)} + \frac{1}{\lambda^2}\left(\|u\|^3_{L^3(Q_1)} + \|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right)\right) \\ &\qquad + \left(\|u\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u\|^2_{L^2_{t,x}(Q_1)}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}(Q_1)}. \end{aligned} \tag{28}$$

Proof. We will first control the kinetic energy and energy dissipation terms using the local energy inequality, then split the cubic term into the average and oscillation.

6.1. **Control of the quadratic velocity terms.** We recall (22) - (26).

We first show that the left-hand side (22) controls the energy of u_λ thanks to (16). For $-\lambda^2 \le s \le \lambda^2$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x &\ge \frac{1}{C\lambda^3}\int_{B_\lambda}\frac{|u(s,x)|^2}{2}\,\mathrm{d}x \\ &= \frac{1}{C\lambda^2}\left\|u_\lambda\left(\frac{s}{\lambda^2}\right)\right\|^2_{L^2(B_1)}, \\ \int_{-1}^{\lambda^2}\int_{\mathbb{R}^3}\psi_\lambda(s,x)\eta(x)|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t &\ge \frac{1}{C\lambda^3}\int_{-\lambda^2}^{\lambda^2}\int_{B_\lambda}|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{C\lambda^2}\|\nabla u_\lambda\|^2_{L^2(Q_1)}. \end{aligned}$$

Taking the supremum for $s \in [-\lambda^2, \lambda^2]$ yields

$$\begin{aligned} &\|u_\lambda\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u_\lambda\|^2_{L^2(Q_1)} \\ &\le C\lambda^2\left(\int_{\mathbb{R}^3}\psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x + \int_{-1}^{s}\int_{\mathbb{R}^3}\psi_\lambda(s,x)\eta(x)|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned}$$

Next, we bound the four terms (23)-(26) on the right hand side. (23) can be controlled using bound (15):

$$\int_{\mathbb{R}^3}\psi_\lambda(-1,x)\eta(x)\frac{|u(-1,x)|^2}{2}\,\mathrm{d}x \le \frac{1}{2}\|u\|^2_{L^\infty_t L^2_x(Q_1)}.$$

For (24), we apply (18) for

$$\begin{aligned} &\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u\left(\frac{|u|^2}{2} - \frac{|\tilde{u}|^2}{2}\right)\mathrm{d}x\right|\mathrm{d}t \\ &\quad \le \frac{C}{\lambda^4}\int_{-1}^{\lambda^2}\int_{B_1}|u||u-\tilde{u}||u+\tilde{u}|\,\mathrm{d}x\,\mathrm{d}t \le \frac{C}{\lambda^4}\|u\|^3_{L^3(Q_1)}. \end{aligned}$$

For (25), we obtain with (17)

$$\int_{-1}^{\lambda^2}\int_{\mathbb{R}^3}\left(|\nabla\eta||\nabla\psi_\lambda|+\psi_\lambda|\Delta\eta|\right)\frac{|u|^2}{2}\,\mathrm{d}x\,\mathrm{d}t\le C\left\|u\right\|_{L_t^\infty L_x^2(Q_1)}^2.$$

The last term (26) is bounded by

$$\begin{aligned}\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u(P-\tilde{P})\,\mathrm{d}x\right|\mathrm{d}t&\le\frac{C}{\lambda^4}\left\|u\right\|_{L^3(Q_1)}\left\|P\right\|_{L^{\frac{3}{2}}(Q_1)}\\&\le\frac{C}{\lambda^4}\left(\left\|u\right\|_{L^3(Q_1)}^3+\left\|P\right\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}\right).\end{aligned}$$

When combining the above estimates for (23)-(26), we always take the smallest power of λ. We conclude the bounds for the kinetic energy and energy dissipation of u_λ by the right-hand side of (28):

$$\begin{aligned}&\left\|u_\lambda\right\|_{L_t^\infty L_x^2(Q_1)}^2+\left\|\nabla u_\lambda\right\|_{L^2(Q_1)}^2\\&\quad\le C\left(\lambda^2\left\|u\right\|_{L_t^\infty L_x^2(Q_1)}^2+\frac{1}{\lambda^2}\left(\left\|u\right\|_{L^3(Q_1)}^3+\left\|P\right\|_{L^3(Q_1)}^{\frac{3}{2}}\right)\right).\end{aligned}$$

6.2. **Control of the cubic velocity term.** Here, we apply an argument from [GKT07, Lemmas 3.2, 3.3].

Lemma 6.2. *Suppose* $\lambda\in(0,\frac{1}{4})$. *Then*

$$\int_{-1}^{\lambda^2}\int_{B_\lambda}|u|^3\,\mathrm{d}x\,\mathrm{d}t\le C\left(\lambda^3\left\|u\right\|_{L^3(Q_1)}^3+\left(\left\|u\right\|_{L_t^\infty L_x^2(Q_1)}^2+\left\|\nabla u\right\|_{L^2(Q_1)}^2\right)\left\|\nabla u\right\|_{L^{\frac{5}{3}}(Q_1)}\right).$$

Proof. We split $|u|\le|\tilde{u}|+|u-\tilde{u}|$. We consider the scaling of the average

$$\int_{B_\lambda}|\tilde{u}(\cdot)|^3\,\mathrm{d}x\le\int_{B_\lambda}\left(\frac{1}{|B_1|}\int_{B_1}|u(\cdot,x')|\,\mathrm{d}x'\right)^3\mathrm{d}x\le C\lambda^3\left(\int_{B_1}|u(\cdot,x)|\,\mathrm{d}x\right)^3,$$

so we integrate in time to get

$$\int_{-1}^{\lambda^2}\int_{B_\lambda}|\tilde{u}|^3\,\mathrm{d}x\,\mathrm{d}t\le C\lambda^3\left\|u\right\|_{L^3(Q_1)}.$$

For the velocity oscillation, we can interpolate for

$$\begin{aligned}\left\|u-\tilde{u}\right\|_{L_{t,x}^3}^3&\le\left\|u-\tilde{u}\right\|_{L_t^\infty L_x^2}^{\frac{6}{5}}\left\|u-\tilde{u}\right\|_{L_t^2L_x^6}^{\frac{4}{5}}\left\|u-\tilde{u}\right\|_{L_t^{\frac{5}{3}}L_x^{\frac{15}{4}}}\\&\le\left\|u\right\|_{L_t^\infty L_x^2}^{\frac{6}{5}}\left\|\nabla u\right\|_{L_{t,x}^2}^{\frac{4}{5}}\left\|\nabla u\right\|_{L_{t,x}^{\frac{5}{3}}},\end{aligned}$$

whichis bounded above by $C\left(\left\|u\right\|_{L_t^\infty L_x^2}^2+\left\|\nabla u\right\|_{L_{t,x}^2}^2\right)\left\|\nabla u\right\|_{L_{t,x}^{\frac{5}{3}}}$. □

We apply Lemma 6.2 to get

$$\begin{aligned}\|u_\lambda\|^3_{L^3(Q_1)} &= \int_{-1}^{1}\int_{B_1} \left|\lambda u(\lambda^2 t, \lambda x)\right|^3 \mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\lambda^2}\int_{-\lambda^2}^{\lambda^2}\int_{B_\lambda} \left(|\tilde{u}|^3 + |u-\tilde{u}|^3\right)\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\lambda^2}\left(\lambda^3\|u\|^3_{L^3(Q_1)} + \left(\|u\|^2_{L^\infty_t L^2_x} + \|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}\right) \\ &\leq C\lambda\|u\|^3_{L^3(Q_1)} + \frac{C}{\lambda^2}\left(\|u\|^2_{L^\infty_t L^2_x} + \|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}.\end{aligned}$$

6.3. **Control of the pressure term.** For the local pressure term P^{loc}, recall that η is supported in B_1. We apply the Riesz Theorem for the singular integral operator $(-\Delta)^{-1}\,\mathrm{div}\,\mathrm{div}$ and conclude

$$\begin{aligned}\left\|P^{\mathrm{loc}}\right\|_{L^{\frac{3}{2}}(Q_1)} &\leq \|\eta(u-\tilde{u})\otimes(u-\tilde{u})\|_{L^{\frac{3}{2}}(Q_1)} \\ &\leq C\|u-\tilde{u}\|^2_{L^3(Q_1)}.\end{aligned}$$

As for the non-local term P^{har}, we apply the mean value property of harmonic functions for

$$\begin{aligned}\int_{B_\lambda}|P^{\mathrm{har}}(x)|^{\frac{3}{2}}\,\mathrm{d}x &= \int_{B_\lambda}\left|\frac{1}{|B_{\frac{1}{8}}|}\int_{B_{\frac{1}{8}}}|P^{\mathrm{har}}(x+y)|\,\mathrm{d}y\right|^{\frac{3}{2}}\mathrm{d}x \\ &\leq \int_{B_\lambda}\frac{1}{|B_{\frac{1}{8}}|}\int_{B_{\frac{1}{8}}}|P^{\mathrm{har}}(x+y)|^{\frac{3}{2}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \frac{1}{|B_{\frac{1}{8}}|}\int_{B_\lambda}\int_{B_{\frac{3}{8}}}|P^{\mathrm{har}}(z)|^{\frac{3}{2}}\,\mathrm{d}z\,\mathrm{d}x \\ &\leq (8\lambda)^3\int_{B_{\frac{3}{8}}}|P^{\mathrm{har}}(z)|^{\frac{3}{2}}\,\mathrm{d}z.\end{aligned}$$

Then we apply the triangle inequality at the larger scale for

$$\left\|P^{\mathrm{har}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_\lambda)} \leq C\lambda^3\left\|P^{\mathrm{har}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} \leq C\lambda^3\left(\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} + \left\|P^{\mathrm{loc}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right).$$

Together with the estimate on P^{loc} we conclude

$$\begin{aligned}\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_\lambda)} &\leq C\lambda^3\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} + C\left\|P^{\mathrm{loc}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} \\ &\leq C\lambda^3\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} + C\|u-\tilde{u}\|^3_{L^3(Q_1)} \\ &\leq C\lambda^3\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} + C\left(\|u\|^2_{L^\infty_t L^2_x} + \|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}.\end{aligned}$$

The pressure term scales in the following way

$$\begin{aligned}\|P_\lambda\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} &= \frac{1}{\lambda^2}\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_\lambda)} \\ &\leq C\lambda\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} + \frac{C}{\lambda^2}\left(\|u\|^2_{L^\infty_t L^2_x} + \|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}.\end{aligned}$$

Together with the control of the velocity terms, we have shown that

$$\begin{aligned}(29)\qquad &\|u_\lambda\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u_\lambda\|^2_{L^2(Q_1)} + \frac{1}{\lambda^2}\left(\|u_\lambda\|^3_{L^3(Q_1)} + \|P_\lambda\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right)\\ &\leq C\lambda\Bigg(\|u\|^2_{L^\infty_t L^2_x(Q_1)} + \frac{1}{\lambda^2}\left(\|u\|^3_{L^3(Q_1)} + \|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right)\Bigg)\\ &\quad + \frac{C}{\lambda^4}\left(\|u\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u\|^2_{L^2_{t,x}(Q_1)}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}(Q_1)}.\end{aligned}$$

The proof is completed by choosing λ small such that $C\lambda \leq \frac{1}{4}$. □

Corollary 6.3. *For every $\varepsilon_2 > 0$, there exists $\varepsilon_{\frac{5}{3}} > 0$, such that if*

$$\limsup_{r\to 0} \|\nabla u_r\|_{L^{\frac{5}{3}}(Q_1)} < \varepsilon_{\frac{5}{3}},$$

then

$$\limsup_{r\to 0} \|\nabla u_r\|_{L^2(Q_1)} < \varepsilon_2.$$

Proof. Define

$$U(r) := \|u_r\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u_r\|^2_{L^2(Q_1)} + \frac{1}{\lambda^2}\left(\|u_r\|^3_{L^3(Q_1)} + \|P_r\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right).$$

Then by applying Proposition 6.1 to u_r, we can find some $\lambda \in (0, \frac{1}{4})$ and obtain the recursive relation

$$U(\lambda r) \leq \frac{1}{4}U(r) + C_\lambda\left(\|u_r\|^2_{L^\infty_t L^2_x} + \|\nabla u_r\|^2_{L^2_{t,x}}\right)\|\nabla u_r\|_{L^{\frac{5}{3}}_{t,x}}.$$

We drop the dependence of C_λ in λ because λ has been fixed.

By our assumption, there exists $r_0 > 0$ such that for all $r < r_0$ we have

$$\|\nabla u_r\|_{L^{\frac{9}{5}}(Q_1)} \leq \varepsilon_{\frac{5}{3}},$$

where the value of $\varepsilon_{\frac{5}{3}}$ will be determined later. Therefore, for all $r \leq r_0$ we have

$$U(\lambda r) \leq \frac{1}{4}U(r) + CU(r)\varepsilon_{\frac{5}{3}} \leq \frac{1}{2}U(r)$$

provided $\varepsilon_{\frac{5}{3}}$ is sufficiently small. Thus we conclude that

$$\limsup_{r\to 0} \|\nabla u_r\|^2_{L^2(Q_1)} \leq \limsup_{r\to 0} U(r) = 0.$$

□

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