APPLICATIONS OF THE POINT-TO-SET PRINCIPLE TO PROJECTION PROBLEMS

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ABSTRACT. This paper surveys methods from computable analysis that have recently been used to solve problems in geometric measure theory. We begin with an introduction to Kolmogorov complexity, optimal oracles, and the point-to-set principle, which relates the Hausdorff dimension of a set $E \subseteq \mathbb{R}^n$ to the complexity of points $x \in E$. We then showcase new proofs of various results regarding the dimension of projected sets, including the Kaufman bounds for exceptional sets of orthogonal projections.

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1. INTRODUCTION

A common goal in mathematics is to formalize intuitive notions of complexity. A familiar example is *dimension*, which measures the complexity of mathematical spaces in terms of the *degrees of freedom* of points that move through the space. Another such formalization of complexity is Hausdorff dimension, which is used to measure the complexity of fractals and other pathological sets in Euclidean space. The study of Hausdorff dimension is one of the primary focuses of geometric measure theory. While the problems discussed in this paper are motivated by geometric measure theory, we approach them through a seemingly unrelated perspective on complexity: computability theory in the form of *Kolmogorov complexity*. To measure the complexity of an object X in this context, we give a Turing machine – a rigorous model of a computer – a list of instructions to compute (output) X and consider the shortest list of instructions that is required. Objects that require more instructions to compute are regarded as more complex. We often interpret the list of instructions as information, and say that more complex objects require more information to compute. A recent breakthrough of J. Lutz and N. Lutz relates the computability-theoretic notion of complexity to the Hausdorff dimension of sets in

Euclidean space. Dubbed the *point-to-set principle*, the theorem has given rise to exciting new methods in geometric measure theory. We introduce these methods and show how they are applied to projection problems, from simple exercises and toy problems to current research.

2. Kolmogorov Complexity

In this section, we give a brief introduction to Kolmogorov complexity and construct the upper and lower algorithmic dimensions of points in $\mathbb{R}^n$. We then state the point-to-set principle, proved in 2016 by J. Lutz and N. Lutz [2]. A more thorough introduction to Kolmogorov complexity can be found in [1]. An introduction to the basics of computability theory can be found in [7].

We denote the set of finite binary strings as $\{0,1\}^*$. For our purposes, a Turing machine is an algorithm which takes a finite input string and performs a predetermined (possibly infinite) sequence of operations. If M is a Turing machine and $x \in \{0,1\}^*$, we write $M(x)$ for the output of M given input x. A universal Turing machine is one which can simulate any other Turing machine by taking a second finite string called a *program*, analogous to a programming language. If U is a universal Turing machine, $\pi \in \{0,1\}^*$ a program and $x \in \{0,1\}^*$, we write its output as $U(\pi, x)$. If $\tilde{M}$ is the Turing machine defined by the program π, $U(\pi, x)$ is equivalent to $\tilde{M}(x)$.

Definition 2.1. The *conditional Kolmogorov complexity* of a finite binary string $x \in \{0,1\}^*$ given a finite binary string $y \in \{0,1\}^*$ is the length of the shortest program $\pi \in \{0,1\}^*$ that returns x on the input y. Formally, the conditional Kolmogorov complexity of x given y is

$$K(x \mid y) = \min_{\pi \in \{0,1\}^*} \{l(\pi) : U(\pi, y) = x\}$$

where U is a fixed universal Turing machine and $l(\pi)$ is the length of the program π. Any such π which achieves this minimum is called a *witness* and said to *testify to* the value $K(x \mid y)$. The *Kolmogorov Complexity* of $x \in \{0,1\}^*$ is defined to be $K(x \mid \epsilon)$, where ϵ is the empty string (later, ϵ will be a small real number. Whether ϵ denotes the empty string or a small real number should be clear from the context). The notation $K(x, y)$ is used for $K(\langle x, y \rangle)$, where $\langle x, y \rangle$ denotes the standard bijecive map from $\{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$.

The Kolmogorov complexity of $x \in \{0,1\}^*$ can be thought of as the minimum amount of information required to compute x. The Kolmogorov complexity of x given y is the minimum amount of information required to compute x, having been given the information of y. In this sense, it is easy to see that $K(x \mid y) \leq K(x) + O(1)$ for any $y \in \{0,1\}^*$. There are many such cases where a technical error term $O(1)$ is required due to the nature of working with Turing machines; in particular, it describes the difference in length between the descriptions of two different Turing machines. These constants can often be ignored when thinking in terms of information.

The above definition would seem to depend on the choice of Turing machine used to compute x. However, there are *optimal* Turing machines which minimize the complexity for every string up to a constant error term. We take U in the above definition to be such an optimal Turing machine.

Example 2.2. Let $x \in \{0,1\}^* = 1010\ldots10$, where 10 is repeated n times. Then $K(x) \leq O(\log n)$. To see this, we can define a Turing machine M which takes a string π and repeats it n times on input $\langle \pi, n\rangle$. Let c_M be the constant description of M, that is, the program that describes M which is independent of π and n. Then, given input $\langle c_M, 10, n\rangle$, the fixed universal Turing machine U simulates M on input $\langle 10, n\rangle$ and outputs x. Therefore, $K(x) \leq l(\langle c_M, 10, n\rangle) \leq \log(n)+2l(10)+2l(c_M) \leq O(\log n) + 4 + O(1) = O(\log n)$.

From the information-theoretic perspective, having information of the string 10 and the number of times it is repeated, n, is sufficient to compute x. Hence, the Kolmogorov complexity of x is less than the length of the information needed to specify 10 and n (plus some fixed constants).

Remark 2.3. Technical details on how to compute the length of the string $\langle 10, n\rangle$ can be found in [2], [7]. However, it is sufficient to know the following:

- The length of the information required to specify a string of length n is at most n, since it is a program for itself given a Turing machine which simply outputs its input.
- The length of the information required to specify the number n is on the order of $\log n$.

The concept of Kolmogorov can be naturally extended to other finite data objects via the standard binary encoding. For example, it is valid to talk about the Kolmogorov complexity of vectors $q \in \mathbb{Q}^n$. Similarly, we can define the Kolmogorov complexity at precision r to further extend the concept to points $x \in \mathbb{R}^n$ using the standard infinite binary expansion of x.

Definition 2.4. The *Kolmogorov complexity at precision* r of $x \in \mathbb{R}^n$ is the length of the shortest program π which outputs a rational number contained in a ball of radius 2^{-r} around x. Formally, this is given by

$$K_r(x) = \min_{\pi \in \{0,1\}^*} \{l(\pi) : U(\pi) \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\}.$$

The *conditional Kolmogorov complexity at precision* r of $x \in \mathbb{R}^n$ given $y \in \mathbb{R}^m$ at precision s is

$$K_{r,s}(x \mid y) = \max\{\min\{K_r(p \mid q) : p \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\} : q \in B_{2^{-s}}(y) \cap \mathbb{Q}^m\}.$$

We abbreviate $K_{r,r}(x \mid y)$ to $K_r(x \mid y)$.

One useful result about Kolmogorov complexity is the *symmetry of information*. In words, it says that finding the Kolmogorov complexity of strings x and y is approximately the same as first finding the Kolmogorov complexity of x, and then finding the Kolmogorov complexity of y given the information of x and $K(x)$ obtained from finding $K(x)$.

Proposition 2.5. (Symmetry of information) *Let* $x, y \in \{0,1\}^*$. *Then*

$$K(x, y) = K(x) + K(y \mid x, K(x)) + O(1).$$

A slightly weaker form of symmetry of information holds in $\mathbb{R}^n$.

Proposition 2.6. ([3])*Let* $x, y \in \mathbb{R}^n$ *and* $r \in \mathbb{N}$. *Then*

$$|K_r(x \mid y) - K_r(y) - K_r(x, y)| \leq O(\log r) + O(\log\log \|y\|).$$

The proofs of Propositions 2.4 and 2.5 are rather technical, and thus omitted. Next, we define the algorithmic dimension of points in Euclidean space, which characterizes the regularity of their infinite binary expansions.

Definition 2.7. The *lower* and *upper algorithmic dimension* of $x \in \mathbb{R}^n$ are given by

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}, \text{ and}$$
$$\mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}.$$

Note that we have the trivial estimates $0 \leq \dim(x) \leq \mathrm{Dim}(x) \leq 1$ for all infinite *binary* strings x, and $0 \leq \dim(x) \leq \mathrm{Dim}(x) \leq c$ for all $x \in \mathbb{R}^n$, where c is an explicit constant depending only on n. To see this, note that the first r digits of x is a program for itself (on the machine that outputs its input), and if $x \in \mathbb{R}^n$, then the first r digits of x can be encoded in $n \cdot r \log_2 10$ binary bits.

The last step in developing the machinery required to formulate the point-to-set principle is relativizing to an arbitrary *oracle* set $A \subseteq \mathbb{N}$. An oracle can be thought of as countable information, and an oracle Turing machine is a Turing machine M with "access" to this information. A formal definition of oracle machines can be found in [7].

Definition 2.8. (Informal) An *oracle Turing machine* M^A is a Turing machine M and countable set $A \subseteq \{0,1\}^*$ called an *oracle*, where M is able to access and use the information stored in A. Slightly more formally, the machine M^A operates as M with the added ability to "query" the oracle in the following sense: at computation step, M^A is allowed to ask if the current contents of M^A is contained in A. We often write $A \subseteq \mathbb{N}$, and assume that A is encoded in binary.

If we take the fixed universal Turing machine in definitions 2.1, 2.4 and 2.7 to be a universal oracle machine, we can adjust the definitions relative to an arbitrary oracle set $A \subseteq \mathbb{N}$. Thus, $K^A(x \mid y)$, $K^A_r(x)$, $K^A_{r,s}(x \mid y)$, $\dim^A(x)$, and $\mathrm{Dim}^A(x)$ are well defined.

Definition 2.9. The conditional Kolmogorov complexity of x given y *relative to the oracle* $A \subseteq \mathbb{N}$ is

$$K^A(x \mid y) = \min_{\pi\in\{0,1\}^*} \{l(\pi) : U^A(\pi, y) = x\},$$

where U^A denotes the universal Turing machine U with access to the oracle A. The definitions of $K^A_r(x)$, $K^A_{r,s}(x \mid y)$ follow similarly from Definition 2.4.

The lower and upper algorithmic dimension of $x \in \mathbb{R}^n$ *relative to the oracle* $A \subseteq \mathbb{N}$ are given by

$$\dim^A(x) = \liminf_{r\to\infty} \frac{K^A_r(x)}{r}, \text{ and}$$
$$\mathrm{Dim}^A(x) = \limsup_{r\to\infty} \frac{K^A_r(x)}{r}.$$

The following examples illustrates the importance these relativized definitions.

Example 2.10. Let $x = x_1x_2\ldots x_n \in \{0,1\}^*$ and $A = \{x\} \subseteq \mathbb{N}$. Then $K^A(x) \leq c_M$, where c_M is a constant independent of n. To see this, we can define an oracle

Turing machine M^A which does the following on the input ϵ: M^A enumerates over all strings in lexicographic order, and for each string y, M^A asks if $y \in A$. Once the answer is yes, M^A finishes its computation. Of course, the answer is only yes if $y = x$, thus $K^A(x) \leq l(\epsilon) + c_M = c_M$, where c_M is the constant description of M^A.

Example 2.11. Let $x \in \{0,1\}^{\mathbb{N}}$ now be a random infinite binary string, in the sense that $\dim(x) = 1$. There are many other characterizations of "random" which result in $\dim(x) = 1$, however for simplicity we take this to be the definition. let $A = \{x \upharpoonright r : r \in \mathbb{N}\}$, where $x \upharpoonright r$ denotes the truncation of the string x to its first r digits. We define the oracle Turing machine M^A which on input ϵ enumerates over all strings in lexicographic order, and for each string y, M^A asks if $y \in A$ (as in the previous example). If the answer is yes and $l(y) = r$, then M^A outputs y. Again, we have that $y = x$, and thus $K^A_r(x) \leq l(\epsilon) + c_M$ for all $r \in \mathbb{N}$, so $\dim^A(x) = \liminf_{r\to\infty} \frac{K^A_r(x)}{r} \leq \liminf_{r\to\infty} \frac{c_M}{r} = 0$.

The process of enumerating all strings up to some length n in these examples takes exponential time, however we emphasize that we are interested in the minimal length over programs, rather than the time complexity of these programs. To further illustrate the power of oracles in this setting, consider the following variation of Example 2.11: Since any infinite string x is countable, we could have taken $X = \{x_1, \dots\}$ to be a countable set of random infinite strings, and defined $A' = \{x_i \upharpoonright r : i, r \in \mathbb{N}\}$. Thus the oracle A' reduces the algorithmic dimension of countably many random strings to 0.

Before concluding this section, we make the following remark. Many of the bounds on the Kolmogorov complexity of a point $x \in \mathbb{R}^n$ proved in this paper can also be relativized to an oracle set, and we make use of this fact in the proofs of some of the main theorems. In particular, oracles can only reduce complexity, thus if $K(x)$ is bounded above by some expression f, then so is $K^A(x)$ for any oracle A.

3. Fractal Dimension

For the sake of self-containment, we review the definitions of Hausdorff and packing dimensions for sets in Euclidean space. A thorough introduction to the subject can be found in [5].

Definition 3.1. Let $E \subseteq \mathbb{R}^n$ and $s \geq 0$. Let $d(E)$ denote the diameter of E. For $\delta > 0$, define

$$H^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} d(E_i)^s \,:\, E \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } d(E_i) \leq \delta\right\}.$$

We define the *s-dimensional Hausdorff measure* to be

$$H^s(E) = \lim_{\delta\to 0} H^s_\delta(E).$$

Definition 3.2. The *Hausdorff dimension* of $E \subseteq \mathbb{R}^n$ is

$$\dim_H(E) = \inf\{s \,:\, H^s(E) < \infty\} = \sup\{s \,:\, H^s(E) > 0\}.$$

Definition 3.3. Let $E \subseteq \mathbb{R}^n$ and $s \geq 0$. For $\delta > 0$, write

$$P^s_\delta(E) = \sup \sum_{i=1}^{\infty} d(B_i)^s$$

where the supremum is taken over all disjoint families of closed balls $\{B_1, B_2, \dots\}$ such that $d(B_i) \leq \delta$ and the centers of the B_i's are in E. Set

$$P^s(E) = \lim_{\delta \to 0} P^s_\delta(E).$$

Since P^s as defined above is not countably subadditive, it is not a measure. Thus, following a standard procedure in measure theory, we define the the *s-dimensional packing measure* to be

$$\mathcal{P}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} P^s(E_i) \,:\, E = \bigcup_{i=1}^{\infty} E_i \right\}.$$

Definition 3.4. The *packing dimension* of $E \subseteq \mathbb{R}^n$ is

$$\dim_p(E) = \inf\{s \,:\, \mathcal{P}^s(E) < \infty\} = \sup\{s \,:\, \mathcal{P}^s(E) > 0\}.$$

We conclude this section by noting that for any $E \subseteq \mathbb{R}^n$, we have $\dim_H(E) \leq \dim_p(E)$.

4. The Point-to-Set Principle

We are now ready to state the point-to-set principle due to J. Lutz and N. Lutz, which bridges the fields of geometric measure theory and computability theory.

Theorem 4.1. (Point-to-set principle [2]). *Let $E \subseteq \mathbb{R}^n$. Then,*

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and}$$
$$\dim_p(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x).$$

To get a taste for how Theorem 4.1 might be applied to the study of fractal geometry, we prove the following well known fact about Lipschitz computable functions (a subset of Lipschitz continuous functions).

Proposition 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be K-Lipschitz computable. That is, there is a Turing machine M such that for all $x \in \mathbb{R}^n$, $r \in \mathbb{N}$, and $q \in B_{2^{-(r+K)}}(x) \cap \mathbb{Q}^n$, $\|M(q,r) - f(x)\| < 2^{-r}$. Then for any $E \subseteq \mathbb{R}^n$,*

$$\dim_H f(E) \leq \dim_H E.$$

Proof. Let f be Lipschitz computable, $A \subseteq \mathbb{N}$ be an oracle, and fix $x \in E \subseteq \mathbb{R}^n$, $r \in \mathbb{N}$, and $q \in B_{2^{-(r+K)}}(x) \cap \mathbb{Q}^n$. Then

$$\begin{aligned} K^A_r(f(x)) &\leq K^A(M(q,r)) \\ &\leq K^A(q,r) + c_M \\ &\leq K^A(q) + \log r + c_M \\ &\leq K^A_{r+K}(x) + \log r + c_M \\ &\leq K^A_r(x) + \log r + c_M. \end{aligned}$$

Therefore,

$$\begin{aligned}\dim^A(f(x)) &= \liminf_{r\in\mathbb{N}} \frac{K_r(f(x))}{r} \\ &\leq \liminf_{r\in\mathbb{N}} \frac{K_r(x) + \log r + c_M}{r} \\ &= \liminf_{r\in\mathbb{N}} \frac{K_r(x)}{r} + \liminf_{r\in\mathbb{N}} \frac{\log r}{r} + \liminf_{r\in\mathbb{N}} \frac{c_M}{r} \\ &= \dim^A(x).\end{aligned}$$

Since $A \subseteq \mathbb{N}$ was arbitrary, we can apply Theorem 4.1 to get

$$\dim_H f(E) \leq \dim_H E.$$

□

5. Optimal Oracles

Note that the minimum over oracle sets in Theorem 4.1 is achieved, i.e., there exists an oracle $A \subseteq \mathbb{N}$ such that $\dim_H(E) = \sup_{x\in E} \dim^A(x)$. In fact, for any set E, there are infinitely many. Such an oracle is called a *Hausdorff oracle* for E. It is natural to ask if there is a Hausdorff oracle for E which minimizes the complexity of all points in E. Generally, there is not. If we weaken the condition to minimizing the complexity of a *single* point, however, the resulting theory is surprisingly fruitful. An oracle satisfying this condition is called *Hausdorff optimal* for E.

Definition 5.1. Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. We say A is Hausdorff optimal for E if the following hold:

(1) A is a Hausdorff oracle for E, i.e.,

$$\dim_H(E) = \sup_{x\in E} \dim^A(x).$$

(2) For every $B \subseteq \mathbb{N}$ and every $\epsilon > 0$, there is a point $x \in E$ such that $\dim^{A,B}(x) \geq \dim_H(E) - \epsilon$ and for almost every $r \in \mathbb{N}$, we have

$$K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r,$$

where $K_r^{A,B}$ denotes the complexity with respect to the oracle $A, B = A \cup B$.

While the second condition only guarantees a single point whose complexity is unaffected by B, it can be shown that if A satisfies conditions (1) and (2), then "most" points in E are unaffected by B.

Proposition 5.2. ([8]) *Let $E \subseteq \mathbb{R}^n$ and A be an oracle. Define the set*

$$N(A, B, \epsilon) = \{x \in E : (\forall^\infty) K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r\}.$$

Then A is Hausdorff optimal for E if and only if A is a Hausdorff oracle for E and $\dim_H N(A, B, \epsilon) = \dim_H E$ for every $B \subseteq \mathbb{N}$ and $\epsilon > 0$.

Proof. Let $A \subseteq \mathbb{N}$ be Hausdorff optimal for E, $B \subseteq \mathbb{N}$, $\epsilon > 0$, and let $C \subseteq \mathbb{N}$ be a Hausdorff oracle for $N(A, B, \epsilon)$. For the sake of contradiction, suppose

$$\dim_H N(A, B, \epsilon) < \dim_H E - \delta$$

for some $0 < \delta < \epsilon$. Then by theorem 4.1, for any $x \in N(A, B, \epsilon)$, we have

$$\begin{aligned}\dim^{A,(B,C)}(x) &\leq \dim^{C}(x)\\ &\leq \dim_H N(A, B, \epsilon)\\ &< \dim_H E - \delta.\end{aligned}$$

Since A is Hausdorff optimal for E, there is an $x \in E$ such that

$$\dim^{A,(B,C)}(x) \geq \dim_H(E) - \epsilon$$

and for almost all $r \in \mathbb{N}$,

$$K_r^{A,(B,C)}(x) \geq K_r^A(x) - \delta r.$$

By our previous discussion, x cannot be in $N(A, B, \epsilon)$. However, this implies that for infinitely many r,

$$K_r^{A,(B,C)}(x) < K_r^A(x) - \epsilon r,$$

hence such an x cannot exist, proving the forward direction. For the reverse direction, let A be a Hausdorff oracle for E such that $\dim_H N(A, B, \epsilon) = \dim_H E$ for every $B \subseteq \mathbb{N}$ and $\epsilon > 0$. By the point-to-set principle, we have

$$\begin{aligned}\dim_H E &= \dim_H N(A, B, \epsilon)\\ &\leq \sup_{x \in N(A,B,\epsilon)} \dim^{A,B}(x).\end{aligned}$$

Therefore, there is an $x \in N(A, B, \epsilon) \subseteq E$ such that

$$K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r$$

for almost all $r \in \mathbb{N}$. □

The point-to-set principle has been used to extend several classical theorems in geometric measure theory to certain types of non-analytic sets. For example, Lutz and Stull proved Marstrand's projection theorem for sets with equal Hausdorff and packing dimension in [4] (we'll return to this in section 7). More recent research has shown that many of these algorithmic proofs can be generalized to sets with optimal oracles. Thus, for the purposes of this paper, we want to know which sets have optimal oracles. The following theorem shows that three common classes of sets have optimal oracles. Proofs can be found in [8].

Theorem 5.3. *Let $E \subseteq \mathbb{R}^n$. If at least one of the following holds, then E has an optimal Hausdorff oracle.*

- *E is analytic.*
- *$\dim_H(E) = \dim_p(E)$.*
- *$\dim_H(E) = s$, and there is a metric outer measure ν such that ν is absolutely continuous with respect to $\mathcal{H}^s$ and $0 < \nu(E) < \infty$.*

6. Radial Projections

In this section, we showcase an application of the point-to-set principle to the study of radial projections in $\mathbb{R}^n$.

Definition 6.1. The *radial projection* with respect to $x \in \mathbb{R}^n$ is a map $\pi_x : \mathbb{R}^n \backslash \{x\} \to S^{n-1}$ given by

$$\pi_x(y) = \frac{y - x}{|y - x|}.$$

The *radial projection of* $E \subseteq \mathbb{R}^n$ with respect to x is $\pi_x(K\backslash\{x\})$.

Our investigation was initially motivated by a theorem of Orponen that characterizes the exceptional set of $x \in \mathbb{R}^n$ for which the radial projection of a set E with respect to x has Hausdorff dimension 0.

Definition 6.2. The *invisible s-set* of K is the set

$$\mathrm{Inv}_s(K) = \{x : \dim_H \pi_x(K\backslash\{x\}) \leq s\}.$$

We drop the subscript and write $\mathrm{Inv}(K)$ if $s = 0$.

Theorem 6.3. (Orponen [6]) *Let $E \subseteq \mathbb{R}^n$ be Borel with $\dim_H E > 0$. Then, one of the following holds:*

- $\dim_H \mathrm{Inv}(E) = 0$.
- $\mathrm{Inv}(E)$ *is contained on a line.*

It turns out that the point-to-set principle can be used to give a short proof of a slight variation of this theorem.

Theorem 6.4. *Let $E \subseteq \mathbb{R}^n$. If $0 < s < \dim_H(E) - 1$, then $\mathrm{Inv}_s(E) = \emptyset$.*

The proof relies on a geometric observation, formalized by the following lemma. Given $x \in \mathbb{R}^n$, the radial projection of some point $z \in \mathbb{R}^n$ with respect to x, and the distance between x and z, $\|x - z\|$, we can compute the original point z by finding the intersection between the line $\{tx + (1-t)z : t \in [0,1]\}$ and the $n-1$ sphere of radius $\|x - z\|$.

Lemma 6.5. *Let $x, z \in \mathbb{R}^n$ and $r \in \mathbb{N}$. Then there exists a Turing machine M such that*

$$M(x \upharpoonright r, \pi_x(z) \upharpoonright r, \|x - z\| \upharpoonright r) = z \upharpoonright r.$$

In particular, we have

$$K_r(z) \leq K_r(x, \pi_x(z), \|x - z\|) + O(1)$$

and

$$K_r^x(z) \leq K_r^x(\pi_x(z), \|x - z\|) + O(1). \tag{6.6}$$

The construction of M is fairly routine. However, familiarity with the language of Turing machines is not meant to be a prerequisite for this paper, thus we omit the proof.

Proof of Theorem 6.4. To reach a contradiction, suppose that $\mathrm{Inv}_s(E)$ is not empty. Let $x \in \mathrm{Inv}_s(E)$, $z \in E\backslash\{x\}$, and $A \subseteq \mathbb{N}$ be a Hausdorff oracle for $\pi_x(E\backslash\{x\})$. Then

$$\begin{aligned} K_r^{A,x}(z) &\leq K_r^{A,x}(\|x - z\|, \pi_x(z)) + O(1) \\ &\leq K_r^{A,x}(\|x - z\|) + K_r^{A,x}(\pi_x(z)) + O(1). \end{aligned} \tag{eq. 6.6}$$

Since $\|x - z\| \in \mathbb{R}$, it takes at most r bits of data to compute $\|x - z\|$ up to precision r. Since $\dim_H(\pi_x(E)) \leq s$, by Theorem 4.1, there are infinitely many r such that

$$\frac{K_r^A(\pi_x(z))}{r} \leq s + \epsilon.$$

Hence, there are infinitely many r such that $K_r^{A,x}(z) \leq r + r(s+\epsilon) + O(1)$. Therefore,

$$\dim^{A,x}(z) = \liminf_{r\to\infty} \frac{K_r^{A,x}(z)}{r} \leq 1 + s + \epsilon.$$

By letting ϵ tend to 0 and applying Theorem 4.1, we have $\dim_H(E) \leq 1 + s$, contradicting our assumption about the dimension of E. □

7. Orthogonal Projections

Definition 7.1. The *orthogonal projection* of a set $E \subseteq \mathbb{R}^n$ with respect to the direction $e \in S^{n-1}$ is

$$P_e(E) \coloneqq \{e \cdot z : z \in E\}.$$

A landmark theorem of Marstrand says that for almost all directions e, the Hausdorff dimension of $P_e(E)$ is maximal for analytic sets E.

Theorem 7.2. (Marstrand's projection theorem) *Let $E \subseteq \mathbb{R}^n$ be analytic. Then for almost all $e \in \mathcal{S}^{n-1}$,*

$$\dim_H P_e(E) = \min\{n-1, \dim_H E\}.$$

Lutz and Stull used the point-to-set principle to give a new proof of Marstrand's projection theorem for sets with equal Hausdorff and packing dimension in [4], and later extended this proof to work for all sets with optimal oracles. It is natural to ask about the set of directions of measure zero for which the projection is not maximal. The goal of this section is to prove the Kaufman bounds on the dimension of this exceptional set for sets with optimal oracles.

Theorem 7.3. (Kaufman Bounds) *Let $E \subseteq \mathbb{R}^2$ with optimal Hausdorff oracle A. If $\alpha \leq \dim_H E \leq 1$, then*

$$\dim_H\{e \in \mathcal{S}^1 : \dim_H P_e(E) < \alpha\} \leq \alpha.$$

Note that we recover Marstrand's projection theorem from the Kaufman bounds. Theorem 7.3 has not appeared in this form before, and it is stronger than its classical counterpart proven by Kaufman for analytic sets. Our approach follows that of Lutz and Stull in their proof of Marstrand's projection theorem. Lemmas 7.4, 7.10 and 7.12 are theirs, as well as the proofs. Lemma 7.7 and Theorem 7.16 are variations of results in [3] and [4], as well as the proofs. In particular, we add an $0 \leq \alpha \leq 1$ term to Theorem 4.1 of [4] corresponding to the dimension of exceptional sets of directions $e \in \mathcal{S}^1$, which allows us to formulate and prove Theorem 7.3. The intuition for doing so is relatively straightforward: Theorem 4.1 of [4] provides a lower bound on projected points $z \cdot e$, for $z \in E$ in the case where $\alpha = 1$. While sufficient to prove Marstrand's projection theorem, this gives a trivial estimate on the dimension of exceptional sets of directions. Allowing α to vary, we are able to quantify how the lower bound on complexity of the projected point $e \cdot z$ weakens, and leverage this to get a bound on the dimension of exceptional sets. The aforementioned leveraging follows Lutz and Stull's proof of Marstrand's projection theorem, which uses the Theorem 4.1 of [4].

The idea for the proof of our first lemma is as follows: we assume that the complexity of the point z is not too large relative to the dimension of the set which contains it. We also impose a condition that says if z and w lie on a line which is orthogonal to e, then either the complexity of w is large or w is close to z.

The result is a lower bound on the complexity of the projected point. The intuition is that under these conditions, an approximation of the projected point and the direction e is sufficient to recover the original point by enumerating over points u of low complexity such that $e \cdot z = e \cdot u$. Our conditions ensure that this

u is a good approximation of z. Since the information of the projected point and e is sufficient to compute z, we can reason that the information required to compute the projected point given e is at least the information required to compute z. Thus, we obtain a lower bound on the complexity of $e \cdot z$.

Lemma 7.4. ([4]) *Let $z \in \mathbb{R}^2$, $e \in \mathcal{S}^1$, $r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\epsilon, \eta \in \mathbb{Q}_+$ satisfy $r \geq \log(2\|z\| + 5) + 1$. Suppose the following conditions hold:*

(1) $K_r(z) \leq (\eta + \epsilon)r$

(2) For every $w \in B_1(z)$ such that $e \cdot w = e \cdot z$,

$$K_r(w) \geq (\eta - \epsilon)r + (r - t)\delta$$

whenever $t = -\log \|z - w\| \in (0, r]$

Then for every oracle $A \subset \mathbb{N}$,

$$K_r^{A,e}(e \cdot z) \geq K_r^{A,e}(z) - \frac{4\epsilon}{\delta} r - K(\epsilon) - K(\eta) - O(\log r)$$

where the constant implied by the big-O bound depend only on z and e.

Proof. Let z, e, r, δ, ϵ, and η satisfy the hypothesis. Let (A, e) be an oracle and $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5 \in \{0,1\}^*$ be a program such that $U^{A,e}(\pi_1) = q \in \mathbb{Q}$, $U(\pi_2) = h \in \mathbb{Q}^2$, $U(\pi_3) = s \in \mathbb{N}$, $U(\pi_4) = \tau \in \mathbb{Q}$ and $U(\pi_5) = \iota \in \mathbb{Q}$. Define an oracle Turing machine M which given oracle (A, e) and input π, simulates in parallel $U(\sigma)$ for every program $\sigma \in \{0,1\}^*$ such that $l(\sigma) \leq (\iota + \tau)s$. If a simulation halts on $p \in \mathbb{Q}^2 \cap B_{2^{-1}}(h)$ with $|e \cdot p - q| \leq 2^{-s}$, then $M^{A,e}$ halts and returns p. Let c_M be the constant description of M.

Let π_1, π_2, π_3, π_4, and π_5 testify to $K_r^{A,e}(e \cdot z)$, $K_1(z)$, $K(r)$, $K(\epsilon)$, and $K(\eta)$ respectively, and let $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$. Since $K_r(z) \leq (\eta + \epsilon)r$, there is a $\sigma \in \{0,1\}^*$ such that $l(\sigma) \leq (\eta + \epsilon)r$ and $\|p - z\| \leq 2^{-r}$, where $U(\sigma) = p$. Since the function $x \mapsto e \cdot x$ is 1-Lipschitz, $|e \cdot z - e \cdot p| \leq 2^{-r}$, thus $M^{A,e}$ is guaranteed to halt on π.

Let $M^{A,e}(\pi) = p \in \mathbb{Q}^2$, and let $w = p + [e \cdot (z - p)]e$. Then

$$\begin{aligned}
\|p - w\| &= \|p - (p + [e \cdot (z - p)]e)\| \\
&= \|[e \cdot (z - p)]e\| \\
&= |e \cdot (z - p)| \\
&= |e \cdot z - e \cdot p| \\
&\leq 2^{-r}
\end{aligned}$$

and

$$\begin{aligned}
e \cdot w &= e \cdot (p + [e \cdot (z - p)]e) \\
&= e \cdot p + (e \cdot e)[e \cdot (z - p)] \\
&= e \cdot p + e \cdot (z - p) \\
&= e \cdot z.
\end{aligned}$$

Thus, there is a $w \in B_{2^{-r}}(p) \subseteq B_{2^{-1}}(p) \subseteq B_1(z)$ with $e \cdot w = e \cdot z$, and

$$\begin{aligned}
K_r^{A,e}(w) &\leq l(\pi) \\
&\leq K_r^{A,e}(e \cdot z) + K_1(z) + K(r) + K(\epsilon) + K(\eta) + c_M \\
&= K_r^{A,e}(e \cdot z) + K(\epsilon) + K(\eta) + O(\log r).
\end{aligned}$$

Rearranging gives

$$K_r^{A,e}(e \cdot z) \geq K_r^{A,e}(w) - K(\epsilon) - K(\eta) - O(\log r). \tag{7.5}$$

Let $t = -\log \|z - w\|$. If $t \geq r$, then $\|z - w\| \leq 2^{-r}$ and $K_r^{A,e}(z) = K_r^{A,e}(w)$, so the proof is complete. If $t < r$, then $w \in B_{2^{1-t}}(z)$ implies that $K_r^{A,e}(w) \geq K_{t-1}^{A,e}(z)$. Therefore,

$$K_r^{A,e}(w) \geq K_r^{A,e}(z) - 2(r - t) - O(\log r). \tag{7.6}$$

By our construction of M,

$$\begin{aligned}(\eta + \epsilon)r &\geq K(p) \\ &\geq K_r(w) - O(\log r)\end{aligned}$$

By condition (2) of the hypothesis,

$$(\eta + \epsilon)r \geq (\eta - \epsilon)r + (r - t)\delta - O(\log r),$$

which implies

$$r - t \leq \frac{2\epsilon}{\delta} r + O(\log r).$$

Combining this with (7.5) and (7.6) completes the proof.

□

Next, we want to give a lower bound on the complexity of points w such that $e \cdot w = e \cdot z$. First note that the information of z and w is sufficient to compute the direction e which is orthogonal to the line between w and z. This shows that given z, the complexity of e cannot be much larger than the complexity of w. Then, we use a symmetry of information argument to rearrange the resulting inequality to obtain a bound on the complexity of w based on the complexity of e given z. However, some care is required because we need to compute *approximations* of e given precision-r approximations of the points w and z. Thus the following lemma uses a simple geometric argument to show that the error in the approximation of e is linearly correlated with the error in approximations of w and z. We then apply the symmetry of information to obtain a lower bound on the complexity of w. A version of this lemma was proved by Lutz and Stull in the context of bounding the complexity of points on a shared line in [3].

Lemma 7.7. *Let $z \in \mathbb{R}^2$, $e \in \mathcal{S}^1$, and $r \in \mathbb{N}$. Let $w \in \mathbb{R}^2$ such that $\|z - w\| \geq 2^{-r}$, e is nonzero in all coordinates, and $e \cdot z = e \cdot w$. Then,*

$$K_r(w) \geq K_t(z) + K_{r-t,r}(e \mid z) + O(\log r)$$

where $t = -\log \|z - w\|$.

Proof. Let z, w, e, and r satisfy the hypothesis. If $r - t = O(\log r)$, then $K_{r-t,r}(e \mid z) = O(\log r)$, and the result follows trivially. Otherwise, let $q = (q_1, q_2) \in B_{2^{-r}}(z) \cap \mathbb{Q}^2$ and π_q testify to

$$\hat{K}_r(w \mid q) \coloneqq \min\{K(p \mid q) : p \in B_{2^{-r}}(w) \cap \mathbb{Q}^n\}.$$

Let M be a Turing machine which, given input (π_q, q), first computes $p = (p_1, p_2) = U(\pi_q)$. Then, M computes the unit vector $e' \in \mathcal{S}^1$ such that $e' \cdot q = e' \cdot p$ using the following formula:

$$e' = \frac{\left(1, \frac{p_1 - q_1}{q_2 - p_2}\right)}{\left\|\left(1, \frac{p_1 - q_1}{q_2 - p_2}\right)\right\|}.$$

The following geometric argument will show that $|e' - e| \leq 2^{2+t-r}$. Let θ be the angle between e' and e. Since $e' \perp p - q$ and $e \perp w - z$, the angle of intersection between the lines generated by the vectors $p-q$ and $w-z$, $\angle(L_{p-q}, L_{w-z})$, is equal to θ. We now bound $2\sin(\theta)$. For $0 \leq \theta \leq \frac{\pi}{2}$, $\sin(\theta)$ is increasing. Thus, without loss of generality, we can assume that p and q fall on opposite sides of L_{w-z} (if not, θ can be increased reflecting one point across L_{w-z}). If $d = d(w,z)$, $r_1 = d(w,p)$ and $r_2 = d(z,q)$, then

$$\begin{aligned} 2\sin(\theta) &\leq 2\frac{r_1 + r_2}{d} \\ &\leq 2\frac{2^{1-r}}{2^{-t}} \\ &= 2^{2+t-r}. \end{aligned}$$

Since $|e'| = |e| = 1$, by the law of cosines, for $0 \leq \theta \leq \frac{\pi}{2}$ we have

$$\begin{aligned} |e' - e| &= \sqrt{2 - 2\cos(\theta)} \\ &= \sqrt{2 \cdot 2\sin^2\left(\frac{\theta}{2}\right)} \\ &= 2\sin\left(\frac{\theta}{2}\right) \\ &\leq 2\sin(\theta) \\ &\leq 2^{2+t-r}. \end{aligned}$$

Thus $|M(\pi_q, q) - e| \leq 2^{2+t-r}$ implies that $K_{r-t}(e \mid z) \leq K_{r-t-2}(e \mid z) + O(1) \leq |\pi_q| + O(1)$. Or, equivalently

$$K_{r-t}(e \mid z) \leq K_r(w \mid z) + O(1). \tag{7.8}$$

We complete the proof by using the symmetry of information to note that

$$\begin{aligned} K_r(w \mid z) &\leq K_{r,t}(w \mid z) + O(\log r) \\ &= K_{r,t}(w \mid w) + O(\log r) \\ &= K_r(w) - K_t(w) + O(\log r) \\ &= K_r(w) - K_t(z) + O(\log r). \end{aligned}$$

Combining this with (7.8) gives the desired inequality. □

To ensure that the second condition of Lemma 7.4 is satisfied, we define an oracle that reduces the complexity of the point z without affecting other points. The construction can be found in [3], however it is rather technical, and thus omitted. Recall that for $z \in \mathbb{R}^n$ and $r \in \mathbb{N}$, we denote the concatenation of the first r bits of the binary expansion of z as $z \upharpoonright r$. As expected, the complexity of $z \upharpoonright r$ is very close to $K_r(z)$. In particular, we have

$$K_r(z) = K(z \upharpoonright r) + O(\log r). \tag{7.9}$$

Lemma 7.10. ([3]). *Let $z \in \mathbb{R}^n$, $\eta \in \mathbb{Q} \cap [0, \dim(z)]$, and $r \in \mathbb{N}$. Then there is an oracle $D = D(r, z, \eta)$ with the following properties:*

(1) For every $t \leq r$,

$$K_t^D(z) = \min\{\eta r, K_t(z)\} + O(\log r).$$

(2) For every $m, t \in \mathbb{N}$ and $y \in R^m$,

$$K^D_{t,r}(y \mid z) = K_{t,r}(y \mid z) + O(\log r),$$

and

$$K^{z,D}_t(y) = K^z_t(y) + O(\log r).$$

In particular, the oracle D encodes σ, the lexicographically-first time-minimizing witness to $K(z \uparrow r \mid z \uparrow s)$, where $s = \max\{t \leq r : K_{t-1}(z) \leq \eta r\}$, in the sense that for all $x \in \{0,1\}^*$ and oracles $B \subseteq \mathbb{N}$, we have

$$K^{B,D}(x) = K^B(x \mid \sigma) + O(1). \tag{7.11}$$

We require one extra property of the oracle D in Lemma 7.10 to prove the following theorems: if we allow access to any other oracle B, then allowing further access to D does not significantly decrease the complexity of z (any more than has already been done by B). Thus, we are able to take advantage of the properties of D in the presence of other oracles. The following lemma makes this precise.

Lemma 7.12. ([4]). *Let $z \in \mathbb{R}^n$, $B \subseteq \mathbb{N}$, $\eta \in \mathbb{Q} \cap [0, \dim(z)]$, $\epsilon > 0$, and $r \in \mathbb{N}$. Let $D = D(r, z, \eta)$ be the oracle defined in Lemma 7.9. If*

$$K^B_r(z) \geq K_r(z) - \epsilon r, \tag{7.13}$$

then

$$K^{B,D}_r(z) \geq K^D_r(z) - \epsilon r - O(\log r). \tag{7.14}$$

Proof. Let $s = \max\{t \leq r : K_{t-1}(z) \leq \eta r\}$. Let σ be the lexicographically-first time-minimizing witness to $K(z \uparrow r \mid z \uparrow s)$ (that is, σ is the witness given in Lemma 7.10). Then

$$K(\sigma \mid z \uparrow r, s, K(z \uparrow r \mid z \uparrow s)) = O(1).$$

This is because given the first r bits of z, the value $K(z \uparrow r \mid z \uparrow s)$ and the parameter s, we are able to execute an exhaustive search for the first time-minimizing witness to $K(z \uparrow r \mid z \uparrow s)$. Since $K(z \uparrow r \mid z \uparrow s) \leq r + O(\log r)$ and $s \leq r$, we have $K(K(z \uparrow r \mid z \uparrow s), s) = O(\log r)$. Applying the symmetry of information, we get

$$K(z \mid z \uparrow r) = O(\log r). \tag{7.15}$$

Moreover, this argument can be relativized so equation (7.15) holds for any oracle. The rest of the proof is a series of calculations using the symmetry of information

and several tagged equations:

$$
\begin{aligned}
&K_r^{B,D}(z) = K^{B,D}(z \upharpoonright r) + O(\log r) && \text{(eq. 7.9)}\\
&= K_r^{B}(z \upharpoonright r \mid \sigma) + O(\log r) && \text{(eq. 7.11)}\\
&= K^{B}(z \upharpoonright r) + K^{B}(\sigma \mid z \upharpoonright r) - K^{B}(\sigma) + O(\log r) && \text{(symmetry of info.)}\\
&= K^{B}(z \upharpoonright r) - K^{B}(\sigma) + O(\log r) && \text{(eq. 7.15)}\\
&= K_r^{B}(z) - K^{B}(\sigma) + O(\log r) && \text{(eq. 7.9)}\\
&\geq K_r(z) - K^{B}(\sigma) - \epsilon r - O(\log r) && \text{(ineq. 7.13)}\\
&\geq K_r(z) - K(\sigma) - \epsilon r - O(\log r) && \text{(oracles decrease complexity)}\\
&= K(z \upharpoonright r) - K(\sigma) - \epsilon r - O(\log r) && \text{(eq. 7.9)}\\
&= K(z \upharpoonright r) + K(\sigma \mid z \upharpoonright r) - K(\sigma) - \epsilon r - O(\log r) && \text{(eq. 7.15)}\\
&= K(z \upharpoonright r \mid \sigma) - \epsilon r - O(\log r) && \text{(symmetry of info.)}\\
&= K^{D}(z \upharpoonright r) - \epsilon r - O(\log r) && \text{(eq. 7.11)}\\
&= K_r^{D}(z) - \epsilon r - O(\log r). && \text{(eq. 7.9)}
\end{aligned}
$$

□

The following theorem combines the results of Lemmas 7.4, 7.7, 7.10 and 7.12. Variations of this theorem have been used to prove several results about the Hausdorff and Packing dimensions of projections, see [3], [4].

Theorem 7.16. *Let $z \in \mathbb{R}^2$, $e \in \mathcal{S}^1$ with all coordinates nonzero, $A \subseteq \mathbb{N}$, $0 < \alpha \leq 1$, $\eta \in \mathbb{Q} \cap (0, \min\{\dim(z), \alpha\})$, $\epsilon > 0$, and $r \in \mathbb{N}$. Assume the following conditions are satisfied.*

(1) For every $s \leq r$, $K_s(e) \geq \alpha s - O(\log s)$.
(2) $K_r^{A,e}(z) \geq K_r(z) - \epsilon r$,

Then,

$$K_r^{A,e}(e \cdot z) \geq \eta r - \epsilon r - \frac{4\epsilon}{\alpha - \eta} r - K(\epsilon) - K(\eta) - O(\log r).$$

Proof. Assume the hypothesis. Let $D = D(r, z, \eta)$ be the oracle as defined in Lemma 7.10.

First assume that the conditions of Lemma 7.4, relative to D, hold for z, e, r, η, 2ϵ, and $\delta = \alpha - \eta$. Then, by Lemmas 7.4, 7.10 and 7.12,

$$
\begin{aligned}
K_r^{A,e,D}(e \cdot z) &\geq K_r^{A,e,D}(z) - \frac{4\epsilon}{1-\eta} r - K^{D}(2\epsilon) - K^{D}(\eta) - O(\log r)\\
&\geq K_r^{D}(z) - \epsilon r - \frac{4\epsilon}{\delta} r - K(\epsilon) - K(\eta) - O(\log r)\\
&= \eta r - \epsilon r - \frac{4\epsilon}{\delta} r - K(\epsilon) - K(\eta) - O(\log r).
\end{aligned}
$$

Note that the $O(\log r)$ term depends only on z. To complete the proof, we show that the conditions of Lemma 7.4 hold relative to D.

By property (1) of Lemma 7.10,

$$K_r^{D}(z) \leq \epsilon r + O(\log r)\,,$$

so for all sufficiently large r,

$$K_r^{D}(z) \leq (\eta + \epsilon) r\,,$$

which gives condition (1) of Lemma 7.4, relative to D. To show that condition (2) of Lemma 7.4 also holds relative to D, let $w \in B_1(z)$ such that $e \cdot w = e \cdot z$ and $\|z - w\| \geq 2^{-r}$, and let $t = -\log \|z - w\|$. Then, by Lemma 7.7, relative to D, we have

$$\begin{aligned}
K_r^D(w) &\geq K_t^D(z) + K_{r-t,r}^D(e \mid z) + O(\log r) \\
&= K_t^D(z) + K_{r-t,r}(e \mid z) - O(\log r) && \text{(Lemma 7.10 (2))} \\
&\geq K_t^D(z) + K_{r-t,r}(e) - \epsilon r - O(\log r) && \text{(condition (2))} \\
&\geq \eta t + K_{r-t,r}(e) - \epsilon r - O(\log r) && \text{(Lemma 7.10 (1))} \\
&\geq \eta t + r - t - \epsilon r - O(\log r) && \text{(condition (1))} \\
&= t(\eta - 1) + r(1 - \epsilon) - O(\log r) \\
&\geq (\eta - \epsilon) r + (1 - \eta)(r - t) \\
&\geq (\eta - \epsilon) r + \delta(r - t)\,.
\end{aligned}$$

Hence, the conditions of Lemma 7.4 hold relative to D, and the proof is complete. □

Proof of Theorem 7.3. Assume the hypothesis. Let $e \in \mathcal{S}^1$ satisfy condition (1) of Theorem 7.16, and let B be an oracle that satisfies

$$\dim_H P_e(E) = \sup_{z \in P_e(E)} \dim^B(z).$$

Since A can encode only a countable set of $e \in \mathcal{S}^1$, almost all directions e are random relative to A, and therefore satisfy condition (1). Since A is Hausdorff optimal for E, for each $n \in \mathbb{N}$, there is a point $z_n \in E$ such that

- $\dim^{A,B,e}(z_n) \geq \dim_H E - \frac{1}{2n}$, and
- $K_r^{A,B,e}(z_n) \geq K_r^A(z_n) - \frac{r}{2n}$.

By fixing sufficiently large n and letting $\epsilon = \frac{1}{2n}$, by Theorem 7.16, we have

$$K_r^{A,B,e}(e \cdot z_n) \geq \eta r - \epsilon r - \frac{4\epsilon}{\alpha - \eta} r - K(\epsilon) - K(\eta) - O(\log r)$$

for all sufficiently large r. Therefore,

$$\begin{aligned}
\dim^{A,B}(e \cdot z_n) &\geq \dim^{A,B,e}(e \cdot z_n) \\
&= \liminf_{r \in \mathbb{N}} \frac{K_r^{A,B,e}(e \cdot z_n)}{r} \\
&\geq \liminf_{r \in \mathbb{N}} \frac{\eta r - \epsilon r - \frac{4\epsilon}{\alpha - \eta} r - K(\epsilon) - K(\eta) - O(\log r)}{r} \\
&= \eta - \epsilon - \frac{4\epsilon}{\alpha - \eta}.
\end{aligned}$$

By letting η approach $\min\{\dim_H E, \alpha\}$ and n approach infinity, we see that

$$\sup_{z \in E} \dim^{A,B}(e \cdot z) \geq \min\{\dim_H E, \alpha\}.$$

By applying Theorem 4.1, we get

$$\dim_H P_e(E) \geq \min\{\dim_H E, \alpha\}.$$

Thus, if $\alpha \leq \dim_H E \leq 1$ and $\dim_H P_e(E) < \alpha$,

$$K_r(e) < \alpha r - O(\log r)$$

for infinitely many r and

$$\begin{aligned} \dim(e) &= \liminf_{r\in\mathbb{N}} \frac{K_r(e)}{r} \\ &\leq \liminf_{r\in\mathbb{N}} \frac{\alpha r - O(\log r)}{r} \\ &= \alpha. \end{aligned}$$

Since oracles cannot increase the dimension of points, the proof is complete by applying theorem 4.1.

Acknowledgments

I am more grateful that words can express to my mentors Iqra Altaf and Donald Stull. All of my future efforts are to retroactively earn the endless time that they dedicated to my learning and their genuine enthusiasm for teaching. I am lucky to have experienced this sort of student-mentor relationship at such an early stage. On that note, I'd like to thank Peter May for providing this opportunity to myself and hundreds of other young mathematics students each year. There will be a generation of mathematicians who owe their start to Peter May.

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