LEBESGUE INTEGRATION

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Abstract. An Introduction to Lebesgue Integration, with the Monotone Convergence Theorem and Fatou's Lemma.

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1. Introduction

The Lebesgue integral is an essential extension of the Riemann integral that addresses several shortcomings of the latter. While the Riemann integral is invaluable for elementary purposes, it fails in more complex scenarios, particularly when dealing with functions defined on intricate sets or sequences of functions. This paper discusses both conceptual and practical motivations for the development of the Lebesgue integral, highlighting why it supersedes the Riemann integral.

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1.1. **Riemann vs Lebesgue.** To illustrate the limitations of the Riemann integral, consider the function $f: [0,1] \to \mathbb{R}$, defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function, known as the Dirichlet function, takes the value 1 for rational numbers and 0 for irrationals. Due to the density of both rational and irrational numbers in $\mathbb{R}$, any partition of $[0,1]$ will contain both types of numbers in every subinterval. Consequently, the upper Riemann sum for any partition will be 1, while the lower Riemann sum will be 0. Since the Riemann integral is defined only when the upper and lower sums converge to the same value, the Riemann integral cannot be defined for this function.

This example demonstrates a critical limitation of the Riemann integral: it is not equipped to handle functions that are highly discontinuous or have complex behavior over densely populated sets.

One of the primary motivations for the Lebesgue integral arises from the need to properly integrate functions defined on sets that are "ill-behaved" under the Riemann framework. In particular, certain pathological sets, such as the Vitali set, are unmeasurable under standard Riemann integration. Although I will not address the creation of the Vitali set, it utilizes the axiom of choice to illustrate that no countably additive, translation-invariant set function can assign a finite nonzero measure to every subset of $[0,1]$. This forces us to restrict the domains and functions that can be integrated under Riemann's definition, leading to undesirable exclusions.

In addition to the conceptual limitations, Riemann integration is also ill-behaved with respect to limits. While the Riemann integral of a uniformly convergent sequence of functions is equal to the limit of the integrals of the individual functions, this property does not hold for other types of limits. In particular, it is challenging to prove results that allow the interchange of a Riemann integral with a limit of a sequence of functions.

Measure theory, which underlies the Lebesgue integral, resolves these issues by providing powerful theorems such as the Dominated Convergence Theorem, Monotone Convergence Theorem, and Fatou's Lemma. These results enable us to interchange integrals and limits in many more cases, thus extending the applicability of integration to a wider variety of problems. Although some of these results can be proved within the Riemann framework, doing so would be extremely cumbersome due to the need to rule out numerous counterexamples.

2. σ-Algebras

2.1. **Definition of σ-Algebras.** In developing the foundations of the Lebesgue integral, it is crucial to introduce a suitable framework for defining and measuring subsets of a given set. In various applications, the set X may represent different spaces, such as the real numbers $\mathbb{R}$, the unit interval $[0,1]$, or the natural numbers $\mathbb{N}$. The concept of a σ-algebra provides a systematic way to handle these subsets, ensuring closure under common set operations. This serves as a foundation for measure theory, which generalizes the classical notions of length, area, and volume.

Definition 2.1. A family $\mathcal{A}$ of subsets of a set X is called a σ-algebra (or σ-field) if it satisfies the following properties:

(i) The empty set $\varnothing$ belongs to $\mathcal{A}$,
(ii) If $A \in \mathcal{A}$, then the complement $A^c = X \setminus A$ also belongs to $\mathcal{A}$,
(iii) If $A_1, A_2, \cdots \in \mathcal{A}$ is a countable collection of sets, then the union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

The closure properties of σ-algebras under countable unions and complements imply closure under countable intersections, using De Morgan's laws.

A σ-algebra provides the basic structure required to define measurable sets, and hence measurable functions and integrals. The pair $(X, \mathcal{A})$ is called a measurable space.

2.2. Examples of σ-Algebras.

Definition 2.2 (Trivial σ-algebra)**.** Let X be any set. The collection $\{\varnothing, X\}$ forms a trivial σ-algebra. It contains only two elements: the empty set and the entire set X, and satisfies all the properties of a σ-algebra.

Definition 2.3 (Power set σ-algebra)**.** The power set $\mathcal{P}(X)$, which consists of all subsets of X, is a σ-algebra. This example demonstrates that any set can be decomposed into its smallest and largest possible σ-algebra, though the power set often contains too many elements to be useful in practice.

Definition 2.4 (Countable and co-countable σ-algebra)**.** For $X = \mathbb{R}$, consider the collection of sets:

$$\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ is countable or } A^c \text{ is countable}\}.$$

This forms a σ-algebra, as the union of countable sets is countable, and the complement of a countable set is uncountable, satisfying the conditions for closure.

3. Borel Sets

If we are given an arbitrary collection $\mathcal{C}$ of subsets of a set X, we can form the smallest σ-algebra containing all sets in $\mathcal{C}$, called the σ-algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$. The Borel σ-algebra, a central concept in analysis, is an important example. It is generated by the collection of open intervals on the real line $\mathbb{R}$. Formally,

$$\mathcal{B}(\mathbb{R}) = \sigma(\{(a, b) : a, b \in \mathbb{R}\}), \tag{1}$$

is the Borel σ-algebra on $\mathbb{R}$. It contains open sets, closed sets, and any countable unions or intersections of these sets.

4. Lemmas on σ-Algebras

Theorem [Intersections of σ-algebras] Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be a family of σ-algebras on a set X. Then, the intersection $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is also a σ-algebra on X.

Proof. By definition, each $\mathcal{A}_\alpha$ satisfies the properties of a σ-algebra. Since the intersection of any family of σ-algebras retains the properties of closure under complements, countable unions, and countable intersections, it follows that the intersection itself forms a σ-algebra. □

5. Measure

5.1. Definition of Measure.

Definition 5.1. A measure is a function μ defined on a σ-algebra $\mathcal{A}$ of subsets of a set X, where the function $\mu : \mathcal{A} \to [0, \infty]$ satisfies the following properties:

(i) $\mu(\varnothing) = 0$,
(ii) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$,
(iii) μ is *countably additive*, meaning that for any sequence $\{A_n\}$ of disjoint sets in $\mathcal{A}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{2}$$

These properties guarantee that a measure provides a consistent way of assigning sizes to sets, and its countable additivity ensures that it behaves well under unions of disjoint sets.

6. Finite Measures and σ-Finite Measures

Definition 6.1 (Finite Measure)**.** A measure μ on a measurable space $(X, \mathcal{A})$ is said to be *finite* if the measure of the entire space X is finite, i.e.,

$$\mu(X) < \infty.$$

In this case, the measure μ assigns a finite size or volume to the entire set X. For example, if μ is the Lebesgue measure on the interval $[0, 1]$, then $\mu([0, 1]) = 1$, which means the measure is finite.

Definition 6.2 (σ-Finite Measure)**.** A measure μ is called σ-finite if the space X can be written as a countable union of measurable sets, each with finite measure. That is, $X = \bigcup_{n=1}^{\infty} A_n$, where each $A_n \in \mathcal{A}$ and $\mu(A_n) < \infty$ for all n. Formally,

$$X = \bigcup_{n=1}^{\infty} A_n \quad \text{with} \quad \mu(A_n) < \infty \text{ for all } n.$$

This property generalizes the notion of finiteness and is particularly useful because many measures, like the Lebesgue measure on $\mathbb{R}$, are not finite but are σ-finite. For example, the Lebesgue measure on $\mathbb{R}$ is σ-finite because $\mathbb{R}$ can be expressed as the countable union of bounded intervals, each of which has finite Lebesgue measure:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n] \quad \text{with} \quad \lambda([-n, n]) = 2n < \infty.$$

Thus, while $\mu(\mathbb{R}) = \infty$, the measure is σ-finite because $\mathbb{R}$ can be covered by a countable collection of sets with finite measure.

Note that finite is stronger than σ-finite for all measures.

7. Lemmas on Measures

We will now explore some key lemmas that clarify the behavior of measures in various situations. These lemmas will prove essential in further discussions of integrals and measurable sets.

Theorem [Monotonicity of Measures] Let μ be a measure on a σ-algebra $\mathcal{A}$, and let $E \subseteq F$ with $E, F \in \mathcal{A}$. Then,

$$\mu(E) \leq \mu(F).$$

Proof. Since $F = E \cup (F \setminus E)$, and the sets E and $F \setminus E$ are disjoint, by countable additivity,

$$\mu(F) = \mu(E) + \mu(F \setminus E).$$

Since $\mu(F \setminus E) \geq 0$, it follows that $\mu(E) \leq \mu(F)$. □

Theorem [Continuity from Below] Let μ be a measure on a σ-algebra $\mathcal{A}$, and let $\{E_n\}$ be an increasing sequence of sets in $\mathcal{A}$ such that $E_n \subseteq E_{n+1}$ for all n. Then,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Let $A_1 = E_1$ and define the disjoint sets $A_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then,

$$E_n = \bigcup_{k=1}^{n} A_k \quad \text{and} \quad \bigcup_{n=1}^{\infty} E_n = \bigcup_{k=1}^{\infty} A_k.$$

By countable additivity of μ,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Since $\mu(E_n) = \sum_{k=1}^{n} \mu(A_k)$, the right-hand side converges to $\lim_{n\to\infty} \mu(E_n)$, completing the proof. □

Theorem [Continuity from Above [Bar, Lemma 3.4, p. 21]] Let $\{F_n\}$ be a decreasing sequence of measurable sets with F_1 of finite measure. Then

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n).$$

The proof follows similarly by considering disjoint unions and telescoping sums.

8. Examples of Measures

We now provide several examples of measures to illustrate the above concepts.

Definition 8.1 (Counting Measure)**.** Let $X = \mathbb{N}$ be the set of natural numbers, and let $\mathcal{A}$ be the power set of X. Define the counting measure μ by

$$\mu(A) = |A|,$$

for any subset $A \subseteq X$, where $|A|$ denotes the number of elements in A. This measure is finite for finite sets and σ-finite for countably infinite sets, as $\mathbb{N}$ can be written as a countable union of finite sets.

Definition 8.2 (Dirac Measure)**.** Let X be any set and fix an element $p \in X$. The Dirac measure δ_p is defined by

$$\delta_p(A) = \begin{cases} 1 & \text{if } p \in A, \\ 0 & \text{if } p \notin A. \end{cases}$$

The Dirac measure is finite, as it takes values in $\{0, 1\}$.

Definition 8.3 (Outer Measure)**.** The outer measure m^* of a set $A \subseteq \mathbb{R}$ generalizes the concept of length by considering all possible coverings of A with open intervals. It is defined as:

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n \text{ are open intervals}\right\},$$

where $\ell(I_n)$ is the length of the interval I_n. This approach seeks the smallest total length of intervals that cover A.

Key Properties: The outer measure m^* retains the essential properties of a measure:

- **Monotonicity:** If $A \subseteq B$, then $m^*(A) \leq m^*(B)$.
- **Countable Subadditivity:** For any countable collection of sets $\{A_n\}$,

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

- **Translation Invariance:** If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then $m^*(A + x) = m^*(A)$.

Example (Outer Measure of a Point): For a single point $A = \{0\}$, the outer measure is zero because it can be covered by intervals of arbitrarily small length:

$$m^*(\{0\}) = 0.$$

More generally, any countable set, including $\mathbb{Q}$, has outer measure zero.

Definition 8.4 (Lebesgue Measure)**.** For $X = \mathbb{R}$, the Lebesgue measure λ assigns a measure to each interval according to its length. For any interval $I = [a, b]$,

$$\lambda(I) = b - a.$$

The Lebesgue measure extends to a wide class of sets in $\mathbb{R}$, including open sets, closed sets, and countable unions of such sets. The collection of sets measurable under λ forms the Borel σ-algebra, and λ is σ-finite but not finite. Note that Lebesgue measure and outer measure coincide on all Lebesgue measurable sets.

9. Measurable Sets

Definition 9.1 (Measurable Set)**.** Any set $A \subseteq X$ is called X-measurable or, if the σ-algebra is known, simply measurable.

Example (Borel Measurable Sets): In $\mathbb{R}$, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open sets. Any set in $\mathcal{B}(\mathbb{R})$ is called Borel measurable, and these sets include intervals, countable unions of intervals, and more. The Lebesgue measure assigns finite measures to bounded intervals and is countably additive on disjoint unions of such intervals.

Definition 9.2 (Lebesgue Measurable Sets)**.** A set $E \subseteq \mathbb{R}$ is said to be *Lebesgue measurable* if, for every set $A \subseteq \mathbb{R}$, the following condition holds:

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where m^* denotes the outer measure, and E^c is the complement of E in $\mathbb{R}$.

In other words, a set E is measurable if it "divides" any subset A of $\mathbb{R}$ in such a way that the outer measure of A is equal to the sum of the outer measures of the intersections of A with E and A with E^c. This property ensures that the set E behaves well with respect to the notion of measure and can be consistently assigned a size by the Lebesgue measure λ.

10. The Set of Lebesgue Measurable Sets as a σ-Algebra

Theorem The collection of all Lebesgue measurable sets in $\mathbb{R}$, denoted $\mathcal{L}(\mathbb{R})$, forms a σ-algebra. This means that the set of Lebesgue measurable sets satisfies the three key properties required for a σ-algebra:

(i) **The empty set is measurable:** By definition, the Lebesgue measure λ assigns zero measure to the empty set, i.e., $\lambda(\varnothing) = 0$. Therefore, $\varnothing \in \mathcal{L}(\mathbb{R})$.

(ii) **Closure under complements:** If $A \in \mathcal{L}(\mathbb{R})$, then its complement $A^c = \mathbb{R} \setminus A$ is also Lebesgue measurable. This follows from the fact that the outer measure m^*, which defines Lebesgue measurability, satisfies:

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c) \quad \text{for any set } B \subseteq \mathbb{R}.$$

Thus, the complement of any Lebesgue measurable set is itself measurable.

(iii) **Closure under countable unions:** If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of Lebesgue measurable sets, then their union $\bigcup_{n=1}^{\infty} A_n$ is also Lebesgue measurable. This property follows from the *countable additivity* of the Lebesgue measure:

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Since each A_n is measurable and Lebesgue measure is countably additive, the union of a countable collection of measurable sets remains measurable.

These three properties—containing the empty set, closure under complements, and closure under countable unions—ensure that the set of all Lebesgue measurable sets forms a σ-algebra in $\mathbb{R}$. This σ-algebra is essential for the development of integration theory, as it provides a structured collection of sets over which the Lebesgue measure can be applied.

11. Measurable Functions

Definition 11.1 (Measurable Function)**.** A function $f : X \to \mathbb{R}$ is said to be *measurable* with respect to a σ-algebra $\mathcal{A}$ on X (or $\mathcal{A}$-measurable) if for every real number α, the set

$$\{x \in X : f(x) > \alpha\} \in \mathcal{A}.$$

In other words, the preimage of any interval of the form (α, ∞) under f must be a measurable set in $\mathcal{A}$. This condition ensures that the function f interacts well with the measure on the space X, and it allows us to extend the notion of integration to more general functions.

Theorem [Criteria for Measurability] The following statements are equivalent for a function $f : X \to \mathbb{R}$:

(a) For every $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ is measurable.
(b) For every $\alpha \in \mathbb{R}$, the set $B_\alpha = \{x \in X : f(x) \geq \alpha\}$ is measurable.
(c) For every $\alpha \in \mathbb{R}$, the set $C_\alpha = \{x \in X : f(x) < \alpha\}$ is measurable.
(d) For every $\alpha \in \mathbb{R}$, the set $D_\alpha = \{x \in X : f(x) \leq \alpha\}$ is measurable.

Proof. We will show that statement (a) implies (b), (b) implies (c), (c) implies (d), and (d) implies (a), thereby proving the equivalence of all four statements.

(a) implies (b): The sets $B_\alpha = \{x \in X : f(x) \geq \alpha\}$ and $A_\alpha = \{x \in X : f(x) > \alpha\}$ are related as follows:

$$B_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha - 1/n}.$$

Since $A_{\alpha-1/n}$ is measurable for each n, and measurable sets are closed under countable intersections, it follows that B_α is measurable. Thus, (a) implies (b).

(b) implies (c): The sets $C_\alpha = \{x \in X : f(x) < \alpha\}$ and $B_\alpha = \{x \in X : f(x) \geq \alpha\}$ are complements of each other:

$$C_\alpha = X \setminus B_\alpha.$$

Since B_α is measurable by (b) and the complement of a measurable set is also measurable, it follows that C_α is measurable. Thus, (b) implies (c).

(c) implies (d): Similarly, the sets $D_\alpha = \{x \in X : f(x) \leq \alpha\}$ and $C_\alpha = \{x \in X : f(x) < \alpha\}$ are related by:

$$D_\alpha = \bigcup_{n=1}^{\infty} C_{\alpha + 1/n}.$$

Since $C_{\alpha+1/n}$ is measurable for each n, and measurable sets are closed under countable unions, it follows that D_α is measurable. Thus, (c) implies (d).

(d) implies (a): Finally, the sets $A_\alpha = \{x \in X : f(x) > \alpha\}$ and $D_\alpha = \{x \in X : f(x) \leq \alpha\}$ are complements of each other:

$$A_\alpha = X \setminus D_\alpha.$$

Since D_α is measurable by (d), and the complement of a measurable set is measurable, it follows that A_α is measurable. Thus, (d) implies (a).

Since all implications have been established, we conclude that the four conditions (a), (b), (c), and (d) are equivalent. □

Example (Constant Functions): Any constant function $f(x) = c$ for all $x \in X$ is measurable. If $\alpha \geq c$, then

$$\{x \in X : f(x) > \alpha\} = \varnothing,$$

which is measurable. If $\alpha < c$, then

$$\{x \in X : f(x) > \alpha\} = X,$$

which is also measurable. Thus, constant functions are always measurable.

12. Lemmas on Measurable Functions

In this section, we prove important lemmas regarding measurable functions, along with defining the key concepts of lim inf and lim sup for sequences of functions.

Definition 12.1 (Limit Inferior and Limit Superior)**.** Let $\{f_n\}$ be a sequence of real-valued functions defined on a set X. The *limit inferior* and *limit superior* of the sequence are defined pointwise as follows:

$$\liminf_{n\to\infty} f_n(x) = \sup_{m\geq 1} \left(\inf_{n \geq m} f_n(x) \right),$$

$$\limsup_{n\to\infty} f_n(x) = \inf_{m\geq 1} \left(\sup_{n \geq m} f_n(x) \right).$$

In words, the lim inf is the largest lower bound of the tail of the sequence, while the lim sup is the smallest upper bound of the tail of the sequence. These

concepts provide useful tools for analyzing the behavior of sequences of functions, particularly in the context of measure theory.

Theorem [Measurability of Limit Functions] Let $\{f_n\}$ be a sequence of measurable functions in $M(X, \mathcal{A})$, where $M(X, \mathcal{A})$ denotes the set of measurable functions on a measurable space $(X, \mathcal{A})$. Define the functions:

$$f(x) = \inf_{n\geq 1} f_n(x), \quad F(x) = \sup_{n\geq 1} f_n(x),$$

$$f^*(x) = \liminf_{n\to\infty} f_n(x), \quad F^*(x) = \limsup_{n\to\infty} f_n(x).$$

Then the functions f, F, f^*, and F^* are measurable functions.

Proof. We need to show that each of these functions is measurable, given that the sequence $\{f_n\}$ consists of measurable functions.

- For $f(x) = \inf_{n\geq 1} f_n(x)$, observe that:

$$\{x \in X : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) \geq \alpha\}.$$

Since each set $\{x \in X : f_n(x) \geq \alpha\}$ is measurable by the measurability of f_n, the countable intersection of measurable sets is also measurable. Therefore, f is measurable.

- Similarly, for $F(x) = \sup_{n\geq 1} f_n(x)$, we have:

$$\{x \in X : F(x) \geq \alpha\} = \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) \geq \alpha\}.$$

Since countable unions of measurable sets are measurable, it follows that F is measurable.

- For $f^*(x) = \liminf_{n\to\infty} f_n(x)$, we express it as:

$$f^*(x) = \sup_{m\geq 1} \inf_{n\geq m} f_n(x).$$

The infimum of measurable functions is measurable, and the supremum of measurable functions is also measurable, so f^* is measurable.

- Similarly, for $F^*(x) = \limsup_{n\to\infty} f_n(x)$, we have:

$$F^*(x) = \inf_{m\geq 1} \sup_{n\geq m} f_n(x),$$

which is also measurable by the same reasoning.

Thus, f, F, f^*, and F^* are all measurable functions. □

Theorem [Measurability of Pointwise Limits] If $\{f_n\}$ is a sequence of measurable functions in $M(X, \mathcal{A})$ that converges pointwise to a function f, then f is also measurable.

Proof. Assume that $f_n(x) \to f(x)$ for all $x \in X$. We can express the pointwise limit as:

$$f(x) = \lim_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x).$$

Since we have already established that $\liminf_{n\to\infty} f_n(x)$ is measurable from the previous theorem, it follows that the limit function f is measurable. □

13. Simple Functions and the Lebesgue Integral

Definition 13.1 (Simple Functions)**.** A real-valued function $\varphi : X \to \mathbb{R}$ is called a *simple function* if it takes on only a finite number of distinct values. A simple measurable function can be expressed in the form:

$$\varphi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $a_j \in \mathbb{R}$, $E_j \subseteq X$ are measurable sets, and χ_{E_j} is the indicator function of the set E_j, defined as:

$$\chi_{E_j}(x) = \begin{cases} 1 & \text{if } x \in E_j, \\ 0 & \text{if } x \notin E_j. \end{cases}$$

Any simple function φ can be uniquely represented by grouping points that share identical function values into distinct and disjoint measurable subsets. In this standard form, the coefficients a_j are distinct and the sets E_j are pairwise disjoint.

Definition 13.2 (Lebesgue Integral of Simple Functions)**.** Let φ be a simple function in $M^+(X, \mathcal{A})$ with the standard representation:

$$\varphi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $a_j \geq 0$ for all j. The *Lebesgue integral* of φ with respect to a measure μ is defined as:

$$\int_X \varphi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

This definition is well-posed, meaning that if the same simple function φ is represented in multiple ways, the value of the integral remains the same. The integral sums the products of the distinct values a_j and the measures $\mu(E_j)$ of the corresponding measurable sets E_j.

If the measure of the space X is infinite or if the function takes infinite values, the integral may be infinite. In particular, if $\varphi = 0$ identically, we define:

$$\int_X \varphi \, d\mu = 0.$$

Theorem

[Linearity [Bar, Lemma 4.3, p. 28]] For non-negative simple functions φ, ψ and any $c \geq 0$,

$$\int c\varphi \, d\mu = c \int \varphi \, d\mu, \qquad \int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu.$$

Definition 13.3 (Measure Defined by the Integral of a Simple Function)**.** Let φ be a simple function in $M^+(X, \mathcal{A})$, and let μ be a measure on X. We can define a new measure λ on X by:

$$\lambda(E) = \int_E \varphi \, d\mu.$$

This measure λ is non-negative, countably additive, and satisfies the same properties as μ. As a special case, if φ is identically zero, we define:

$$\lambda(E) = \mu(E).$$

This definition of a measure via the integral of a simple function will be useful in proving convergence theorems, such as the Monotone Convergence Theorem.

Definition 13.4 (Lebesgue Integral for Non-Negative Measurable Functions)**.** Let $f : X \to [0, \infty]$ be a non-negative measurable function, and let $E \subseteq X$ be a measurable set. The *Lebesgue integral* of f over E with respect to a measure μ is defined as:

$$\int_E f \, d\mu = \sup \left\{ \int_E \varphi \, d\mu : 0 \leq \varphi \leq f \text{ and } \varphi \text{ is a simple function} \right\}.$$

In this definition, the supremum is taken over all simple functions φ that satisfy $0 \leq \varphi(x) \leq f(x)$ for all $x \in E$.

Theorem [Monotonicity of the Lebesgue Integral] Let f and g be two functions in $M^+(X, \mathcal{A})$ such that $0 \leq f(x) \leq g(x)$ for all $x \in X$. Then,

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

Proof. If $f \leq g$ and both functions are measurable, then for all simple functions φ_n such that $\varphi_n \leq f$, we have that $\varphi_n \leq g$. Thus,

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

□

Theorem [Monotone Convergence Theorem (MCT)] Let $\{f_n\}$ be a monotone increasing sequence of functions in $M^+(X, \mathcal{A})$ such that $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Assume $f_n \to f$ pointwise and define $f(x) = \lim_{n\to\infty} f_n(x)$. Then:

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. Since the sequence f_n is increasing, by the monotonicity of the Lebesgue integral, we have for all n:

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu.$$

Thus,

$$\lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

To complete the proof, we need to show that:

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu.$$

Let φ be a simple function such that $0 \leq \varphi(x) \leq f(x)$ for all $x \in X$. Define the sets

$$A_n = \{x \in X : f_n(x) \geq \alpha\varphi(x)\} \quad \text{for } 0 < \alpha < 1.$$

Since $f_n(x)$ is increasing, we have $A_n \subseteq A_{n+1}$, and

$$X = \bigcup_{n=1}^{\infty} A_n.$$

By the monotonicity lemma and the fact that $\varphi \leq f_n \leq f$, we have:

$$\alpha \int_{A_n} \varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int_{A_n} f \, d\mu.$$

By continuity from above and the definition of measure via the integral of a simple function, we have:

$$\alpha \int_X \varphi \, d\mu = \lim_{n\to\infty} \int_{A_n} f_n \, d\mu.$$

Taking limits, we get:

$$\alpha \int_X \varphi \, d\mu \leq \lim_{n\to\infty} \int_{A_n} f_n \, d\mu.$$

By the arbitrariness of α, we get:

$$\int_X \varphi \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Thus, we have:

$$\int_X f \, d\mu = \sup \int_X \varphi \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Hence, the Monotone Convergence Theorem is established. □

Theorem [Fatou's Lemma] Let (f_n) be a sequence of non-negative measurable functions on a measure space $(X, \mathcal{M}, \mu)$, such that $f_n \in M^+(X, \mathcal{A})$ for all n. Then Fatou's Lemma states:

$$\int (\liminf_{n\to\infty} f_n) \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

Proof. Define $g_m = \inf\{f_m, f_{m+1}, f_{m+2}, \ldots\}$, for $m \in \mathbb{N}$. Observe that the sequence (g_m) is monotone increasing and non-negative, and we have:

$$\lim_{m\to\infty} g_m = \liminf_{n\to\infty} f_n.$$

By the definition of g_m, we have $g_m \leq f_n$ for all $n \geq m$. Thus,

$$\int g_m \, d\mu \leq \int f_n \, d\mu \quad \text{for all } n \geq m.$$

Taking the limit inferior as $n \to \infty$, we obtain:

$$\int g_m \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

Since (g_m) is a monotone increasing sequence of measurable functions, we apply the Monotone Convergence Theorem to get:

$$\int \left(\lim_{m\to\infty} g_m \right) d\mu = \lim_{m\to\infty} \int g_m \, d\mu.$$

Substituting $\lim_{m\to\infty} g_m = \liminf_{n\to\infty} f_n$, we have:

$$\int (\liminf_{n\to\infty} f_n) \, d\mu = \lim_{m\to\infty} \int g_m \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

This completes the proof of Fatou's Lemma. □

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