HODGE THEORY AND ALGEBRAIC GEOMETRY

KAIYI CHEN

Abstract. In this expository paper, we present two extensions of classical Hodge theory into algebraic geometry. One is the Hodge-de Rham spectral sequence and its degeneration, for which we will present one proof by appealing to analytic results and another pure algebraic proof. The other extension is Deligne's construction of mixed Hodge structures on general complex algebraic varieties.

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1. Introduction

For a compact Kähler manifold X, the Hodge decomposition theorem produces a direct sum decomposition (see [Voi02, Section 6.1.3])

$$\mathrm{H}^n(X, \mathbf{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X)$$

on its cohomology. Such "Hodge structures" brings much information about the complex structure of X. For example, an elliptic curve E is determined by the Hodge structure on $\mathrm{H}^1(E)$ (see [Voi02, Section 7.2.2]). For a family of compact Kähler manifolds, the Hodge structures of the fibers will vary, by analyzing which we can know a lot about the family.

The Hodge decomposition theorem is proved by analytical methods. Algebraic geometers have been trying to adapt this theory to the world of algebraic geometry. Since every complex smooth projective variety is Kähler, the Hodge decomposition holds. However, there are many more objects in algebraic geometry. In one direction, we may work over fields other than $\mathbf{C}$, or as the relative version, work with general morphisms of schemes, to apply the force of Hodge theory to general algebra or even number theory. In the other direction, we want a Hodge theory on noncompact or singular varieties, since for example, there are many singular objects that appear in nature and bring interesting information.

In this paper, we discuss two kinds of work around this topic. One is the degeneration of Hodge-de Rham spectral sequence, which is a weak version of the Hodge decomposition that can be adapted to general fields and morphisms. The other is the construction of mixed Hodge structures on general complex varieties. Along the way, we explore the relevant concepts and theorems, mainly following [Del68], [DI87], [Del71], and [Del74].

In Section 2 we fix some notation and conventions. Then we will recall the classical Hodge theory in Section 3. After that the text will be split into two parts, focus on the Hodge-de Rham spectral sequence and mixed Hodge structures, respectively. The two parts are almost logically independent.

In Part 1, we discuss the degeneration of the Hodge-de Rham spectral sequence. The main result is the following.

Theorem 1.1. *[Del68, Theorem 5.5] Let $f : X \to S$ be a smooth proper family of schemes, with S of characteristic 0, by which we mean that all residue fields of points in S has characteristic 0. Then we have*

(1) The Hodge and de Rham cohomologies $\mathcal{H}^{p,q}_{\mathrm{Hodge}}(X/S)$ and $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ are vector bundles over S. Their formations commute with base change.

(2) The Hodge-de Rham spectral sequence $E(X/S)$ degenerates at the E_1-page.

We will first motivate and introduce the Hodge-de Rham spectral sequence in Section 4. After that, we will provide two proof of the above theorem in the special case where $S = \operatorname{Spec} K$ is the spectrum of a field. The first proof, appeared in [Del68], is by reduction to $\mathbf{C}$ and appealing to classical Hodge theory. The other proof, in [DI87], is by reduction modulo p, which makes no use of analytical result. Finally, we will discuss the proof of the full form of the above theorem in [Del68].

In Part 2, we will introduce mixed Hodge structures, and discuss Deligne's construction of them on noncompact or singular complex algebraic varieties in [Del71] and [Del74]. The main result is the following.

Theorem 1.2. *[Del74, Section 8.2] For any variety X and integer n, and there is a canonical mixed Hodge structure on $\mathrm{H}^n(X)$, which is functorial under pullback by morphisms of varieties. If X is smooth and projective, then this mixed Hodge structure coincide with the classical pure Hodge structure. Moreover, if (p,q) is a pair of number such that $h^{p,q}(\mathrm{H}^n(X)) \neq 0$, then*

(1) $(p,q) \in [0,n] \times [0,n]$;
(2) If $\dim X = N$ and $n \geq N$, then $(p,q) \in [n-N, N] \times [n-N, N]$;
(3) If X is proper, then $p+q \leq n$;
(4) If X is smooth, then $p+q \geq n$.

In Section 8 we will motivate and define mixed Hodge structures. Section 9 contains some results about filtration that are important for the construction. In Section 10 we construct mixed Hodge structures from smooth (noncompact) varieties. Finally, we discuss the idea of the general case in Section 11.

2. Notation and Conventions

1. All filtrations are finite. Increasing filtrations have indices as subscripts, and decreasing filtrations have indices as superscripts.

Shifts of a decreasing filtration $F^\bullet$ are defined by $(F[n])^i = F^{n+i}$. Shifts of an increasing filtration $W_\bullet$ is defined by $(W[n])_i = W_{i-n}$.

A decreasing filtration $F^\bullet$ is sometimes regarded as an increasing filtration with $F_i = F^{-i}$, and vice versa.

If $(A, F^\bullet)$ is a decreasing filtered object in an abelian category, then the associated graded object $\mathrm{Gr}_F^i A = F^i A / F^{i+1} A$. For an increasing filtered object $(A, W_\bullet)$, the associated graded object $\mathrm{Gr}_i^W A = W_i A / W_{i-1} A$.

2. Denote by $\mathrm{Ch}(\mathcal{C})$ the category of chain complexes in an additive category $\mathcal{C}$. Chain complexes have increasing indices, and are written as superscripts.

Shifts of a complex $K^\bullet$ are defined by $K[n]^i = K^{i+n}$ with differential multiplied by $(-1)^n$.

All abelian categories in the paper will have enough injectives. Denote by $\mathrm{D}(\mathcal{C})$ the unbounded derived category of an abelian category $\mathcal{C}$. Similar for symbols like $\mathrm{D}^+(\mathcal{C})$.

For a complex $K^\bullet$, the naive truncations $\sigma^{\geq i} K^\bullet$ and $\sigma^{\leq i} K^\bullet$ are given by

$$(\sigma^{\geq i} K^\bullet)^n = \begin{cases} K^n, & n \geq i \\ 0, & n < i \end{cases} \quad \text{and} \quad (\sigma^{\leq i} K^\bullet)^n = \begin{cases} 0, & n > i \\ K^n, & n \leq i. \end{cases}$$

The truncations $\tau^{\geq i} K^\bullet$ and $\tau^{\leq i} K^\bullet$ are given by

$$(\tau^{\geq i} K^\bullet)^n = \begin{cases} K^n, & n > i \\ \mathrm{coker}(\mathrm{d}^{i-1} : K^{i-1} \to K^i), & n = i \\ 0, & n < i \end{cases}$$

and

$$(\tau^{\leq i} K^\bullet)^n = \begin{cases} 0, & n > i \\ \ker(\mathrm{d}^i : K^i \to K^{i+1}), & n = i \\ K^n, & n < i. \end{cases}$$

We have $\mathrm{H}^n(\tau^{\geq i}K^\bullet) = \begin{cases} \mathrm{H}^n(K^\bullet), & n \geq i \\ 0, & n < i \end{cases}$ and similarly for $\tau^{\leq i}K^\bullet$. Especially, $\tau^{\geq i}, \tau^{\leq i}$ descend to endofunctors of $\mathrm{D}(\mathcal{C})$. However, the naive truncations do not descend.

3. Any sheaf will at least be a sheaf of abelian groups. For a topological space X, denote by $\mathrm{Shv}(X)$ the abelian category of sheaves on X. For a sheaf $\mathcal{F}$ on X, denote the sheaf cohomology by $\mathrm{H}^*(X, \mathcal{F})$. For an abelian group A, denote by A_X the constant sheaf with values in A on X. The subscript will be omitted if there is no ambiguity.

We will mainly encounter two kinds of spaces: schemes with the Zariski topology, and complex manifolds/analytic varieties/complex spaces with the analytic topology.

If X is a complex manifold, denote by $\mathrm{H}^*(X, \mathbf{C})$ its singular/de Rham/sheaf cohomology with coefficients in $\mathbf{C}$, which are canonically identified. If we want to use the de Rham model in particular, we may write $\mathrm{H}^*_{\mathrm{dR}}(X)$ to emphasize.

If X is a scheme, $\mathrm{D}(X)$ will denote the unbounded derived category of quasi-coherent sheaves on X. If X is a complex space, $\mathrm{D}(X)$ will denote the derived category of abelian sheaves on X. This should not cause any confusion. Similarly for notations like $\mathrm{D}^+(X)$.

Derived functors are written as $\mathrm{R}f_*$ and $\otimes^{\mathrm{L}}$ in the usual way. For a complex $\mathcal{F}^\bullet$ of sheaves on a space X, the hypercohomology is defined by $\mathbf{H}^*(X, \mathcal{F}^\bullet) = \mathrm{H}^*(\mathrm{R}\Gamma(X, \mathcal{F}^\bullet))$, where $\mathrm{R}\Gamma(X, -)$ is the derived global section.

4. A filtered complex $(K^\bullet, F^\bullet)$ is a complex of filtered objects. The filtration on each term is required to be finite, but can be infinite globally.

For a filtered complex $(K^\bullet, F^\bullet)$ in an abelian category $\mathcal{C}$, we have an associated spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}_F^p K^\bullet) \Rightarrow \mathrm{H}^{p+q}(K^\bullet)$$

as defined in [Del71, p. 1.3], which is functorial in $(K^\bullet, F^\bullet)$. We may write $E(K^\bullet, F^\bullet)$ when emphasizing this functoriality. For an increasing filtration $W_\bullet$ we obtain the spectral sequence by regarding W as a decreasing filtration.

A special case is that $F^i K^\bullet = \sigma^{\geq i} K^\bullet$ is the naive truncation filtration. In this case the spectral sequence reads

$$E_1^{p,q} = \mathrm{H}^q(K^p) \Rightarrow \mathrm{H}^{p+q}(K^\bullet).$$

For a left exact functor $T : \mathcal{C} \to \mathcal{D}$ between abelian categories and a bounded below filtered complex $(K^\bullet, F^\bullet)$ in $\mathcal{C}$, we have the associated spectral sequence

$$E_1^{p,q} = \mathrm{R}^{p+q}T(\mathrm{Gr}_F^p K^\bullet) \Rightarrow \mathrm{R}^{p+q}T(K^\bullet)$$

that is functorial in $(K^\bullet, F^\bullet)$. It can be constructed by filtered T-acyclic resolutions and apply the construction above, see Section 9.

5. For a commutative ring A, write A/p for the quotient ring $A \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z}$. For an abelian group or a sheaf A, write A/p for $\mathrm{coker}(p : A \to A)$.

3. Review of Classical Hodge Theory

In its classical form, the Hodge decomposition theorem states that the cohomology of a compact Kähler manifold admits a canonical decomposition into bigraded pieces.

Let X be a complex manifold. Denote by $\mathcal{A}_X^n$, $\mathcal{A}_X^{p,q}$, $\mathcal{O}_X$ and Ω_X^n the sheaf of smooth complex n-forms, smooth (p,q)-forms, holomorphic functions and holomorphic n-forms on X, respectively.

The exterior differential d of smooth forms gives rise to a chain complex $\mathcal{A}_X^\bullet$, which is a soft resolution of $\mathbf{C}_X$ by the Poincaré lemma [Voi02, Proposition 4.18]. This leads to the isomorphism

$$\mathrm{H}^n(X, \mathbf{C}_X) = \mathbf{H}^n(X, \mathcal{A}_X^\bullet) = \mathrm{H}^n(A^\bullet(X)) = \mathrm{H}^n_{\mathrm{dR}}(X)$$

between the sheaf cohomology and de Rham cohomology of X, where $A^n(X)$ denotes the space of global n-forms on X.

The decomposition $\mathrm{d} = \partial + \bar{\partial}$ gives rise to a bicomplex $\mathcal{A}_X^{\bullet,\bullet}$. Similarly as above, the $\bar{\partial}$-Poincaré lemma [Voi02, Proposition 2.36] implies that $\mathcal{A}_X^{p,\bullet}$ is a soft resolution of Ω_X^p for each p. On one hand, this leads to the isomorphism

$$\mathrm{H}^q(X, \Omega_X^p) = \mathbf{H}^q(X, \mathcal{A}_X^{p,\bullet}) = \mathrm{H}^q(A^{p,\bullet}(X)) = \mathrm{H}^{p,q}_{\bar{\partial}}(X)$$

between $\mathrm{H}^q(X, \Omega_X^p)$ and the Dolbeault cohomology $\mathrm{H}^{p,q}_{\bar{\partial}}(X)$, where $A^{p,q}(X)$ is the space of global (p,q)-forms on X. On the other hand, this means that the bicomplex $\mathcal{A}_X^{\bullet,\bullet}$ provides a soft resolution of the holomorphic de Rham complex

$$0 \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots.$$

By definition $\mathrm{Tot}(\mathcal{A}_X^{\bullet,\bullet}) = \mathcal{A}_X^\bullet$, thus the complex $\Omega_X^\bullet$ is quasi-isomorphic to $\mathcal{A}_X^\bullet$ via the inclusion map, and

$$\mathbf{H}^n(X, \Omega_X^\bullet) = \mathbf{H}^n(X, \mathcal{A}_X^\bullet) = \mathrm{H}^n(X, \mathbf{C}).$$

The induced fact that $\Omega_X^\bullet$ is a resolution of $\mathbf{C}_X$ is referred to as the holomorphic Poincaré lemma.

Let

$$\mathrm{H}^{p,q}(X) = \{\alpha \in \mathrm{H}^{p+q}_{\mathrm{dR}}(X) \mid \alpha = [\omega] \text{ for } \omega \in A^{p,q}(X)\}$$

be the space of de Rham cohomology classes representable by a (p,q)-form. Then we have the following theorem.

Theorem 3.1 (Hodge Decomposition)**.** *[Voi02, Section 6.1.3] Assume X is compact Kähler.*

(1) There is a direct sum decomposition

$$\mathrm{H}^n_{\mathrm{dR}}(X) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X).$$

Moreover, $\overline{\mathrm{H}^{p,q}(X)} = \mathrm{H}^{q,p}(X)$ under the natural conjugation on $\mathrm{H}^n_{\mathrm{dR}}(X)$.

(2) If $\mathrm{H}^{p,q}_{\bar{\partial}}(X)$ denotes the Dolbeault cohomology of X, then there is a canonical isomorphism $\mathrm{H}^{p,q}(X) \cong \mathrm{H}^{p,q}_{\bar{\partial}}(X)$ preserving the representative (p,q)-forms.

More formally, we can say that $\mathrm{H}^n(X, \mathbf{Q})$ admits a Hodge structure of weight n, in the following sense.

Definition 3.2. A **(pure, rational) Hodge structure** V of weight n consists of:

(1) A finite-dimensional $\mathbf{Q}$-vector space $V_{\mathbf{Q}}$;

(2) A decomposition of $V_{\mathbf{C}} = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ as

$$V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q},$$

such that $\overline{V^{p,q}} = V^{q,p}$ under the conjugation on $V_{\mathbf{C}}$ defined by $v \otimes z \mapsto v \otimes \overline{z}$ for $v \in V$ and $z \in \mathbf{C}$.

A **morphism of Hodge structures** (of weight n) is a map of the underlying $\mathbf{Q}$-vector space preserving the bidegrees.

Remark 3.3. The decomposition of $V_{\mathbf{C}}$ can be equivalently described as a representation on $V_{\mathbf{R}} = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ of the **Deligne torus** $S = \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{\mathrm{m}}$, where $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}$ is the **Weil restriction** functor from $\mathbf{C}$ to $\mathbf{R}$, see [Del71, Section 2.1].

Remark 3.4. The category HS_n of Hodge structures of weight n is abelian. For a morphism $f : V \to V'$ in HS_n, the kernel $\ker(f)$ has underlying vector space $\ker(f)_{\mathbf{Q}} = \ker(V_{\mathbf{Q}} \to V'_{\mathbf{Q}})$, with the induced bidegree decomposition. Similar for cokernels.

Remark 3.5. Hodge structures of different weights are related by shifting the bidegrees by $(1, 1)$, hence in fact we only need to consider Hodge structures of weight 0 and 1. But it is convenient to allow different weights, which is also necessary for defining mixed Hodge structures.

Similarly, a morphism of Hodge structures, as defined above, is called of degree 0. A degree d morphism of Hodge structures, from a Hodge structure of weight n to a Hodge structure of weight $n + 2d$, lifts the bidegrees by (d, d).

The Hodge decomposition theorem is obtained, roughly, by considering the Laplacians on X induced by the Kähler metric and appealing to the Hodge theorem on harmonic forms. Note that the decomposition, though relying on the existence of the Kähler metric to hold, is independent of the Kähler metric.

In this form, the Hodge decomposition is exclusive to the case of compact Kähler manifolds, which only constitute a small part of the realm of algebraic geometry. Namely, every smooth projective complex algebraic variety, after analytification, is a compact Kähler manifold. There thus leave a lot of spaces for algebraic geometers trying to generalize this theorem:

Question 3.6.
(1) Is there a "Hodge decomposition" for smooth projective algebraic varieties over a general field? More generally, what about the relative case, namely for smooth projective (or even proper) families of schemes?
(2) Is there a "Hodge decomposition" for varieties that might be noncompact or singular?

These two questions lead to the notions of the Hodge-de Rham spectral sequence and mixed Hodge structures, respectively.

Part 1. The Hodge-de Rham Spectral Sequence

In this part, we will discuss some extensions of the Hodge theory to base schemes other than $\mathbf{C}$.

4. Basic Definitions

The Hodge decomposition itself cannot be directly generalized to such settings. One evidence is that the definition of a Hodge structure makes use of the complex conjugation. Another evidence comes from the theory of variations of Hodge structure, which is a complex analytic story of how a family of complex Kähler manifolds give rise to a varying family of Hodge structures. In the following we give a brief introduction to the subject. Details are referred to [Voi02].

Let $f : X \to S$ be a family of compact Kähler manifolds, meaning f is a proper submersion of complex manifolds, such that the fiber $X_s = f^{-1}(s)$ admits a Kähler metric for every $s \in S$. The cohomology groups $\mathrm{H}^n(X_s, \mathbf{C})$ fit together to give the local system $\mathrm{H}^n(X/S) = \mathrm{R}^n f_* \mathbf{C}$, the n-th higher pushforward of the constant sheaf $\mathbf{C}$ on X. Extending scalars from $\mathbf{C}$ to $\mathcal{O}_S$ gives a holomorphic vector bundle $\mathcal{H}^n(X/S) = \mathrm{H}^n(X/S) \otimes_{\mathbf{C}} \mathcal{O}_S$ over S. (We identify locally free sheaves with vector bundles.) Inside each fiber $\mathcal{H}^n(X/S)|_s = \mathrm{H}^n(X_s, \mathbf{C})$ are direct sum components $\mathrm{H}^{p,n-p}$ for each p; they fit together to give smooth subbundles $\mathcal{H}^{p,n-p}(X/S) \subset \mathcal{H}^n(X/S)$, by [Voi02, Proposition 9.22]. However, among them only $\mathcal{H}^{n,0}(X/S)$ would be a *holomorphic* subbundle in general. Instead, if we define the Hodge filtration

$$F^i \mathcal{H}^n(X/S) = \bigoplus_{p \geq i} \mathcal{H}^{i,n-i}(X/S),$$

then $F^i \mathcal{H}^n(X/S)$ are holomorphic subbundles of $\mathcal{H}^n(X/S)$ for all i.

There is an algebraic construction of $\mathcal{H}^n$ and the filtration $\mathcal{F}^i \mathcal{H}^n$. For a complex manifold X, we have the holomorphic de Rham complex

$$0 \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots .$$

In the relative case of a submersive family $f : X \to S$, we also have the relative holomorphic de Rham complex $\Omega_{X/S}^\bullet$ defined by

$$0 \to f^* \Omega_S^1 \to \Omega_X^1 \to \Omega_{X/S}^1 \to 0$$

and $\Omega_{X/S}^n = \bigwedge^n \Omega_{X/S}^1$, with differentials induced from that of $\Omega_X^\bullet$. The complex $\Omega_{X/S}^\bullet$ is a resolution of the sheaf $f^{-1}\mathcal{O}_S$ on X. If f is also proper, we can use the projection formula to obtain

$$\mathrm{R}^n f_* \Omega_{X/S}^\bullet = \mathrm{R}^n f_* (\mathbf{C} \otimes_{\mathbf{C}} f^{-1}\mathcal{O}_S) = \mathrm{R}^n f_* \mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_S = \mathcal{H}^n(X/S).$$

Consequently, the spectral sequence associated to $\Omega_{X/S}^\bullet$ and f is given by

$$E_1^{p,q} = \mathrm{R}^q f_* \Omega_{X/S}^p \Rightarrow \mathrm{R}^{p+q} f_* \Omega_{X/S}^\bullet = \mathcal{H}^{p+q}(X/S).$$

Also, one can prove that $\mathrm{R}^q f_* \Omega_{X/S}^p$ is a vector bundle over S such that the fiber $\mathrm{R}^q f_* \Omega_{X/S}^p|_s = \mathrm{H}^q(X_s, \Omega_{X_s}^p)$ by analytical methods.

By definition, this spectral sequence induces the filtration

$$F^i \mathrm{R}^n f_* \Omega_{X/S}^\bullet = \mathrm{Im}(\mathrm{R}^n f_* \sigma^{\geq i} \Omega_{X/S}^\bullet \to \mathrm{R}^n f_* \Omega_{X/S}^\bullet)$$

on the target, where $\sigma^{\geq i}$ is the naive truncation. It turns out this filtration coincide with the Hodge filtration defined before.

When S is point, the spectral sequence looks like

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega_X^p) \Rightarrow \mathrm{H}^{p+q}(X, \mathbf{C}).$$

When X is a compact Kähler manifold, the Hodge decomposition theorem implies that this spectral sequence degenerates at the E_1-page by dimension counting. For a general family of compact Kähler manifolds, the spectral sequence also degenerates at the E_1-page by taking fibers to reduce to the above case.

The above discussion inspires the following definition.

Definition 4.1. Let S be a noetherian scheme, and $f : X \to S$ a scheme of finite type over S. In the remaining part of the paper, we shall abbreviate this as "**let $f : X \to S$ be a family of schemes**". We have the de Rham complex $\Omega^\bullet_{X/S}$ of coherent sheaves on X.

(1) Denote by $\mathcal{H}^{p,q}_{\mathrm{Hodge}}(X/S) = \mathrm{R}^q f_* \Omega^p_{X/S}$ the **Hodge cohomology** of X over S.

(2) Denote by $\mathcal{H}^n_{\mathrm{dR}}(X/S) = \mathrm{R}^n f_* \Omega^\bullet_{X/S}$ the **de Rham cohomology** of X over S.

(3) The spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p,q}_{\mathrm{Hodge}}(X/S) \Rightarrow \mathcal{H}^n_{\mathrm{dR}}(X/S)$$

associated to $\Omega^\bullet_{X/S}$ and f is called the **Hodge-de Rham spectral sequence**. Sometimes we may denote this spectral sequence by $E(X/S)$ for convenience.

The filtration on $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ associated to the Hodge-de Rham spectral sequence is called the **Hodge filtration**. It is given by

$$F^i \mathcal{H}^n(X/S) = \mathrm{Im}(\mathrm{R}^n f_* \sigma^{\geq i} \Omega^\bullet_{X/S} \to \mathrm{R}^n f_* \Omega^\bullet_{X/S}).$$

Similar definitions can be made for a proper submersive family of complex manifolds, or in fact for any map of complex spaces.

If S is a scheme of finite type over $\mathbf{C}$ and $f : X \to S$ is a smooth family of schemes over $\mathbf{C}$, we can talk about their analytification S^{an} and $f^{\mathrm{an}} : X^{\mathrm{an}} \to S^{\mathrm{an}}$ as complex spaces (if S is smooth, then S^{an} and X^{an} are plain complex manifolds). The algebraic de Rham complex $\Omega^\bullet_{X/S}$ has the holomorphic one $\Omega^\bullet_{X^{\mathrm{an}}/S^{\mathrm{an}}}$ as its analytification. Especially, if $S = \operatorname{Spec} \mathbf{C}$ and f is projective, then we can use Serre's GAGA principle ([Ser56]) to identify the algebraic and holomorphic version of Hodge-de Rham spectral sequence, so we obtain the following corollary of the Hodge decomposition theorem.

Corollary 4.2. *If X is a smooth projective complex algebraic variety, then the (algebraic) Hodge-to-de Rham spectral sequence of X degenerates at the E_1-page.*

We are therefore led to ask the following question.

Question 4.3. Let $f : X \to S$ be a family of schemes.

(1) Are the Hodge and de Rham cohomologies $\mathcal{H}^{p,q}_{\mathrm{Hodge}}(X/S)$ and $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ vector bundles over S? Do their formations commute with base change?
(2) Does the Hodge-de Rham spectral sequence $E(X/S)$ degenerates at the E_1-page?

We provide the following affirmative answer to the above question.

Theorem 4.4. *[Del68, Theorem 5.5] Let $f : X \to S$ be a smooth proper family of schemes, with S of characteristic 0, by which we mean that all residue fields of points in S has characteristic 0. Then we have*

(1) The Hodge and de Rham cohomologies $\mathcal{H}^{p,q}_{\mathrm{Hodge}}(X/S)$ and $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ are vector bundles over S. Their formations commute with base change.

(2) The Hodge-de Rham spectral sequence $E(X/S)$ degenerates at the E_1-page.

The theorem no longer holds if S has positive characteristic. In the following, we first discuss the special case where $S = \operatorname{Spec} K$ for a field K. We give two different proofs: One is by reducing to the case $K = \mathbf{C}$ and appealing to the classical Hodge theory; The other, however, is purely algebraic and makes use of positive characteristic!

5. Degeneration over a Field by Reduction to $\mathbf{C}$

We will need the following theorem on cohomology and base change, see [Del68, Theorem 3.3].

Theorem 5.1. *Given a noetherian local ring A and a proper morphism $f : X \to S = \operatorname{Spec} A$, and a bounded complex of coherent sheaves $\mathcal{F}^\bullet$ on X such that each $\mathcal{F}^q$ is flat over A. For any noetherian ring B and a map $A \to B$, there are base change morphisms $\varphi^n_B : \mathrm{H}^n(X, \mathcal{F}^\bullet) \otimes_A B \to \mathrm{H}^n(Y, (g')^*\mathcal{F}^\bullet)$ as in the diagram **??**.*

The followings are equivalent:

(1) The base change morphisms φ^n_B are always an isomorphism.

(2) The cohomology groups $\mathrm{H}^n(X, \mathcal{F}^\bullet)$ are all free as A-modules.

Moreover if A is Artinian with residue field k, we always have

$$l_A(\mathrm{H}^n(X, \mathcal{F}^\bullet)) \leq l_A(A) \cdot \dim_k(\mathrm{H}^n(X_0, \mathcal{F}^\bullet|_{X_0})),$$

whose equality for all n being also equivalent to (1) and (2). Here $X_0 = X \otimes_A k$.

By Theorem 5.1, the two assertions in (1) of Theorem 4.4 are equivalent. When $S = \operatorname{Spec} K$ is the spectrum of a field, they are vacuous, hence we only need to focus on the degeneration of $E(X/K)$.

There is a "Lefschetz principle", which is useful in reducing lots of propositions over a general field of characteristic 0 to $\mathbf{C}$, where we can use analytic methods. More explicitly, we have the following proposition.

Proposition 5.2 (Lefschetz Principle)**.** *Let K be a field of characteristic 0, and X is a scheme separated over K of finite type, then*

(1) There exists a subfield $K_0 \subset K$ finitely generated over $\mathbf{Q}$ such that there exist a scheme X_0 over K_0 which becomes X/K after base change.

(2) There exists an embedding $K_0 \hookrightarrow \mathbf{C}$ of fields.

Proof. (1) Since X is separated of finite type over K, it admits a finite cover by affine open sets, which, together with their intersections, are all of the form $\operatorname{Spec} K[x_1, ..., x_n]/(f_1, ..., f_m)$. We can also express the gluing data of the cover by finitely many polynomial maps. Let $K_0 \subset K$ be the subfield generated over $\mathbf{Q}$ by the coefficients of the polynomials used. Then the same data, viewed as over K_0, can be used to construct X_0/K_0 as desired.

(2) Factor $K_0/\mathbf{Q}$ as a purely transcendental extension $L = \mathbf{Q}(\alpha_1, ..., \alpha_n)$ and a finite extension K_0/L. By choosing n algebraic independent elements in $\mathbf{C}$ (which exist by cardinal consideration), we can embed L into $\mathbf{C}$. Then the embedding can be extended to K_0 since $\mathbf{C}$ is algebraically closed. □

The degeneration of $E(X/K)$ amounts to the equation

$$\sum_{p+q=n} \dim_K \mathrm{H}^q(X, \Omega_X^p) = \dim_K \mathbf{H}^n(X, \Omega_X^\bullet),$$

which is invariant of the base field. Thus we can assume $K = \mathbf{C}$. In this case, the result already holds if X is projective, by Corollary 4.2. For a general proper variety X, we use Chow's lemma to find a projective variety X' with a birational morphism $f : X' \to X$. By resolution of singularity we can moreover assume that X' is smooth. We compare $E(X/\mathbf{C})$ and $E(X'/\mathbf{C})$. The morphism $\Omega_X^\bullet \to f_*\Omega_{X'}^\bullet$ induces a pullback $f^* : E(X/\mathbf{C}) \to E(X'/\mathbf{C})$ of spectral sequences. On the E_1-page, it is given by

$$f^* : \mathrm{H}^q(X, \Omega_X^p) \to \mathrm{H}^q(X, f_*\Omega_{X'}^p) \to \mathrm{H}^q(X', \Omega_{X'}^p).$$

There is also a Gysin map $f_! : \mathrm{H}^q(X', \Omega_{X'}^p) \to \mathrm{H}^q(X, \Omega_X^p)$ defined via Grothendieck-Hartshorne duality, such that $f_! \circ f^* = \mathrm{id}$, see [Del68, Section 4]. Especially, f^* is injective. Then an inductive argument on pages deduces the degeneration of $E(X/\mathbf{C})$ from that of $E(X'/\mathbf{C})$. This finishes the proof.

Remark 5.3. A further analysis [Del68, Section 5.3] reveals that the Hodge decomposition Theorem 3.1 in fact holds for any smooth proper complex algebraic varieties, which may not in general admit any Kähler metric.

6. Degeneration over a Field by Reduction Modulo p

In this section, we present a completely different proof of the degeneration of $E(X/K)$, which is independent of any analytic result. Surprisingly, this is done by reduce to the case where K is a field of characteristic p.

Let K be a field of characteristic 0 and X a smooth proper variety over K. The core technique come into play is the passage to limits. Write $K = \varinjlim A_\alpha$ as the filtered colimit of its finitely generated $\mathbf{Z}$-subalgebras. By [Stacks, Tags 0C0C, 081F], there is a scheme X_α smooth and proper over A_α which becomes X/K after base change. By construction, A_α is an integral domain of finite type over $\mathbf{Z}$, whose quotient field is contained in K. Up to replacing A_α by a localization $A_\alpha[t^{-1}]$ for some nonzero $t \in A_\alpha$, we could moreover assume that A_α is smooth over $\mathbf{Z}$ using generic smoothness [Vak, Theorem 21.3.5] and [Stacks, Tags 0C0C]. From now on we will write A for A_α and $\mathfrak{X}$ for X_α, and $f : \mathfrak{X} \to \operatorname{Spec} A$ for the structure map. By the semi-continuity theorem and Grauert's theorem [Vak, Theorem 28.1.1, 28.1.5] (and their variants for complexes), the Hodge and de Rham cohomologies $\mathrm{R}^q f_*\Omega_{\mathfrak{X}/A}^p$ and $\mathrm{R}^q f_*\Omega_{\mathfrak{X}/A}^\bullet$ are locally free on a nonempty open set of A, thus we may further localize to assume they are locally free on A. By Theorem 5.1, their formations commute with base change. Especially, for any map $A \to k$ into a field k, we have

$$\begin{aligned} \dim_k \mathrm{H}^q(X_k, \Omega_{X_k}^p) &= \operatorname{rank} \mathrm{R}^q f_*\Omega_{\mathfrak{X}/A}^p, \\ \dim_k \mathbf{H}^q(X_k, \Omega_{X_k}^\bullet) &= \operatorname{rank} \mathrm{R}^q f_*\Omega_{\mathfrak{X}/A}^\bullet, \end{aligned} \tag{6.1}$$

where $X_k = \mathfrak{X} \otimes_A k$ is the base change.

The degeneration of $E(X/K)$ amounts to a dimensional equation, which translates into

$$\sum_{p+q=n} \operatorname{rank} \mathrm{R}^q f_*\Omega_{\mathfrak{X}/A}^p = \operatorname{rank} \mathrm{R}^n f_*\Omega_{X/S}^\bullet$$

by (6.1) applied to $k = K$. Again by (6.1), this is equivalent to the degeneration of $E(X_s/\kappa(s))$ for some special fiber $X_s = \mathfrak{X} \otimes_A \kappa(s)$, where $s \in \operatorname{Spec} A$ and $\kappa(s)$ is its residue field. For those s mapped to closed points by the smooth map $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{Z}$, the residue fields $\kappa(s)$ will be of positive characteristic.

Hence we see that degeneration of $E(X/K)$ for fields of characteristic 0 follows from degeneration over fields of characteristic p. However, this only holds conditionally: We should assume that $\dim X < p$ and X is *liftable* to the *truncated Witt ring* $W_2(k)$, which we explains in the following.

For a perfect field k of characteristic p, the (second) truncated Witt ring $W_2(k)$ is the unique flat lifting (up to isomorphism) of k from $\mathbf{F}_p$ to $\mathbf{Z}/p^2\mathbf{Z}$. A concise introduction on Witt rings is given in [Ser95]. The ring $W_2(k)$ is given set-theoretically by ordered pairs $(a_0, a_1) \in k^{\oplus 2}$ with addition and multiplication defined by

$$(a_0, a_1) + (b_0, b_1) = \left(a_0 + b_0, a_1 + b_1 - \frac{(a_0 + b_0)^p - a_0^p - b_0^p}{p}\right)$$

and

$$(a_0, a_1) \cdot (b_0, b_1) = (a_0 b_0, a_0^p b_1 + b_0^p a_1 + p a_1 b_1)\,,$$

respectively. There is a (unique) ring isomorphism $W_2(\mathbf{F}_p) \cong \mathbf{Z}/p^2\mathbf{Z}$, and by functoriality there is a ring map $\mathbf{Z}/p^2\mathbf{Z} = W_2(\mathbf{F}_p) \to W_2(k)$ for every k. We can identify $W_2(k)/p$ with k by sending (a_0, a_1) to a_0.

We say X is *liftable* to $W_2(k)$ if there is a smooth proper morphism $f : \tilde{X} \to \operatorname{Spec} W_2(k)$ such that $\tilde{X} \otimes_{W_2(k)} k = X$. We have the following theorem.

Theorem 6.2. *[DI87, Corollary 2.4] Let k be a perfect field of characteristic p, and X be a smooth proper variety over k with $\dim X < p$. Suppose X is liftable over the truncated Witt ring $W_2(k)$, then $E(X/k)$ degenerates at the E_1-page.*

We can now use Theorem 6.2 to finish the proof of degeneration in characteristic 0. Recall our finite type smooth $\mathbf{Z}$-algebra A with a smooth proper family $f : \mathfrak{X} \to \operatorname{Spec} A$. We may assume that f is of relative dimension n. By smoothness of $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{Z}$, we can find a prime $p > n$ inside the image of $\operatorname{Spec} A$, and a closed point $s \in \operatorname{Spec} A/p$. The residue field $k = \kappa(s)$ is finite with characteristic p, and $\operatorname{Spec} W_2(k)$ is an infinitesimal thickening of $\operatorname{Spec} k$. The fiber X_s satisfy the condition of Theorem 6.6, because the natural map $\mathcal{O}_{\operatorname{Spec} A, s} \to k$ factors through $W_2(k)$ by the infinitesimal lifting property, so the base change of $\mathfrak{X}$ provides the lift of X_s.

In the rest of this section, we discuss the proof of Theorem 6.2. The key ingredients are the Frobenius map, the Cartier isomorphism, and a partial decomposition result related to the de Rham complex.

6.1. The Frobenius and the Cartier Isomorphism.

Here we give a brief introduction to the Frobenius and the Cartier isomorphism. Details are referred to [Kat70].

Definition 6.3. (1) Suppose S is scheme of characteristic p, meaning $p = 0$ in $\mathcal{O}_S$. The **absolute Frobenius** $F_S : S \to S$ is defined by id_S on topological spaces and by $a \mapsto a^p$ on the structure sheaf.

(2) Suppose $f : X \to S$ is a morphism of schemes of characteristic p. The **Frobenius twist** $X^{(p)}$ is the scheme over S defined as the fiber product

$$\begin{array}{ccc} X^{(p)} & \xrightarrow{W} & X \\ {\scriptstyle f^{(p)}}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[F_S]{} & S \end{array}$$

with $f^{(p)}$ being its structure map, as shown in the above diagram. The **relative Frobenius** $F_{X/S}$ is defined as in the following diagram

$$\begin{array}{ccccc} & & F_X & & \\ & \overset{F_{X/S}}{\nearrow} & & \searrow & \\ X & \dashrightarrow & X^{(p)} & \xrightarrow{W} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^{(p)}} & & \downarrow{\scriptstyle f} \\ & & S & \xrightarrow[F_S]{} & S \end{array}$$

using the universal property of fiber products. The map $F_{X/S}$ is a morphism over S.

Remark 6.4. Let k be a perfect field of characteristic p. Then $\operatorname{Spec} W_2(k)$ is not a scheme of characteristic p. However, the absolute Frobenius F_k lifts to $W_2(k)$ by $(a_0, a_1) \mapsto (a_0^p, a_1^p)$.

For a finite type morphism $f : X \to S$ as above, we have that $F_{X/S} : X \to X^{(p)}$ is a finite universal homeomorphism, see [Stacks, Tags 0CCB, 0CCD].

We shall identify the underlying space of X and $X^{(p)}$ if convenient. In this case, the structure sheaf $\mathcal{O}_{X^{(p)}}$ can be identified with $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S, F} f^{-1}\mathcal{O}_S$, where $F : f^{-1}\mathcal{O}_S \to f^{-1}\mathcal{O}_S$ is given by forming p-th power. The morphism $W : X^{(p)} \to X$ induces the canonical map $W^{-1} : \mathcal{O}_X \to \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S, F} f^{-1}\mathcal{O}_S = \mathcal{O}_{X^{(p)}}$ on structure sheaves. The relative Frobenius $F_{X/S}$ induces the map $F_{X/S}^{-1} : \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S, F} f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ given by $x \otimes \lambda \mapsto \lambda x^p$. The pullback $F_{X/S}^{-1} : \Omega^1_{X^{(p)}/S} \to F_*\Omega^1_{X/S}$ is zero, because for any section $x \otimes \lambda$ of $\mathcal{O}_{X^{(p)}/S}$ we have

$$F_{X/S}^{-1}\mathrm{d}(x \otimes \lambda) = \mathrm{d}(\lambda x^p) = \lambda p x^{p-1}\mathrm{d}x = 0.$$

The Cartier isomorphism, as stated below, is also crucial for the proof. It can be verified by a local computation.

Theorem 6.5 (Cartier Isomorphism)**.** *[Kat70, Theorem 7.2] Given a smooth morphism $f : X \to S$ between schemes of characteristic p. There is a unique homomorphism of $\mathcal{O}_{X^{(p)}}$-algebras*

$$C^{-1} : \bigoplus_i \Omega^i_{X^{(p)}/S} \to \bigoplus_i \mathcal{H}^i(F_{X/S,*}\Omega^\bullet_{X/S})$$

such that $C^{-1}\mathrm{d}(x \otimes 1)$ is the class of $x^{p-1}\mathrm{d}x$ for local sections x of $\mathcal{O}_X$. Moreover, C^{-1} is an isomorphism.

6.2. **Decomposition of the de Rham complex.** In this section, we will write F for the relative Frobenius $F_{X/S}$ for convenience. The following theorem readily implies Theorem 6.2.

Theorem 6.6. *[DI87, Theorem 2.1] Suppose k is a perfect field of characteristic $p > 0$, and X is a smooth scheme over $S = \operatorname{Spec} k$. For any lift $\tilde{X}$ of X to $\tilde{S} = \operatorname{Spec} W_2(k)$, there is a canonical isomorphism*

$$\varphi_{\tilde{X}} : \bigoplus_{i \leq p-1} \Omega^i_{X^{(p)}/S}[-i] \xrightarrow{\sim} \tau^{<p} F_* \Omega^\bullet_{X/S}$$

in $D(X^{(p)})$, which induces the Cartier isomorphism C^{-1} on cohomology.

In fact, under the condition of Theorem 6.2, we have an isomorphism

$$\bigoplus \Omega^i_{X^{(p)}/S}[-i] \xrightarrow{\sim} F_* \Omega^\bullet_{X/S}$$

by Theorem 6.6. Since F is finite, we get

$$\bigoplus_i \mathrm{H}^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}/S}) = \mathbf{H}^n(X^{(p)}, F_* \Omega^\bullet_{X/S}) = \mathbf{H}^n(X, \Omega^\bullet_{X/S}).$$

Moreover, since $X^{(p)}$ is obtained from X by base change along a field extension, we have $\dim_k \mathrm{H}^n(X^{(p)}, \Omega^i_{X^{(p)}/S}) = \dim_k \mathrm{H}^n(X, \Omega^i_{X/S})$. Note that F is k-linear, thus the same is the identification $\mathbf{H}^n(X^{(p)}, F_* \Omega^\bullet_{X/S}) = \mathbf{H}^n(X, \Omega^\bullet_{X/S})$. Hence we obtain

$$\sum_i \dim_k \mathrm{H}^{n-i}(X, \Omega^i_{X/S}) = \dim_k \mathbf{H}^n(X, \Omega^\bullet_{X/S}),$$

which equivalently says that $E(X/k)$ degenerates at the E_1-page.

In what follows, we give a proof of Theorem 6.6, following [DI87].

Proof of Theorem 6.6. To give the isomorphism $\varphi_{\tilde{X}}$ it is equivalent to give morphisms $\varphi^i_{\tilde{X}} : \Omega^i_{X^{(p)}/S} \to F_* \Omega^\bullet_{X/S}$ inducing the Cartier isomorphism on cohomology, for $0 \leq i < p$. The morphism $\varphi^0_{\tilde{X}} : \mathcal{O}_{X^{(p)}} \to F_* \Omega^\bullet_{X/S}$ has to be

$$\mathcal{O}_{X^{(p)}} \xrightarrow{C^{-1}} \mathcal{H}^0(F_* \Omega^\bullet_{X/S}) \hookrightarrow F_* \Omega^\bullet_{X/S}.$$

Suppose $\varphi^1_{\tilde{X}} : \Omega^1_{X^{(p)}/S}[-1] \to F_* \Omega^\bullet_{X/S}$ is defined as desired. Then we can define $\varphi^i_{\tilde{X}} (i < p)$ using the diagram

$$\begin{array}{ccc} (\Omega^1_{X^{(p)}/S})^{\otimes i}[-i] & \xrightarrow{(\varphi^1_{\tilde{X}})^{\otimes i}} & (F_* \Omega^\bullet_{X/S})^{\otimes i} \\ {\scriptstyle a_i}\uparrow & & \downarrow{\scriptstyle m} \\ \Omega^i_{X^{(p)}/S}[-i] & \xrightarrow{\varphi^i_{\tilde{X}}} & F_* \Omega^\bullet_{X/S} \end{array},$$

where a_i is the (shift of the) "anti-symmetrization" map defined by

$$\omega_1 \wedge \cdots \wedge \omega_i \mapsto \frac{1}{i!} \sum_{\sigma \in S_i} \operatorname{sgn}(\sigma) \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)},$$

which is the natural way of embedding an exterior algebra into the tensor algebra. However, note that a_i can only be defined for $i < p$ since we are in characteristic p. The map m is defined by the exterior multiplication $\wedge$ on $F_* \Omega^\bullet_{X/S}$. By construction, for local sections $\omega_1, ..., \omega_i (i < p)$ of $\Omega^1_{X^{(p)}/S}$ we would have

$$\varphi^i_{\tilde{X}}(\omega_1 \wedge \cdots \wedge \omega_i) = \varphi^1_{\tilde{X}}(\omega_1) \wedge \cdots \wedge \varphi^1_{\tilde{X}}(\omega_i),$$

thus these $\varphi^i_{\tilde{X}}$ induce the Cartier isomorphism on cohomology.

Then, the strategy is to construct $\varphi^1_{\tilde{X}}$ using a lift of the Frobenius F. Recall we are given a lift $\tilde{X}$ over $\tilde{S}$. The scheme $\tilde{S}$ is not of characteristic p, but it admits a lift $\tilde{F}_S : \tilde{S} \to \tilde{S}$ of the absolute Frobenius F_S, so we can form the fiber product $\tilde{X}^{(p)} = \tilde{X} \times_{\tilde{S}, \tilde{F}_S} \tilde{S}$. By the infinitesimal lifting property, locally there exists a lift $\tilde{F} : \tilde{X} \to \tilde{X}^{(p)}$ of F, but not globally in general.

For now, we assume that a local lift $\tilde{F}$ exists. We identify the underlying spaces of X and $X^{(p)}$. Since the pullback $F^* : \Omega^1_{X^{(p)}/S} \to \Omega^1_{X/S}$ is zero, its lift $\tilde{F}^* : \Omega^1_{\tilde{X}^{(p)}/\tilde{S}} \to \Omega^1_{\tilde{X}/\tilde{S}}$ has image in $p\Omega^1_{\tilde{X}/\tilde{S}}$, where we used the identification $\Omega^1_{\tilde{X}/\tilde{S}}/p = \Omega^1_{X/S}$. Now consider the "multiplication by p" map $p : \Omega^1_{\tilde{X}/\tilde{S}} \to \Omega^1_{\tilde{X}/\tilde{S}}$. Since $p^2\Omega^1_{\tilde{X}/\tilde{S}} = 0$, we get an induced map

$$p : \Omega^1_{X/S} = \Omega^1_{\tilde{X}/\tilde{S}}/p \to p\Omega^1_{\tilde{X}/\tilde{S}},$$

which is an isomorphism. In fact, since $f : X \to S$ is smooth, the sheaf $\Omega^1_{\tilde{X}/\tilde{S}}$ is locally free, and we may thus reduce to showing the same thing for the structure sheaf $\mathcal{O}_{\tilde{X}}$. Then, a local computation gives the result when $X = S$, and also implies the general case using the flatness of the map $f : X \to S$.

Define $\varphi : \Omega^1_{X^{(p)}/S} \to F_*\Omega^1_{X/S}$ by the following diagram

$$\begin{array}{ccc} \Omega^1_{\tilde{X}^{(p)}/\tilde{S}} & \xrightarrow{\tilde{F}^*} & p\Omega^1_{\tilde{X}/\tilde{S}} \\ \downarrow & & p\uparrow\simeq \\ \Omega^1_{X^{(p)}/S} & \xrightarrow{\varphi} & \Omega^1_{X/S} \end{array},$$

where the left vertical map is the quotient map $\Omega^1_{\tilde{X}^{(p)}/\tilde{S}} \to \Omega^1_{\tilde{X}^{(p)}/\tilde{S}}/p = \Omega^1_{X^{(p)}/S}$.

We claim the composition $\Omega^1_{X^{(p)}/S} \xrightarrow{\varphi} \Omega^1_{X/S} \xrightarrow{\mathrm{d}} \Omega^2_{X/S}$ is zero. In fact, take any local section $x_0 \otimes \lambda_0$ of $\mathcal{O}_{X^{(p)}} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S, F} f^{-1}\mathcal{O}_S$, with a lift $x \otimes \lambda$ in $\mathcal{O}_{\tilde{X}^{(p)}}$. By definition of φ,

$$p\varphi(\mathrm{d}(x_0 \otimes \lambda_0)) = \tilde{F}^*\mathrm{d}(x \otimes \lambda) = \mathrm{d}\tilde{F}^*(x \otimes \lambda).$$

We have $F^*(x_0 \otimes \lambda_0) = \lambda_0^p x_0$, thus $\tilde{F}^*(x \otimes \lambda) = \lambda x^p + pu$ for some section u of $\mathcal{O}_X$ (here we used a similar isomorphism $p : \mathcal{O}_X \xrightarrow{\sim} p\mathcal{O}_{\tilde{X}}$ as before). Then

$$\mathrm{d}\tilde{F}^*(x \otimes \lambda) = \lambda p x^{p-1}\mathrm{d}x + p\mathrm{d}u,$$

which implies $\varphi(\mathrm{d}(x_0 \otimes \lambda_0)) = \lambda_0 x_0^{p-1}\mathrm{d}x_0 + \mathrm{d}u_0$, for u_0 the reduction of u mod p. Thus $\mathrm{d}\varphi(\mathrm{d}(x_0 \otimes \lambda_0)) = 0$, as desired. Consequently, φ induces a map

$$\varphi^1_{\tilde{X}} : \Omega^1_{X^{(p)}/S}[-1] \to \Omega^\bullet_{X/S},$$

which can be checked to satisfy the requirements.

For the general case, we glue the local constructions together. For details, see [DI87]. Take a finite affine cover $\mathfrak{U} = \{U_i\}$ of X with lifts $\tilde{F}_i : \tilde{U}_i \to \tilde{U}_i^{(p)}$ as above, we get maps $\varphi^1_{\tilde{U}_i} : \Omega^1_{U_i^{(p)}/S} \to F_*\Omega^1_{U_i/S}$. Over each intersection $U_{ij} = U_i \cap U_j$ there are two liftings $\tilde{F}_i|_{U_{ij}}$ and $\tilde{F}_j|_{U_{ij}}$, using whose difference we can canonically define a map $h_{ij} : \Omega^1_{U_{ij}^{(p)}/S} \to F_*\mathcal{O}_{U_{ij}}$ such that $\varphi^1_{\tilde{U}_i}|_{U_{ij}} - \varphi^1_{\tilde{U}_j}|_{U_{ij}} = \mathrm{d}h_{ij}$ with $\mathrm{d} : \mathcal{O}_{U_{ij}} \to \Omega^1_{U_{ij}/S}$ being the differential; moreover, we have $h_{ij} + h_{jk} + h_{ki} = 0$ on U_{ijk}. These together provide a gluing datum defining a global morphism $\varphi_{\tilde{X}} :$

$\bigoplus_{i\le p-1}\Omega^i_{X^{(p)}/S}[-i]\xrightarrow{\sim}\tau^{<p}F_*\Omega^\bullet_{X/S}$ by derived descent theory. In [DI87, pp. 252-253] a classical descent argument using the Čech resolution is given. Namely, For each p, the Čech complex $\check{\mathcal{C}}^\bullet(\mathfrak{U},\Omega^p_{X/S})$ of sheaves is a resolution of $\Omega^p_{X/S}$ (see [Stacks, Tags 01EO]). Thus the de Rham complex $\Omega^\bullet_{X/S}$ can be resolved by the bicomplex $\check{\mathcal{C}}^\bullet(\mathfrak{U},\Omega^\bullet_{X/S})$, hence quasi-isomorphic to the total complex $K^\bullet=\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathfrak{U},\Omega^\bullet_{X/S}))$. The maps $\varphi^1_{\tilde{U}_i}$ can be regarded as from $\Omega^1_{X^{(p)}/S}$ to $F_*j_{i,*}\Omega^1_{U_i/S}$, where $j_i:U_i\hookrightarrow X$ is the inclusion; similarly for h_{ij}. These maps together define a map $\varphi:\Omega^1_{X^{(p)}/S}\to K^1$ whose composite with $\mathrm{d}^1:K^1\to K^2$ vanishes, hence defining a map $\Omega^1_{X^{(p)}/S}[-1]\to K^\bullet\xleftarrow{\sim}\Omega^\bullet_{X/S}$ in the derived category, which can be checked to meet the requirement on cohomology.

□

Remark 6.7. The appearance of the truncation $\tau^{<p}$ comes from the inability to define the anti-symmetrization map a_i for all i, which results in the dimensional constraint in Theorem 6.2. This constraint is essential. A recent result in [Pet23] shows that for any perfect field k of characteristic p, there is a smooth projective variety X/k with $\dim X=p+1$ admitting a lift to $W_2(k)$ such that $E(X/k)$ does not degenerate at the E_1-page.

7. The Family Case

In this section, we briefly sketch the proof of Theorem 4.4 for a general base S, following [Del68].

For any base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S, \end{array}$$

there are natural base change morphisms

$$g^*\mathcal{H}^{p,q}_{\mathrm{Hodge}}(X/S)=g^*\mathrm{R}^qf_*\Omega^p_{X/S}\to\mathrm{R}^qf'_*(g')^*\Omega^p_{X/S}=\mathrm{R}^qf'_*\Omega^p_{X'/S'}=\mathcal{H}^{p,q}_{\mathrm{Hodge}}(X'/S')$$

and similar ones from $g^*\mathcal{H}^n_{\mathrm{dR}}(X/S)$ to $\mathcal{H}^n_{\mathrm{dR}}(X'/S')$. The latter assertion in (1) of [Del68] says precisely that these base change maps are always isomorphisms. By flat base change this holds at least if g is flat. Moreover, for flat g we indeed have an isomorphism $g^*E(X/S)=E(X'/S')$, since they are both associated to the complex $g^*\Omega^\bullet_{X/S}=\Omega^\bullet_{X'/S'}$.

7.1. **Reduction to a Finite-Dimensional Local Artin C-Algebra.** Again, we can focus on the former assertion by Theorem 5.1. There is a so-called "standard argument" in [Del68], which allows us to reduce the proof to the case where $S=\mathrm{Spec}\,A$ is the spectrum of a finite-dimensional local Artin $\mathbf{C}$-algebra A.

By flat base change and faithfully flat descent, we may assume $S=\mathrm{Spec}\,A$ is the spectrum of a complete noetherian local ring, with residue field of characteristic 0.

The next step is to use the theorem on formal functions to reduce to $A/\mathfrak{m}^n$, where $\mathfrak{m}$ is the maximal ideal of A. The theorem on formal functions [Vak, p. 29.4.2] asserts that for any coherent sheaf $\mathcal{F}$ on

$$\mathrm{H}^i(X,\mathcal{F})=\varprojlim\mathrm{H}^i(X_n,\mathcal{F}|_{X_n}),$$

where $X_n = X \times_S S_n$ with $S_n = \operatorname{Spec} A/\mathfrak{m}^n$.

Suppose (1) and (2) of Theorem 4.4 hold for each base $A/\mathfrak{m}^n$. The theorem on formal functions implies $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X/S) = \varprojlim \mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_n/S_n)$ as A-modules, where the A-module structure on $\varprojlim \mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_n/S_n)$ is induced by the $A/\mathfrak{m}^n$-module structures on $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_n/S_n)$ together. By assumption (2) we also have $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_n/S_n) = \mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_{n+1}/S_{n+1}) \otimes_{A/\mathfrak{m}^{n+1}} A/\mathfrak{m}^n$. Choosing a basis for $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_0/S_0)$ and lifting it inductively to $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X_n/S_n)$, we can find a basis for $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X/S)$, showing its freeness.

The spectral sequence $E(X_n/S_n)$ is associated to the complex $\Omega^\bullet_{X_n/S_n} = \Omega^\bullet_{X/S}|_{X_n}$ on X_n, which can be viewed as the complex $\Omega^\bullet_{X/S}/\mathfrak{m}^n\Omega^\bullet_{X/S}$ on X when taking cohomologies. By functorality there is an inverse system

$$\cdots \to E(X_{n+1}/S_{n+1}) \to E(X_n/S_n) \to \cdots$$

with $E(X/S)$ mapping to all of them. By the theorem on formal functions we have $E_1(X/S) = \varprojlim E_1(X_n/S_n)$, from which we can use the degeneration of $E(X_n/S_n)$ to deduce that of $E(X/S)$. Finally, the degeneration of $E(X/S)$ implies that $\mathrm{H}^*_{\mathrm{dR}}(X/S)$ are extensions of the free A-modules $\mathrm{H}^{p,q}_{\mathrm{Hodge}}(X/S)$, hence they are also free.

Thus we may assume $S = \operatorname{Spec} A$ for an Artin local ring A with characteristic 0, so that especially A contains $\mathbf{Q}$. By [Har77, Theorem 8.25A], we can find a field $k \subset A$ such that $k \to A \to A/\mathfrak{m}$ is an isomorphism. Then A becomes a finite dimensional local Artin k-algebra. The final step is to reduce from k to $\mathbf{C}$ using the Lefschetz principle.

7.2. **Comparison of Hodge-de Rham Spectral Sequences.** For a finite dimensional local Artin $\mathbf{C}$-algebra A and a smooth proper map $f : X \to S = \operatorname{Spec} A$, consider the Cartesian square

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \longrightarrow & S \end{array}$$

where $S_0 = \operatorname{Spec} \mathbf{C}$ is the closed point of S. Since f_0 is smooth, we have that X_0 is reduced, hence can be identified with X^{red}, the reduced scheme structure of X. We have Hodge-de Rham spectral sequences

$$E_1^{p,q}(X/S) = \mathrm{H}^q(X, \Omega^p_{X/S}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X)$$

and

$$E_1^{p,q}(X_0/S_0) = \mathrm{H}^q(X_0, \Omega^p_{X_0/S_0}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X_0).$$

Since X is proper over $\mathbf{C}$, we can apply Serre's GAGA principle (for nonreduced schemes, [Ser56]) to conclude

$$\mathrm{H}^n_{\mathrm{dR}}(X) = \mathbf{H}^n(X, \Omega^\bullet_{X/S}) = \mathbf{H}^n(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}/S^{\mathrm{an}}}) = \mathrm{H}^n(X^{\mathrm{an}}, A),$$

where the last equality is because $\Omega^\bullet_{X^{\mathrm{an}}/S^{\mathrm{an}}}$ is a resolution of the constant sheaf A on X^{an}. Also, we have $\mathrm{H}^n_{\mathrm{dR}}(X_0) = \mathrm{H}^n(X^{\mathrm{an}}, \mathbf{C})$ again by GAGA. Thus by the universal coefficient theorem, we have

$$\dim_{\mathbf{C}} \mathrm{H}^n_{\mathrm{dR}}(X) = \dim_{\mathbf{C}} A \cdot \dim_{\mathbf{C}} \mathrm{H}^n_{\mathrm{dR}}(X_0).$$

We also have

$$\sum_{p+q=n} \dim_{\mathbf{C}} \mathrm{H}^q(X, \Omega^p_{X/S}) \geq \dim_{\mathbf{C}} \mathrm{H}^n_{\mathrm{dR}}(X)$$

and

$$\dim_{\mathbf{C}} \mathrm{H}^q(X, \Omega^p_{X/S}) \leq \dim_{\mathbf{C}} A \cdot \dim_{\mathbf{C}} \mathrm{H}^q(X_0, \Omega^p_{X_0/S_0})$$

by the spectral sequence $E(X/S)$ and Theorem 5.1, respectively. Also, by Section 5 we know $E(X_0/S_0)$ degenerates, thus

$$\sum_{p+q=n} \dim_{\mathbf{C}} \mathrm{H}^q(X_0, \Omega^p_{X_0/S_0}) = \dim_{\mathbf{C}} \mathrm{H}^n_{\mathrm{dR}}(X_0).$$

These formulas put together, we can see that the two equalities above are achieved. Hence $E(X/S)$ also degenerates, and $\mathrm{H}^q(X, \Omega^p_{X/S})$ are all free as A-modules by Theorem 5.1, finishing the proof.

Part 2. Mixed Hodge Structures

From now on, we turn to the question of finding a "Hodge structure" on a complex algebraic variety X which may not be smooth or compact. We will always work over $\mathbf{C}$, use the analytic topology and we do not distinguish between various cohomology theories (de Rham, sheaf, singular) and simply write $\mathrm{H}^*(X)$. We say variety simply for complex algebraic varieties.

8. Motivating Mixed Hodge Structures

First, we provide two simplest examples in the case of curves.

Example 8.1 (a complete nodal curve)**.** Let X be a complete nodal curve, with node p. Let $\pi : \tilde{X} \to X$ be the normalization map, and let $\pi^{-1}(p) = \{p_1, p_2\}$. Let us focus on the cohomology group $\mathrm{H}^1(X)$. We have the following long exact sequence associated to the pair $(\tilde{X}, \{p_1, p_2\})$:

$$\mathrm{H}^0(\tilde{X}) \to \mathrm{H}^0(\{p_1, p_2\}) \to \mathrm{H}^1(\tilde{X}, \{p_1, p_2\}) \to \mathrm{H}^1(\tilde{X}) \to \mathrm{H}^1(\{p_1, p_2\}).$$

We have $\mathrm{H}^1(\tilde{X}, \{p_1, p_2\}) = \mathrm{H}^1(\tilde{X}/\{p_1, p_2\}) = \mathrm{H}^1(X)$ and $\mathrm{H}^1(\{p_1, p_2\}) = 0$, thus there is a short exact sequence

$$0 \to V \to \mathrm{H}^1(X) \xrightarrow{\pi^*} \mathrm{H}^1(\tilde{X}) \to 0, \tag{8.2}$$

where $V = \mathrm{Im}(\mathrm{H}^0(\tilde{X}) \to \mathrm{H}^0(\{p_1, p_2\})) \cong \mathbf{C}$.

Now $\mathrm{H}^*(X)$ and $\mathrm{H}^*(\{p_1, p_2\})$ admit pure Hodge structures, and the pullback $\mathrm{H}^*(X) \to \mathrm{H}^*(\{p_1, p_2\})$ is a morphism of pure Hodge structures (of degree 0), namely, preserving the bidegrees. Then we see that $\mathrm{H}^1(X)$ becomes an "extension" of pure Hodge structures of different weights 0 and 1.

Example 8.3 (a smooth open curve)**.** Let X be a smooth complete curve, with D a set of points on X. Let $U = X - D$, and denote the two inclusions by $i : D \to X$ and $j : U \to X$. Let us focus on the cohomology group $\mathrm{H}^1(U, \mathbf{C})$. We have the following long exact sequence associated to the pair (X, U):

$$\mathrm{H}^1(X, U) \to \mathrm{H}^1(X) \to \mathrm{H}^1(U) \to \mathrm{H}^2(X, U) \to \mathrm{H}^2(X),$$

where we are omitting the coefficient $\mathbf{C}$ for convenience. Taking a sufficiently small open disk $B_x \subset X$ around each point $x \in D$, we would have

$$\mathrm{H}^*(X, U) = \mathrm{H}^*\left(\bigcup_{x\in D} B_x, \bigcup_{x\in D} B_x - D\right)$$

by excision. Moreover, using Thom isomorphism $\mathrm{H}^*(\{x\}) \xrightarrow{\sim} \mathrm{H}^{*+2}(B_x, B_x - \{x\})$ (see [Voi02, Section 7.2.1]), the original long exact sequence becomes

$$\mathrm{H}^{-1}(D) \xrightarrow{i_!} \mathrm{H}^1(X) \xrightarrow{j^*} \mathrm{H}^1(U) \to \mathrm{H}^0(D) \xrightarrow{i_!} \mathrm{H}^2(X).$$

Here $i_! : \mathrm{H}^*(D) \to \mathrm{H}^{*+2}(X)$ is the Gysin homomorphism, which is in fact dual to the pushforward $i_* : \mathrm{H}_*(D) \to \mathrm{H}_*(X)$ under Poincaré duality. Then we have $\mathrm{H}^{-1}(D) = 0$ and $V = \ker(i_! : \mathrm{H}^0(D) \to \mathrm{H}^2(X)) \cong \mathbf{C}^{|D|-1}$, and there is a short exact sequence

$$0 \to \mathrm{H}^1(X) \xrightarrow{j^*} \mathrm{H}^1(U) \to V \to 0. \tag{8.4}$$

Now let us take the Hodge structures into consideration. For any integer n, the cohomology groups $\mathrm{H}^n(D)$ and $\mathrm{H}^n(X)$ are pure Hodge structures of weight n. The Gysin homomorphism is a morphism of pure Hodge structures of degree 1: That is, the map $j_! : \mathrm{H}^*(D) \to \mathrm{H}^{*+2}(X)$ sends $\mathrm{H}^{p,q}(D)$ to $\mathrm{H}^{p+1,q+1}(X)$. If we shift the bidegree on $\mathrm{H}^*(D)$ up by $(1,1)$ such that $j_!$ is of degree 0, then V becomes a pure Hodge structure of weight 2. In this case $\mathrm{H}^1(U)$ becomes an "extension" of pure Hodge structures of weights 1 and 2.

The above examples show that it is necessary to allow "mixed" weights for defining "Hodge structures" on the cohomology of a general variety. Now we give the precise definition.

Recall (Definition 3.2) that a pure Hodge structure of weight n on a finite-dimensional rational vector space $V_{\mathbf{Q}}$ is given by a decomposition

$$V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. If we define the Hodge filtration $F^\bullet$ by $F^i V = \bigoplus_{p\geq i} V^{p,n-p}$, then we can recover the bigraded pieces by $V^{p,q} = F^p V \cap \overline{F^q V}$. In this way, one obtains [Del71, Proposition 2.1.9], namely to give a pure Hodge structure of weight n, it is equivalent to give a filtration $F^\bullet$ such that

$$F^p V \cap \overline{F^{n-p+1} V} = 0.$$

Also, a morphism of Hodge structures is equivalently a morphism of the underlying $\mathbf{Q}$-vector space preserving the Hodge filtration after base change to $\mathbf{C}$.

Definition 8.5. A **(rational) mixed Hodge structure** (V, W, F) is given by the following data:

(1) A finite-dimensional rational vector space $V_{\mathbf{Q}}$.
(2) An *increasing* filtration $W_\bullet$ on $V_{\mathbf{Q}}$ called the **weight filtration**.
(3) A *decreasing* filtration $F^\bullet$ on $V_{\mathbf{C}} = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ called the **Hodge filtration**.

Such that on any graded piece $\mathrm{Gr}^W_n V_{\mathbf{C}}$, the filtration induced by the Hodge filtration $F^\bullet$ makes $\mathrm{Gr}^W_n V_{\mathbf{Q}}$ a pure Hodge structure of weight n. For what it means by the induced filtration, see Section 9.

The **Hodge number** $h^{p,q} = h^{p,q}(V)$ associated to a mixed Hodge structure is defined as the dimension of the (p,q)-component of $\mathrm{Gr}^W_{p+q} V_{\mathbf{C}}$.

A **morphism of mixed Hodge structures** is a $\mathbf{Q}$-linear map between the underlying rational vector space preserving the two filtrations.

Note a pure Hodge structure of weight n can be regarded as a mixed Hodge structure (V, W, F) such that $\mathrm{Gr}^W_i V_{\mathbf{Q}} = 0$ whenever $i \neq n$. In [Del74], Deligne proves the following theorem:

Theorem 8.6. *[Del74, Section 8.2] For any variety X and integer n, and there is a canonical mixed Hodge structure on $\mathrm{H}^n(X)$, which is functorial under pullback by morphisms of varieties. If X is smooth and projective, then this mixed Hodge structure coincide with the classical pure Hodge structure. Moreover, if (p,q) is a pair of number such that $h^{p,q}(\mathrm{H}^n(X)) \neq 0$, then*

(1) $(p,q) \in [0,n] \times [0,n]$;
(2) If $\dim X = N$ and $n \geq N$, then $(p,q) \in [n-N, N] \times [n-N, N]$;
(3) If X is proper, then $p+q \leq n$;
(4) If X is smooth, then $p+q \geq n$.

This is consistent with Example 8.1 and Example 8.3. In fact, the short exact sequences (8.2) and (8.4) are in fact short exact sequences of mixed Hodge structures. Here we are implicitly appealing to the fact Theorem 9.3 that MHS is abelian.

9. Generality on Filtrations

In this section, we provide some algebraic results on filtrations that are going to be used. We will not prove most of them, and will refer to [Del71] and [Del74] instead. We fix an abelian category $\mathcal{C}$.

9.1. **Strict Morphisms.** For convenience, we will use F to denote the filtration on any filtered object. Denote by $\mathrm{Fil}(\mathcal{C})$ the category of filtered objects in $\mathcal{C}$, where a morphism $f : (X,F) \to (Y,F)$ of filtered object is by definition a morphism $f : X \to Y$ with $f(F^iX) \subset F^iY$.

Definition 9.1. Let (X,F) be a filtered object. For a subobject $A \subset X$, the **induced filtration** on A is defined by $F^iA = A \cap F^iX$. For a quotient object X/B of X, the **induced filtration** on X/B is defined by $F^i(X/B) = (F^iX + B)/B$.

By repeated application of the above definition, we can associate a filtration to any subquotient Y of a filtered object X, which is well-defined by [Del71, Lemma 1.1.9]. We also call it the induced filtration.

One can check that the category $\mathrm{Fil}(\mathcal{C})$ is additive, and admits all kernels and cokernels. The kernel of $f : (X,F) \to (Y,F)$ is given by $\ker(f : X \to Y)$ with induced filtration from X, and similarly for Y. In $\mathrm{Fil}(\mathcal{C})$ we have a canonical morphism

$$\mathrm{coim}(f) = \mathrm{coker}(\ker f \to X) \to \ker(Y \to \mathrm{coker}\, f) = \mathrm{Im}(f)$$

for every f, and $\mathcal{C}$ is abelian if and only if it is always an isomorphism. Although $\mathrm{coim}(f) = \mathrm{Im}(f)$ as an object in $\mathcal{C}$, they may have different filtrations. A computation shows that the two filtrations, induced from X and Y respectively, are equal if and only if f is *strict*, in the following sense.

Definition 9.2. A morphism $f : (X,F) \to (Y,F)$ in $\mathrm{Fil}(\mathcal{C})$ is **strict** if $f(F^iX) = f(X) \cap F^iY$. We also say that f is strict with respect to F.

Especially, we see $\mathrm{Fil}(\mathcal{C})$ is usually not abelian, having lots of non-strict morphisms.

However, an interesting and important fact about Hodge theory is that morphisms of Hodge structures are always strict with the related filtrations.

Theorem 9.3. *[Del71, Theorem 2.3.5]*

(1) The category of mixed Hodge structures MHS *is abelian.*

(2) The kernel $\ker(f)$ *of a morphism* $f : (V, W, F) \to (V', W, F)$ *has underlying vector space* $\ker(f)_{\mathbf{Q}} = \ker(V_{\mathbf{Q}} \to V'_{\mathbf{Q}})$*, with induced filtration* W *and* F*. Similarly for cokernels.*

(3) Morphisms of mixed Hodge structures are strict with respect to W *and* F*.*

(4) The functor $\mathrm{Gr}_F^n : \mathrm{MHS} \to \mathrm{HS}_n$ *is exact.*

(5) The functor $\mathrm{Gr}_n^W : \mathrm{MHS} \to \mathrm{Vect}_{\mathbf{Q}}$ *is exact.*

For the category HS_n, the analogue of the above theorem follows quite directly, because instead of the Hodge filtrations, we can equivalently consider the direct sum decompositions. The proof of the above theorem is also similar, by constructing functorially a direct sum decomposition $V_{\mathbf{C}} = \bigoplus V^{p,q}$ for any mixed Hodge structure (V, F, W) such that $W_i V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=i} V^{p,q}$ and $F^i V_{\mathbf{C}} = \bigoplus_{p \geq i} V^{p,q}$, see [Del71, Lemma 1.2.8, 1.2.11].

9.2. **Bifiltered Objects and Associated Spectral Sequences.** We will also need bifiltered objects, which are objects with two different filtrations. For notational consistency with mixed Hodge structures, we require that one filtration is increasing and the other is decreasing, denoted universally by W and F respectively. (This is not essential since we can transform between increasing and decreasing filtrations.) A morphism of bifiltered objects is a morphism of the underlying objects preserving both W and F.

Let $(K^\bullet, W, F)$ be a bifiltered object, and consider the spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}_{-p}^W K^\bullet) \Rightarrow \mathrm{H}^{p+q}(K^\bullet)$$

associated to $K^\bullet$ and W. The filtration F can be induced to ${}_W E$ in a quite subtle way, which we explain below.

By inducing filtrations from $K^\bullet$ to its subquotient, we obtain a filtration on $E_0^{p,q} = \mathrm{Gr}_{-p}^W K^{p+q}$. Since E_{r+1} is the cohomology of E_r, we can inductively define filtrations on each page E_r such that the filtration on E_{r+1} is induced from E_r.

Definition 9.4. The filtration defined above is called the **recurrence filtration**. Denote it by F.

However, it turns out that for $r \geq 2$, the differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ does not preserve F in general. Moreover, F also induces a filtration on the limit $H^{p+q}(K^\bullet)$, which is further induced to E_∞ by taking graded pieces, which may not be equal to F. These problems lead to Deligne's careful analysis in [Del71, Section 1.3], where he defined two more filtrations on E_r for $1 \leq r \leq \infty$, as given below. Here we follow [Del74, Section 7.2] instead, where Deligne gives a simplified discussion of these.

Definition 9.5. Notations as above. Let $0 \leq r \leq \infty$ be an integer.

(1) The **first direct filtration** F_{d1} is defined by

$$F_{\mathrm{d1}}^i(E_r(K^\bullet, W)) = \mathrm{Im}(E_r(F^i K^\bullet, W) \to E_r(K^\bullet, W)).$$

(2) The **second direct filtration** $F_{\mathrm{d}2}$ is defined by

$$F^i_{\mathrm{d}2}(E_r(K^\bullet, W)) = \ker(E_r(K^\bullet, W) \to E_r(K^\bullet/F^iK^\bullet, W)).$$

In the above definitions we are using the induced filtration W on $F^iK^\bullet$ and $K^\bullet/F^iK^\bullet$.

We have the following basic properties of these three filtrations.

Proposition 9.6. *[Del71, Proposition 1.3.13] Notations as above.*

(1) We have $F_{\mathrm{d}1} \subset F \subset F_{\mathrm{d}2}$ on $E_r (1 \leq r \leq \infty)$. For $r \leq 1$ these three filtrations coincide.

(2) The differential $\mathrm{d}_r (1 \leq r < \infty)$ is compatible with both $F_{\mathrm{d}1}$ and $F_{\mathrm{d}2}$.

(3) Let F' be the filtration on E_∞ induced from $H^*(K^\bullet)$ by taking graded pieces. Then $F_{\mathrm{d}1} \subset F' \subset F_{\mathrm{d}2}$.

The following theorem provides the condition under which everything about filtration works out perfectly as wanted.

Theorem 9.7 (Lemma of Two Filtrations)**.** *[Del71, Theorem 1.3.16] Notations as above.*

Suppose for some integer $r_0 \geq 0$, we have the following condition:

For all $0 \leq r \leq r_0$, the differential d_r is strictly compatible with the recurrence filtration E_r.

Then, for all $r \leq r_0 + 1$, we have $F_{\mathrm{d}1} = F = F_{\mathrm{d}2}$ on E_r.

Remark 9.8. Letting $r_0 \to \infty$, if the condition is always satisfied, then we will have $F_{\mathrm{d}1} = F = F' = F_{\mathrm{d}2}$ on E_∞ by (3) of Proposition 9.6.

The proof of Theorem 9.7 is basically tricky homological algebra, see [Del71, Section 1.3] or [Del74, Section 7.2]. We are going to use it in together with Theorem 9.3, which offers strictness freely.

9.3. **Filtered Derived Category.** In this section, we briefly discuss derived functors in the content of filtered objects. We also need the bifiltered version. The references are [Del71], [Del74] and [Stacks]. We assume familiarity with usual derived categories, for which one can refer to [Stacks].

9.3.1. Suppose $T : \mathcal{C} \to \mathcal{D}$ is a left exact functor between abelian categories, and $K^\bullet$ be a bounded below complex in $\mathcal{C}$. By the usual formalism of derived categories, we obtain $\mathrm{R}T(K^\bullet) \in \mathrm{D}^+(\mathcal{C})$, which is a complex up to quasi-isomorphism.

Suppose $K^\bullet$ has a filtration F, then we want to bring in F so that we obtain a filtration on $\mathrm{R}T(K^\bullet)$. This is where filtered derived categories come in.

Recall that $\mathrm{Fil}(\mathcal{C})$ is the category of filtered objects in $\mathcal{C}$, which is additive. We denote by $\mathrm{Ch}(\mathrm{Fil}(\mathcal{C}))$ the category of chain complexes in $\mathcal{C}$. Although $\mathcal{C}$ is not abelian, there is no obstacle in forming the homotopy category $K(\mathrm{Fil}(\mathcal{C}))$ obtained by identifying homotopic chain maps.

Definition 9.9. Notations as above.

A morphism $f : (K^\bullet, F) \to (L^\bullet, F)$ in $\mathrm{Ch}(\mathrm{Fil}(\mathcal{C}))$ is called a **filtered quasi-isomorphism** if $\mathrm{Gr}^i_F f : \mathrm{Gr}^i_F K^\bullet \to \mathrm{Gr}^i_F L^\bullet$ is a quasi-isomorphism for all i.

Remark 9.10. Filtered quasi-isomorphic complexes give rise to isomorphic associated spectral sequences. Especially, they are quasi-isomorphic in the usual sense by looking at the E_∞-page.

Then, as in the case of usual derived categories, one show that filtered quasi-isomorphisms give a multiplicative system in $K(\mathrm{Fil}(\mathcal{C}))$.

Definition 9.11. The **filtered derived category** $\mathrm{DF}(\mathcal{C})$ is the localization of $K(\mathrm{Fil}(\mathcal{C}))$ at the filtered quasi-isomorphisms.

Remark 9.12. We get functors $\mathrm{Gr}_F^i : \mathrm{DF}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ by definition.

Next we come to the construction of filtered derived functors. Suppose $T : \mathcal{C} \to \mathcal{D}$ is a left exact functor, and $(K^\bullet, F) \in \mathrm{DF}^+(\mathcal{C})$. We need to take a filtered T-acyclic resolution of $(K^\bullet, F)$, in the following sense.

Definition 9.13. An object $(X, F) \in \mathrm{Fil}(\mathcal{C})$ is called **filtered T-acyclic** if $\mathrm{Gr}_F^p X$ is T-acyclic for all p (that is, $\mathrm{R}^q T \,\mathrm{Gr}_F^p X = 0$ for all p, q with $q > 0$).

If there is a filtered quasi-isomorphism $(K^\bullet, F) \to (L^\bullet, F)$ such that each (L^n, F) is filtered T-acyclic, we let

$$\mathrm{R}T(K^\bullet, F) = (TL^\bullet, F)$$

where the filtration on $TL^\bullet$ is defined by $F^p(TL^q) = T(F^pL^q)$ using left exactness of T. This produces a well-defined object in $\mathrm{DF}^+(\mathcal{D})$. If such resolution is always available, we obtain the filtered derived functor $\mathrm{R}T : \mathrm{DF}^+(\mathcal{C}) \to \mathrm{DF}^+(\mathcal{D})$.

The spectral sequence associated to $(K^\bullet, F)$ and T is then defined by the spectral sequence associated to $(TL^\bullet, F)$, for some filtered T-acyclic resolution $L^\bullet$.

Remark 9.14. In our case $\mathcal{C}$ will always be the category of sheaves on a space, and T will be the global section functor. Then filtered T-acyclic resolutions always exist, for example using the Godement resolution.

There is a similar story of bifiltered objects. For a bifiltered object (X, W, F), we can form iterated graded pieces $\mathrm{Gr}_W^p \mathrm{Gr}_F^q X$ and $\mathrm{Gr}_F^q \mathrm{Gr}_W^p X$. It turns out that they are canonically isomorphic by the butterfly lemma, see [Del71, Section 1.2.1]. We shall denote any of them by $\mathrm{Gr}^{p,q} X$. A morphism $f : (K^\bullet, W, F) \to (L^\bullet, W, F)$ is called a bifiltered quasi-isomorphism if $\mathrm{Gr}^{p,q} f$ is a quasi-isomorphism for any p, q. The bifiltered derived category $\mathrm{DF}_2(\mathcal{C})$ is defined by the localization of the homotopy category of bifiltered complexes at bifiltered quasi-isomorphisms. Bifiltered derived functors are defined using bifiltered acyclic resolutions.

If $(K^\bullet, W, F)$ is a bounded below bifiltered complex, then we can induce the filtration F to the associated spectral sequence

$${}_W E_1^{p,q} = \mathrm{R}^{p+q} T(\mathrm{Gr}_{-p}^W K^\bullet) \Rightarrow \mathrm{R}^{p+q} T(K^\bullet)$$

as in Section 9.2. They are defined by choosing a bifiltered T-acyclic resolution.

10. Construction on Smooth varieties

In this section, we discuss the construction of the Hodge structure on smooth varieties in [Del71]. Another reference is [Voi02, Chapter 8].

Let U be a smooth variety. By Nagata's compactification theorem [Stacks, 0F3T], we can find a proper variety X that contains U as an open subset. By a further application of resolution of singularity (say, of de Jong [Jon96]), we can assume that X is smooth and that $D = X - U$ is a global normal crossing divisor, that is, $D = \bigcup D_i$ is a finite union of smooth irreducible divisors D_i which intersect transversally. Denote by $i : D \to X$ and $j : U \to X$ the inclusion maps.

The key is to use the log de Rham complex to compute the cohomology $\mathrm{H}^*(U)$, as defined follows.

Definition 10.1. Let X be a smooth variety and $D = \bigcup D_i$ be a normal crossing divisor. Let $\Omega^i_X(*D)$ denote the sheaf of holomorphic i-forms possibly having poles at D. The **(holomorphic) log de Rham complex** $\Omega^\bullet_X(\log D)$ is the subcomplex of $\Omega^\bullet_X(*D)$ generated under exterior product by $\Omega^\bullet_X$ and $\frac{df}{f}$ for defining equations of D_i.

We give a specific local description of $\Omega^\bullet_X(\log D)$. For any point $x \in X$, take a polydisk V around x with coordinates $z_1, ..., z_n$, such that $D \cap V = \{(z_1, ..., z_n) \in V \mid z_1 = \cdots = z_k = 0\}$ for some $0 \leq k \leq n$. In the rest of the paper, we call this a standard chart. Then a log differential form on V has the form

$$\sum_{I,J} f_{I,J}(z_1, ..., z_n) \frac{dz_I}{z_I} \wedge dz_J$$

for subsets $I \subset \{1, ..., k\}$, $J \subset \{k+1, ..., n\}$ and holomorphic functions $f_{I,J}$, with $\frac{dz_I}{z_I} = \frac{\bigwedge_{i\in I} dz_i}{\prod_{i\in I} z_i}, dz_J = \bigwedge_{j\in J} dz_j$. (If f, f' are both defining equations of D_i, then $\frac{df}{f} - \frac{df'}{f'}$ would be holomorphic.) From this description, we have $\Omega^i_X(\log D)|_V$ is a free $\mathcal{O}_V$ module with basis $\frac{dz_I}{z_I} \wedge dz_J$ for $|I| + |J| = i$. Thus $\Omega^i_X(\log D)$ is locally free on X.

The two filtrations on $\mathrm{H}^*(U)$ come from the corresponding filtrations on $\Omega^\bullet_X(\log D)$.

Definition 10.2. Notations as above.

(1) The **Hodge filtration** $F^\bullet$ on $\Omega^\bullet_X(\log D)$ is given by the naive truncation

$$F^i\Omega^\bullet_X(\log D) = \sigma_{\geq i}\Omega^\bullet_X(\log D).$$

(2) The **weight filtration** $W_\bullet$ on $\Omega^\bullet_X(\log D)$ is defined by

$$W_i\Omega^\bullet_X(\log D) = \left(\bigwedge^i \Omega^1_X(\log D)\right) \wedge \Omega^{\bullet-i}_X.$$

More specifically, on standard charts $W_i\Omega^\bullet_X(\log D)$ has a basis $\frac{dz_I}{z_I} \wedge dz_J$ with $|I| \leq i$.

The utility of $\Omega^\bullet_X(\log D)$ comes from the following proposition.

Proposition 10.3. *[Del71, Proposition 3.1.8] Let τ denote the increasing truncation filtration of a complex $K^\bullet$ defined by $\tau_i K^\bullet = \tau^{\leq i} K^\bullet$. Then the following natural morphisms of filtered complexes*

$$(\Omega^\bullet_X(\log D), W) \xleftarrow{\alpha} (\Omega^\bullet_X(\log D), \tau) \xrightarrow{\beta} (j_*\mathcal{A}^\bullet_U, \tau)$$

are filtered quasi-isomorphisms.

Before its proof, we make some preparation. Let $D = \bigcup_{i\in A} D_i$ be the decomposition into smooth irreducible divisors. For any subset $K \subset A$ let $D_K = \bigcap_{i\in K} D_i$, and for any integer m let $D^{(m)} = \bigsqcup_{\substack{K\subset A \\ |K|=m}} D_K$ be the *disjoint union* of D_K with $|K| = m$. Especially, $D^{(0)} = X$. Let $i_m : D^{(m)} \to X$ be the disjoint union of inclusion maps.

There is a Poincaré residue map $\mathrm{Res} : W_m\Omega_X^\bullet(\log D) \to i_{m,*}\Omega_{D^{(m)}}^\bullet[-m]$ of complexes of $\mathcal{O}_X$-modules as follows. On standard charts, it is given on the basis $\frac{dz_I}{z_I} \wedge z_J (|I| \le m)$ by

$$\frac{dz_I}{z_I} \wedge dz_J \mapsto \begin{cases} dz_J|_{D_I}, & |I| = m \\ 0, & |I| < m \end{cases}.$$

One can check the independence of the above formula on coordinates. Then Res induces a chain map

$$\mathrm{R}^\bullet : \mathrm{Gr}_m^W \Omega_X^\bullet(\log D) \to i_{m,*}\Omega_{D^{(m)}}^\bullet[-m]. \tag{10.4}$$

The levelwise map R^i is an isomorphism, which can be seen from the local description: The surjectivity is clear, and if $f\frac{dz_I}{z_I} \wedge z_J (|I| = m)$ is mapped to 0, then $f = 0$ on D_I so that $f\frac{dz_I}{z_I} \wedge dz_J$ actually lies in $W_{m-1}\Omega_X^\bullet(\log D)$.

Remark 10.5. It is important here $\mathrm{R}^\bullet$ is a literal isomorphism, not a quasi-isomorphism.

Proof of Proposition 10.3. Note that

$$\tau^{\le m}\Omega_X^\bullet(\log D) \subset W_m\Omega_X^\bullet(\log D)$$

by definition, so the map α is well-defined.

For β, being a filtered isomorphism just means being a quasi-isomorphism, which is shown in [Voi02, Proposition 8.18].

For α, we need to show that $\mathcal{H}^m(\Omega_X^\bullet(\log D))[-m] \xrightarrow{\mathrm{Gr}_m \alpha} \mathrm{Gr}_m^W \Omega_X^\bullet(\log D)$ is a quasi-isomorphism. By (10.4) and the holomorphic Poincaré lemma we have quasi-isomorphisms

$$\mathrm{Gr}_m^W \Omega_X^\bullet(\log D) \xrightarrow[\mathrm{R}^\bullet]{\sim} i_{m,*}\Omega_{D^{(m)}}^\bullet[-m] \xleftarrow{\sim} i_{m,*}\mathbf{C}_{D^{(m)}}[-m].$$

Here we used that the exactness of the functor $i_{m,*}$. On the other hand, by β we have

$$\mathcal{H}^m(\Omega_X^\bullet(\log D)) \xrightarrow[\beta]{\sim} \mathcal{H}^m(j_*\mathcal{A}_U) = \mathrm{R}^m j_*\mathbf{C}_U.$$

For any point x on X, take a standard chart V centered at x. Let the coordinates be $(z_1, ..., z_n)$ and D be defined by $z_1 = \cdots = z_k = 0$. We have $(\mathrm{R}^m j_*\mathbf{C}_U)_x = \mathrm{H}^m(U \cap V)$. Since $U \cap V \cong (D^\times)^k \times D^{n-k}$ for the unit disk D and the punctured disk $D^\times$, by the Künneth formula $\mathrm{H}^*(U \cap V)$ is an exterior algebra with k generators on degree 1, which geometrically correspond to the smooth divisors passing through x. Thus we have an isomorphism $(\mathrm{R}^m j_*\mathbf{C}_U)_x \cong (i_{m,*}\mathbf{C}_{D^{(m)}})_x$, and letting x vary we obtain an isomorphism $\mathrm{R}^m j_*\mathcal{A}_U \cong i_{m,*}\mathbf{C}_{D^{(m)}}$ of sheaves. The fact that $\mathrm{Gr}_m \alpha$ is a quasi-isomorphism follows from the commutativity of the following diagram, which follows from definition.

$$\begin{array}{ccccc} \mathcal{H}^m(\Omega_X^\bullet(\log D)) & & \xrightarrow{\quad\mathrm{Gr}_m^\alpha\quad} & & \mathrm{Gr}_m^W \Omega_X^\bullet(\log D)[m] \\ \downarrow{\scriptstyle\beta}\wr & & & & \wr\downarrow{\scriptstyle\mathrm{R}^\bullet} \\ \mathrm{R}^m j_*\mathbf{C}_U & \xrightarrow{\sim} & i_{m,*}\mathbf{C}_{D^{(m)}} & \xrightarrow{\sim} & i_{m,*}\Omega_{D^{(m)}}^\bullet \end{array}$$

□

Especially, we have $\mathbf{H}^*(X,\Omega_X^\bullet(\log D)) = \mathbf{H}^*(X,j_*\mathcal{A}_U^\bullet) = \mathbf{H}^*(X,\mathrm{R}^\bullet j_*\mathbf{C}_U) = \mathrm{H}^*(U)$ is the cohomology of U. Consequenly, the two filtrations W and F on $\Omega_X^\bullet(\log D)$ would induce filtrations, still denote by W and F, on $\mathrm{H}^*(U)$, given by the spectral sequences

$$ {}_WE_1^{p,q} = \mathbf{H}^{p+q}(X,\mathrm{Gr}_{-p}^W\Omega_X^\bullet(\log D)) \Rightarrow \mathbf{H}^{p+q}(X,\Omega_X^\bullet(\log D)) $$

and

$$ {}_FE_1^{p,q} = \mathbf{H}^{p+q}(X,\mathrm{Gr}_F^p\Omega_X^\bullet(\log D)) \Rightarrow \mathbf{H}^{p+q}(X,\Omega_X^\bullet(\log D)). $$

By (10.4), we have

$$ {}_WE_1^{p,q} = \mathbf{H}^{p+q}(X,i_{-p,*}\Omega_{D^{(-p)}}^\bullet[p]) = \mathbf{H}^{2p+q}(D^{(-p)},\Omega_{D^{(-p)}}^\bullet) = \mathrm{H}^{2p+q}(D^{(-p)}). $$

Moreover, the differential $\mathrm{d}_1 : {}_WE_1^{p,q} \to {}_WE_1^{p+1,q}$, viewed as a map $\mathrm{H}^{2p+q}(D^{(-p)}) \to \mathrm{H}^{2p+q+2}(D^{(-p-1)})$, is given by the direct sum of Gysin homomorphisms $\mathrm{H}^{2p+q}(D_K) \to \mathrm{H}^{2p+q+2}(D_L)$ with $K \supset L$ and $|K| = -p, |L| = -p-1$ up to sign, see [Voi02, Proposition 8.34].

Consider the recurrent filtration F on ${}_WE_1^{p,q}$. By Proposition 9.6 and the definition of $F_{\mathrm{d}1}$ we have

$$ \begin{aligned} F^i({}_WE_1^{p,q}) &= \mathrm{Im}(\mathbf{H}^{p+q}(X,\sigma^{\geq i}\mathrm{Gr}_{-p}^W\Omega_X^\bullet(\log D)) \to \mathbf{H}^{p+q}(X,\mathrm{Gr}_{-p}^W\Omega_X^\bullet(\log D))) \\ &= \mathrm{Im}(\mathbf{H}^{p+q}(X,\sigma^{\geq i}(i_{-p,*}\Omega_{D^{(-p)}}^\bullet[p])) \to \mathbf{H}^{p+q}(X,i_{-p,*}\Omega_{D^{(-p)}}^\bullet[p])) \\ &= \mathrm{Im}(\mathbf{H}^{2p+q}(D^{(-p)},\sigma^{\geq i+p}\Omega_{D^{(-p)}}^\bullet) \to \mathbf{H}^{2p+q}(D^{(-p)},\Omega_{D^{(-p)}}^\bullet)), \end{aligned} $$

which is just the Hodge filtration of the Hodge structure on $\mathrm{H}^{2p+q}(D^{(-p)})$ up to a shift. We view all terms ${}_WE_1^{p,q}$ as pure Hodge structures with this shift, so that F^i becomes the Hodge filtration and ${}_WE_1^{p,q}$ has weight q. Then the differentials, given by Gysin homomorphisms, are morphisms of pure Hodge structures (of degree 0), hence are strict with respect to F by Theorem 9.3. Finally, we have the following theorem.

Theorem 10.6. *[Del71, Section 3.2] Notations as above.*

(1) On each term ${}_WE_r^{p,q}$, the first direct filtration, the recurrence filtration and the second direct filtration coincide.

(2) The spectral sequence ${}_WE$ degenerates at the E_2-page.

(3) The spectral sequence ${}_FE$ degenerates at the E_1-page.

(4) The filtration W on $\mathrm{H}^n(U)$ descends to a filtration on $\mathrm{H}^n(U,\mathbf{Q})$.

(5) The filtrations $W[n]$ and F defines a mixed Hodge structure on $\mathrm{H}^n(U,\mathbf{Q})$.

Partial proof. We have seen that d_1 is strictly compatible with F. In this paragraph we show that the same for d_0. In fact, the page ${}_WE_0$ is given by ${}_WE_0^{p,q} = \Gamma(X,\mathrm{Gr}_{-p}^W K^{p+q})$, where we have taken a bifiltered Γ-acyclic resolution $(\Omega_X^\bullet(\log D),W,F) \to (K^\bullet,W,F)$. We have to show that $\Gamma(X,\mathrm{Gr}_{-p}^W K^\bullet)$ has differentials strictly compatible with F. By [Del71, Proposition 1.3.2], it is equivalent to the degeneration of the spectral sequence $E(\Gamma(X,\mathrm{Gr}_{-p}^W K^\bullet),F)$ at the E_1-page. Since $(\mathrm{Gr}_{-p}^W K^\bullet,F)$ is filtered quasi-isomorphic to $(\mathrm{Gr}_{-p}^W\Omega_X^\bullet(\log D),F)$, which is moreover isomorphic to $(i_{-p,*}\Omega_{D^{(-p)}}^\bullet[p],\sigma^{\geq\bullet})$ by (10.4), we find that $E(\Gamma(X,\mathrm{Gr}_{-p}^W K^\bullet),F)$ is isomorphic, up to a shift, to the Hodge-de Rham spectral sequence of $D^{(-p)}$. Then the degeneration follows from Theorem 4.4.

Since ${}_WE_1$ consists of complexes of pure Hodge structures, the terms in ${}_WE_2$ can be given pure Hodge structures, with F as their Hodge filtrations. By Theorem 9.7

for $r_0 = 1$, we have $F_{\mathrm{d}1} = F = F_{\mathrm{d}2}$ on ${}_WE_2$. Especially, the differential d_2 preserves F. However, note $\mathrm{d}_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ is from one pure Hodge structure of weight q to the other of weight $q-1$. Since $E_2^{p,q} = \bigoplus_i F^i E_2^{p,q} \cap \overline{F^{q-i}E_2^{p,q}}$ and $F^i E_2^{p+2,q-1} \cap \overline{F^{q-i}E_2^{p+2,q-1}} = 0$, we must have $\mathrm{d}_2 = 0$. Especially, it is strict with respect to F.

Then we can continue this argument on ${}_WE_3$. An induction then proves (1) and (2).

We omit the proof of (3), which is done by relating various spectral sequences to obtain dimensional equalities.

(4) comes from Proposition 10.3. In fact, by these filtered quasi-isomorphisms, the spectral sequence ${}_WE$ is identified with that associated to $(j_*\mathcal{A}_U^\bullet, \tau) = (\mathrm{R}j_*\mathbf{C}_U, \tau)$, where we can use $\mathbf{Q}_U$ instead of $\mathbf{C}_U$.

From Remark 9.8, the Hodge filtration F on $H^*(U)$, after passing to graded pieces, become the recurrent (or any other two) filtration on ${}_WE_\infty$, which are pure Hodge structures of the desired weight. This shows (5). □

Theorem 10.6 gives a mixed Hodge structure on $\mathrm{H}^*(U)$. There are some things left to do, such as checking the independence on the compactification and functorality. They are done in [Del71]. Also, the spectral sequence ${}_WE$ gives a explicit way to compute the mixed Hodge structure on $\mathrm{H}^*(U)$ from the compactification and the normal crossing divisor. For example, we can see it implies (4) of Theorem 8.6.

11. Construction on General Varieties

In this section, we briefly sketch the construction of the mixed Hodge structures on general varieties, possibly singular, in [Del74]. We will mainly be discussing the ideas, keeping ourselves apart from the technical details.

11.1. **Simplicial Resolution and Cohomological Descent.** Recall how we produced a mixed Hodge structure on a nonproper smooth variety U: Choose a compactification X with global normal crossing divisor $D = X - U$, and represent the cohomology $\mathrm{H}^*(U)$ by the cohomology of the log de Rham complex $\Omega_X^\bullet(\log D)$, which further reduces by (10.4) to the cohomology of the smooth proper varieties $D^{(p)} (p \geq 0)$.

Return to the current situation, where X is an arbitrary variety. If X is not proper, we also need to take a compactification. Moreover, to deal with the singularity, we will need *desingularization*. For simplicity, assume X is proper for now. By resolution of singularity, there is a proper smooth variety X' with a proper birational morphism $f : X' \to X$. Then, the next question is to express $\mathrm{H}^*(X)$ using X' and f, which leads to the notion of cohomological descent.

Recall how does descent theory work in the simplest setting. Let X be a topological space and $\mathfrak{U} = \{U_i\}_{i\in I}$ be a cover of X. If $\mathcal{F}$ is a sheaf on X, then there is an equalizer diagram

$$\Gamma(X, \mathcal{F}) \to \prod_{i\in I} \Gamma(U_i, \mathcal{F}) \rightrightarrows \prod_{i,j\in I} \Gamma(U_i \cap U_j, \mathcal{F}) \tag{11.1}$$

so that we can recover the global section $\Gamma(X, \mathcal{F})$ of $\mathcal{F}$ by its restriction to the covers. However, in our setting the mission is more complicated; we want to descent the cohomology groups $\mathrm{H}^i(X, \mathbf{C}) = \mathrm{R}^i\Gamma(X, \mathbf{C})$. It turns out that we have to

upgrade the equalizer diagram above to a simplicial diagram, and we will inevitably use the language of derived categories. A reference will be [Con].

Definition 11.2 (Simplicial objects)**.** Let Δ denote the **simplex category**. Its objects are $[n]$ for $n \geq 0$, where $[n]$ denotes the set $\{0, 1, ..., n\}$ with usual order. Morphisms are non-decreasing maps between ordered sets.

A **simplicial object** in a category $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. A **cosimplicial object** in $\mathcal{C}$ is a functor $X : \Delta \to C$.

We will say simplicial space in short for a simplicial object in the category of topological spaces.

Remark 11.3. Name some special morphisms in Δ as follows:

$$\partial_j^n : [n-1] \to [n], \quad \partial_j^n(m) = \begin{cases} m, & m \leq j-1 \\ m+1, & m \geq j \end{cases}$$

and

$$\sigma_j^n : [n+1] \to [n], \quad \sigma_j^n(m) = \begin{cases} m, & m \leq j \\ m-1, & m \geq j+1 \end{cases}$$

for $0 \leq j \leq n$. They generate all morphisms in Δ with some specified relations, see [Con, Lemma 2.2].

Thus a simplicial object $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ is equivalent an object $X_n = X([n])$ for each $n \geq 0$, with structure morphisms $d_n^j = X(\partial_j^n) : X_n \to X_{n-1}$ and $s_n^j = X(\sigma_j^n) : X_n \to X_{n+1}$ satisfying the corresponding relations.

Remark 11.4. We will also need the **augmented simplex category** Δ_+ in the future. It has $[n]$ for $n \geq -1$ as objects, where $[-1]$ is the empty set. Morphisms are non-decreasing maps between ordered sets. Thus Δ_+ is Δ added an initial object $[-1]$. A functor $X : \Delta_+^{\mathrm{op}} \to \mathcal{C}$ is called an **augmented simplicial object** in $\mathcal{C}$. Similarly for augmented cosimplical objects.

A augmented simplicial object has an underlying simplicial object $X_\bullet$ by forgetting $[-1]$. We say that $X_\bullet$ admits an augmentation to X_{-1}, and formally write $X_\bullet \to X_{-1}$ for the augmented simplicial object.

Example 11.5 (the Čech nerve)**.** Let $\mathcal{C}$ be a category with fiber product, and $U \to X$ a morphism in $\mathcal{C}$. The **Čech nerve** $\check{N}_\bullet(U/X)$ is the simplicial object as follows:

$$\cdots U \times_X U \times_X U \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; U \times_X U \;\substack{\longleftarrow\\ \longleftarrow\\ \longrightarrow}\; U$$

Where we only draw the structure maps d_n^j and s_n^j, which are defined by formulas similar to ∂_j^n and σ_j^n. For example, $s_1^0 : U \times_X U \to U \times_X U \times_X U$ is given by $(a, b) \mapsto (a, a, b)$. The Čech nerve $\check{N}_\bullet(U/X)$ admit a canonical augmentation to X.

Note if $\mathcal{C}$ is the category of topological spaces, X is a space and $U = \bigsqcup U_i$ for some open cover $\mathfrak{U} = \{U_i\}$, then a *truncation* of $\check{N}_\bullet(U/X)$ gives

$$\prod_{i,j} U_i \cap U_j \;\substack{\longleftarrow\\ \longleftarrow\\ \longrightarrow}\; \prod_i U_i \ ,$$

which is used to express the classical descent data (11.1). (More precisely, we have an addition arrow s_0^0 here, which essentially causes no effect due to *cofinal* property of certain functor.)

Given a cover $U \to X$ of spaces as above, and a sheaf $\mathcal{F}$ on X. The theory of cohomological descent says that we can recover the cohomology $\mathcal{F}$ from its pullback to the simplicial space $\check{N}_\bullet(U/X)$. To make this precise, we first need to formulate what a "sheaf" is on $\check{N}_\bullet(U/X)$.

Definition 11.6. Given a simplicial space $X_\bullet$. A sheaf $\mathcal{F}^\bullet$ on $X_\bullet$ is given by a sheaf $\mathcal{F}^n$ on X_n for each n, together with structure maps

$$\partial_j^n : (d_n^j)^* \mathcal{F}^{n-1} \to \mathcal{F}^n, \quad \sigma_j^n : (\sigma_j^n) \mathcal{F}^{n+1} \to \mathcal{F}^n,$$

satisfying the cosimplicial relations.

The category $\mathrm{Shv}(X_\bullet)$ of sheaves on $X_\bullet$ is an abelian category with enough injectives. Denote by $\mathrm{D}(X_\bullet)$ its derived category.

If $f : X_\bullet \to Y_\bullet$ is a morphism of simplicial spaces (that is, a natural transformation between functors), then there are levelwise pullback and pushforward functors f^*, f_* which satisfy the usual properties.

We can also define pushforward and pullback of augmentation maps. Suppose $a : X_\bullet \to S$ is a augmented simplicial space.

Definition 11.7. Notations as above.

(1) For a sheaf $\mathcal{F}$ on S, its pullback $a^*\mathcal{F}$ is given by the sheaf $\mathcal{F}^\bullet$ on $X_\bullet$ defined by $\mathcal{F}^i = a_i^* \mathcal{F}$, where $a_i : X_i \to S$ is the canonical map. The structure map of $\mathcal{F}^\bullet$ are the identity maps.

(2) For a sheaf $\mathcal{F}^\bullet$ on X, its pushforward $a_* \mathcal{F}^\bullet$ is given by

$$a_* \mathcal{F}^\bullet = \ker(a_{0,*} \mathcal{F}^0 \rightrightarrows a_{1,*} \mathcal{F}^1),$$

where the two arrows are induced by ∂_0^1 and ∂_1^1.

Remark 11.8. Suppose $f : X_\bullet \to Y_\bullet$ is a morphism of simplicial spaces and $a : Y_\bullet \to S$ is an augmentation, then they can compose to give an augmentation $a \circ f : X_\bullet \to S$. We have $(a \circ f)_* \simeq a_* \circ f_*$ and $(f \circ a)^* \simeq f^* \circ a^*$ by definition.

Similar conclusion holds for the composition of an augmentation $a : X_\bullet \to S$ and a morphism $f : S \to T$.

Remark 11.9. Denote by pt the singleton space. Any simplicial space $X_\bullet$ admits an augmentation $a : X_\bullet \to \mathrm{pt}$. In this case a_* is called the global section functor, denoted by $\Gamma(X_\bullet, -)$.

Denote by $\mathrm{pt}_\bullet$ the constant simplicial space with a single point at each level. A sheaf on $\mathrm{pt}_\bullet$ is nothing other than a cosimplicial abelian group. We identify $\mathrm{Shv}(\mathrm{pt}_\bullet)$ with the category $\mathrm{Ab}^{\Delta^{\mathrm{op}}}$ of cosimplicial abelian groups. As in the above paragraph, the pushforward $a_{\bullet,*}$ for the canonical map $a_\bullet : X_\bullet \to \mathrm{pt}_\bullet$ is denoted by $\Gamma^\bullet(X_\bullet, -)$. As a cosimplicial abelian group, the n-th level of $\Gamma^\bullet(X_\bullet, \mathcal{F}^\bullet)$ is $\Gamma(X_n, \mathcal{F}^n)$.

As usual, pullback is exact and pushforward is left exact, and they are adjoint to each other. We can derive these functors to obtain adjunction pairs between derived categories. Still, the derived pushforward (pullback) along a composition is isomorphic to the composition of the corresponding derived pushforward (pullback).

Definition 11.10. An augmentation $a : X_\bullet \to S$ is said to be of **cohomological descent** if the unit $\mathrm{id}_{\mathrm{D}(S)} \to \mathrm{R}a_* \circ a^*$ is an isomorphism.

If a is of cohomological descent, then for $\mathcal{F}^\bullet \in \mathrm{D}(S)$, we have

$$\mathrm{R}\Gamma(S, \mathcal{F}^\bullet) = \mathrm{R}\Gamma(S, \mathrm{R}a_*(a^*\mathcal{F}^\bullet)) = \mathrm{R}\Gamma(X_\bullet, a^*\mathcal{F}^\bullet),$$

so that the cohomology group $\mathbf{H}^i(S, \mathcal{F})$ can be recovered from $X_\bullet$ and $a^*\mathcal{F}$.

Return to our goal, where we have an arbitrary variety X. The mixed Hodge structure on $\mathrm{H}^*(X)$ is constructed as follows.

First, find a compactification $X \hookrightarrow \overline{X}$ and an augmented simplicial variety $\overline{X}_\bullet \to \overline{X}$, such that the base change $\overline{X}_\bullet \times_{\overline{X}} X \to X$ is of cohomological descent. Also, for each n we require the variety $\overline{X}_n$ to be smooth and proper, and the closed subvariety $a_n^{-1}(\overline{X} - X) \subset \overline{X}_n$ to be a global normal crossing divisor.

For the cohomological descent property, we need the following fact, which is essentially a consequence of the proper base change theorem.

Theorem 11.11. *[Con, Theorem 7.9] Any proper hypercover $X_\bullet \to S$ is of cohomological descent.*

We will not give the precise definition of a proper hypercover ([Con, Definition 4.1]), which is a special kind of augmentation $X_\bullet \to S$ looking like a bunch of proper maps one followed by another. Admitting Theorem 11.11, the existence of the desired simplicial resolution follows from [Con, Theorem 4.16], which provides a proper hypercover $\overline{X}_\bullet \to \overline{X}$ (whose base change are also proper hypercovers) meeting the requirements on $\overline{X}_n$ and $a^{-1}(\overline{X}_n - X_n)$.

We give a slight indication of how to construct such a proper hypercover. For simplicity assume X is proper. Take a desingularization $a_0 : X_0 \to X$ of X. We cannot take $\check{N}_\bullet(X_0/X)$ as $X_\bullet$ because $X_0 \times_X X_0$ might be singular; instead, take a desingularization $X_1 \to X_0 \times_X X_0$, so that we have the following diagram

$$X_1 \sqcup X_0 \rightrightarrows X_0 \longrightarrow X$$

with natural arrows. Somehow continuing this process we will be able to obtain $X_\bullet$. If X is not proper we will need to use Nagata's compactification theorem and embedded resolution of singularity.

Then, we only need to define a mixed Hodge structure on $\mathrm{H}^*(X) = \mathrm{H}^*(X_\bullet, \mathbf{C})$, which we briefly discuss in the next section.

11.2. **Mixed Hodge Structures on Simplicial Varieties.** For simplicity we change the notations a bit. Let $X_\bullet$ be a simplicial variety such that each X_n is smooth and proper, with a global normal crossing divisor $D_n \subset X_n$. Let $U_n = X_n - D_n$, the open sets U_n should be preserved by the structure maps of $X_\bullet$. We will define a mixed Hodge structure on $\mathrm{H}^*(U_\bullet) = \mathrm{H}^*(U_\bullet, \mathbf{C})$.

We will encounter to kinds of indices below. One comes from simplicial structure, and the other comes from chain complexes. We will denote by $\circ$ the chain complex index and by $\bullet$ the simplicial index.

Again, we will need to consider the simplicial de Rham complex $\Omega^\circ_{X_\bullet}(\log D_\bullet)$, whose structure maps come from pullback of differential forms. Let $j_\bullet : U_\bullet \to X_\bullet$ denote the open inclusion. We have

$$\mathrm{R}\Gamma(U_\bullet, \mathbf{C}_U) = \mathrm{R}\Gamma(X_\bullet, \mathrm{R}j_*\mathbf{C}_U) = \mathrm{R}\Gamma(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet)).$$

A further step is to analyze the derived global section functor $\mathrm{R}\Gamma(X_\bullet, -)$ more carefully, so that we can relate $\mathrm{R}\Gamma(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet))$ to $\mathrm{R}\Gamma(X_n, \Omega^\circ_{X_n}(\log D_n))$ for

individual n. For a simplicial space $X_\bullet$, factor the augmentation $a : X_\bullet \to \mathrm{pt}$ as $X_\bullet \xrightarrow{a_\bullet} \mathrm{pt}_\bullet \xrightarrow{a_{\mathrm{pt}}} \mathrm{pt}$, then

$$\mathrm{R}\Gamma(X_\bullet, \mathcal{F}^\bullet) = \mathrm{R}a_{\mathrm{pt},*}\mathrm{R}\Gamma^\bullet(X_\bullet, \mathcal{F}^\bullet).$$

The functor $\mathrm{R}a_{\mathrm{pt},*} : \mathrm{D}(\mathrm{Ab}^{\Delta^{\mathrm{op}}}) \to \mathrm{D}(\mathrm{Ab})$ can be computed as follows. Given a complex $A^{\circ,\bullet}$ of simplicial abelian groups. We can turn a cosimplicial abelian group into a complex by forming alternating sums of the boundary operators d_n^j, so that $A^{\circ,\bullet}$ becomes a bicomplex, and we can take the total complex $\mathrm{Tot}(A^{\circ,\bullet})$. This operation $s : \mathrm{Ch}(\mathrm{Ab}^{\Delta^{\mathrm{op}}}) \to \mathrm{Ch}(\mathrm{Ab})$ preserves quasi-isomorphisms, hence pass to the derived category. We claim $s = \mathrm{R}a_{\mathrm{pt},*}$. In fact, for a complex $I^{\circ,\bullet}$ of injective cosimplicial abelian groups, we have $\mathrm{R}a_{\mathrm{pt},*}I^{\circ,\bullet} = \ker(I^{\circ,0} \to I^{\circ,1})$ by definition, which is quasi-isomorphic to $\mathrm{Tot}(I^{\circ,\bullet})$, since each complex $I^{n,\bullet}$ is acyclic in positive degrees by [Con, Corollary 2.13].

Thus $\mathrm{R}\Gamma(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet)) = s\mathrm{R}\Gamma^\bullet(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet))$. Take an injective resolution $\Omega^\circ_{X_\bullet}(\log D_\bullet) \to \mathcal{I}^{\circ,\bullet}$ in $\mathrm{D}(X_\bullet)$, then $\mathrm{R}\Gamma^\bullet(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet))$ is given by the complex of cosimplicial abelian groups with n-th simplicial level $\Gamma(X_n, \mathcal{I}^{\circ,n})$. By [Con, Lemma 6.4], the map $\Omega^\circ_{X_n}(\log D_n) \to \mathcal{I}^{\circ,n}$ is an injective resolution for each n, thus $\Gamma(X_n, \mathcal{I}^{\circ,n}) = \mathrm{R}\Gamma(X_n, \Omega^\circ_{X_n}(\log D_n))$. We can upgrade the derived functor $\mathrm{R}\Gamma(X_n, -) : \mathrm{D}(X_n) \to \mathrm{D}(\mathrm{Ab})$ to its bifiltered version, so that the weight filtration W and the Hodge filtration F on $\Omega^\circ_{X_n}(\log D_n)$ can be induced to $\mathrm{R}\Gamma(X_n, \Omega^\circ_{X_n}(\log D_n))$. Consequently, the bicomplex $K^{\circ,\bullet} = \mathrm{R}\Gamma^\bullet(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet))$ admits filtrations W and F.

The total complex $K^\bullet = \mathrm{Tot}(K^{\circ,\bullet})$ can be equipped with filtrations $\delta(W, L)$ and F defined by

$$\delta(W, L)_n(K^\bullet) = \bigoplus_{p,q} W_{n+q}K^{p,q}$$

and

$$F^n(K^\bullet) = \bigoplus_{p,q} F^n K^{p,q}.$$

Finally, we do the same thing to $(K^\bullet, \delta(W, L), F)$ as we did to $(\Omega^\bullet_X(\log D), W, F)$. The spectral sequence ${}_{\delta W,L}E = E(K^\bullet, \delta(W, L))$ converges to $\mathrm{H}^*(K^\bullet) = \mathrm{H}^*(X_\bullet, \Omega^\circ_{X_\bullet}(\log D_\bullet)) = \mathrm{H}^*(U_\bullet)$. Let $D_n^{(m)}$ be defined as in Section 10 for $D_n \subset X_n$, then the E_1-page is ([Del74, Section 8.1.19])

$${}_{\delta(W,L)}E_1^{p,q} = \bigoplus_r \mathrm{H}^{q-2r}(D^{(r)}_{p+r}) \Rightarrow \mathrm{H}^{p+q}(U_\bullet).$$

Again, this spectral sequence degenerates at E_2, from which one deduces that $\delta(W, L)[n]$ and F induced on $\mathrm{H}^n(U_\bullet)$ makes it into a mixed Hodge structure, which is well-defined and functorial. For the details, see [Del74, Section 8.1].

When X is already proper, we have $D_n = \varnothing$ for all n, and the spectral sequence becomes

$${}_{\delta(W,L)}E_1^{p,q} = \mathrm{H}^q(X_p) \Rightarrow \mathrm{H}^{p+q}(X_\bullet).$$

From this we can see (3) of Theorem 8.6.

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