Glimpses of equivariant algebraic topology

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Most of the talk is a review of an old subject

What are equivariant cohomology theories?

What are they good for?

Explosion of interest in the last decade:

The Kervaire invariant one problem
Hill, Hopkins, Ravenel

Topological cyclic homology
Nikolaus, Scholze, many others

Chromatic homotopy theory, nonequivariant calculations
HHR, many Chicago students and postdocs, many others

New directions in equivariant stable homotopy theory
Mathew, M, and our collaborators; many others

OUTLINE: EQUIVARIANT COHOMOLOGY THEORY

- Two classical definitions: Borel and Bredon
- P.A. Smith theory on fixed point spaces
- The Conner conjecture on orbit spaces
- The Oliver transfer and $RO(G)$-graded cohomology
- Mackey functors for finite and compact Lie groups
- Extending Bredon cohomology to $RO(G)$-grading
- A glimpse of the modern world of spectra and G-spectra

Borel's definition (1958):

G a topological group, X a (left) G-space, action $G \times X \longrightarrow X$

$$g(hx) = (gh)x, ex = x$$

EG a contractible (right) G-space with free action

$$yg = y \ \text{ implies } \ g = e$$

$$EG \times_G X = EG \times X / \sim \qquad (yg, x) \sim (y, gx)$$

"homotopy orbit space of X"

A an abelian group

$$H^*_{Bor}(X; A) = H^*(EG \times_G X; A)$$

Characteristic classes in Borel cohomology (M, 1987, 3 pages)

$B(G, \Pi)$: classifying G-space for principal (G, Π)-bundles, principal Π-bundles with G-acting through bundle maps.

Theorem

$$EG \times_G B(G, \Pi) \simeq BG \times B\Pi$$

(over BG). Therefore, with field coefficients,

$$H^*_{Bor}(B(G, \Pi)) = H^*(EG \times_G B(G, \Pi)) \cong H^*(BG) \otimes H^*(B\Pi)$$

as an $H^*(BG)$-module.

Not very interesting theory of equivariant characteristic classes!

Bredon's definition (1967):

Slogan: "orbits are equivariant points" since $(G/H)/G = *$.

A coefficient system $\mathscr{A}$ is a contravariant functor

$$\mathscr{A} : h\mathscr{O}_G \longrightarrow \mathscr{A}b$$

$\mathscr{O}_G$ is the category of orbits G/H and G-maps,

$h\mathscr{O}_G$ is its homotopy category ($= \mathscr{O}_G$ if G is discrete)

$$H^*_G(X; \mathscr{A})$$

satisfies the Eilenberg-Steenrod axioms plus

"the equivariant dimension axiom":

$$H^0_G(G/H; \mathscr{A}) = \mathscr{A}(G/H), \quad H^n_G(G/H; \mathscr{A}) = 0 \text{ if } n \neq 0$$

Bredon de Rham cohomology? Bredon characteristic classes?

Axioms for reduced cohomology theories

Cohomology theory $\widetilde{E}^*$ on based G-spaces (G-CW $\simeq$ types):

Contravariant homotopy functors $\widetilde{E}^n$ to Abelian groups, $n \in \mathbb{Z}$.

Natural suspension isomorphisms

$$\widetilde{E}^n(X) \xrightarrow{\cong} \widetilde{E}^{n+1}(\Sigma X)$$

For $A \subset X$, the following sequence is exact:

$$\widetilde{E}^n(X/A) \longrightarrow \widetilde{E}^n(X) \longrightarrow \widetilde{E}^n(A)$$

The following natural map is an isomorphism:

$$\widetilde{E}^n(\bigvee_{i \in I} X_i) \longrightarrow \prod_{i \in I} \widetilde{E}^n(X_i)$$

$$E^n(X) = \widetilde{E}^n(X_+), \quad X_+ = X \amalg \{*\}; \quad E^n(X, A) = \widetilde{E}^n(X/A)$$

Borel vs Bredon:

$\underline{A}$ = the constant coefficient system, $\underline{A}(G/H) = A$

$$H^*(X/G; A) \cong H^*_G(X; \underline{A})$$

since both satisfy the dimension axiom and Bredon is unique.
Therefore

$$H^*_{Bor}(X; A) \equiv H^*(EG \times_G X; A) \cong H^*_G(EG \times X; \underline{A})$$

On "equivariant points", $EG \times_G (G/H) \cong EG/H = BH$, hence

$$H^*(EG \times_G (G/H); A) = H^*(BH; A).$$

Cellular (or singular) cochain construction:

G-CW complex X, cells of the form $G/H \times D^n$:

$X = \cup X^n$, X^0 = disjoint union of orbits, pushouts

$$\begin{array}{ccc} \coprod_i G/H_i \times S^n & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ \coprod_i G/H_i \times D^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

$$X^\bullet \colon \mathscr{O}_G^{op} \longrightarrow Spaces, \quad X^\bullet(G/H) = X^H$$

Chain complex $C_*(X)$ of coefficient systems:

$$C_n(X)(G/H) = C_n((X^n/X^{n-1})^H; \mathbb{Z})$$

Cochain complex of abelian groups:

$$C^*(X; \mathscr{A}) = \mathrm{Hom}_{Coeff}(C_*(X), \mathscr{A})$$

P.A. Smith theory (1938) (M, 1987, 3 pages)

G a finite p-group, X a finite dimensional G-CW complex.

Consider mod p cohomology. Assume that $H^*(X)$ is finite.

Theorem

If $H^*(X) \cong H^*(S^n)$, then $X^G = \emptyset$ or $H^*(X^G) \cong H^*(S^m)$

for some $m \leq n$.

If H is a normal subgroup of G, then $X^G = (X^H)^{G/H}$.

Finite p-groups are nilpotent.

By induction on the order of G,

we may assume that G is cyclic of order p.

The Bockstein exact sequence

A short exact sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$$

of coefficient systems implies a short exact sequence

$$0 \longrightarrow C^*(X;\mathscr{A}) \longrightarrow C^*(X;\mathscr{B}) \longrightarrow C^*(X;\mathscr{C}) \longrightarrow 0$$

of cochain complexes, which implies a long exact sequence

$$\cdots \longrightarrow H^q_G(X;\mathscr{A}) \longrightarrow H^q_G(X;\mathscr{B}) \longrightarrow H^q_G(X;\mathscr{C}) \longrightarrow \cdots$$

Connecting homomorphism

$$\beta\colon H^q_G(X;\mathscr{C}) \longrightarrow H^{q+1}_G(X;\mathscr{A})$$

is called a "Bockstein operation".

Smith theory

Let $FX = X/X^G$. Define $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ so that

$$H^*_G(X;\mathscr{A}) \cong \widetilde{H}^*(FX/G),$$
$$H^*_G(X;\mathscr{B}) \cong H^*(X),$$
$$H^*_G(X;\mathscr{C}) \cong H^*(X^G)$$

On orbits $G = G/e$ and $* = G/G$,

$$\mathscr{A}(G) = \mathbb{F}_p, \quad \mathscr{A}(*) = 0$$
$$\mathscr{B}(G) = \mathbb{F}_p[G], \quad \mathscr{B}(*) = \mathbb{F}_p$$
$$\mathscr{C}(G) = 0, \quad \mathscr{C}(*) = \mathbb{F}_p$$

Let

$$a_q = \dim \widetilde{H}^q(FX/G), \quad b_q = \dim H^q(X), \quad c_q = \dim H^q(X^G)$$

Proof of Smith theorem; $p = 2$ only for brevity

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{A} \oplus \mathscr{C} \longrightarrow 0$$

On G, $\quad 0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_2[G] \longrightarrow \mathbb{F}_2 \oplus 0 \longrightarrow 0.$

On $*$, $\quad 0 \longrightarrow 0 \longrightarrow \mathbb{F}_2 \longrightarrow 0 \oplus \mathbb{F}_2 \longrightarrow 0.$

$$H^*(X; \mathscr{A} \oplus \mathscr{C}) \cong H^*(X; \mathscr{A}) \oplus H^*(X; \mathscr{C})$$

Bockstein long exact sequence implies

$$\chi(X) = \chi(X^G) + 2\widetilde{\chi}(FX/G)$$

and

$$a_q + c_q \leq b_q + a_{q+1}$$

Inductively, for $q \geq 0$ and $r \geq 0$,

$$a_q + c_q + \cdots + c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1}.$$

Let $n = \dim(X)$. With $q = n+1$ and $r > n$, get $c_i = 0$ for $i > n$. With $q = 0$ and $r > n$, get

$$\sum c_q \leq \sum b_q.$$

So far, all has been general. If $H^*(X) \cong H^*(S^n)$, then $\sum b_q = 2$.

$\chi(X) \equiv \chi(X^G)$ mod 2 implies $\sum c_q = 0$ ($X^G = \emptyset$) or $\sum c_q = 2$.

The Conner conjecture (1960); first proven by Oliver (1976)

G a compact Lie group, X a finite dimensional G-CW complex with finitely many orbit types, A an abelian group.

Theorem

If $\widetilde{H}^*(X;A) = 0$, then $\widetilde{H}^*(X/G;A) = 0$.

Conner: True if G is a finite extension of a torus.

If H is a normal subgroup of G, then $X/G = (X/H)/(G/H)$.

Reduces to $G = S^1$ and G finite. Smith theory methods apply.

General case: let N be the normalizer of a maximal torus T in G.

Then $\chi(G/N) = 1$ and $\widetilde{H}^n(X/N;A) = 0$.

The Oliver transfer

Theorem

Let $H \subset G$, $\pi\colon X/H \longrightarrow X/G$. For $n \geq 0$, there is a transfer map

$$\tau\colon \widetilde{H}^n(X/H; A) \longrightarrow \widetilde{H}^n(X/G; A)$$

such that $\tau \circ \pi^*$ is multiplication by $\chi(G/H)$.

Proof of the Conner conjecture.

Take $H = N$. The composite

$$\widetilde{H}^n(X/G; A) \xrightarrow{\pi^*} \widetilde{H}^n(X/N; A) \xrightarrow{\tau} \widetilde{H}^n(X/G; A)$$

is the identity and $\widetilde{H}^n(X/N; A) = 0$. □

How do we get the Oliver transfer?

$RO(G)$-GRADED COHOMOLOGY

$$X \wedge Y = X \times Y / X \vee Y$$

V a representation of G, S^V its 1-point compactification.

$$\Sigma^V X = X \wedge S^V, \quad \Omega^V X = \mathrm{Map}_*(S^V, X)$$

Suspension axiom on an "$RO(G)$-graded cohomology theory E^*":

$$\widetilde{E}^{\alpha}(X) \cong \widetilde{E}^{\alpha+V}(\Sigma^V X)$$

for all $\alpha \in RO(G)$ and all representations V.

Theorem

If $\mathscr{A} = \underline{A}$, then $H^*_G(-;\mathscr{A})$ extends to an $RO(G)$-graded cohomology theory.

$\underline{A} = \underline{\mathbb{Z}} \otimes A$: $A = \mathbb{Z}$ suffices.

Construction of the Oliver transfer

Let $X_+ = X \amalg \{*\}$. Consider $\varepsilon\colon (G/H)_+ \longrightarrow S^0$.

Theorem

For large enough V, there is a map

$$t\colon S^V = \Sigma^V S^0 \longrightarrow \Sigma^V G/H_+$$

such that $\Sigma^V \varepsilon \circ t$ has (nonequivariant) degree $\chi(G/H)$.

The definition of $\tau\colon \widetilde{H}^n(X/H;A) \longrightarrow \widetilde{H}^n(X/G;A)$.

$$\widetilde{H}^n(X/H;A) \cong \widetilde{H}^n_G(X \wedge G/H_+;\underline{A}) \cong \widetilde{H}^{n+V}_G(X \wedge \Sigma^V G/H_+;\underline{A})$$

$$\widetilde{H}^n(X/G;A) \cong \widetilde{H}^n_G(X;\underline{A}) = \widetilde{H}^{n+V}_G(X \wedge S^V;\underline{A})$$

Smashing with X, t induces τ. □

How do we get the map t?

Generalizing, let M be a smooth G-manifold.

Embed M in a large V. The embedding has a normal bundle ν.

The embedding extends to an embedding of the total space of ν as a tubular neighborhood in V.

The Pontryagin Thom construction gives a map $S^V \longrightarrow T\nu$, where $T\nu$ is the Thom space of the normal bundle.

Compose with $T\nu \longrightarrow T(\tau \oplus \nu) \cong T\varepsilon = M_+ \wedge S^V$.

The composite is the transfer $t\colon S^V \longrightarrow \Sigma^V M_+$.

Atiyah duality: M_+ and $T\nu$ are Spanier-Whitehead dual.
This is the starting point for equivariant Poincaré duality,
for which $RO(G)$-grading is essential.

$RO(G)$-graded Bredon cohomology

Theorem

$H^*_G(-;\mathscr{A})$ extends to an $RO(G)$-graded theory if and only if the coefficient system $\mathscr{A}$ extends to a Mackey functor.

Theorem

$\underline{\mathbb{Z}}$, hence $\underline{A}$, extends to a Mackey functor.

What is a Mackey functor?

First definition, for finite G

Let $G\mathscr{S}$ be the category of finite G-sets. A Mackey functor $\mathscr{M}$ consists of covariant and contravariant functors

$$\mathscr{M}^*, \mathscr{M}_*\colon G\mathscr{S} \longrightarrow \mathscr{A}b,$$

which are the same on objects (written M) and satisfy:

$$M(A \amalg B) \cong M(A) \oplus M(B)$$

and a pullback of finite sets gives a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{g} & T \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ S & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} M(P) & \xrightarrow{g_*} & M(T) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ M(S) & \xrightarrow[f_*]{} & M(B) \end{array}$$

Suffices to define on orbits.

Pullback condition gives the "double coset formula".

Example: $\mathscr{M}(G/H) = R(H)$ (representation ring of H).

Restriction and induction give $\mathscr{M}^*$ and $\mathscr{M}_*$.

Second definition, for finite G

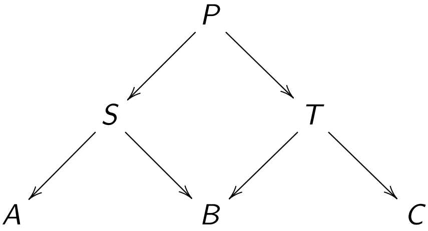
Category G-$Span^+$ of "spans" of finite G-sets.

Objects are finite G-sets. Morphisms $A \longrightarrow B$ are diagrams

$$A \longleftarrow S \longrightarrow B$$

Really equivalence classes: $S \sim S'$ if $S \cong S'$ over A and B.

Composition by pullbacks:



G-$Span^+(A, B)$ is an abelian monoid under disjoint union of spans. Let G-$Span(A, B)$ be its Grothendieck group, getting the category G-$Span$. A Mackey functor $\mathscr{M}$ is a (contravariant) functor

$$\mathscr{M} : G\text{-}Span \longrightarrow \mathscr{A}b,$$

written M on objects and satisfying $M(A \amalg B) \cong M(A) \oplus M(B)$.

Lemma

A Mackey functor is a Mackey functor.

Given $\mathscr{M}$,

$$A \xleftarrow{=} A \longrightarrow B, \quad A \longleftarrow B \xrightarrow{=} B$$

give $\mathscr{M}^*$ and $\mathscr{M}_*$. Given $\mathscr{M}^*$ and $\mathscr{M}_*$, composites give $\mathscr{M}$.

Topological reinterpretation: third definition

For based G-spaces X and Y with X a finite G-CW complex,

$$\{X, Y\}_G \equiv \mathrm{colim}_V [\Sigma^V X, \Sigma^V Y]_G$$

"Stable orbit category" or "Burnside category" $\mathscr{B}_G$:
objects G/H, abelian groups of morphisms

$$\mathscr{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G$$

A Mackey functor is a contravariant additive functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$.
This is THE definition if G is a compact Lie group.

Theorem

If G is finite, a Mackey functor is a Mackey functor. $\mathscr{B}_G$ is isomorphic to the full subcategory of orbits G/H in G-Span.

The Mackey functor $\underline{\mathbb{Z}}$

Define

$$\mathscr{A}_G(G/H) = \mathscr{B}_G(G/H, *) \cong \{S^0, S^0\}_H = A(H).$$

This gives the Burnside ring Mackey functor $\mathscr{A}_G$.

Augmentation ideal sub Mackey functor $\mathscr{I}_G(G/H) = IA(H)$.

The quotient Mackey functor $\mathscr{A}_G/\mathscr{I}_G$ is $\underline{\mathbb{Z}}$.

How can we extend $\mathbb{Z}$-grading to $RO(G)$-grading?

Represent ordinary $\mathbb{Z}$-graded theories on G-spectra by Eilenberg-MacLane G-spectra, which then represent $RO(G)$-graded theories!

What are spectra?

- Prespectra (or spectra): sequences of spaces T_n and maps $\Sigma T_n \longrightarrow T_{n+1}$
- Ω-(pre)spectra: Adjoints are equivalences $T_n \xrightarrow{\simeq} \Omega T_{n+1}$
- Spectra: Spaces E_n and homeomorphisms $E_n \longrightarrow \Omega E_{n+1}$
- Spaces to prespectra: $\{\Sigma^n X\}$ and $\Sigma(\Sigma^n X) \xrightarrow{\cong} \Sigma^{n+1} X$
- Prespectra to spectra, when $T_n \xrightarrow{\subset} \Omega T_{n+1}$:

$$(LT)_n = \operatorname{colim} \Omega^q T_{n+q}$$

- Spaces to spectra: $\Sigma^\infty X = L\{\Sigma^n X\}$
- Spectra to spaces: $\Omega^\infty E = E_0$
- Coordinate-free: spaces T_V and maps $\Sigma^W T_V \longrightarrow T_{V \oplus W}$

What are spectra good for?

- First use: Spanier-Whitehead duality [1958]
- Cobordism theory [1959] (Milnor; MSO has no odd torsion)
- Stable homotopy theory [1959] (Adams; ASS for spectra)
- Generalized cohomology theories [1960] (Atiyah-Hirzebruch; K-theory, AHSS)
- Generalized homology theories [1962] (G.W. Whitehead)
- Stable homotopy category [1964] (Boardman)

Representing cohomology theories

Fix Y. If $Y \simeq \Omega^2 Z$, then $[X, Y]$ is an abelian group.

For $A \subset X$, the following sequence is exact:

$$[X/A, Y] \longrightarrow [X, Y] \longrightarrow [A, Y]$$

The following natural map is an isomorphism:

$$[\bigvee_{i \in I} X_i, Y] \longrightarrow \prod_{i \in I} [X_i, Y]$$

For an Ω-spectrum $E = \{E_n\}$,

$$\widetilde{E}^n(X) = \begin{cases} [X, E_n] & \text{if } n \geq 0 \\ [X, \Omega^{-n} E_0] & \text{if } n < 0 \end{cases}$$

Suspension:

$$\widetilde{E}^n(X) = [X, E_n] \cong [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] = \widetilde{E}^{n+1}(\Sigma X)$$

What are classical G-spectra (any G)?

- Classical G-spectra: spectra with G-action
- G-spaces T_n and G-maps $\Sigma T_n \longrightarrow T_{n+1}$
- Classical Ω-G-spectra: $T_n \xrightarrow{\simeq} \Omega T_{n+1}$

Classical Ω-G-spectra $E = \{E_n\}$ represent $\mathbb{Z}$-graded cohomology.

$$\widetilde{E}^n_G(X) = \begin{cases} [X, E_n]_G & \text{if } n \geq 0 \\ [X, \Omega^{-n} E_0]_G & \text{if } n < 0 \end{cases}$$

Ordinary theories

Eilenberg-Mac Lane spaces:

$$\pi_n K(A,n) = A, \quad \pi_q K(A,n) = 0 \ \text{ if } \ q \neq n.$$

$$\widetilde{H}^n(X;A) = [X, K(A,n)]$$

Based G-spaces X have homotopy group coefficient systems

$$\underline{\pi}_n(X) = \pi_n(X^\bullet); \quad \underline{\pi}_n(X)(G/H) = \pi_n(X^H).$$

Eilenberg-Mac Lane G-spaces:

$$\underline{\pi}_n K(\mathscr{A},n) = \mathscr{A}, \quad \underline{\pi}_q K(\mathscr{A},n) = 0 \ \text{ if } \ q \neq n.$$

$$\widetilde{H}^n_G(X;\mathscr{A}) = [X, K(\mathscr{A},n)]_G$$

What are genuine G-spectra (G compact Lie)?

- G-spaces T_V, G-maps $\Sigma^W T_V \longrightarrow T_{V\oplus W}$
  where V, W are real representations of G
- Ω-G-spectra: G-equivalences $T_V \xrightarrow{\simeq} \Omega^W T_{V\oplus W}$

Genuine Ω-G-spectra E represent $RO(G)$-graded theories.

Imprecisely,

$$E_G^{V-W}(X) = [\Sigma^W X, E_V].$$

Ordinary? Need genuine Eilenberg-Mac Lane G-spectra.

A quick and dirty construction (1981)

Build a good "equivariant stable homotopy category" of G-spectra.

Use sphere G-spectra $G/H_+ \wedge S^n$ to get a theory of G-CW spectra.

Mimic Bredon's construction of ordinary $\mathbb{Z}$-graded cohomology, but in the category of G-spectra, using Mackey functors instead of coefficient systems.

Apply Brown's representability theorem to represent the 0th term by a G-spectrum $H\mathscr{M}$: for G-spectra X,

$$H^0_G(X;\mathscr{M}) \cong \{X, H\mathscr{M}\}_G.$$

Then $H\mathscr{M}$ is the required Eilenberg-Mac Lane G-spectrum.

What are G-spectra good for?

- Equivariant K-theory [1968] (Atiyah, Segal)
- Equivariant cobordism [1964] (Conner and Floyd)
- $RO(G)$-graded homology and cohomology theories
- Equivariant Spanier-Whitehead and Poincaré duality
- Equivariant stable homotopy category (Lewis-M)
- Completion theorems (KU_G, π_G^*, MU_G-modules): (Atiyah-Segal, Segal conjecture, Greenlees-M)
- Nonequivariant applications!!!

Kervaire invariant one problem (if time permits)

Framed manifold M: trivialization of its (stable) normal bundle.

Ω_n^{fr}: Cobordism classes of (smooth closed) framed n-manifolds.

Is every framed n-manifold M, $n = 4k + 2$, framed cobordant to a homotopy sphere (a topological sphere by Poincaré conjecture)?

$$\kappa\colon \Omega_{4k+2}^{fr} \longrightarrow \mathbb{F}_2$$

$\kappa[M]$ is the Kervaire invariant, the Arf invariant of a quadratic refinement of the cup product form on $H^{2k+1}(M; \mathbb{F}_2)$ that is determined by the given framing.

$\kappa[M] = 0$ if and only if $[M] = [\Sigma]$ for some homotopy sphere Σ.

History

$n = 2, 6, 14$: $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ have $\kappa = 1$ framings.

Kervaire (1960): PL, non-smoothable, 10-manifold M with $\kappa = 1$.

Kervaire and Milnor (1963): maybe $\kappa = 0$ for $n \neq 2, 6, 14$?

Browder (1969): $\kappa = 0$ unless $n = 2^{j+1} - 2$ for some j, and then $\kappa = 0$ if and only if h_j^2 does not survive in the ASS, $h_j \leftrightarrow Sq^{2^j}$.

Calculation/construction (Barratt, Jones, Mahowald, Tangora: h_4^2 and h_5^2 survive the ASS. (h_6^2 doable? Zhouli Xu et al!)

Hill, Hopkins, Ravenel

Theorem (2009)

$\kappa = 0$ *unless n is* 2, 6, 14, 30, 62, *or maybe* 126*:*
h_j^2 *has a non-zero differential in the ASS,* $j \geq 7$*.*

Calculations of $RO(G)$-graded groups $H_G^*(*;\underline{\mathbb{Z}})$ are critical!

Haynes Miller quote (Bourbaki Séminaire survey):

Hill, Hopkins, and Ravenel marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

- The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers.
- The theory of structured ring spectra, implemented by M and many others; and
- Equivariant stable homotopy theory, as developed by M and collaborators.

Structured ring spectra and structured ring G-spectra

E_∞ ring spectra (M-Quinn-Ray [1972])

E_∞ ring G-spectra (Lewis-M [1986])

Paradigm shift in stable homotopy theory.

Symmetric monoidal category of spectra $\mathscr{S}$ under $\wedge$;

E_∞ ring spectra are just commutative monoids in $\mathscr{S}$.

Elmendorf-Kriz-Mandell-M [1997]: S-modules, operadic $\wedge$

Hovey-Shipley-Smith [2000]: Symmetric spectra, categorical $\wedge$

Mandell-M-Shipley-Schwede [2001]: Orthogonal, comparisons

Mandell-M [2002]: Orthogonal G-spectra and S_G-modules

"Brave new" nonequivariant subjects:

"Brave new algebra" (Waldhausen's name, 1980's; now apt)

"Brave new algebraic geometry" (Toen-Vezzosi's name; Lurie; also apt)

Revitalized brave new equivariant areas:

Equivariant ∞ loop space theory

Equivariant algebraic K-theory

Prospective applications to algebraic K-theory of number rings?

Theorem

Let L be a Galois extension of a field F with Galois group G.

There is an E_∞ ring G-spectrum $K_G(L)$ such that

$$(K_G(L))^H = K(L^H) \quad \textit{for} \quad H \subset G$$

where $\pi_* K(R) =$ Quillen's algebraic K-groups of R.

Many altogether new directions just this past decade!

Amusing recent results: categorical G-homotopy theory

Definition A (small) G-category $\mathscr{C}$ has a G-set of objects,

a G-set $\mathscr{C}(x, y)$ of morphisms $x \to y$ for each pair of objects,

G-fixed identity morphisms $\mathrm{id}_x \colon x \to x$,

and composition G-maps

$$\mathscr{C}(y, z) \times \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z)$$

for triples (x, y, z). Composition must be associative and unital.

A G-category $\mathscr{C}$ has a classifying G-space $B_G\mathscr{C}$.

Topological G-categories $\mathscr{C}$: G-spaces of objects and morphisms.

They also have classifying G-spaces $B_G\mathscr{C}$.

Anna Marie Bohmann, Kristen Mazur, Angelica Osorno, Viktoriya Ozornova, Kate Ponto, and Carolyn Yarnall:

Can do all of algebraic topology of G-spaces with G-categories

A G-poset (partially ordered set) is a G-category with at most one morphism, denoted $x \leq y$, between any two objects. Thus each $g \in G$ acts by an order-preserving map.

M, Marc Stephan, and Inna Zakharevich (2016):

Can do all algebraic topology of G-spaces with G-posets