PROGRAM NOTES AND ABSTRACTS FOR WEEK 1

Apprentice Program: Daniil Rudenko

Continuing. We will cover a variety of topics in algebra, geometry and combinatorics.

Probability: Xuan Wu

TITLE: Determinants in probability

ABSTRACT: We aim to give an introduction to integrable probability and its applications in statistical physics.

The tool we focus on is known as determinantal point processes. We will illustrate this method through two well-studied models - one from random matrix theory and another from statistical mechanics

Analysis: Takis Souganidis and Beniada Shabani

TITLE (Takis Souganidis): An introduction to partial differential equations

ABSTRACT: I will present an example of ballistic deposition for surface growth and I will use this as a "vehicle" to introduce a class of nonlinear partial differential equations. Then I will spend the rest of time to explain how one can make some sense of such equations and the types of mathematical questions that arise.

TITLE (Beniada Shabani): Hamilton-Jacobi equations

ABSTRACT: Hamilton-Jacobi equations are first order differential equations that arise naturally in optimal control theory, finance, game theory, physics, imaging, etc. In the first talk we will formally derive the equations from a deterministic control problem, using Bellman's dynamic programming principle. Next we describe a different approach to Hamilton-Jacobi equations in terms of calculus of variations and derive the Hopf-Lax formula for the solution of the problem. The final part will be dedicated to a weaker notion of solutions called the viscosity solutions. A Calculus sequence should suffice for understanding most of the concepts.

Number Theory: Zijian Yao and Zhilin Luo

TITLE (Zijian Yao): Dirichlet?s theorem on the infinitude of primes in arithmetic progression.

ABSTRACT: Continued from last week.

TITLE (Zhilin Luo): In how many different ways can one represent a positive integer n as the sum of two squares?

ABSTRACT: We will first recall the work of B. Riemann on the meromorphic continuation of Riemann zeta function, which is ultimately related to the Poisson summation formula of Jacobi theta function. It turns out that the Fourier expansion of the square of Jacobi theta function is related to the following classical problem in number theory:

In how many different ways can one represent a positive integer n as the sum of two squares?

Let this number be $r(n)$. C. Jacobi first expressed $r(n)$ as generalized divisor sums which is easy to compute. We will establish the result of Jacobi via an identity between the square of Jacobi theta function and a suitable Eisenstein series.

The remarkable identity between theta function and Eisenstein series was later generalized by C. Siegel, polished by A. Weil, and refined by S. Kudla-S. Rallis and many others. We hope our talk can be a motivational introduction to the formula of Siegel-Weil and its variants.

Dynamics: Danny Calegari

TITLE: Introduction to complex dynamics

ABSTRACT: We discuss the Mandelbrot set, and its significance in the theory of dynamics of rational maps of the Riemann sphere and more broadly. A solid background in complex analysis would be desirable.

Geometry: Michael Klug

TITLE (from last week) : Circle packings

ABSTRACT: What could be more central than a circle? Laying a bunch of nonoverlapping coins out on a table, we obtain a planar embedding of a graph with vertices at the centers and edges between tangent circles. We will prove the Koebe-Andreev-Thurston circle packing theorem which states that all such graphs can be realized through this construction and that for triangulations this realization is unique up to Möbius transformations.

Algebraic Topology: Peter May and others

Continuing: This week will introduce monads and their relationship to adjoint functors, culminating in the Beck monadicity theorem, which says that for some adjoint pairs of functors (F, G) from $\mathcal{T}$ to $\mathcal{S}$, the category $\mathcal{S}$ is equivalent to the category of algebras over the monad associated to the adjunction. This says that many algebraic concepts, such as groups, are completely characterized as being quotients of free algebras of the relevant kind. The notion of quotient is cleanly encapsulated by the notion of an algebra over a monad. This idea will later be developed in homotopical analogues that are very important in algebraic topology.

TITLE: Coambiguous concepts

ABSTRACT: We will start with some categorical linguistics, introducing the categorical language required for comparisons between subjects and comparisons between concepts. "Coambiguous concepts" are different definitions that non-obviously have the same or equivalent content. The idea leads to seriously interesting comparisons, either relating different areas of mathematics or giving alternative perspectives on a single area. The first week, we will introduce the categorical language that allows us to make such comparisons. Don't be put off by the undefined terms below!

Example: Finite T_0 topological spaces and finite posets are isomorphic concepts By contrast, what do we mean by equivalent concepts?

Example: Finite dimensional real vector spaces and their maps are equivalent to finite real matrices and matrix multiplication

Example: Finite sets are equivalent to the set of sets $\mathbf{n} = \{1, \cdots, n\}$

Example: Groups are equivalent to free groups modulo relations

Generalization next week: Monads and their algebras; the categorical "monadicity theorem".

TITLE: Homotopically coambiguous definitions

More deeply, what do we mean by concepts that are homotopically coambiguous, by which we mean that different definitions give homotopically equivalent concepts?

Example: Topological spaces, simplicial sets, small categories, and posets are homotopically equivalent concepts.

Example: Simplicial abelian groups and chain complexes are homotopically equivalent concepts.

Possible topics to be explored in more depth later on

Topic 1: Finite spaces and larger contexts (book in progress)

http://math.uchicago.edu/~may/REU2020/FINITEBOOK.pdf

Topic 2: An introduction to stable homotopy theory and spectra

Slide talk: https://www.youtube.com/watch?v=vRsrCNLkSA0

Topic 3: An introduction to equivariant homotopy and cohomology theory

Slide talk: https://www.cornell.edu/video/peter-may-equivariant-cohomology

Topic 4: Operads and their algebras

Old new example: n-connective spaces are homotopically equivalent to "monadically coequalized" n-fold suspensions of E_n-spaces.

Old new example: Connective spectra are homotopically equivalent to "monadically coequalized" suspensions of E_∞-spaces.