

HYPERBOLIC 3-MANIFOLDS

JINGXUAN SONG

ABSTRACT. The complementary space of a hyperbolic knot admits a hyperbolic structure. Therefore, examining the properties of hyperbolic 3-manifolds helps deepen our understanding of hyperbolic knots. In particular, isometries and elementary groups that act on hyperbolic 3-manifolds reveal the important structure of hyperbolic 3-manifolds. Investigation on special isometry groups leads to the construction of quotient hyperbolic space and the discovery of universal elementary neighborhoods. These topics are included in *Hyperbolic Knot Theory* by Jessica S. Purcell, and this paper focuses on providing a comprehensive and especially graphical explanation that serves to compensate for the concise analysis in the book.

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1. PRELIMINARY

Definition 1.1. A *manifold* X is a topological space that is locally Euclidean.

Definition 1.2. Let X be a manifold. We say that a *chart* (U, ϕ) is a homeomorphic map $\phi : U \rightarrow \phi(U) \subset X$ where U is an open subset of some manifold M .

Definition 1.3. Let X be a manifold, and G be a group acting on X . We say a manifold M has a (G, X) -*structure* if for every point $x \in M$, there exists a chart (U, ϕ) where $U \subset M$ is a small neighborhood of x . Moreover, if two charts (U, ϕ) and (V, ψ) overlaps, that is, $U \cap V$ is nonempty, then the transition map

$$\gamma = \phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V)$$

is an element of G .

Definition 1.4. Let X be a 2-dimensional Euclidean space $\mathbb{E}^2$. Let G be isometries of Euclidean space $\text{Isom}(\mathbb{E}^2)$. When a 2-manifold admits an $(\text{Isom}(\mathbb{E}^2), \mathbb{E}^2)$ -structure, we say the manifold admits a *Euclidean structure*, or is Euclidean. Similarly, let $\mathbb{H}^2$ denotes the hyperbolic 2-space, that is, the upper half plane of $\mathbb{R}^2$

equipped with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

If a 2-manifold admits an $(\text{Isom}(\mathbb{H}^2), \mathbb{H}^2)$ -structure, we say the manifold admits a *hyperbolic structure*, or is *hyperbolic*. In general, an n -manifold that admits an $(\text{Isom}(\mathbb{H}^n), \mathbb{H}^n)$ -structure admits a *hyperbolic structure*, or is *hyperbolic*.

Theorem 1.5. *The group of orientation preserving isometries of $\mathbb{H}^3$ is $PSL(2, \mathbb{C})$. It acts on the boundary $\partial\mathbb{H}^3$ via Möbius transformations. That is, for*

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$$

the action of A on $\partial\mathbb{H}^3$ is given by

$$A(z) = \frac{az + b}{cz + d}, \text{ for } z \in \partial\mathbb{H}^3,$$

and it extends uniquely to the interior of hyperbolic 3-space.

The proof of the theorem is omitted here since it is a standard proof in hyperbolic geometry and it is not a major focus of this paper. The details of the proof can be found in Marden's *Outer circles: An introduction to hyperbolic 3-Manifolds* [2].

This result on isometry group of hyperbolic 3-manifold is crucial in our following discussion. By studying specific discrete elementary subgroup of $PSL(2, \mathbb{C})$, we will be able to specify the structure of the ϵ -thin part in any complete orientable hyperbolic 3-manifold. In particular, we will show that the value of ϵ is universal for all such manifolds. The detail of this results will be discussed in section 5.

2. CLASSIFICATION OF ISOMETRY

In this section, we will focus on the classification of elements in $PSL(2, \mathbb{C})$. We will define three types of isometry in $PSL(2, \mathbb{C})$ based on their fixed point(s) and show that every element falls into exactly one of the three categories.

Definition 2.1. Let $A \in PSL(2, \mathbb{C})$, we say that

- (1) A is *parabolic* if it fixes a single point on $\partial\mathbb{H}^3$;
- (2) A is *elliptic* if it fixes two points on $\partial\mathbb{H}^3$ and rotates about the geodesic axis between them in $\mathbb{H}^3$, fixing the axis point-wise;
- (3) A is *loxodromic* if it fixes two points on $\partial\mathbb{H}^3$ and dilates and rotates about the axis between them.

From definition 2.1, we know that in the spherical model of hyperbolic 3-space, an elliptic isometry can be characterized by a rotation around some axes, and a loxodromic isometry can be understood as a rotation composed by dilation. If we shifted to the standard model of hyperbolic 3-space, i.e. the upper half space of $\mathbb{R}^3$ with hyperbolic metric, the translation that maps z to $z + 1$ gives an example of parabolic isometry. Indeed, this transformation has no fixed point other than the infinity on the boundary. These examples are illustrated in Figure 1.

Now let us deduce the trichotomic classification. We know from Brouwer fixed-point theorem that a continuous map from a compact convex set to itself must has a fixed point. Since the hyperbolic space with its boundary $\mathbb{H}^3 \cup \partial\mathbb{H}^3$ is a compact convex set, then every $A \in PSL(2, \mathbb{C})$ fixes at least one point.

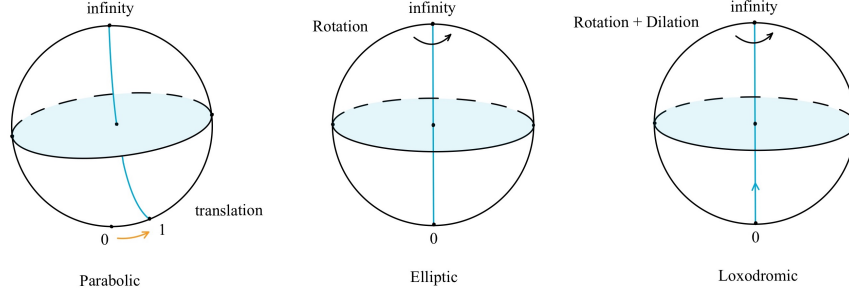


FIGURE 1. Classification of isometry

We also know from the complex variables studies that a linear fractional transformation, i.e. Möbius transformation, can be determined by three points. This means a map with three fixed points can only be the identity map, otherwise there will be two distinct transformations that share the same value on these three points. Therefore, a non-trivial map in $\mathrm{PSL}(2, \mathbb{C})$ can have at most two fixed points.

Suppose A is an element of $\mathrm{PSL}(2, \mathbb{C})$ acting on hyperbolic 3-space. We first ask if there are any fixed points in the interior of the space $\mathbb{H}^3$. If this is true, then we say that A is elliptic. If this is not true, then A must fix either one or two points on the boundary $\partial\mathbb{H}^3$. We say that A is parabolic if it has one fixed point on $\partial\mathbb{H}^3$, or loxodromic if it has two fixed points on $\partial\mathbb{H}^3$. This analysis is in accordance with the definition, hence we can conclude that for all $A \in \mathrm{PSL}(2, \mathbb{C})$, A should lie in exactly one of the three categories: parabolic, elliptic, or loxodromic.

Lemma 2.2. *For $A \in \mathrm{PSL}(2, \mathbb{C})$,*

- (1) *A is parabolic if and only if A conjugates to*

$$z \mapsto z + 1$$

- (2) *A is elliptic if and only if A conjugates to*

$$z \mapsto e^{2i\theta} z \quad \text{with } 2\theta \neq 2\pi n \text{ for any } n \in \mathbb{Z}$$

- (3) *A is loxodromic if and only if A conjugates to*

$$z \mapsto \zeta^2 z \quad \text{with } |\zeta| > 1$$

Proof. We will only give the proof for the parabolic case, and a similar argument can be used to prove the other two cases.

Suppose $A \in \mathrm{PSL}(2, \mathbb{C})$ conjugates to the map B that send z to $z + 1$. Note that B is an example of parabolic isometry and it only fixes the point of infinity on $\partial\mathbb{H}^3$. Since all elements in $\mathrm{PSL}(2, \mathbb{C})$ maps boundary to boundary, then all conjugates of B , in particular A , should have exactly one fixed point on $\partial\mathbb{H}^3$. By definition, this shows that A is parabolic.

Suppose $A \in \mathrm{PSL}(2, \mathbb{C})$ is parabolic, then it fixes a single point $x_0 \in \partial\mathbb{H}^3$. Since A is not the identity map, we can choose $x_1, y_1 \in \partial\mathbb{H}^3$ such that A maps x_1 to y_1

and x_0, x_1, y_1 are three distinct points. Choose $U \in \text{PSL}(2, \mathbb{C})$ determined by

$$\begin{aligned} 0 &\mapsto x_1 \\ y_1 &\mapsto 1 \\ \infty &\mapsto x_0 \end{aligned}$$

so that $B = U^{-1}AU$ fixes the infinity and maps 0 to 1. By theorem 1.5, we have

$$B(z) = \frac{a}{b} \cdot z + 1$$

If $a \neq b$, then B fixes $\frac{b}{b-a}$ in addition to the point of infinity. However, this is a contradiction since being a conjugate of a parabolic map A , the map B can only have one fixed point on the boundary. Therefore, we must have $a = b$ so that

$$B(z) = z + 1$$

Therefore, we conclude that A is a conjugate to $z \mapsto z + 1$. $\square$

3. ACTION OF DISCRETE SUBGROUP AND CORRESPONDENCE

In this section, we will focus on the action of discrete $\text{PSL}(2, \mathbb{C})$ subgroup on hyperbolic 3-manifolds. We will try to understand the importance of discreteness in constructing quotient manifold that preserve the hyperbolic structure.

Definition 3.1. A subgroup of $\text{PSL}(2, \mathbb{C})$ is *discrete* if it contains no sequence of distinct elements converging to the identity element.

Definition 3.2. The action of $G \leq \text{PSL}(2, \mathbb{C})$ on $\mathbb{H}^3$ is *properly discontinuous* if for every closed ball $B \subset \mathbb{H}^3$, the set $\{\gamma \in G \mid \gamma(B) \cap B \neq \emptyset\}$ is a finite set.

Proposition 3.3. A subgroup of $\text{PSL}(2, \mathbb{C})$ is discrete if and only if its action on $\mathbb{H}^3$ is properly discontinuous.

Proof. If the subgroup of $\text{PSL}(2, \mathbb{C})$ fails to be discrete, then there exists a sequence A_n converging to the identity map. Since the identity map fixes every point, then the distance between x and its image $A_n x$ should converge to 0 as n goes to ∞ . This means a closed ball centered at any fixed point of radius R must intersect with its images $A_n(B_{x,R})$ for all large enough n . This is illustrated in the left figure of Figure 2. Hence the action of G fails to be properly discontinuous.

For the opposite direction, the infinite intersections of a closed ball B centered at x with its images $A_n(B_{x,R})$ force the image of x to accumulate around it. This implies for large enough n , $A_n(x)$ should lie in an ϵ -neighborhood of x . In particular, $A_n(x)$ does not converge to any point in the boundary $\partial\mathbb{H}^3$, as illustrated in the right figure of Figure 2. Meanwhile, each A_n have at least one fixed point on $\partial\mathbb{H}^3$. Since $\partial\mathbb{H}^3$ is compact, then the sequence of fixed points $\{p_n\} \cup \{q_n\}$ should give convergent subsequences. If there is one accumulation point in $\{p_n\} \cup \{q_n\}$, then we can use parabolic model $a_n x + b_n$ after conjugation; if there are two distinct accumulation points, then we can use the $a_n x$ model (elliptic or loxodromic) after conjugation. We can see that in both cases the isolation of $A_n(B_{x,R})$ with the boundary gives a convergent subsequence of A_n . $\square$

The proof examines why convergent isometry subsequences could force the properly discontinuous condition to fail, and vice versa, by analyzing the fixed points of $A \in \text{PSL}(2, \mathbb{C})$ on $\partial\mathbb{H}^3$. It also gives a useful criteria for showing discreteness of a

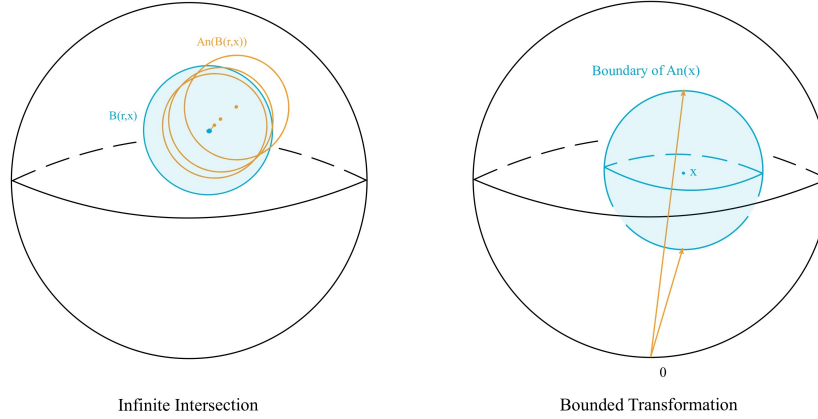


FIGURE 2. Illustration of the proof for proposition 3.3

subgroup $G \leq \text{PSL}(2, \mathbb{C})$ – such G is discrete if and only if it does not contain an infinite sequence of distinct elements converging to some elements $A \in \text{PSL}(2, \mathbb{C})$.

Definition 3.4. The action of a group $G \leq \text{PSL}(2, \mathbb{C})$ is *free* if the identity element of G is the only element that have fixed points in $\mathbb{H}^3$.

Remark 3.5. A group $G \leq \text{PSL}(2, \mathbb{C})$ that acts freely on $\mathbb{H}^3$ should not contains elliptic isometries, since they fix a geodesic axis in the interior $\mathbb{H}^3$.

The following proposition is important since it gives a way to construct new hyperbolic manifolds. It also helps us to understand the correspondence between the structure of group $G \leq \text{PSL}(2, \mathbb{C})$ and the covering spaces (quotient space) of hyperbolic manifolds.

Definition 3.6. Let X be a topological space, and G is a group that acts on X . We define the quotient space as

$$X/G = \{G_x : x \in X\}$$

where G_x denotes the orbit of x under the action of G , with quotient topology: $U \subset X/G$ is open if and only if $\psi^{-1}(U)$ is open in X . We call $\psi : X \rightarrow X/G$ that maps $x \mapsto G_x$ the covering projection, or the covering map.

Proposition 3.7. *The action of a Group $G \leq \text{PSL}(2, \mathbb{C})$ on $\mathbb{H}^3$ is free and properly discontinuous if and only if $\mathbb{H}^3/G$ is a 3-manifold with a complete hyperbolic structure and with covering projection $\mathbb{H}^3 \rightarrow \mathbb{H}^3/G$.*

We omit the proof here since it is carefully discussed in the book [1]. However, we shall focus on the connection between discreteness and a complete hyperbolic structure. The requirement of free and being properly discontinuous aims to preserve a nice environment around each point $G_x \in X/G$. Here, "nice" means a neighborhood that resembles its pre-image in X . For every point $G_x \in X/G$, if it is contained in a neighborhood that is isometric to all copies of its pre-image in the original manifold X , then we can deduce that X/G admits a complete hyperbolic structure.

If the action of group G is not free, then we can not isolate each translation of x . Hence, their neighborhood overlaps with each other. If the action is not properly discontinuous, then the closed ball $B(x, R)$ intersects with its images infinitely many times even if the radius R is arbitrarily small. Therefore, we cannot isolate a ϵ -neighborhood around x that does not overlap with its images. In both cases, a neighborhood of G_x cannot be isometric to all copies of its pre-image in the original manifold X .

Definition 3.8. Let H denotes the closed horoball about infinity in $\mathbb{H}^3$:

$$H = \{(x, y, z) \mid z \geq 1\}$$

Let G be an infinite discrete group in $\mathrm{PSL}(2, \mathbb{C})$ fixing infinity (after conjugation). We will show in the next section that G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

If $G \cong \mathbb{Z}$, then H/G is isomorphic to $A \times [1, \infty)$, i.e. the space between two vertical plane with proper identification of the two planes. In this case, we say that H/G is a *rank-1 cusp*.

If $G \cong \mathbb{Z} \times \mathbb{Z}$, then H/G is isomorphic to $T \times [1, \infty)$, i.e. a column bounded by four vertical planes with proper identification on its vertical faces. In this case, we say that H/G is a *rank-2 cusp*.

The space H/G is illustrated in Figure 3 for both $G \cong \mathbb{Z}$ and $G \cong \mathbb{Z} \times \mathbb{Z}$.

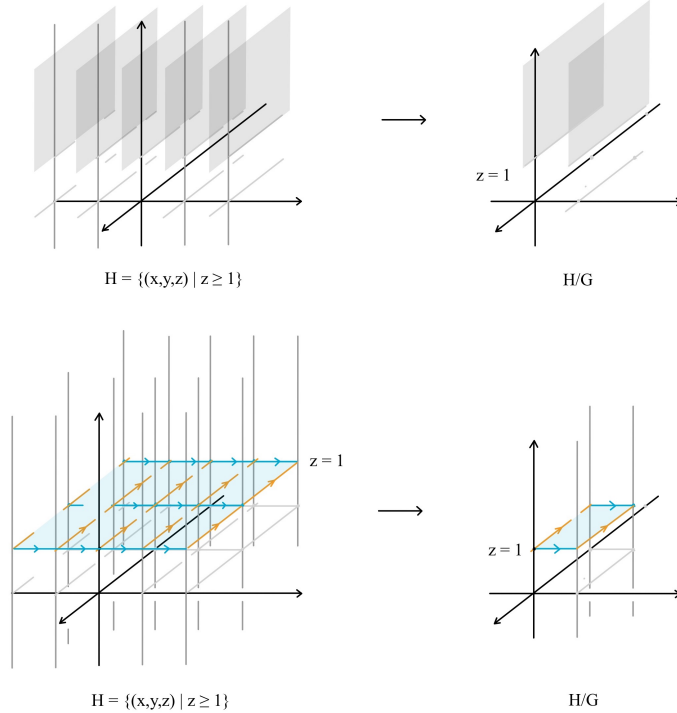


FIGURE 3. Rank-1 cusp (top) and Rank-2 cusp (bottom) in upper half space model of hyperbolic 3-space

Observe that there is an inverse correspondence between the quotient manifold and the subgroups of G . This relationship is illustrated in the figure 4.

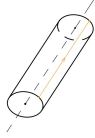
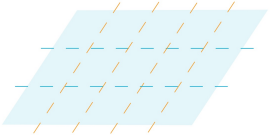
	Quotient Space	Group
	$\partial H / \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$
	$\partial H / \mathbb{Z}$	$\mathbb{Z}$
	∂H	Identity

FIGURE 4. Inverse Correspondence Response

4. ELEMENTARY GROUP

In this section, we introduce the notion of elementary group. The understanding of elementary group lays an important foundation for section 5.

Definition 4.1. We say that a subgroup $G \subset \text{PSL}(2, \mathbb{C})$ is *elementary* if one of the following holds:

- (1) The union of all fixed points on $\partial \mathbb{H}^3$ of all nontrivial elements of G is a single point on $\partial \mathbb{H}^3$.
- (2) The union of all fixed points on $\partial \mathbb{H}^3$ of all nontrivial elements of G consists of exactly two points on $\partial \mathbb{H}^3$.
- (3) there exists $x \in \mathbb{H}^3$ such that for all $g \in G$, $g(x) = x$.

The group is *nonelementary* if it is not elementary.

The definition of an elementary group is inconvenient to use as a condition for an elementary group. Therefore, we are motivated to find a more concrete description of the group structure of an elementary group. One way to approach this problem is by looking for possible generators of elementary group, as well as their relation with the trichotomy classification of isometries in $\text{PSL}(2, \mathbb{C})$.

Let us first examine the third case. From the definition, we know that neither parabolic nor loxodromic transformation fixes any points in the interior $\mathbb{H}^3$. Therefore, the third case elementary group should contain only elliptic transformations.

Using a similar argument, we can also deduce that the first case in the definition should only contain parabolic transformations.

The second case is a little tricky, since at the first sight it possibly contains both parabolic and loxodromic transformations. However, if the group G contains a

parabolic with fixed point $x \in \partial\mathbb{H}^3$ and a loxodromic with fixed points $x, y \in \partial\mathbb{H}^3$, then composing the two translations gives a transformation that fixes $x, z \in \partial\mathbb{H}^3$ with z distinct from y . Therefore, the union of all fixed points on $\partial\mathbb{H}^3$ consists at least three point. Hence G is not a second-case elementary group. This argument shows that a second G contains only loxodromic transformations.

Proposition 4.2. *If G is discrete and the union of all fixed points of nontrivial elements in G is a single point on $\partial\mathbb{H}^3$, then G is generated by parabolic elements in $PSL(2, \mathbb{C})$ and is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.*

Proof. We have already explained that a first case elementary group G only contains parabolics so that it must be generated by parabolic elements. By conjugation, we can assume that all elements in G fixes infinity ∞ , then all elements should also fix horospheres about ∞ . Since horospheres about ∞ in $\mathbb{H}^3$ are horizontal planes that are isometric to Euclidean plane, then G acts on the Euclidean plane by Euclidean translations, i.e. isometries on $\mathbb{R}^2$ of the form $ax + b$. This means G can be generated by either one element, in which case $G \cong \mathbb{Z}$; or G is generated by two linearly independent elements, in which case $G \cong \mathbb{Z} \times \mathbb{Z}$. If G have more than two generators, we can always find their "largest common divisor" and reduce the number of generators since G is discrete. This process is illustrated in Figure 5. $\square$

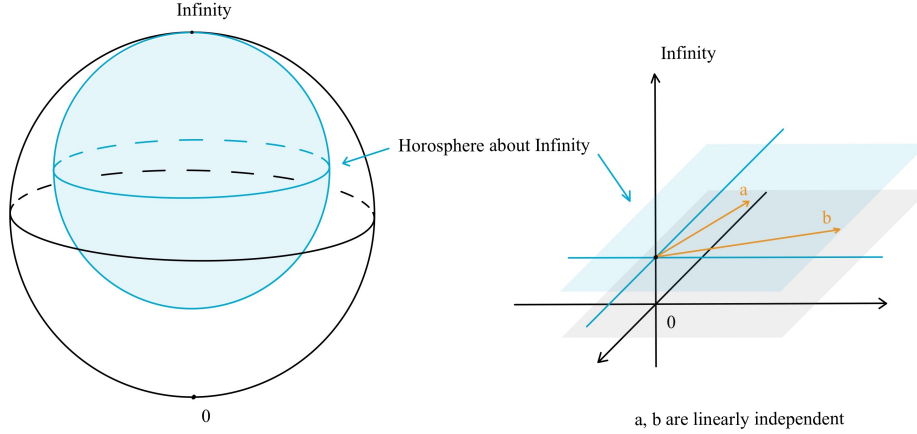


FIGURE 5. Illustration of the proof for proposition 4.2

Proposition 4.3. *If G is discrete the union of all fixed points of nontrivial elements in G consists of exactly two points on $\partial\mathbb{H}^3$, then G is generated by a single loxodromic element and is isomorphic to $\mathbb{Z}$.*

Proof. We have already explained that the second case in the definition should only contain loxodromics, hence it must be generated by loxodromic transformations. Observe that if all elements in G fix exactly two points on $\partial\mathbb{H}^3$, then they also fix the geodesic axis that connects the two points. This means that the image of the axis under each transformation is itself, regardless of variations within the axis. Since G is discrete, then the minimal distance d between each point with its

translates exist, and it is realized by some elements in G . Therefore, we conclude that G is generated by some elements A that fix the geodesic axis and $d(x, Ax)$ realize the minimal distance. This process is illustrated in Figure 6. $\square$

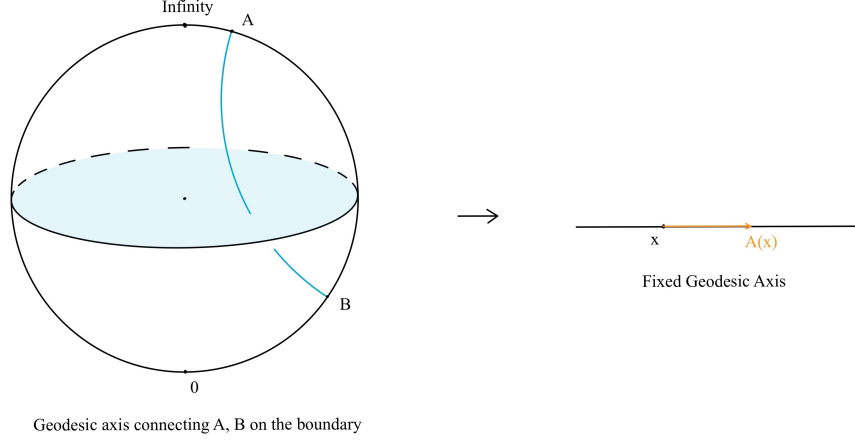


FIGURE 6. Illustration of the proof for proposition 4.3

5. UNIVERSAL ELEMENTARY NEIGHBORHOOD

In this section, we will prove the theorem of universal elementary neighborhoods.

Theorem 5.1 (Universal Elementary Neighborhoods). *There is a universal constant $\epsilon_3 > 0$ such that for all $x \in \mathbb{H}^3$, and for any discrete group $G \leq PSL(2, \mathbb{C})$ without elliptics, if H denotes the subgroup of G generated by all elements of G that translate x distance less than ϵ_3 , then H is elementary.*

Before we formulate the proof, we want to introduce the following theorem derived from the work of Jorgensen and Klein in 1982 [3]. We will also discuss some properties of a non-elementary group. These results will help us construct a contradiction in the proof.

Theorem 5.2. *Suppose $\{\langle A_n, B_n \rangle\}$ is a sequence of non-elementary discrete subgroups of $PSL(2, \mathbb{C})$ such that $\lim A_n = A$ and $\lim B_n = B$. Then $\langle A, B \rangle$ is a non-elementary discrete subgroup of $PSL(2, \mathbb{C})$.*

This theorem also claims that if A and B are non-elementary discrete subgroup of $PSL(2, \mathbb{C})$, then they cannot fix a same point in the interior $\mathbb{H}^3$. Otherwise, the group $\langle A, B \rangle$ would be elementary by case III in the definition 4.1. Moreover, suppose A_n and B_n are sequences of non-elementary discrete $PSL(2, \mathbb{C})$ that converges to A and B respectively. If A and B fix the same point in the interior $\mathbb{H}^3$, then the group $\langle A_n, B_n \rangle$ eventually fails to be non-elementary and discrete.

Lemma 5.3. *If G is a non-elementary discrete subgroup of $PSL(2, \mathbb{C})$ that contains no elliptics, then the following properties holds:*

- (1) G is infinite;

- (2) for any nontrivial $A \in G$, there exists a loxodromic $B \in G$ that has no common fixed points with A ;
- (3) If $B \in G$ is loxodromic, then there is no nontrivial $C \in G$ that has exactly one fixed point in common with B ;
- (4) G contains two loxodromic elements with no fixed points in common.

The proof of the lemma is quite technical so we omit it here. Also it can be found in the original book [1]. The idea that underlies those properties is to avoid cases I and II in the definition 4.1. Since we know exactly that case I elementary group is generated by parabolics fixing the same point and case II elementary group is generated by a single loxodromic transformation, we shall add additional transformations with different fixed points to make the group non-elementary. Properties (2), (3), and (4) summarize different results after this addition procedure.

Now we are ready to prove the universal elementary neighborhood.

Proof. We use the same notation with the book: for fixed $r > 0$, let $G(r, x)$ denotes the subgroup of G generated by all elements of G that translate x distance less than r , i.e.

$$G(r, x) = \{A \in G \mid d(x, Ax) < r\}$$

The group generated by $G(r, x)$ is denoted by $\langle G(r, x) \rangle$.

The first step is to observe that every point has a elementary neighborhood since G is discrete. That is, for a fixed point $x \in \mathbb{H}^3$, if r is sufficiently small, then $\langle G(r, x) \rangle$ is elementary. Here we allow r to be dependent on x . Since G is discrete, then we have minimal distance $\inf d(x, Ax) > 0$ so that eventually $\langle G(r, x) \rangle$ contains only the identity map, which is elementary.

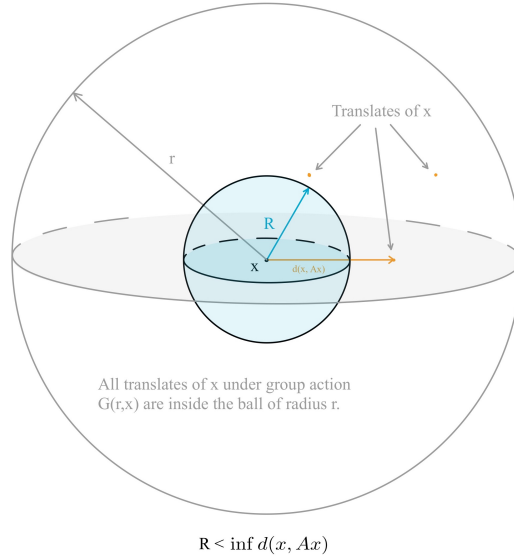


FIGURE 7. Illustration for the first step

The second step is to show the lower bound of r , so we know that $\langle G(r, x) \rangle$ becoming elementary is independent with the choice of G and x .

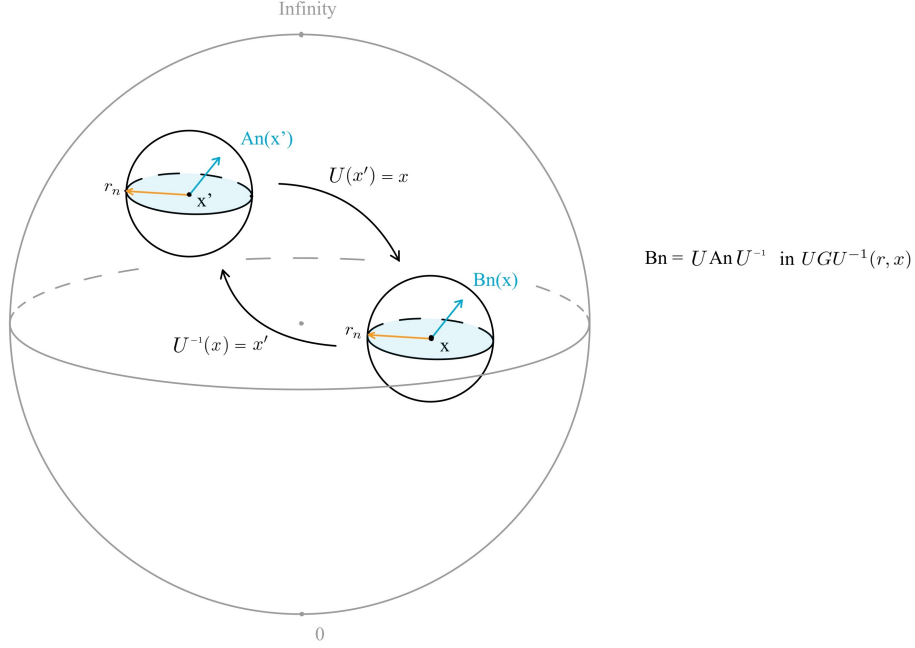


FIGURE 8. Illustration for the conjugation in the second step

Observe that the freedom in the choice of x can be reflected the freedom in determining G by conjugation. This is because $\langle G'(r, x') \rangle$ is elementary if and only if $\langle UGU^{-1}(r, x) \rangle$ is elementary, where U is the isometry that maps $x' \mapsto x$.

Now we are ready to show that the lower bound of r is independent of G . We will prove it by contradiction. Fix $x \in \mathbb{H}^3$. Suppose the statement is not true, then we can find sequences of r_n and G_n such that $r_n \rightarrow 0$ and $\langle G_n(r_n, x) \rangle$ is non-elementary. Next, we want to find sequences $A_n, B_n \in G_n(r_n, x)$ such that $\langle A_n, B_n \rangle$ is non-elementary. We will formulate the construction later using proposition 5.4, but for now let us assume the construction is true. Since $r_n \rightarrow 0$ so that $A_n(x)$ converges to x but not some points on the boundary $\partial\mathbb{H}^3$, then A_n should have a convergent subsequence that converges to some $A \in \text{PSL}(2, \mathbb{C})$. Note that A should fix x since

$$A(x) = \lim_{n \rightarrow \infty} A_n(x) = x$$

Similar situation applies to the sequence $\{B_n\}$: there is a subsequence of $\{B_n\}$ converges to some $B \in \text{PSL}(2, \mathbb{C})$ such that B fixes x as well. Since $x \in \mathbb{H}^3$ is fixed by all elements in $\langle A, B \rangle$, then by definition $\langle A, B \rangle$ should be elementary. However, we also have $\langle A, B \rangle$ is non-elementary according to theorem 5.2. This gives a contradiction.

The construction of $\{A_n\}$ and $\{B_n\}$ are as follow: we choose r_n such that

$$r_n = \inf\{r > 0 \mid \langle G_n(r, x) \rangle \text{ is nonelementary}\}$$

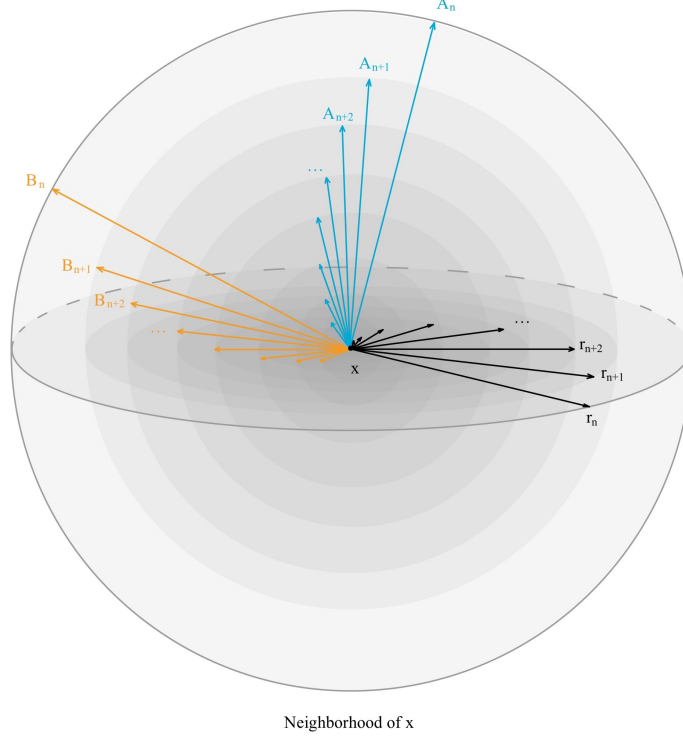


FIGURE 9. Illustration for the convergence of A_n and B_n in the third step.

We know that $r_n > 0$ from the first step. Also, our choice of r_n does not affect our construction since it depends on G_n . Take $d_n < r_n$ for all $n \in \mathbb{N}$, then $\langle G_n(d_n, x) \rangle$ should be elementary case I or II. In both situations lemma 5.4 allows us to find $A_n, B_n \in G_n(r_n, x)$ such that $\langle A_n, B_n \rangle$ is nonelementary. $\square$

The existence of universal elementary neighborhood allows us to separate a hyperbolic 3-manifolds into thick parts and thin parts according to the universal constant. It also gives a concrete description of the structure of each ϵ -thin part. This result is summarized in the following theorem. The proof of the theorem can be found in *Hyperbolic Knot Theory* [1].

Theorem 5.4. *There exists a universal constant $\epsilon_3 > 0$ such that for $0 < \epsilon < \epsilon_3$, the ϵ -thin part of any complete orientable, hyperbolic 3-manifold M consists of tubes around short geodesics, rank-1 cusps, and/or rank-2 cusps.*

6. ACKNOWLEDGMENTS

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